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SHORT COMMUNICATIONS

A Classical Solution, Weakened on the Axis, of a Centrally Symmetric Mixed Problem for a Three-Dimensional Hyperbolic Equation in Hölder Spaces

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In the cylinder $Q = G \times (0,T)$, $G = \{x \in \mathfrak{R}^3 : \ r = |x| < R\}$, we consider the centrally symmetric (with respect to space variables) mixed problem

$$\begin{gathered}\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) + b(r,t)\sum_{i=1}^{3} x_i \frac{\partial u(x,t)}{\partial x_i} \\ + c(r,t)\frac{\partial u(x,t)}{\partial t} + q(r,t)u(x,t) = 0, \quad (x,t) \in Q,\end{gathered} \tag{1}$$

$$u(x,0) = \varphi(r), \qquad \partial u(x,0)/\partial t = \psi(r), \qquad 0 \le r \le R, \tag{2}$$

$$u(x,t)|_\Gamma = 0, \qquad 0 \le t \le T, \tag{3}$$

where

$$\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2, \qquad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and $\Gamma = \left\{(x,t) \in \bar{Q} : \ r = R\right\}$ is the lateral surface of the cylinder.

The central symmetry of the problem with respect to the space variables exhibits itself in the dependence of the initial functions φ and ψ and the coefficients of Eq. (1) on x, in the Laplace operator, and in the geometry of Q. For the Cauchy problem (1), (2) with $b \equiv c \equiv q \equiv 0$, it was shown in [1, pp. 326, 327] that, in view of the focusing singularity at the point $r = 0$, one must impose the conditions $\varphi \in C^3$ and $\psi \in C^2$ to provide the existence of a classical solution. Similar examples were also constructed, e.g., in [2, p. 213; 3, pp. 247, 251–254]. In these examples, the conditions imposed on the smoothness of the functions φ and ψ to guarantee the existence of a classical solution were stated in various classes but cannot be made necessary. To ensure that the conditions imposed on the smoothness of the functions φ and ψ and providing the existence of a solution coincide with necessary conditions, one has to modify [4] the notion of a classical solution, so that some growth of second-order derivatives as $|x| \to 0$ is allowed instead of continuity. Let us recall the corresponding definition and the main result of [4].

Definition. A *classical solution of problem* (1)–(3) *weakened on the axis* $r = 0$ is a function $u \in C^1\left(\bar{Q}\right) \cap C^2\left(\bar{Q} \backslash \{0\} \times [0,T]\right)$ that makes Eq. (1) an identity in the cylinder $\bar{Q}\backslash\{0\} \times [0,T]$ with the deleted axis and satisfies conditions (2) and (3) in the ordinary sense and the conditions

$$\lim_{|x|\to 0} |x|\Delta u(x,t) = 0, \qquad \lim_{|x|\to 0} \sum_{i=1}^{3} x_i \frac{\partial^2 u}{\partial x_i \partial t} = 0 \tag{4}$$

on the axis $r = 0$.

The notion of a classical solution weakened on the axis $r = 0$ proves to be convenient, and it was shown in [4] that if the coefficients b, c, and q of Eq. (1) are continuous in Q, their derivatives

$\partial b/\partial r$, $\partial c/\partial r$, and $\partial q/\partial r$ are bounded in $\bar{Q}$, and the coefficient b satisfies the matching condition $b(R,t)=0$, then for the existence of a classical solution weakened on the axis $r=0$, it is necessary and sufficient that the initial functions φ and ψ satisfy the conditions

$$\varphi(r)\in C^1[0,R]\cap C^2(0,R], \qquad \varphi(R)=\Delta_r\varphi(R)=0,$$
$$\lim_{r\to 0} r\Delta_r\varphi(r)=0 \qquad \left(\Delta_r\varphi=\frac{d^2\varphi}{dr^2}+2r^{-1}\frac{d\varphi}{dr}\right), \tag{5}$$
$$\psi(r)\in C[0,R]\cap C^1(0,R], \qquad \psi(R)=0, \qquad \lim_{r\to 0}(r\,d\psi/dr)=0. \tag{6}$$

The aim of the present research is to derive similar results for a weakened classical solution of Eq. (1) with coefficients $b=c=0$ in Hölder spaces. Such results obtained for the case $q=0$ were reported at the VI Conference of Mathematicians of Belarus [5]. Recall that a function $f\in C_\alpha^m(\Omega)$ belongs to the Hölder space $C_\alpha^m(\Omega)$ with exponent α, $0<\alpha\le 1$, if the inequality

$$\left|\frac{\partial^m f(x'')}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}-\frac{\partial^m f(x')}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}\right|\le C\,|x''-x'|^\alpha$$

is valid for any points $x', x''\in\Omega$, where $\beta_1+\beta_2+\dots+\beta_n=m$ and C is a constant independent of x' and x''.

In problem (1)–(3), we pass to spherical coordinates. Since the weakened (on the axis $|x|=r=0$) classical solution $u(x,t)$ of the mixed problem (1)–(3) is unique, and since the problem data are symmetric with respect to x, it follows that the solution is also symmetric with respect to x, i.e., $u(x,t)=u(r,t)$. As a result, since $b=c=0$ in Eq. (1), we have the following mixed problem for the function $u(r,t)$ in the rectangle $\bar{Q}=[0,R]\times[0,T]$:

$$\partial^2u(r,t)/\partial t^2-\Delta_r u(r,t)+q(r,t)u(r,t)=0, \tag{7}$$
$$u(r,0)=\varphi(r), \qquad \partial u(r,0)/\partial t=\psi(r), \qquad 0\le r\le R, \qquad u(R,t)=0. \tag{8}$$

Condition (4) acquires the form

$$\lim_{r\to 0} r\Delta_r u(r,t)=0, \qquad \lim_{r\to 0}\left(r\,\partial^2u(r,t)/\partial r\,\partial t\right)=0. \tag{9}$$

Theorem. *Let the derivative $\partial q(r,t)/\partial r$ belong to $C_\alpha\left(\bar{Q}\right)$ and tend to zero at the rate of r^α and $(R-r)^\alpha$ as $r\to 0$ and $r\to R$, respectively. A weakened (on the axis $|x|=r=0$) classical solution of problem* (7), (8) *exists, belongs to the Hölder space $C_\alpha^2((0,R]\times[0,T])$, and satisfies the conditions*

$$\sup_{0\le r\le R,\,0\le t\le T}\left|r^{1-\alpha}\Delta_r u(r,t)\right|<\infty, \qquad \sup_{0\le r\le R,\,0\le t\le T}\left|r^{1-\alpha}\partial^2u/\partial r\,\partial t\right|<\infty \tag{10}$$

if and only if the functions φ and ψ satisfy conditions (5) *and* (6), *belong to the Hölder spaces $\varphi\in C_\alpha^2(0,R]$ and $\psi\in C_\alpha^1(0,R]$, and satisfy the conditions*

$$\sup_{0\le r\le R}\left|r^{1-\alpha}\Delta_r\varphi(r)\right|<\infty, \qquad \sup_{0\le r\le R}\left|r^{1-\alpha}\psi'(r)\right|<\infty. \tag{11}$$

Proof. Necessity. This is straightforward.

Sufficiency. As was mentioned above, it had been shown in [4] that if the functions φ and ψ satisfy conditions (5) and (6), then there exists a weakened (on the axis $|x|=r=0$) classical solution of problem (7), (8) in the form $u(r,t)=v(r,t)/r$, where $v(r,t)$ is a classical solution of the mixed problem

$$\partial^2v(r,t)/\partial t^2-\partial^2v(r,t)/\partial r^2+q(r,t)v(r,t)=0, \qquad (r,t)\in(0,R)\times(0,T), \tag{12}$$
$$v(r,0)=\Phi(r)=r\varphi(r), \qquad \partial v(r,0)/\partial t=\Psi(r)=r\psi(r), \qquad 0\le r\le R, \tag{13}$$
$$v(0,t)=0, \qquad v(R,t)=0, \tag{14}$$

which admits the representation

$$v(r,t)=\frac{\tilde\Phi(r+t)+\tilde\Phi(r-t)}{2}+\frac{1}{2}\int\limits_{r-t}^{r+t}\tilde\Psi(\xi)d\xi+\frac{1}{2}\int\limits_{r-(t-\tau)}^{r+(t-\tau)}\hat q(\xi,\tau)\tilde v(\xi,\tau)d\xi\,d\tau. \tag{15}$$

Here the symbols $\tilde\Phi$ and $\tilde\Psi$ stand for the functions obtained from the functions Φ and Ψ as odd extensions from the closed interval $[0,R]$ to $[-R,R]$ and further as $2R$-periodic extension to the entire real line. Likewise, $\tilde v$ stands for the function obtained as the odd $2R$-periodic extension of the function v with respect to the variable r from the rectangle $\bar Q$ to $\mathfrak{R}\times[0,T]$. The coefficient q has been extended as an even function with respect to the variable r from $\bar Q$ to $[-R,R]\times[0,T]$ and then as a $2R$-periodic function of r to the entire strip $\mathfrak{R}\times[0,T]$. This even $2R$-periodic (with respect to r) extension of the function q is denoted by $\hat q$.

Since $\Phi\in C^2[0,R]$, $\Psi\in C^1[0,R]$, $\varphi\in C^2_\alpha(0,R]$, $\psi\in C^1_\alpha(0,R]$, and condition (11) is satisfied, we have $\tilde\Phi\in C^2_\alpha(\mathfrak{R})$ and $\tilde\Psi\in C^1_\alpha(\mathfrak{R})$. In addition, the last term on the right-hand side in (15) belongs to $C^2_\alpha(\mathfrak{R}\times[0,T])$. Consequently, it follows from (5) that $v\in C^2_\alpha\left(\bar Q\right)$.

We use the representation (15) and show that the function $u(r,t)=v(r,t)/r$ belongs to $C^2_\alpha((0,R]\times[0,T])$ and satisfies condition (10).

Since $v\in C^2_\alpha\left(\bar Q\right)$, we have $u\in C^2_\alpha((0,R]\times[0,T])$. Let us now prove the validity of condition (10). It follows from the representation (15) that

$$\begin{aligned}\left|r^{1-\alpha}\Delta_r u(r,t)\right|&=\left|\frac{1}{r^\alpha}\frac{\partial^2 v(r,t)}{\partial r^2}\right|\le\left|\frac{\tilde\Phi''(t+r)-\tilde\Phi''(t-r)}{2r^\alpha}\right|+\left|\frac{\tilde\Psi'(t+r)-\tilde\Psi'(t-r)}{2r^\alpha}\right|\\&\quad+\frac{1}{2r^\alpha}\int\limits_0^t\left|\frac{\partial}{\partial r}\left[\hat q(r+(t-\tau),\tau)\tilde v(r+(t-\tau),\tau)\right]\right.\\&\quad\left.-\frac{\partial}{\partial r}\left[\hat q(r-(t-\tau),\tau)\tilde v(r-(t-\tau),\tau)\right]\right|d\tau,\end{aligned}\tag{16}$$

$$\begin{aligned}\left|r^{1-\alpha}\frac{\partial^2 u(r,t)}{\partial r\,\partial t}\right|&=\frac{1}{r^\alpha}\left|\frac{\partial^2 v(r,t)}{\partial r\,\partial t}-\frac{1}{r}\frac{\partial v(r,t)}{\partial t}\right|\le\left|\frac{\tilde\Phi''(t+r)+\tilde\Phi''(t-\tau)}{2r^\alpha}-\frac{\tilde\Phi'(t+r)-\tilde\Phi'(t-r)}{2r^{1+\alpha}}\right|\\&\quad+\left|\frac{\tilde\Psi'(t+r)+\tilde\Psi'(t-r)}{2r^\alpha}-\frac{\tilde\Psi(t+r)-\tilde\Psi(t-r)}{2r^{1+\alpha}}\right|\\&\quad+\frac{1}{2r^\alpha}\int\limits_0^t\left|\frac{\partial}{\partial r}\left[\hat q(r+(t-\tau),\tau)\tilde v(r+(t-\tau),\tau)+\hat q(r-(t-\tau),\tau)\tilde v(r-(t-\tau),\tau)\right]\right.\\&\quad\left.-\frac{1}{r}\left[\hat q((t-\tau)+r,\tau)\tilde v((t-\tau)+r,\tau)+\hat q((t-\tau)-r,\tau)\tilde v((t-\tau)-r,\tau)\right]\right|d\tau.\end{aligned}\tag{17}$$

Since $\tilde\Phi\in C^2_\alpha(\mathfrak{R})$, $\tilde\Psi\in C^1_\alpha(\mathfrak{R})$, and $\partial\left(\hat q\tilde v\right)/\partial r\in C_\alpha(R\times[0,T])$, it follows from the Taylor formula with remainder in Peano's form that

$$\begin{aligned}&\tilde\Phi'(t\pm r)=\tilde\Phi'(t)\pm\tilde\Phi''(t)r+\Theta_1\left(t,r^\alpha\right),\qquad\tilde\Psi(t\pm r)=\tilde\Psi(t)\pm\tilde\Psi'(t)r+\Theta_2\left(t,r^\alpha\right),\\&\hat q((t-r)\pm r,\tau)\tilde v((t-\tau)\pm r,\tau)=\hat q((t-\tau),\tau)\tilde v((t-\tau),\tau)\\&\qquad\qquad\pm(\partial/\partial t)\left[\hat q((t-\tau),\tau)\tilde v((t-\tau),\tau)\right]r+\Theta_3\left(t,r^\alpha\right),\end{aligned}\tag{18}$$

$$\sup_{0\le t\le T,\,0\le r\le R}\frac{\Theta_i(t,r^\alpha)}{r^\alpha}\le C,\qquad i=1,2,3,\tag{19}$$

for small r. Consequently, relations (17) and (16), together with (18) and (19), imply condition (10). The proof of the theorem is complete.

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