ALGORITHMS FOR CONSTRUCTION OF OPTIMAL AND SUBOPTIMAL SOLUTIONS IN NETWORK OPTIMIZATION PROBLEMS

L.A. Pilipchuk[1], A.S. Pilipchuk[2], Y.H. Pesheva[3]

[1]L.A. Pilipchuk Belarussian State University,
Nezalezhnosti Ave.4, 220050 Minsk, BELARUS
E-mail: pilipchuk@bsu.by

[2]A.S. Pilipchuk State Scientific Institution
Academician A.K. Krasin Str. 99
220109 Minsk, BELARUS
e-mail: pilip74@rambler.ru

[3]Y.H. Pesheva Technical University of Sofia
P.O. Box 384 1000 Sofia, BULGARIA
e-mail:yhp@vmei.acad.bg

For a distributive flow programming optimization problem of a special structure direct and dual algorithms are constructed. These algorithms are based on a research of the theoretical and graph properties of the solution space bases. Optimality conditions are received, that allow to calculate a part of the components of the Lagrange vector. Algorithms that decompose calculation systems for pseudo-plans of the problem are presented. Suitable directions for change of the dual criterion function are constructed.

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Introduction

Many problems in linear and nonlinear programming [1,2] have in details developed theory and comprehensively investigated methods of solution. However the process of allocation of special problems in separate classes [3] and the development of new solution methods for them is a problem of interest. It is connected with the fact that each class of problems is characterized by a special structure of the set of parameters and restrictions. The investigated problem concerns the class of network optimization problems and has additional parameters, such as the variables of intensity of units, transformation factors of an arc flow (the generalized network), interrelation of the plan components (additional restrictions). Application to the solution of network optimization problems of special algorithms which are taking into account features of a problem, its theoretical and graph properties [3,4], modern achievements in numerical realization of flow programming problems allows to solve high dimensional problems.

1. Statement of the problem, basic concepts and definitions.

Consider the following mathematical model of an extreme network problem

$$\varphi(x) = \sum_{(i,j)\in U} c_{ij}x_{ij} + \sum_{j\in I^*} c_i x_i \longrightarrow \min, \tag{1}$$

$$\sum_{(i,j)\in I_i^+(U)} x_{ij} - \sum_{(i,j)\in I_i^-(U)} \mu_{ji}x_{ji} = \begin{cases} x_i \mathrm{sign}(i), & i \in I^*, \\ a_i, & i \in I \setminus I^*, \end{cases} \tag{2}$$

$$\sum_{(i,j)\in U} \lambda_{ij}^p x_{ij} + \sum_{i\in I^*} \lambda_i^p x_i = \beta^p, \quad p = \overline{1,q}, \tag{3}$$

$$0 \le x_{ij} \le d_{ij}, (i,j) \in U, \tag{4}$$

$$b_{*i} \le x_i \le b_i^*, i \in I^*, \tag{5}$$

where $G = (I, U)$ – a finite orientated generalized network without multiple arcs and loops with set of nodes I and set of arcs U , $|U| \gg | I |$, $I_i^+(U) = \{j : (i, j) \in U\}$, $I_i^-(U) = \{j : (i, j) \in U\}$, x_{ij} – a flow along the arc (i, j); μ_{ij} – transformation factor of the flow along arc (i, j), $\mu_{ij} \in]0, 1]$, I^* – a subset of I,$I^* \neq \oslash$. Nodes $i \in I^*$ are named dynamic, $sign(i) = 1$, if $i \in I^*_+$; $sign(i) = -1$, if $i \in I^*_-$; $I^*_+, I^*_- \subseteq I^*, I^*_+ \bigcap I^*_- = \oslash$

Any node is a point of production (or consumption) of some item, x_{ij} – a flow on arc $(i, j), \mu_{ij}$ – factor of transportation losses of goods from node i to node j on arc $(i, j), I^*$ – set of nodespoints of manufacture (consumption) with a preliminary unknown volume x_i of the manufactured (consumed) goods, a_i – volume of production (consumption) at node $i \in I \setminus I^*$.

Vector $x = (x_{ij}, (ij) \in U, x_i, i \in I^*)$ – is a plan, if on it restrictions (2) – (5) of the problem are carried out. A plan $x^0 \in X$ is optimal, if $c'x^0 = \min c'x, x \in X$, where X is the set of plans. If $\varepsilon \geq 0$, a suboptimal (ε-optimal) plan x^ε is defined by an inequality

$$c'x^\varepsilon - c'x^0 \leq \varepsilon, \quad x^\varepsilon \in X.$$

We shall name restrictions (2) a network part of system of basic restrictions (2), (3), restrictions (3) – additional basic restrictions, restrictions (4), (5) – direct restrictions.

2. A network part of the system of basic restrictions, support criterion.

Let's define a support of a network $G = (I, U)$ for system (2).

Definition 1. *The set of sets $R = \{U_R, I^*_R\}, U_R \subseteq U, I^*_R \subseteq I^*)$ is named a support of a network $G = (I, U)$ for system (2), if at $\widetilde{U}_R = U_R, \widetilde{I}^*_R = I^*_R$ system*

$$\sum_{j \in I_i^+(\widetilde{U}_R)} x_{ij} - \sum_{j \in I_i^-(\widetilde{U}_R)} \mu_{ji} x_{ji} = \begin{cases} x_i sign(i), i \in \widetilde{I}^*_R, \\ 0, i \in I \setminus \widetilde{I}^*_R \end{cases} \tag{6}$$

has only a zero solution, and has a nonzero solution in each of the following cases:

1) $\widetilde{U}_R = U_R \bigcup (i, j), \widetilde{I}^*_R = I^*_R$, *for any arc* $(i, j) \in U \setminus U_R$;

2) $\widetilde{U}_R = U_R, \widetilde{I}^*_R = I^*_R \bigcup \{i\}$, *for any node* $i \in I^* \setminus I^*_R$.

Theorem 1. *(Network support criterion). The set of sets* $R = \{U_R, I_R^*\}$ *is a support of a network* $G = (I, U)$ *for system (2), if and only if the following conditions are fulfilled:*

1) $|U_R| + |I_R^*| = |I|$;

2) *any component of the connectivity* $G_R^l = \{I^l, U_R^l\}, I^l = I(U_R^l), l = \overline{1, s}$, *(s-number of the components of connectivity) of the network* $G_R = (I, U_R)$ *is a network of one of the following types:*

a) does not contain cycles (is a tree), and contains a unique node from set I_R^l, $|I_R^l \bigcap I_R^*| = 1$;

b) Contains a unique nondegenerate cycle and does not contain nodes from set I^*, $I_R^l \bigcap I_R^* = \oslash; l = \overline{1, s}$.

Proof of theorem 1 is given in [5].

Definition 2. *A cycle is called nonsingular (nondegenerate), if the product of transformation factors of arc flows for its forward arcs is not equal to the product of factors of transformation of arc flows for its backward arcs.*

3. Fundamental system of elements.

On the basis of theorem 1 we shall choose a support $R = \{U_R, I_R^*\}$ of network G for system (2). Consider the construction of the generated characteristic vectors [4], which make fundamental system of solutions of system (7):

$$\sum_{j \in I_i^+(U)} x_{ij} - \sum_{j \in I_i^-(U)} \mu_{ji} x_{ji} = \begin{cases} x_i sign[i], i \in I^*, \\ 0, i \in I \setminus I^*. \end{cases} \tag{7}$$

The theoretical and graph structure of support $R = \{U_R, I_R^*\}$ of network G for the network part of system of basic restrictions (2) is a set of s – components of connectivity

$$G^l = (I^l, U_R^l), \bigcup_{l=1}^{s} I^l = I, I^l = I(U_R^l), \quad l = \overline{1, s} \quad I_R^* = \bigcup_{l=1}^{s} I_R^l,$$

each of which is either a tree containing a unique dynamic node from set $i \in I_R^*$, $|I_R^l \bigcap I^l| = 1$, or does not contain dynamic nodes $I_R^l \bigcap I^l = \oslash$ and contains a unique nondegenerate cycle. On the basis of network properties of the support (theorem 1) theoretical and graph properties of the fundamental system of solutions of system (7) are considered in work [4] and the general solution of linear system (2) is constructed. We shall designate through $\delta_{\tau\rho} = (\delta_{ij}^{\tau\rho}, (i,j) \in U; \delta_i^{\tau\rho}, i \in I^*)$ and $\delta_\gamma = (\delta_{ij}^{\gamma}, (i,j) \in U; \delta_i^{\gamma}, i \in I^*)$ the characteristic vectors generated by arcs $(\tau,\rho), (\tau,\rho) \in U \backslash U_R$, and nodes $\gamma, \gamma \in I^* \backslash I_R^*$ accordingly, making fundamental system of solutions of system (7) [4,6].

Consider the graph $\widetilde{G} = (I, U_R \bigcup (\tau,\rho)), (\tau,\rho) \in U \backslash U_R, I_R^* \subseteq I$, where set of sets $R = \{U_R, I_R^*\}$ forms a support of network $G = (I,U)$ for system (2). Among the components of connectivity of network $\widetilde{G} = (I, U_R \bigcup (\tau,\rho))$ there is a unique component of connectivity (we shall designate it) which is a network of one of the following types:

a) contains a unique nondegenerate cycle and a unique node from set I_R^*, $|I^l \bigcap I_R^*| = 1$;

b) contains a circuit connecting two nodes from set I_R^*, $|I_R^l \bigcap I_R^*| = 2$;

c) contains two cycles, at least one of which is non degenerate, and does not contain nodes from set I_R^*, $|I_R^l \bigcap I_R^*| = \oslash$.

Theorem 2. *Among the components of the characteristic vector $\delta_{\tau\rho}$ generated by arc $(\tau,\rho) \in U \backslash U_R$ or components of the characteristic vector δ_γ generated by node $\gamma \in I^* \backslash I_R^*$, nonzero can be only those components of the specified vectors which correspond to elements (arcs and dynamic nodes) of sets $U_R \bigcup (\tau,\rho)$ and I_R^* and belong to the following network structures:*

a) the unique circuit connecting a dynamic node (we shall designate it u) from set I_R^* and a node υ of a cycle;

b) the unique circuit connecting nodes u and υ from set I_R^*;

c) two cycles and a unique circuit which connects nodes of the specified cycles.

The proof of the theorem is given in [4].

4. Decomposition of system of restrictions.

Theorem 3. *The general solution of system (2) can be presented [4] in the following form:*

$$x_{ij} = \sum_{(\tau,\rho) \in U \backslash U_R} x_{\tau\rho} \delta_{ij}^{\tau\rho} + \sum_{\gamma \in I^* \backslash I_R^*} x_\gamma \delta_{ij}^{\gamma} + \widetilde{x}_{ij}, (i,j) \in U_R, \tag{8}$$

$$x_i = \sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho}\delta_i^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \delta_i^\gamma + \widetilde{x}_i, i \in I_R^*,$$

where $\widetilde{x} = (\widetilde{x}_{ij}, (i,j) \in U; \widetilde{x}_i, i \in I^*)$ *– any particular solution of system (2).*

Theorem 4. *(network support criterion). The aggregation of sets* $K = \{U_K, I_K^*\}$ *is a support of a network* $G = (I,U)$ *for system (2), (3) if and only if the following conditions are executed:*

1) The aggregation of sets $K = \{U_K, I_K^*\}$ *can be presented as an association of not intersecting sets of sets* $R = \{U_R, I_R^*\}$, $W = \{U_W, I_W^*\}$ *and* $U_K = U_R \cup U_W$ $U_R \cap U_W = \oslash$ $I_K^* = I_R^* \cup I_W^*, I_R^* \cap I_W^* = \oslash$;

2) The aggregation of sets $R = \{U_R, I_R^*\}$ *– is a support of the network* $G = (I,U)$ *for system (2) (theorem 1).*

3) $|W| = q$ *, where* q *– the number of linearly independent equations of system (3).*

4) The matrix of determinants $\Lambda_W = (\Lambda_{W_1}, \Lambda_{W_2})$*, is nonsingular,*

$$\Lambda_{\tau\rho}^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \sum_{i)\in I_R^*} \lambda_i^p \delta_i^{\tau\rho} + \lambda_{\tau\rho}^p,$$

$$\Lambda_{W_1} = (\Lambda_{\tau\rho}^p, p = \overline{1,q}; (\tau,\rho) \in U_W), \; \Lambda_{W_2} = (\Lambda_\gamma^p, p = \overline{1,q}; \gamma \in I_W^*),$$

$$\Lambda_\gamma^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^\gamma + \sum_{i)\in I_R^*} \lambda_i^p \delta_i^\gamma + \lambda_\gamma^p.$$

On the basis of theorem 4 we execute decomposition of system (2),(3):

$$\sum_{(\tau,\rho)\in U\setminus U_R} \Lambda_{\tau\rho}^p x_{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} \Lambda_\gamma^p x_\gamma = \widetilde{\beta}^p, p = \overline{1,q}, \tag{9}$$

$$\widetilde{\beta}^p = \beta^p - \sum_{(i,j)\in U_R} \lambda_{ij}^p \widetilde{x}_{ij} - \sum_{(k)\in U_R} \lambda_k^p \widetilde{x}_k;$$

$$\Lambda_{\tau\rho}^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \sum_{i)\in I_R^*} \lambda_i^p \delta_i^{\tau\rho} + \lambda_{\tau\rho}^p,$$

(10)

$$\Lambda^p_\gamma = \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^\gamma_{ij} + \sum_{i)\in I^*_R} \lambda^p_i\delta^\gamma_i + \lambda^p_\gamma.$$

We shall name quantities $\Lambda^p_{\tau\rho}$, Λ^p_γ determinants of the network structures making fundamental system of solutions of system (7).They are generated by arcs $(\tau,\rho) \in U\backslash U_R$ and nodes $\gamma \in I^*\backslash I^*_R$ accordingly, corresponding to the $p-th$ restriction of system of additional restrictions (3), $p = \overline{1,q}$.
Consider variables corresponding to aggregation of sets W. The system (9) will become

$$\Lambda_W x_W = \widetilde{\beta} - \Lambda_N x_N, \tag{11}$$

$$x_W = (x_{\tau\rho}, (\tau,\rho) \in U_W; x_\gamma, \gamma \in I^*_W),$$

$$x_N = (x_{\tau\rho}, (\tau,\rho) \in U_N; x_\gamma, \gamma \in I^*_N).$$

The system (11) has a unique solution due to nonsingularity of the matrix Λ_W.

The support components are calculated according to (8).

5. Increment of the objective function.

Pair $\{x, K\}$ of a plan and a support is called a support plan. Support plan $\{x, K\}$ is nonsingular if it satisfies the following conditions:

$$0 < x_{ij} < d_{ij},\ (i,j) \in U_K;\ a_{*i} < x_i < a^*_i,\ i \in I^*_K \tag{12}$$

Alongside with plan x we consider plan $\overline{x}$, $\Delta x = \overline{x} - x$. From [4,7] it follows that for the increment $\Delta x = (\Delta x_{ij}, (i,j) \in U; \Delta x_i$, $i \in I^*)$ the following relations are carried out:

$$\Delta x_{ij} = \sum_{(\tau,\rho)\in U\backslash U_R} \delta^{\tau\rho}_{ij}\Delta x_{\tau\rho} + \sum_{\gamma\in I^*\backslash I^*_R} \delta^\gamma_{ij}\Delta x_\gamma, (i,j) \in U_R,$$

$$\Delta x_i = \sum_{(\tau,\rho)\in U\backslash U_R} \delta^{\tau\rho}_i\Delta x_{\tau\rho} + \sum_{\gamma\in I^*\backslash I^*_R} \delta^\gamma_i\Delta x_\gamma, i \in I^*_R. \tag{13}$$

On the basis of (13) for the increment Δx we receive:

$$\sum_{(\tau,\rho)\in U\backslash U_R} \Lambda^p_{\tau\rho}\Delta x_{\tau\rho} + \sum_{\gamma\in I^*\backslash I^*_R} \Lambda^p_{\gamma}\Delta x_{\gamma} = 0, p = \overline{1,q}, \tag{14}$$

Let's write down (13), (14) in a matrix form:

$$\Delta x_R = S_W \Delta x_W + S_N \Delta x_N;\ \Delta_W \Delta x_W = -\Delta_N \Delta x_N, \tag{15}$$

where

$$\Delta x_W = \{\Delta x_{ij}, (i,j) \in U_W; \Delta x_i, i \in I^*_W\},$$
$$\Delta x_R = (\Delta x_{ij}, (i,j) \in U_R; \Delta x_i, i \in I^*_R),$$
$$\Delta x_N = \{\Delta x_{ij}, (i,j) \in U_N; \Delta x_i, i \in I^*_N\},$$
$$U_K = U_R \cup U_W, I^*_K = I^*_R \cup I^*_W,$$
$$U_N = U\backslash U_K, I^*_N = I^*\backslash I^*_K,$$
$$S_W = (\delta_{\tau\rho}, (\tau,\rho) \in U_W; \delta_\gamma, \gamma \in I^*_W),$$
$$S_N = (\delta_{\tau\rho}, (\tau,\rho) \in U_N; \delta_\gamma, \gamma \in I^*_N).$$

Matrices Δ_W and Δ_N consist of determinants of the structures generated by components of sets $W = \{U_W, I^*_W\}$, $N = \{U_N, I^*_N\}$ respectively. Set $W = \{U_W, I^*_W\}$ on the basis of which components of the vector Δx_W are constructed, is chosen so that $|\Delta_W| \neq 0$. As K is a support, from theorem 4, then $|\Delta_W| \neq 0$. Hence, from (15) vector Δx_W is calculated according to (16):

$$\Delta x_W = -\Delta_W^{-1}\Delta_N \Delta x_N \tag{16}$$

Let's calculate the increment of objective function (1):

$$\Delta\varphi(x) = \sum_{(ij)\in U} c_{ij}\Delta x_{ij} = \sum_{(\tau,\rho)\in U\backslash U_R} \Delta_{\tau\rho}\Delta x_{\tau\rho} + \sum_{\gamma\in I^*\backslash I^*_R} \Delta_\gamma \Delta x_\gamma,$$

using analytical expressions (13) for components of vector Δx_R .

Let's $r' = \Delta'_W \Lambda_W^{-1}, r = (r_1, r_2, \dots r_q)$,

$$\widetilde{\Delta}_N = (\widetilde{\Delta}_{\tau\rho}, (\tau,\rho) \in U_N; \widetilde{\Delta}_\gamma, \gamma \in I^*_N),$$

$$\widetilde{\Delta}_{\tau\rho} = \Delta_{\tau\rho} - \sum_{p=1}^{q} r_p \Lambda^p_{\tau\rho},\ (\tau,\rho) \in U_N,$$

$$\widetilde{\Delta}_\gamma = \Delta_\gamma - \sum_{p=1}^{q} r_p \Lambda_\gamma^p,\ \gamma \in I_N^*.$$

We shall name components of vector $\widetilde{\Delta}_N = (\widetilde{\Delta}_{\tau\rho}, (\tau,\rho) \in U_N;\ \widetilde{\Delta}_\gamma$, $\gamma \in I_N^*)$ estimations. On construction, $\widetilde{\Delta}_K = 0$, $K = \{U_K, I_K^*\}$. So, the increment of objective function (1) of problem (1) – (5) looks like

$$\Delta\varphi(x) = \sum_{(\tau,\rho)\in U_N} \widetilde{\Delta}_{\tau\rho} \Delta x_{\tau\rho} + \sum_{\gamma \in I_N^*} \widetilde{\Delta}_\gamma \Delta x_\gamma \tag{17}$$

6. Optimality criterion. An estimation of suboptimality.

Theorem 5. *For optimality of plan x it is necessary and enough that there exists such a support $K = \{U_K, I_K^*\}$ of network G for system (2) – (3) at which on the support plan $\{x, K\}$ the following conditions of minimum are satisfied*

$$\widetilde{\Delta}_{ij} x_{ij} = \min_{0 \le \omega \le d_{ij}} \widetilde{\Delta}_{ij}\, \omega,\ (i,j) \in U_N,$$

$$\tag{18}$$

$$\widetilde{\Delta}_i x_i = \min_{a_{*i} \le \upsilon \le a_i^*} \widetilde{\Delta}_i\, \upsilon,\ i \in I_N^*$$

The support on which the criterion of optimality (18) is carried out, is called optimal. The support is referred to as regular, if

$$\widetilde{\Delta}_{ij} \neq 0, (i,j) \in U_N, \widetilde{\Delta}_i \neq 0, i \in I_N^*.$$

Consider formula (17) for the increment of the objective function:

$$\Delta\varphi(x) = \sum_{(i,j)\in U_N} \widetilde{\Delta}_{ij} \Delta x_{ij} + \sum_{i \in I_N^*} \widetilde{\Delta}_i \Delta x_i =$$

$$= \sum_{(i,j)\in U_N} \widetilde{\Delta}_{ij} (\overline{x}_{ij} - x_{ij}) + \sum_{i \in I_N^*} \widetilde{\Delta}_i (\overline{x}_i - x_i)$$

Let's find the maximal decrease of the objective function:

$$\Delta\varphi(x) = \sum_{(i,j)\in U_N} \widetilde{\Delta}_{ij} (\overline{x}_{ij} - x_{ij}) + \sum_{i \in I_N^*} \widetilde{\Delta}_i (\overline{x}_i - x_i)$$

on the variables $\overline{x}_{ij}, (i,j) \in U_N, \overline{x}_i, i \in I_N^*$ satisfying the following restrictions:

$$0 \leq \overline{x}_{ij} \leq d_{ij}, (i,j) \in U_N;\ a_{*i} \leq \overline{x}_i \leq a_i^*, i \in I_N^*. \qquad (19)$$

We shall designate this minimum as follows $\beta(x,K) = \min \Delta\varphi(x)$ and name it an estimation of suboptimality (ε-optimality) [1,2] of the support plan $\{x, K\}$:

$$\beta(x,K) = \min_{(19)} \Delta\varphi(x) = \sum_{\substack{(i,j)\in U_N,\\ (19),\widetilde{\Delta}_{ij}<0}} \widetilde{\Delta}_{ij}(\overline{x}_{ij} - x_{ij}) +$$

$$+ \sum_{\substack{(i,j)\in U_N,\\ (19),\widetilde{\Delta}_{ij}>0}} \widetilde{\Delta}_{ij}(\overline{x}_{ij} - x_{ij}) + \sum_{\substack{i\in I^*,\\ (19),\widetilde{\Delta}_i<0}} \widetilde{\Delta}_i(\overline{x}_i - x_i) + \sum_{\substack{i\in I^*,\\ (19),\widetilde{\Delta}_i>0}} \widetilde{\Delta}_i(\overline{x}_i - x_i) =$$

$$= \sum_{(i,j)\in U_N,(19)} \widetilde{\Delta}_{ij}(z_{ij} - x_{ij}) + \sum_{i\in I_N^*,(19)} \widetilde{\Delta}_i(z_i - x_i) = c'z - c'x$$

The solution of the problem of minimization of $\triangle\varphi(x)$ under conditions (19) is a pseudo-plan $z = (z_{ij}, (i,j) \in U; z_i, i \in I^*)$ of problem (1) – (5). Components of vector z are calculated by rules: on arcs $(i,j) \in U_N$ and nodes $i \in I_N^*$ from the following conditions of optimality criterion:

$$z_{ij} = 0, \quad \text{if} \quad \widetilde{\Delta}_{ij} > 0; \qquad z_i = a_{*i}, \quad \text{if} \quad \widetilde{\Delta}_i > 0;$$

$$z_{ij} = d_{ij}, \quad \text{if} \quad \widetilde{\Delta}_{ij} < 0; \qquad z_i = a_i^* \quad \text{if} \quad \widetilde{\Delta}_i > 0;$$

$$z_{ij} \in [0, d_{ij}], \text{ if } \widetilde{\Delta}_{ij} = 0; (i,j) \in U_N; \ \ z_i \in [a_{*i}, a_i^*], \text{ if } \widetilde{\Delta}_i = 0, i \in I_N.$$

Components of vector $z_W = (z_{\tau\rho}, (\tau,\rho) \in U_W; z_\gamma, \gamma \in I_W^*)$ are calculated from the system

$$\Lambda_W z_W = \overline{\beta}, \quad \overline{\beta} = (\overline{\beta}^p, p = \overline{1,q}),$$

$$\overline{\beta}^p = \overline{\alpha}^p - \sum_{(\tau,\rho)\in U_N} \Lambda_{\tau\rho}^p z_{\tau\rho} - \sum_{\gamma\in I_N^*} \Lambda_\gamma^p z_\gamma,$$

$$\overline{\alpha}^p = \beta^p - \sum_{(i,j)\in U_R} \lambda_{ij}^p \widetilde{x}_{ij} - \sum_{i\in I_R^*} \lambda_i^p \widetilde{x}_i.$$

Components $z_R = (z_{\tau\rho}, (\tau,\rho) \in U_R; z_\gamma, \gamma \in I_R^*)$ are calculated from the system:

$$z_{ij} = \sum_{(\tau,\rho)\in U\setminus U_R} z_{\tau\rho}\delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} z_\gamma \delta_{ij}^{\gamma} + \widetilde{z}_{ij}, (i,j) \in U_R,$$

$$z_i = \sum_{(\tau,\rho)\in U\setminus U_R} z_{\tau\rho}\delta_{i}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} z_\gamma \delta_{i}^{\gamma} + \widetilde{z}_{i}, i \in I_R^*,$$

where $\widetilde{z} = (\widetilde{z}_{ij}, (i,j) \in U; \widetilde{z}_i, i \in I^*)$ – any particular solution of the network part (1.2) of system (2), (3) for the conjugate flow $\widetilde{z}$.

Theorem 6. *(ε - optimality criterion). At any $\varepsilon \geq 0$ for ε – optimality of plan x it is necessary and enough that a support K exists, at which $\beta(x,K) \leq \varepsilon$.*

As shown in [5], the estimation of suboptimality allows the following decomposition $\beta(x,K) = \mu(x)+\mu(K)$, where $\mu(x) = \varphi(x)-\varphi(x^0)$ – a measure for nonoptimality of plan x, $\mu(K) = \psi(\lambda^0)-\psi(\lambda)$ – a measure for nonoptimality of the support. Thus, it is possible to improve an estimation of ε - optimality by means of independent improvement of the flow and the support. We'll do the improvement of the support with the help of the dual support method.

7. The dual problem. Decomposition of the linear system for calculation of the conjugate flow.

The dual problem for problem (1) - (5) looks like:

$$\sum_{i\in I\setminus I^*} a_i y_i + \sum_{p=1}^{q} \beta^p \tau_p + \sum_{i\in I^*} a_{*i}\omega_i - \sum_{i\in I^*} a_i^* t_i - \sum_{(i,j)\in U} d_{ij} \upsilon_{ij} \longrightarrow \max,$$

$$y_i - \mu_{ij} y_j - \upsilon_{ij} + \sum_{p=1}^{q} \lambda_{ij}^p \tau_p \leq c_{ij}, (i,j) \in U,$$

$$-y_i sign(i) + \omega_i - t_i + \sum_{p=1}^{q} \lambda_i^p \tau_p = c_i, i \in I^*, \tag{20}$$

$$\upsilon_{ij} \geq 0, (i,j) \in U,$$

$$t_i \geq 0, \omega_i \geq 0, i \in I^*.$$

A vector

$$\lambda = (y, \tau, t, \omega, \upsilon) = (y_i, i \in I; \tau_k, k = \overline{1, q}; t_i, \omega_i, i \in I^*; \upsilon_{ij}, (i,j) \in U),$$

which components satisfy all restrictions of dual problem (20), is referred to as a dual plan of problem (1) - (5). We shall calculate the conjugate flow $\delta = (\delta_{ij}, (i,j) \in U; \delta_i, i \in I^*)$ corresponding to the dual plan λ by:

$$\delta_{ij} = c_{ij} - y_i + \mu_{ij} y_j - \sum_{p=1}^{q} \lambda_{ij}^p \tau_p \leq c_{ij}, (i,j) \in U,$$

$$\delta_i = c_i + y_i sign(i) - \sum_{p=1}^{q} \lambda_i^p \tau_p, i \in I^*.$$

If for the support non degenerate [2] conjugate flow $\{\delta, K\}$ support K is not optimum, then in case of dual nondegeneracy [2] there exists a variation $\Delta\delta$ of conjugate flow δ, which conducts to an increase in the dual objective function. Two variants for construction of a direction are possible.

1) Let (i_0, j_0) – an arc on which the limit of flow $\overline{x}_{ij}$ (0 or $d_{i_0 j_0}$) is achieved. We shall construct a direction $\Delta\delta_K = (\Delta\delta_{ij}, (i,j) \in U_K; \Delta\delta_i, i \in I_K^*)$ as follows:

$$\Delta\delta_{i_0 j_0} = \begin{cases} -1, & \text{if} \quad \overline{x}_{i_0 j_0} = d_{i_0 j_0}, \\ \;\;1, & \text{if} \quad \overline{x}_{i_0 j_0} = 0, \end{cases}$$

$$\Delta\delta_{ij} = 0, (i,j) \in U_K \backslash (i_0 j_0),$$

$$\Delta\delta_i = 0, i \in I_K^*.$$

2) Let i_0 – is a dynamic node on which the limit of component $\overline{x}_{i_0} (\overline{x}_{i_0} = b_{*i} \vee b_i^*)$ is achieved.

We shall construct a direction $\Delta\delta_K = (\Delta\delta_{ij}, (i,j) \in U_K; \Delta\delta_i, i \in I_K^*)$ as follows:

$$\Delta\delta_{i_0} = \begin{cases} -1, \text{ if} & \overline{x}_{i_0} = b_{i_0}^*, \\ \;\;1, \text{ if} & \overline{x}_{i_0} = b_{*i_0}, \end{cases}$$

$$\Delta\delta_{ij} = 0, (i,j) \in U_K, \quad \Delta\delta_i = 0, i \in I_K^* \backslash i_0$$

Arc (i_0, j_0) and dynamic node i_0 are named critical.

Let's consider case 1). We shall designate $\alpha = \Delta\delta_{i_0 j_0}$, $\tau_p = r_p$, $p = \overline{1,q}$; $y_i = u_i$, $i \in I$. Support elements of an increment of the conjugate flow satisfy the system:

$$\Delta\delta_{i_0 j_0} = -(\Delta u_{i_0} - \mu_{i_0 j_0}\Delta u_{j_0} + \sum_{p=1}^{q} \lambda_{ij}^{p} \Delta r_p) = \alpha,$$

$$\Delta\delta_{ij} = -(\Delta u_i - \mu_{ij}\Delta u_j + \sum_{p=1}^{q} \lambda_{ij}^{p} \Delta r_p) = 0, (i,j) \in U_K \backslash (i_0, j_0), \quad (21)$$

$$\Delta\delta_i = \Delta u_i sign(i) - \sum_{p=1}^{q} \lambda_i^p \Delta r_p = 0, i \in I_K^*.$$

Let's consider any element (arc or node) from set $W = (U_W, I_W^*)$. To each element from set of sets W there corresponds the characteristic vector $\delta(\tau,\rho) \vee \delta(\gamma)$ generated by this element. We multiply each nonzero element of the characteristic vector $\delta(\tau,\rho)$, $(\tau,\rho) \in U_W$ by the equation of system (21), corresponding to this element, and we sum up the received equalities:

$$\sum_{p=1}^{q} \Lambda_{\tau\rho}^{p} \Delta r_p = -\alpha \delta_{i_0 j_0}^{\tau\rho}, (\tau,\rho) \in U_W. \quad (22)$$

Consider set I_W^*. To each element $\gamma \in I_W^*$ there corresponds a characteristic vector $\delta(\gamma) = (\delta_{ij}^{\gamma}, (i,j) \in U; \delta_i^{\gamma}, i \in I^*)$. We multiply every nonzero component of vector $\delta(\gamma)$ by corresponding to this component equation of system (21). We execute the specified transformations for all elements of set I_W^*:

$$\sum_{p=1}^{q} \Lambda_{\gamma}^{p} \Delta r_p = -\alpha \delta_{i_0 j_0}^{\gamma}, \gamma \in I_W^*. \quad (23)$$

In a matrix-vector form relations (22), (23) will become:

$$\Lambda_W^{'} \Delta r = \overline{\alpha}, \ \overline{\alpha} = (-\alpha\delta_{i_0 j_0}^{\tau\rho}, (\tau,\rho) \in U_W; -\alpha\delta_{i_0 j_0}^{\gamma}, \gamma \in I_W^*). \quad (24)$$

The aggregation of sets $K = \{U_K, I_K^*\}$ is a support of network $G = (I, U)$ for system (2), (3) and hence, matrix Λ_W is nonsingular.

We shall calculate vector δr :

$$\delta r = (\Lambda_W^{'})^{-1} \overline{\alpha}. \quad (25)$$

Components $\Delta\delta_N = (\Delta\delta_{\tau\rho}, (\tau,\rho) \in U_N; \Delta\delta_\gamma, \gamma \in I_N^*)$ of direction $\Delta\delta$ are:

$$\Delta\delta_{\tau\rho} = \begin{cases} -\sum_{p=1}^{q} \Lambda_{\tau\rho}^p \Delta r_p - \Delta\delta_{i_0 j_0}\delta_{i_0 j_0}^{\tau\rho}, \text{if } \ (i_0, j_0) \ - \ \text{critical arc;} \\ -\sum_{p=1}^{q} \Lambda_{\tau\rho}^p \Delta r_p - \Delta\delta_{i_0}\delta_{i_0}^{\tau\rho}, \quad \text{if } \ i_0 \ - \ \text{critical node;} \end{cases}$$

$$\Delta\delta_{\gamma} = \begin{cases} -\sum_{p=1}^{q} \Lambda_{\tau\rho}^p \Delta r_p - \Delta\delta_{i_0 j_0}\delta_{i_0 j_0}^{\gamma}, \text{if } \ (i_0, j_0) \ - \ \text{critical arc;} \\ -\sum_{p=1}^{q} \Lambda_{\gamma}^p \Delta r_p - \Delta\delta_{i_0}\delta_{i_0}^{\gamma}, \qquad \text{if } i_0 \ - \ \text{critical node.} \end{cases}$$

In a matrix-vector form last relations look like

$$\Delta\delta_N = -\Lambda_N^{'}\Delta r - \alpha t,$$

where vector $t = (\delta_{i_0 j_0}^{\tau\rho}, (\tau,\rho) \in U_N; \delta_{i_0 j_0}^{\gamma}, \gamma \in I_N^*)$ if arc (i_0, j_0) is critical, and $t = (\delta_{i_0}^{\tau\rho}, (\tau,\rho) \in U_N; \delta_{i_0}^{\gamma}, \gamma \in I_N^*)$, if dynamic node i_0 is critical. The dual step along the constructed direction $\Delta\delta$ is calculated by standard rules [1,2].

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