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THE EXPLICIT FORM OF NO ARBITRAGE CONDITION WHEN THE TERM STRUCTURE MODEL IS MULTI-FACTOR

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The no arbitrage conditions are derived in the explicit form for the market, where the zero coupons bonds of various maturities are accessible for the investors to draw up the portfolios. It is supposed, that the investor at any moment of time has a possibility to make the self-financed portfolio of given value. It is considered that the processes of the short interest rate and rates of inflation follow the stochastic differential equations. The known result for a portfolio with two assets is extended on case of any number of assets and inflation. The no arbitrage condition for multi-factor models of a term structure of the interest rates is considered. The condition of existence of a risk free self-financed portfolio is obtained at first, and then for want of it fulfillment the no arbitrage condition is derived.

Keywords: no arbitrage condition, market price of risk, bond price, risk free self-financed portfolio, inflation, segmented market, multi-factor model.

Introduction

In the financial market the term *arbitrage* refers to the possibility of making a trading gain with no chance of loss. The idea expressed by the no arbitrage condition consists that in the equilibrium market two portfolios of securities, which ensure identical payments, should have in each instant the identical price. Intuitively it is clear that such definition of the price excludes the arbitrage. The arbitrage theory of market asset pricing is recently popular. It bases on the assumption that the financial market is arbitrage-free. To check the fulfillment of such assumption it is necessary to have a no arbitrage conditions. This explains the interest to derive such conditions.

Under consideration of the continuous time mathematical models of the price dynamics in the financial market it is usually assumed that the processes of the interest rates in the market follow the stochastic differential equations. It results that the process of the price of market asset also follows the stochastic differential equation. These random processes are described by some objective probability measure. The no arbitrage condition, usually used in such situation, is the assumption about existence of the equivalent martingale measure (Panjer et al., 1998). This condition is rather general but difficult checked because at present the problem on construction equivalent martingale (risk-neutral) measure waits for its solution. Therefore deriving of the no arbitrage conditions in the explicit form, without resorting to construction of equivalent martingale measure, is useful.

The no arbitrage conditions for the continuous time one-factor models of the term structure of interest rate are most known (see for example, Black and Scholes, 1973). Usually these conditions are received for a portfolio with two financial assets in the financial market, in which there is also a risk free asset. Such condition states that the excess expected asset's return over the short rate divided by the asset's return volatility is independent of the asset's maturity. The extension of this no arbitrage condition to the market with inflation (Richard (1978)) is known also. There the no arbitrage condition is formulated (without the proof) for a portfolio with three assets. It states that the excess expected asset's return over the nominal short rate should be a linear combination of the asset's return volatilites that are appropriate to both a stochastic dynamics of the real short interest rate and the rate of inflation respectively. The factors of this linear combination should not depend on the maturity and make sense of "market prices of risk" because of a stochastic dynamics of the real interest rate and rate of inflation.

In the present paper the no arbitrage conditions are derived in the explicit form for the market, where the zero coupons bonds of n various maturities are accessible for the investors to draw up the portfolios. It is supposed, that the investor at any moment of time t has a possibility to make the self-financed portfolio of value $S(t)$, by including in this portfolio of the bonds with maturities T_j, $1 \le j \le n$, on the sum S_j. $S(t) = \sum_{j=1}^{n} S_j$. It is considered that the processes of the short interest rate and rates of inflation follow the stochastic differential equations, and the prices of the bonds are expressed by functions that have the mathematical derivatives of the necessary orders.

The article is organized as follows. In Section 1 the no arbitrage condition in an one-factor model is derived. There the known result for a portfolio from two assets is extended to case of any number of assets. In

Section 2 adding of inflation complicates the statement of a problem of the previous section. The no arbitrage condition in the market with inflation is obtained also for a portfolio with any number of assets. Section 3 is devoted to the no arbitrage condition in the segmented market (without inflation), where it is considered, that in the market simultaneously there are some segments, in which the bonds with hardly distinguishing maturities enter, and each segment has its own risk free interest rate. It is considered that in this situation the investor has a possibility to purchase the bonds of any segment. In Section 4 the no arbitrage condition for multi-factor models of a term structure of the interest rates is considered. In this section the vector process is accepted as a basis for the short interest rate process. A condition of existence of a risk free self-financed portfolio is obtained at first and then under its fulfillment the no arbitrage condition is derived. In this model the inflation is equivalent to some additional factor. In Section 5 some important special cases are discussed.

1. The no arbitrage condition in an one-factor model

In this section we shall consider the trade the default-free discount bonds with various dates of maturity in the market, which is described by an one-factor model. To exclude the arbitrage possibilities the portfolio, consisting of any combination of the bonds, in each instant should earn the same income, as risk-free asset of the same value. In this case for any number (greater two) bonds, sold in the market, the general no arbitrage condition at the determination of the bond prices it is possible to obtain. It states that the excess of expected bond's return over the short interest rate divided by the bond's return volatility is independent of the bond's maturity.

To demonstrate it mathematically for each $T \geq 0$ we shall assume, that the process of the price $\{P\ (t,\ T),\ t \leq T\}$ of the default-free discount bonds maturing at time T, is Itô process (Björk, 1996)

$$dP(t,T) = \mu^T(P(t,T),\ t)\ dt + \sigma^T(P(t,T),\ t)\ dW(t).$$

The superscript T emphasizes the dependence of a drift and volatility on the bond maturity date T. Dividing this equation by the bond price $P\ (t,\ T)$, we have

$$\frac{dP(t,T)}{P(t,T)} = \frac{\mu^T\left(P(t,T),t\right)}{P(t,T)}dt + \frac{\sigma^T\left(P(t,T),t\right)}{P(t,T)}dW(t).$$

The left-hand side is equal to the instantaneous yield interest rate of the bond. For a simplicity we use the symbols $\mu^{T}(t)$ and $\sigma^{T}(t)$ to denote $\dfrac{\mu^T\left(P(t,T),t\right)}{P(t,T)}$ and $\dfrac{\sigma^T\left(P(t,T),t\right)}{P(t,T)}$ respectively and obtain

$$\frac{dP(t,T)}{P(t,T)} = \mu^T(t)dt + \sigma^T(t)dW(t). \tag{1}$$

That is $\mu^{T}(t)$ and $\sigma^{T}(t)$ are respectively drift and volatility of the instantaneous yield interest rate of the bond.

Now we shall consider case, when in the market one trades securities with n maturity dates $T_j, j = 1, ..., n$. Let investor has some money sum $S(t)$, which can be spent for purchasing of securities in this market, and spends the sum $S_j = N_j\, P(t,T_j)$ for purchasing N_j of securities with maturity date T_j, so that

$$S(t) = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} N_j P(t,T_j).$$

The increment of this portfolio value for an infinitesimal time interval is determined by equality

$$dS(t) = \sum_{j=1}^{n} N_j dP(t,T_j) = \sum_{j=1}^{n} S_j \frac{dP(t,T_j)}{P(t,T_j)},$$

that with regard for equation (1) results in a relation

$$dS(t) = \sum_{j=1}^{n} S_j \left(\mu^{(j)}(t)dt + \sigma^{(j)}(t)dW(t)\right) =$$

$$= \sum_{j=1}^{n} S_j \mu^{(j)}(t)dt + \left(\sum_{j=1}^{n} S_j \sigma^{(j)}(t)\right)dW(t),$$

where $\mu^{(j)}(t)$ and $\sigma^{(j)}(t)$ are a drift and a volatility of yield of with maturity dates T_j, $j = 1, ... , n$. To obtain a risk-free return it is necessary, that the sum in brackets in a stochastic term was equal to zero. That is for getting the risk-free profits it need to distribute the available sum $S(t)$ so that to fulfill the equality $\sum_{j=1}^{n} S_j \sigma^{(j)}(t) = 0$.

Without loss of generality, suppose that $\sigma^{(n)}(t) \neq 0$, so that

$$S_n = -\frac{1}{\sigma^{(n)}(t)} \sum_{j=1}^{n-1} S_j \sigma^{(j)}(t).$$

Such choice S_n ensures for a time interval $(t, t+dt)$ the risk-free deriving of the profits that is equal to

$$dS(t) = \sum_{j=1}^{n-1} S_j \mu^{(j)}(t)dt - \frac{\mu^{(n)}(t)}{\sigma^{(n)}(t)} \sum_{j=1}^{n-1} S_j \sigma^{(j)}(t)dt =$$

$$= \left[\sum_{j=1}^{n-1} S_j \left(\mu^{(j)}(t) - \frac{\mu^{(n)}(t)}{\sigma^{(n)}(t)} \sigma^{(j)}(t) \right) \right] dt.$$

Under no arbitrage the risk-free portfolio should earn interests according to the short rate $r = r(t)$. It means, that in the market the no arbitrage conditions will be held if only for any distribution of the investor's money $\{S_j\}$ on types of securities for deriving of the risk-free profit the increment of the sum $S(t)$ will be exactly equal to an increment of risk-free asset of the same value. In other words it should be fulfilled the equality

$$dS(t) = S(t)\, r(t)\, dt =$$

$$= \sum_{j=1}^{n} S_j r(t)dt = \sum_{j=1}^{n-1} S_j r(t)dt - \frac{r(t)}{\sigma^{(n)}(t)} \sum_{j=1}^{n-1} S_j \sigma^{(j)}(t)dt =$$

$$= \left[\sum_{j=1}^{n-1} S_j \left(r(t) - \frac{r(t)}{\sigma^{(n)}(t)} \sigma^{(j)}(t) \right) \right] dt. \qquad (2)$$

Equating the obtained formulae for an risk-free increment of cost $S(t)$, we come to equality

$$\left[\sum_{j=1}^{n-1} S_j\left(\mu^{(j)}(t)-\frac{\mu^{(n)}(t)}{\sigma^{(n)}(t)}\sigma^{(j)}(t)\right)\right]dt=\left[\sum_{j=1}^{n-1} S_j\left(r(t)-\frac{r(t)}{\sigma^{(n)}(t)}\sigma^{(j)}(t)\right)\right]dt\,.$$

Hence

$$\sum_{j=1}^{n-1} S_j\left(\mu^{(j)}(t)-\frac{\mu^{(n)}(t)}{\sigma^{(n)}(t)}\sigma^{(j)}(t)-r(t)+\frac{r(t)}{\sigma^{(n)}(t)}\sigma^{(j)}(t)\right)=$$

$$=\sum_{j=1}^{n-1} S_j\sigma^{(j)}(t)\left(\frac{\mu^{(j)}(t)-r(t)}{\sigma^{(j)}(t)}-\frac{\mu^{(n)}(t)-r(t)}{\sigma^{(n)}(t)}\right)=0\,.$$

As this equality should be fulfilled for any distribution $\{S_j,\ 1 \le j \le n\}$ of the available sum $S(t)$ on types of securities, each term of this sum should be equal to zero. From here we come to the following condition of the no arbitrage condition

$$\frac{\mu^{(j)}(t)-r(t)}{\sigma^{(j)}(t)}=\frac{\mu^{(n)}(t)-r(t)}{\sigma^{(n)}(t)}\,, \qquad 1 \le j \le n-1. \tag{3}$$

Thus, the ratio of excess expected bond's return $\mu^{(j)}(t)$) of the securities with maturity dates T_j over the short interest rate $r(t)$ to the volatility $\sigma^{(j)}(t)$ of process of yield interest rate of this securities under no arbitrage condition should not depend on maturity date and should be identical for all maturity dates. We shall designate this function through $\lambda(r, t)$, and we shall obtain for any $T \ge t$ the known equality

$$\frac{\mu^{T}(t)-r(t)}{\sigma^{T}(t)}=\lambda(r(t),t),$$

which is considered as definition of function $\lambda(r, t)$ called a market price of risk or market risk premium. It means, that it is true a following

Proposition 1. In case of no arbitrage possibilities, there is a function $\lambda(r, t)$ such, that it is valid the equality

$$\mu^T(t) = r(t) + \lambda(r(t), t)\ \sigma^T(t) \tag{4}$$

for any maturity dates T.

This equality is named the (local) no arbitrage condition.

2. The no arbitrage condition in the market with inflation

If an inflation occurs in the market then a nominal interest rate $R(t)$, used for determination of the discount price of the bond, is determined not only actual short interest rate $r(t)$ but also rate of inflation $i(t)$, which reflects a relative changes in the consumer price index of consumer goods and services. Usually connection between these rates is described by the so-called Fisher equation (Fabozzi, 1995)

$$1 + R(t) = (1 + r(t))(1 + i(t)). \tag{5}$$

We take that the process of the actual interest rate $r(t)$ follows the stochastic differential equation

$$dr(t) = \mu_r(r(t), t)\, dt + \sigma_r(r(t), t)\, dW_r(t). \tag{6}$$

Similarly, we suppose that the process of the rate of inflation $i(t)$ follows the following stochastic differential equation

$$di(t) = \mu_i(i(t), t)\, dt + \sigma_i(i(t), t)\, dW_i(t). \tag{7}$$

The subscripts in these equations show what process is characterized by the appropriate functions of drift and volatility, and also the Wiener processes. As the mechanisms underlying a stochastic change of the processes $r(t)$ and $i(t)$ are generally various and in the certain degree are independent, the processes $W_r(t)$ and $W_i(t)$ are various also and can be dependent only somewhat. Therefore in the general case the Wiener processes $W_r(t)$ and $W_i(t)$ can be presented as

$$W_r(t) = \rho\, W_0(t) + \sqrt{1-\rho^2}\, W_1(t), \quad W_i(t) = \rho\, W_0(t) + \sqrt{1-\rho^2}\, W_2(t),$$

where $W_0(t)$, $W_1(t)$, and $W_2(t)$ are independent standard Wiener processes, and ρ represents a coefficient of correlation between processes $W_r(t)$ and $W_i(t)$. In this section for a simplicity will be assumed that $\rho = 0$. The result in more general case, when $\rho \neq 0$, can be obtained as a special case from results of the Section 4.

As in considered case the price of the discount bond is determined by the nominal interest rate, for maturity date T it is described by function

$P(r, i, t, T)$. If we assume that function $P(r, i, t, T)$ is differentiable on t and twice differentiable on r and i then the equation (1) by application of the Itô derivation formula is transformed to a form

$$\frac{dP(t,T)}{P(t,T)} = \mu^T(t)dt + \sigma_r^T(t)dW_1(t) + \sigma_i^T(t)dW_2(t), \tag{8}$$

where the arguments r and i at functions μ and σ are omitted for brevity. These functions are determined by the formulae

$$\sigma_r^T(t) = \sigma_r(r,t)\frac{1}{P(r,i,t,T)}\frac{\partial P}{\partial r}, \tag{9}$$

$$\sigma_i^T(t) = \sigma_i(i,t)\frac{1}{P(r,i,t,T)}\frac{\partial P}{\partial i}, \tag{10}$$

$$\mu^T(t) = \frac{1}{P}\left(\frac{\partial P}{\partial t} + \mu_r\frac{\partial P}{\partial r} + \mu_i\frac{\partial P}{\partial i} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma_i^2\frac{\partial^2 P}{\partial i^2}\right). \tag{11}$$

The actual price of the bond $B(r, i, t, T)$ can be determined by division of a nominal price $P(r, i, t, T)$ on a level of consumer prices $C(t)$, which grows according to the rate of inflation $i(t)$ (Richard, 1978). This growth is determined by the equation

$$dC(t) = i(t)C(t)dt.$$

Thus, $B(r, i, t, T) = P(r, i, t, T)/C(t)$. Applying again formula Itô, we shall obtain the equation for process of the actual price of the bond with maturity date T as

$$\frac{dB(t,T)}{B(t,T)} = \left(\mu^T(t) - i(t)\right)dt + \sigma_r^T(t)dW_r(t) + \sigma_i^T(t)dW_i(t). \tag{12}$$

Now again, as well as in preceding section, we shall consider case, when in the market one trades the bonds with n maturity dates T_j, $j = 1, \dots, n$, $n > 2$. Let investor spends for purchasing of bonds the sum $S(t)$, purchasing N_j of bonds with maturity dates T_j, so that $S_j = N_j\,P(t,T_j)$, i.e.

$$S(t) = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} N_j P(t, T_j).$$

The increment of portfolio value of these bonds for an infinitesimal time interval is determined as well as above by equality

$$dS(t) = \sum_{j=1}^{n} N_j dP(t, T_j) = \sum_{j=1}^{n} S_j \frac{dP(t, T_j)}{P(t, T_j)}.$$

Let's assume now, that the processes of the bond prices with any maturity term are generated by the same short interest rate that follows process (6). Then with regard for equation (8) it is possible to derive the relation

$$\begin{aligned} dS(t) &= \sum_{j=1}^{n} S_j \left(\mu^{(j)}(t)dt + \sigma_r^{(j)}(t)dW_r(t) + \sigma_i^{(j)}(t)dW_i(t) \right) = \\ &= \sum_{j=1}^{n} S_j \mu^{(j)}(t)dt + \left(\sum_{j=1}^{n} S_j \sigma_r^{(j)}(t) \right) dW_r(t) + \left(\sum_{j=1}^{n} S_j \sigma_i^{(j)}(t) \right) dW_i(t), \end{aligned} \tag{13}$$

where $\mu^{(j)}(t)$ and $\sigma^{(j)}(t)$ are a drift and a volatility of yield of the bonds with maturity date T_j , $j = 1, \ldots, n$. To obtain the risk-free return it is necessary, that the sum into brackets in stochastic terms was equal to zero. It means that for deriving the risk-free profits it is necessary to distribute the available sum $S(t)$ so that to fulfill the equality

$$\sum_{j=1}^{n} S_j \sigma_r^{(j)}(t) = 0, \qquad \sum_{j=1}^{n} S_j \sigma_i^{(j)}(t) = 0. \tag{14}$$

The equalities (14) are the existence condition of risk-free self-financed portfolio. In order to obtain the no arbitrage condition it is necessary to add a demand that the self-financed portfolio has to earn at the risk-free nominal interest rate $R(t)$, i.e.

$$\sum_{j=1}^{n} S_j \mu^{(j)}(t) = S(t)R(t) = \sum_{j=1}^{n} S_j R(t). \tag{15}$$

Thus the no arbitrage condition is held if for any set $\{S_j\}$ the following equalities are simultaneously fulfilled

$$\sum_{j=1}^{n}\left(\mu^{(j)}(t)-R(t)\right)\,S_j=0,\;\sum_{j=1}^{n}\sigma_r^{(j)}(t)\;S_j=0,\;\;\sum_{j=1}^{n}\sigma_i^{(j)}(t)\;S_j=0.\quad(16)$$

The equalities (16) can be written in the matrix form

$$\begin{pmatrix}\mu^{(1)}(t)-R(t) & \mu^{(2)}(t)-R(t) & \dots & \mu^{(n)}(t)-R(t)\\ \sigma_r^{(1)}(t) & \sigma_r^{(2)}(t) & \dots & \sigma_r^{(n)}(t)\\ \sigma_i^{(1)}(t) & \sigma_i^{(2)}(t) & \dots & \sigma_i^{(n)}(t)\end{pmatrix}\begin{pmatrix}S_1\\ S_2\\ \dots\\ S_n\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \dots\\ 0\end{pmatrix}.$$

This can be considered as a system of equations with respect to $\{S_j\}$. In order that this system would have nontrivial solution it is necessary that a matrix rank would be less than n, $n > 2$, i.e. we obtain the equivalent condition

$$\det\begin{pmatrix}\mu^{(1)}(t)-R(t) & \mu^{(2)}(t)-R(t) & \dots & \mu^{(n)}(t)-R(t)\\ \sigma_r^{(1)}(t) & \sigma_r^{(2)}(t) & \dots & \sigma_r^{(n)}(t)\\ \sigma_i^{(1)}(t) & \sigma_i^{(2)}(t) & \dots & \sigma_i^{(n)}(t)\end{pmatrix}=0.\quad(17)$$

From this it follows that the rows of matrix are linear dependent (Horn and Johnson, 1986). Hence we have for each component of the rows of matrix the relations

$$\mu^{(j)}(t)-R(t)=\lambda_r(t,r,i)\;\sigma_r^{(j)}(t)+\lambda_i(t,r,i)\sigma_i^{(j)}(t),\;\;1\le j\le n.\quad(18)$$

The factors $\lambda_r(t, r, i)$ and $\lambda_i(t, r, i)$ are independent on maturity date and make sense of market prices of risk in connection with a stochastic changes interest rates and inflation respectively. Note that for case $n = 3$ this result is contained in Richard (1978).

3. The no arbitrage condition in the segmented market

The arguments of the previous sections is agreed with the theory of the term structure of the interest rates based on a so-called expectations hypothesis. According to this theory the forward interest rates are considered as unbiased estimates of expected future short interest rates. Therefore is natural the supposition that it is possible in the equations for processes of the bond prices with any maturity to use the same equation of process of the short interest rate. However not always the results of this theory will be agreed with market realities. In this connection there are also other theories of term structure (Hull, 1993). According to the theory of market segmentation there are simultaneously some independent processes of the short interest rates appropriate to various maturity terms and controlled by the supply and demand for assets with these terms. More often assets, which are traded in the market, are divided into three segments: short-term, intermediate term and long-term assets. Because of independence of mechanisms of installation of the prices on assets into each from segments, it is possible to assume that the independent Wiener processes generate the equilibrium processes of the short-term interest rates inside of various segments. Consider no arbitrage conditions in this case.

Still we shall assume, that in the financial market there is traded the zero coupon bonds with maturity dates that form a set $\{T_j, 1 \leq j \leq n\}$. Let's assume also, that this maturity set is divided on m of nonintersecting segments T_k, $1 \leq k \leq m$, $m < n$. Inside k-th of segment the actual short interest rate $r_k(t)$ follows the stochastic differential equation

$$dr_k(t) = \mu_{kr}(r(t), t)\, dt + \sigma_{kr}(r(t), t)\, dW_k(t), \quad 1 \leq k \leq m. \quad (19)$$

According to this the yield of the bond with maturity date $T_j \in \mathrm{T}_k$ is determined by the stochastic differential equation

$$\frac{dP(t, T_j)}{P(t, T_j)} = \mu^{(j)}(r_k, t)dt + \sigma^{(j)}(r_k, t)dW_k(t). \quad (20)$$

Under the inflation instead of the actual interest rate $r_k(t)$ in the equation (20) it is necessary to use the nominal interest rate $R_k(t) = (1 + r_k(t))(1 + i(t)) - 1$, where the rate of inflation varies according to process (7). In this case equation (20) should be modified to a form (8) - (11), taking into account a stochastic behavior of the rate of inflation. As the purpose of the present section is to find a no arbitrage for the segmented market, we shall not take into account inflation.

As well as until now we assume, that the investor makes a portfolio of the bonds of value $S(t)$ purchasing N_j bonds with maturity term T_j on the price $P(t,T_j)$ so that $S_j = N_j P(t,T_j)$,

$$S(t) = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} N_j P(t, T_j).$$

Using method, which was used in the previous sections, it is possible to obtain an increment of this portfolio value of the bonds for an infinitesimal time interval as

$$dS(t) = \sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj}\mu^{(j)}(r_k, t) S_j dt + \sum_{k=1}^{m}\sum_{j=1}^{n} I_{kj}\sigma^{(j)}(r_k, t) S_j dW_k(t), \qquad (21)$$

where I_{kj} is the indicator

$$I_{kj} = \begin{cases} 1, & if \quad T_j \in \mathbf{T}_k, \\ 0, & if \quad T_j \notin \mathbf{T}_k. \end{cases}$$

Again for deriving of the risk-free profit it is necessary, that $\{S_j\}$ were selected so that the stochastic terms in (21) were equal to zero. Therefore we have m conditions of risk-free deriving of interests

$$\sum_{j=1}^{n} \left(I_{kj}\sigma^{(j)}(r_k, t) \right) S_j = 0, \qquad 1 \le k \le m. \qquad (22)$$

The equalities (22) are the existence conditions of risk-free self-financed portfolio. As well as in the previous section in order to obtain the no arbitrage condition, it is necessary to add a demand that the self-financed portfolio can earn inside each of m market segments only at the risk-free interest rate $r_k(t)$, $1 \le k \le m$, i.e.

$$\sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj}\mu^{(j)}(r_k, t) S_j = \sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj} r_k(t) S_j,$$

hence

$$\sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj}\left(\mu^{(j)}(r_k,t)-r_k(t)\right)S_j = 0. \qquad (23)$$

Thus the no arbitrage condition is held if for any set $\{S_j\}$ the equalities (22) and (23) are simultaneously fulfilled. In vector-matrix notation this means

$$\begin{pmatrix} \sum_{k=1}^{m} I_{k1}\left(\mu^{(1)}(r_k,t)-r_k(t)\right) & \dots & \sum_{k=1}^{m} I_{kn}\left(\mu^{(n)}(r_k,t)-r_k(t)\right) \\ \sigma^{(1)}(r_1,t) & \dots & \sigma^{(n)}(r_1,t) \\ \dots. & \dots & \dots \\ \sigma^{(1)}(r_m,t) & \dots & \sigma^{(n)}(r_m,t) \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_n \end{pmatrix} = 0.$$

Because this equality must be fulfilled for any $\{S_j\}$ therefore determinant of matrix must be equal to zero. Then the rows of matrix are linear dependent, i.e.

$$\sum_{k=1}^{m}\left(\mu^{(j)}(r_k,t)-r_k(t)\right)I_{kj} = \sum_{k=1}^{m}\lambda_k(r_k,t)\sigma^{(j)}(r_k,t)I_{kj}, \ \ 1 \le j \le n. \qquad (24)$$

As the bond with some specific maturity date T_j can belong only to one segment, the sums in equality (24) contain only a single nonzero term for this maturity. Let us determine a set J of pairs (j, k) for witch $I_{kj}=1$, i.e. $(j, k) \in J$ if a security with maturity date T_j is traded at the k-th segment of financial market. Then we can formulate the no arbitrage condition in the form

Proposition 2. The no arbitrage conditions are held in the segmented market if for each $(j, k) \in J$ the equality is fulfilled

$$\frac{\mu^{(j)}(r_k,t)-r_k}{\sigma^{(j)}(r_k,t)} = \lambda_k(r_k,t).$$

It means, that for each segment of the financial market there is a market price of risk, identical for the bonds of all maturity dates of this segment, but in general various for various segments.

4. The no arbitrage condition for multi-factor models

In multi-factor models of a term structure of the interest rates it is supposed that the price P is some function of several state variables of interest rate (two factor model see Moreno (1996) or Richard (1978), general case see Duffie and Kan (1992)). Process of the short interest rate is determined as a vector stochastic process that is determined by a system of the stochastic differential equations concerning a m-vector $\bar{r}(t)$ with components $r_k(t)$, $1 \le k \le m$,

$$d\bar{r}(t) = \bar{\mu}(\bar{r},t)dt + \bar{\sigma}(\bar{r},t)dW(t), \tag{25}$$

where $\bar{\mu}(\bar{r},t)$ is a m-vector of a drift with components $\mu_k(r_k, t)$, $\bar{\sigma}(\bar{r},t)$ is $(m\times q)$-matrix of volatilities with components $\sigma_{kl}(r_k, t)$, $1 \le k \le m$, $1 \le l \le q$, $W(t)$ − q-dimensional stochastic process with components that are scalar independent standard Wiener processes $W_l(t)$. It should be noted that one of component of vector $\bar{r}$ could be the rate of inflation. In these conditions the stochastic differential equation for the price $P(\bar{r}, t, T_j)$ of the zero coupon bond with maturity term T_j has a form

$$\frac{dP_j}{P_j} = \mu^{(j)}(t)dt + \sigma^{(j)}(t)dW(t), \tag{26}$$

where the simplified designations $P_j = P(\bar{r}, t, T_j)$ are used, and

$$\mu^{(j)}(t) = \frac{1}{P_j}\left(\frac{\partial P_j}{\partial t} + \frac{\partial P_j}{\partial r}\bar{\mu}(\bar{r},t) + \frac{1}{2}tr\left(\frac{\partial^2 P_j}{\partial r^2}\bar{\sigma}(\bar{r},t)\bar{\sigma}^T(\bar{r},t)\right)\right), \tag{27}$$

$$\sigma^{(j)}(t) = \frac{1}{P_j}\frac{\partial P_j}{\partial r}\sigma(\bar{r},t), \tag{28}$$

It should be noted for clearness that here $\frac{\partial P_j}{\partial r}$ is the row vector with components $\frac{\partial P_j}{\partial r_k}$, $1 \le k \le n$, and $\frac{\partial^2 P_j}{\partial r^2}$ is the $(n\times n)$-matrix with elements

$\frac{\partial^2 P_j}{\partial r_i \partial r_k}$, $1 \le i, k \le n$; $tr\,(A)$ designs a trace of matrix A.

Again we assume, that the bonds of n various maturities T_j, $1 \le j \le n$, are accessible for the investor to draw up a self-financing portfolio of value $S(t)$. In each instant he uses the sum S_j to purchase the bonds with maturity term T_j on the price P_j. The increment of value of such portfolio of the bonds for an infinitesimal time interval is determined by equation

$$dS(t) = \sum_{j=1}^{n} \frac{dP_j}{P_j} S_j = \sum_{j=1}^{n} \left(\mu^{(j)}(t)dt + \bar{\sigma}^{(j)}(t)dW(t) \right) S_j =$$

$$= \left(\sum_{j=1}^{n} \mu^{(j)}(t) S_j \right) dt + \left(\sum_{j=1}^{n} \sigma^{(j)}(t) S_j \right) dW(t). \tag{29}$$

For deriving of the risk-free profit it is necessary that it is fulfilled the equalities

$$\sum_{j=1}^{n} \sigma^{(j)}(t) S_j = 0. \tag{30}$$

The no arbitrage condition implies the addition to (30) the demand that the self-financed portfolio earns with the risk free nominal interest rate $R(t)$, i.e.

$$\sum_{j=1}^{n} \mu^{(j)}(t) S_j = R(t)S(t) = \sum_{j=1}^{n} R(t) S_j. \tag{31}$$

Combining the equations (30) and (31) gives the systems

$$\begin{pmatrix} \mu^{(1)}(t) - R(t) & \mu^{(2)}(t) - R(t) & \dots & \mu^{(n)}(t) - R(t) \\ \sigma_1^{(1)}(t) & \sigma_1^{(2)}(t) & \dots & \sigma_1^{(n)}(t) \\ \dots & \dots & \dots & \dots \\ \sigma_q^{(1)}(t) & \sigma_q^{(2)}(t) & \dots & \sigma_q^{(n)}(t) \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_n \end{pmatrix} = 0. \tag{32}$$

This system must be satisfied for any admissible $\{S_j\}$ therefore determinant of matrix must be equal to zero. It means that the rows of matrix are linear dependent. This allows formulating the no arbitrage condition as

Proposition 3. In order to the no arbitrage conditions are held in the financial market that is described by multi-factor model (25) − (26) it is necessary that the following equalities be fulfilled

$$\mu^{(j)}(t) - R(t) = \sum_{l=1}^{q} \sigma_l^{(j)}(t)\lambda_l(t, \bar{r}), \qquad 1 \le j \le n. \tag{33}$$

The variables $\lambda_l(t, \bar{r})$, $1 \le l \le q$, in expression (33) make sense of the market prices of risk because of a stochastic behavior l-th component of stochastic term of factor increment in the equation (25).

The substitution in equality (33) the expressions for $\mu^{(j)}(t)$ and $\sigma^{(j)}(t)$ in the forms (27) and (28) gives after appropriate rearrangement the partial differential equation with respect to price $P(\bar{r}, t, T_j)$.

5. Some important special cases

It should be noted that in literature the special cases are often occurred where the bond prices depend on some factors only through the single variable. This variable is usually the nominal interest rate. Above in Section 2 such example was presented. The other examples are described in Langetieg (1980), Cox, Ingerssol, and Ross (1985), Chaplin (1987), Longstaff and Schwartz (1992), Chaplin and Sharp (1993) (see also Duffie and Kan (1992) and Vetzal (1994)). For these cases it is possible to assume that the nominal interest rate $R(t)$ is determined by factors $r_k(t)$, $1 \le k \le m$, as the weighted sum $R(t) = a\bar{r}(t) = \sum_{k=1}^{m} a_k r_k(t)$. Then the price of the discount bond is determined for maturity date T is described by function $P(R, t, T)$. At the same time the case considered is a special case of the multi-factor model and the results of previous section are valid here too. Therefore the equation (26) is right here also but designations (27) and (28) can be made more specified.

$$\mu^{(j)}(t) = \frac{1}{P_j}\left(\frac{\partial P_j}{\partial t} + \frac{\partial P_j}{\partial R} a\bar{\mu}(\bar{r}, t) + \frac{1}{2}\frac{\partial^2 P_j}{\partial R^2}\left[a\bar{\sigma}(\bar{r}, t)(a\bar{\sigma}(\bar{r}, t))^T\right]\right), \tag{34}$$

$$\sigma^{(j)}(t) = \frac{1}{P_j}\frac{\partial P_j}{\partial R} a\sigma(\bar{r},t), \tag{35}$$

where $a = (a_1\ a_2\ \ldots\ a_m)$ is a row vector, $P_j = P(R,\ t,\ T_j)$, and the (mathematical) derivatives $\frac{\partial P_j}{\partial R}$ and $\frac{\partial^2 P_j}{\partial R^2}$ are scalars. Then the expression (33) can be written in a form

$$\frac{\mu^{(j)}(t) - R(t)}{\frac{1}{P_j}\frac{\partial P_j}{\partial R}} = a\bar{\sigma}(\bar{r},t)\lambda(\bar{r},t), \qquad 1 \le j \le n. \tag{36}$$

Here $\lambda(\bar{r},t)$= $(\lambda_1(\bar{r},t)\ \ \lambda_2(\bar{r},t)\ \ \ldots\ \ \lambda_q(\bar{r},t))^{\mathrm{T}}$ is a column vector of the market prices of risk because of a stochastic behavior components of stochastic term $\bar{\sigma}(\bar{r},t)dW(t)$ of factor increment in the equation (25). Note that $\frac{1}{P_j}\frac{\partial P_j}{\partial R} = \frac{\partial \ln P_j}{\partial R}$. Thus we derive the following interpretation of the relation (33).

Proposition 4. The no arbitrage condition for bond pricing states that the excess of expected bond's return $\mu^{(j)}(t)$ over the nominal interest rate $R(t)$ divided by the (mathematical) derivative of logarithm of bond's price P_j with respect to the nominal interest rate $R = R(t)$ is independent of the bond's maturity T_j.

The relation (36) is the base to formulate the partial differential equation for the bond price $P(R,\ t,\ T_j)$. For this it is sufficient to substitute the explicit form $\mu^{(j)}(t)$ from (34) and to rearrange the terms of obtained expression. However it should be noted that the bond price will have the form $P(R,\ t,\ T)$ if only expressions $[a\bar{\mu}(\bar{r},t)]$, $[a\bar{\sigma}(\bar{r},t)(a\bar{\sigma}(\bar{r},t))^{\mathrm{T}}]$, and $[a\bar{\sigma}(\bar{r},t)\lambda(\bar{r},t)]$ are the functions of $R(t) = a\bar{r}(t)$. This is possible for example if a, $\bar{\mu}(\bar{r},t), \bar{\sigma}(\bar{r},t)$, $\lambda(\bar{r},t)$ are determined in the following way

a – a row vector with components $a_k = 1$, $1 \le k \le m$;

$\bar{\mu}(\bar{r},t)$ – a vector with components $(\alpha\, r_k(t) + \beta_k)$, $1 \le k \le m$;

$\bar{\sigma}(\bar{r},t)$ – a diagonal matrix with elements $\sigma\sqrt{r_k(t) + \delta_k}$,$1 \le k \le m = q$;

$\lambda(\bar{r},t)$ – a vector with components $\lambda\sqrt{r_k(t) + \delta_k}$, $1 \le k \le q = m$.

At these functions the solution $P(R, t, T)$ of the bond price equation belongs to the affine class. As illustration of this version we shall note that it is usual in the markets of stable economy the values of the short interest rate $r(t)$ and the rate of inflation $i(t)$ are rather small, and it is possible to use linear approximation of formula (5) (Bodie et al., 1996)

$$R(t) \approx r(t) + i(t). \tag{37}$$

In this case $m = 2$, $r_1(t) = r(t)$, $r_2(t) = i(t)$.

The question, whether there are other possible ways of determination of these functions to ensure the solution of the bond price equation in the form $P(R, t, T_j)$, remains open.

Conclusions

It is known that in financial market there is no arbitrage if and only if there exists an equivalent martingale probability measure (Duffie, 1992). This condition is rather general but difficult checked because at present the problem of construction of equivalent martingale (risk-neutral) measure is waiting for its solution. Therefore deriving of the no arbitrage conditions in the explicit form, without resorting to construction of equivalent martingale measure, is useful. This paper has presented some cases when this manages to be made in financial market where the discount bonds with some arbitrary number of maturities are traded.

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