SPECTRAL ANALYSIS OF MARKOV CHAINS

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Abstract

The paper deals with the problem of a statistical analysis of Markov chains connected with the spectral density. We present the expressions for the function of spectral density. These expressions may be used to estimate the parameter of the Markov chain.

1 Introduction

We observe $\Xi = \{\xi_t, \quad t = 0, 1, 2, \ldots\}$ a Markov chain with a finite states space $S = \{0, 1\}$. Suppose that initial state probabilities and transition probability matrix are known.

The purpose of this paper is to derive the expression for the spectral density function of Ξ and to investigate it's properties.

2 Spectral density function of the first order Markov chain

Consider $\Xi = \{\xi_t, \quad t = 0, 1, 2, \ldots\}$ a homogeneous Markov chain with a finite states space $S = \{0, 1\}$, transition probability matrix

$$P = \begin{pmatrix} \varepsilon & 1-\varepsilon \\ 1-\varepsilon & \varepsilon \end{pmatrix}, \tag{1}$$

where $0 \le \varepsilon \le 1$, and initial state probabilities:

$$P\{\xi_0 = 0\} = P\{\xi_0 = 1\} = \frac{1}{2}. \tag{2}$$

Theorem. *Let Ξ be a homogeneous Markov chain satisfying (1), (2). Then spectral density function of Ξ is*

$$f(\lambda) = \frac{1}{8\pi} \frac{1-(2\varepsilon-1)^2}{1-2(2\varepsilon-1)\cos\lambda + (2\varepsilon-1)^2}, \tag{3}$$

where $\lambda \in [-\pi, \pi]$.

Proof. The spectral density function, defined as $f(\lambda)$, may be written as [1]

$$f(\lambda) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r(\tau) \exp\{-i\lambda\tau\}, \tag{4}$$

where

$$r(\tau) = cov\{\xi_t, \xi_{t+\tau}\} = E\{\xi_t \xi_{t+\tau}\} - E\{\xi_t\}E\{\xi_{t+\tau}\} \tag{5}$$

and $\lambda \in (-\pi, \pi)$.

Under the conditions (1) and (2) we have

$$E\xi_t = P\{\xi_t = 1\} = \frac{1}{2}, \tag{6}$$

$$E\{\xi_{t+\tau}\} = P\{\xi_{t+\tau} = 1\} = \frac{1}{2} \tag{7}$$

and

$$E\{\xi_t\xi_{t+\tau}\}P\{\xi_t = 1, xi_{t+\tau} = 1\} = P\{\xi_{t+\tau} = 1|\xi_t = 1\}P\{\xi_t = 1\} = \\ = \frac{1}{2}P\{\xi_{t+\tau} = 1|\xi_t = 1\},$$

where the conditional probability $P\{\xi_{t+\tau} = 1|\xi_t = 1\}$ can be defined from the transition probability matrix pro τ steps [2], [3], which in our case equals:

$$P^{(\tau)} = \frac{1}{2}\begin{pmatrix} (2\varepsilon - 1)^\tau + 1 & (2\varepsilon - 1)^\tau - 1 \\ (2\varepsilon - 1)^\tau - 1 & (2\varepsilon - 1)^\tau + 1 \end{pmatrix}.$$

Consequently we can write down

$$P\{\xi_{t+\tau} = 1|\xi_t = 1\} = \frac{1}{2}((2\varepsilon - 1)^\tau + 1)$$

and

$$E\{\xi_t\xi_{t+\tau}\} = \frac{1}{4}(1 + (2\varepsilon - 1)^\tau). \tag{8}$$

Then, after substituting (6), (7) and (8) to (5), we obtain

$$r(\tau) = \frac{1}{4}(2\varepsilon - 1)^\tau.$$

Hence,

$$f(\lambda) = \frac{1}{8\pi}(1 + 2\sum_{\tau=1}^{\infty}(2\varepsilon - 1)^\tau \cos \lambda\tau). \tag{9}$$

It is easy to show, that for $|q| < 1$

$$\sum_{\tau=1}^{\infty} q^\tau \cos \tau\lambda = \frac{q \cos \lambda - q^2}{1 - 2q \cos \lambda + q^2}. \tag{10}$$

Let $q = 2\varepsilon - 1$. Since $|2\varepsilon - 1| < 1$ when $0 \leq \varepsilon \leq 1$, we can write down

$$\sum_{\tau=1}^{\infty} (2\varepsilon - 1)^{\tau} \cos \tau\lambda = \frac{(2\varepsilon - 1)\cos\lambda - (2\varepsilon - 1)^2}{1 - 2(2\varepsilon - 1)\cos\lambda + (2\varepsilon - 1)^2}. \tag{11}$$

After substituting (11) to (9) we obtain

$$f(\lambda) = \frac{1}{8\pi}\left(1 + 2\frac{(2\varepsilon - 1)\cos\lambda - (2\varepsilon - 1)^2}{1 - 2(2\varepsilon - 1)\cos\lambda + (2\varepsilon - 1)^2}\right) =$$

$$= \frac{1}{8\pi}\frac{1 - (2\varepsilon - 1)^2}{1 - 2(2\varepsilon - 1)\cos\lambda + (2\varepsilon - 1)^2}.$$

□

Corollary. *If* $\varepsilon \in \left[0, \frac{1}{2}\right[$ *then the frequency value* $\lambda_1 = \pi$ *being the maximum point of the spectral density function* $f(\lambda)$ *and*

$$f(\lambda_1) = \frac{1}{8\pi}\frac{1 - \varepsilon}{\varepsilon}.$$

If $\varepsilon \in \left]\frac{1}{2}, 1\right]$ *then the frequency value* $\lambda_2 = 0$ *being the maximum point of the spectral density function* $f(\lambda)$ *and*

$$f(\lambda_2) = \frac{1}{8\pi}\frac{\varepsilon}{1 - \varepsilon}.$$

Proof. Consider the function $f(\lambda)$, defined by equation (3). It is easy to obtain that for the $\varepsilon \in \left[0, \frac{1}{2}\right[$ the frequency value $\lambda_1 = \pi$ being the maximum point of the function $f(\lambda)$, and the frequency value $\lambda_2 = 0$ being the minimum point of $f(\lambda)$. Also for the $\varepsilon \in \left]\frac{1}{2}, 1\right]$ we can obtain that the frequency value $\lambda_1 = \pi$ being the minimum point and $\lambda_2 = 0$ – maximum point of the function $f(\lambda)$.

Having calculated the function $f(\lambda)$ in the points λ_1 and λ_2 we get

$$f(\lambda_1) = f(\pi) = \frac{1}{8\pi}\frac{1 - \varepsilon}{\varepsilon}, \tag{12}$$

$$f(\lambda_2) = f(0) = \frac{1}{8\pi}\frac{\varepsilon}{1 - \varepsilon}. \tag{13}$$

□

References

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