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ONE APPROACH TO THE ANALYTICAL SOLUTION OF A TWO-DIMENSIONAL NONSTATIONARY PROBLEM OF HEAT CONDUCTION IN REGIONS WITH MOVING BOUNDARIES ON THE MODEL OF A HALF-SPACE

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With the use of the solution of the Dirichlet nonstationary problem with discontinuous unmixed boundary conditions on the surface of an isotropic half-space a two-dimensional model of the problem with a moving phase boundary is considered. The problem models, for example, the processes of freezing of moist ground or the processes of formation of ice in stagnant water if a temperature lower than the freezing temperature is prescribed on the boundary surface in a circular region of finite radius. The classical one-dimensional result follows as a particular case from solution of this problem for an infinite radius of the circle.

Many processes of heat exchange are associated with a change in the physicochemical nature of a material or a substance in regions with moving boundaries. The thermophysical coefficients of transfer can change in steps, and a certain heat of transformation (in freezing, melting, evaporation, and crystallization) or heat of chemical reactions is required for thermodynamic transformations (transitions). Rather voluminous references on this subject are given in the review work [1] devoted to the methods of solution of the boundary-value problems of heat conduction in regions with moving boundaries, which is attributed, first of all, to the importance of such investigations for different practical applications in nuclear power engineering, space and laser technology, ecology, medicine, structural thermal physics, etc.

In the monographs [2, p. 421, and 3, p. 256], the one-dimensional nonstationary problem of Stefan is investigated in studying the processes of freezing of moist ground and the processes of formation of ice in stagnant water. The model of a half-space is used as the physical model of the body under study, while the mathematical model is based on a system of two one-dimensional differential equations of nonstationary heat conduction in the presence of a moving boundary of phase transformation on a plane interface of the corresponding phases and in the presence of the corresponding conditions. Thus, for example, for the one-dimensional problem of freezing of moist ground the conditions at the phase boundary are written in the form [2]

$$T_1(\xi, \tau) = T_2(\xi, \tau) = T_{\mathrm{fr}}\,, \quad \lambda_1 \frac{\partial T_1(\xi, \tau)}{\partial x} - \lambda_2 \frac{\partial T_2(\xi, \tau)}{\partial x} = \rho^* W \gamma_2 \frac{d\xi}{d\tau}\,,$$

where $T_1(x, \tau)$ and $T_2(x, \tau)$ are the temperature fields of the frozen $(0 < x < \xi)$ and moist $(\xi < x < \infty)$ regions of the ground respectively, λ_1 and λ_2 are the coefficients of thermal conductivity in the corresponding regions, T_{fr} = const is the freezing temperature, W is the moisture content of the ground, γ_2 is the density of

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the ground, ρ^* is the heat of phase transformation, $\xi = \xi(\tau)$ is the law of movement (generally speaking, unknown) of the phase boundary, x is the space variable, and $\tau \geq 0$ is the time variable. In [2], the solution is considered and an analysis is made of the corresponding temperature fields in the case of freezing of moist ground, while in [3] the corresponding process of formation of ice in stagnant water ($T_{fr} \equiv 0$) is described.

Let us consider the formulation and solution of the corresponding problem with a moving boundary for a two-dimensional case. We assume that the moist ground is in a melted (thawed) state and has a uniform (throughout the volume) initial temperature distribution $T(r, z, 0) = T_0 > 0$, where $r > 0$ and $z > 0$ are the cylindrical coordinates. At the initial time $\tau = 0$, a certain temperature of the medium T_m = const is instantaneously established in the circular region $0 < r < R$ on the surface of the moist ground $z = 0$; with all the changes this temperature is lower than the freezing temperature ($T_m < T_{fr}$). As a result, for certain values of R, T_m, T_0, and T_{fr} a frozen layer can be formed; this layer is related to the velocity of motion of the phase boundary along the normal to the limiting isothermal surface with a temperature $T(\eta, \xi, \tau)$, where $\eta = \eta(\tau)$ and $\xi = \xi(\tau)$. We note that the limiting moving boundary $0 < \eta < R$, $\xi > 0$ has the freezing temperature T_{fr}. At this moving boundary, the transition from one aggregative state to another occurs, which requires a certain heat of phase transformation ρ^*. We will assume that when $z, r \to \infty$ the boundaries of the melted zone have a certain constant temperature of the ground and the transfer coefficients (a, thermal diffusivity, c, specific heat, λ, thermal conductivity, and $b = \lambda/\sqrt{a}$, thermal activity) of the frozen and melted zones are different. We also assume that the transfer of heat in the ground occurs only due to the process of heat conduction.

In a locally frozen ground, there can be two zones (the zone of frozen ground $r < \eta < R$, $z < \xi$, i.e., the zone with index 1, and the zone of melted ground $\eta < r < R$, $z > \xi$, i.e., the zone with index 2). The temperature change in these zones is described by the heat-conduction equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1(r, z, \tau)}{\partial r}\right) + \frac{\partial^2 T_1(r, z, \tau)}{\partial z^2} = \frac{1}{a_1}\frac{\partial T_1(r, z, \tau)}{\partial \tau}, \quad 0 < r < \eta < R, \quad 0 < z < \xi, \quad \tau > 0; \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2(r, z, \tau)}{\partial r}\right) + \frac{\partial^2 T_2(r, z, \tau)}{\partial z^2} = \frac{1}{a_2}\frac{\partial T_2(r, z, \tau)}{\partial \tau}, \quad 0 < \eta < r < \infty, \quad \xi < z < \infty, \quad \tau > 0, \tag{2}$$

with the boundary conditions

$$T_1(r, 0, \tau) = T_m, \quad 0 < r < R, \quad \tau > 0; \tag{3}$$

$$T_2(r, 0, \tau) = T_0, \quad R < r < \infty, \quad \tau > 0; \tag{4}$$

$$T_1(r, z, 0) = T_2(r, z, 0) = T_0, \quad r > 0, \quad z > 0; \tag{5}$$

$$\frac{\partial T_2(r, \infty, \tau)}{\partial z} = 0, \quad r > 0, \quad \tau > 0; \tag{6}$$

$$\frac{\partial T_2(\infty, z, \tau)}{\partial r} = 0, \quad z > 0, \quad \tau > 0, \tag{7}$$

and the symmetry condition

$$\frac{\partial T_2(0, z, \tau)}{\partial r} = \frac{\partial T_1(0, z, \tau)}{\partial r} = 0\,, \quad z > 0\,, \quad \tau > 0\,. \tag{8}$$

At the interface between the frozen ground and the melted ground we have the condition

$$T_1(\eta, \xi, \tau) = T_2(\eta, \xi, \tau) = T_{\text{fr}} \tag{9}$$

and the nonlinear boundary condition of freezing

$$\lambda_1 (\nabla T_1 \cdot \mathbf{n}) - \lambda_2 (\nabla T_2 \cdot \mathbf{n}) = \rho^* W \gamma_2 (\mathbf{v} \cdot \mathbf{n})\,, \tag{10}$$

where $\nabla T_i = \frac{\partial T_i}{\partial r}\mathbf{i} + \frac{\partial T_i}{\partial z}\mathbf{j}$, $\mathbf{v} = \frac{d\eta(\tau)}{d\tau}\mathbf{i} + \frac{d\xi(\tau)}{d\tau}\mathbf{j}$ is the vector of the velocity of motion of the phase boundary and n is the vector of the normal to the isothermal surface of phase transformation. Under certain assumptions in which the vector of the velocity of motion of the phase boundary is taken into account in the multidimensional case we can write condition (10) as the system of two equations

$$\lambda_1 \frac{\partial T_1(\eta, \xi, \tau)}{\partial z} - \lambda_2 \frac{\partial T_2(\eta, \xi, \tau)}{\partial z} = \rho^* W \gamma_2 \frac{d\xi}{d\tau}\,, \tag{11}$$

$$\lambda_1 \frac{\partial T_1(\eta, \xi, \tau)}{\partial r} - \lambda_2 \frac{\partial T_2(\eta, \xi, \tau)}{\partial r} = \rho^* W \gamma_2 \frac{d\eta}{d\tau}\,. \tag{12}$$

Thus, the problem of local freezing of moist ground in the cylindrical region $0 < r < R$, $z > 0$ for $\tau > 0$ can be formulated as the problem of conjugation of the corresponding temperature fields in the presence of special boundary conditions at moving phase boundaries.

Let us use the known general solution of the two-dimensional nonstationary problem of Dirichlet for an orthotropic half-space with discontinuous boundary conditions of the first kind [4]; we will seek the solutions of the differential equations (1) and (2) with boundary conditions (3)–(8) in the form

$$T_1(r, z, \tau) = A_1 - \frac{B_1}{2}\int_0^\infty \left[\Phi_1^{(-)}(R, z, \tau, x) + \Phi_1^{(+)}(R, z, \tau, x)\right] J_{1,0}(R, x, r)\, dx\,, \tag{13}$$

$$T_2(r, z, \tau) = A_2 - \frac{B_2}{2}\int_0^\infty \left[\Phi_2^{(-)}(R, z, \tau, x) + \Phi_2^{(+)}(R, z, \tau, x)\right] J_{1,0}(R, x, r)\, dx\,, \tag{14}$$

where $J_{k,m}(R, x, r) = J_k(x) J_m\left(\frac{r}{R}x\right)$ and $J_k(u)$ and $J_m(u)$ are the Bessel functions of the real argument of the kth and mth order respectively [5, p. 526];

$$\Phi_1^{(-)}(R, z, \tau, x) = \exp\left(-\frac{z}{R}x\right)\text{erfc}\left(\frac{z}{2\sqrt{a_1\tau}} - \frac{\sqrt{a_1\tau}}{R}x\right);$$

$$\Phi_1^{(+)}(R, z, \tau, x) = \exp\left(\frac{z}{R}x\right)\text{erfc}\left(\frac{z}{2\sqrt{a_1\tau}} + \frac{\sqrt{a_1\tau}}{R}x\right);$$

$$\Phi_2^{(-)}(R, z, \tau, x) = \exp\left(-\frac{z}{R}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_2\tau}} - \frac{\sqrt{a_2\tau}}{R}x\right);$$

$$\Phi_2^{(+)}(R, z, \tau, x) = \exp\left(\frac{z}{R}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_2\tau}} + \frac{\sqrt{a_2\tau}}{R}x\right);$$

A_i and B_i, $i = 1, 2$, are the unknown constants, and the additional probability integral is

$$\operatorname{erfc}(u) = 1 - \operatorname{erf}(u) = \frac{2}{\sqrt{\pi}}\int_u^\infty \exp(-t^2)\, dt\,.$$

We find the constants A_1 and A_2 from boundary conditions (3)–(5). Since erfc $(\infty) = 0$ and the discontinuous integral is equal to [5, p. 79; 6, p. 209]

$$\int_0^\infty J_1(x)\, J_0\left(\frac{r}{R}x\right) dx = \int_0^\infty J_{1,0}(R, x, r)\, dx = U(R - r) = \begin{cases} 1\,, & r < R\,, \\ 1/2\,, & r = R\,, \\ 0\,, & r > R\,, \end{cases}$$

we have $A_1 = T_{\rm m} + B_1 U(R - r)$ and $A_2 = T_0$, where $U(x)$ is the symmetric unit function.

Consequently, the solutions (13) and (14) will take the form

$$T_1(r, z, \tau) = T_{\rm m} + B_1 U(R - r) - \frac{B_1}{2}\int_0^\infty \left[\Phi_1^{(-)}(R, z, \tau, x) + \Phi_1^{(+)}(R, z, \tau, x)\right] J_{1,0}(R, x, r)\, dx\,; \tag{15}$$

$$T_2(r, z, \tau) = T_0 - \frac{B_2}{2}\int_0^\infty \left[\Phi_2^{(-)}(R, z, \tau, x) + \Phi_2^{(+)}(R, z, \tau, x)\right] J_{1,0}(R, x, r)\, dx\,, \quad r < R\,. \tag{16}$$

We note that the solutions (15) and (16) satisfy the heat-conduction equations (1) and (2) and conditions (3)–(5).

From condition (9) it follows that

$$T_{\rm m} + B_1 U(R - \eta) - \frac{B_1}{2}\int_0^\infty \left[\Phi_1^{(-)}(R, \xi, \tau, x) + \Phi_1^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\, dx =$$

$$= T_0 - \frac{B_2}{2}\int_0^\infty \left[\Phi_2^{(-)}(R, \xi, \tau, x) + \Phi_2^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\, dx = T_{\rm fr}\,. \tag{17}$$

Considering that the last equation is fulfilled for any $\tau > 0$, we find

$$B_1 = \frac{2(T_{\rm fr} - T_{\rm m})}{2U(R - \eta) - \int_0^\infty \left[\Phi_1^{(-)}(R, \xi, \tau, x) + \Phi_1^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\, dx}\,, \tag{18}$$

$$B_2 = \frac{2\,(T_0 - T_{fr})}{\int_0^\infty \left[\Phi_2^{(-)}(R, \xi, \tau, x) + \Phi_2^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\,dx}\,. \tag{19}$$

Thus, in the case of finite $R > 0$ and $\tau > 0$ the values of B_1 and B_2 are not constant and depend on ξ and η, which in turn are functions of time. Consequently, the solutions $T_1(r, z, \tau)$ and $T_2(r, z, \tau)$ can be written in the form

$$T_1(r, z, \tau) = T_m + \frac{(T_{fr} - T_m)\left\{2U(R - r) - \int_0^\infty \left[\Phi_1^{(-)}(R, z, \tau, x) + \Phi_1^{(+)}(R, z, \tau, x)\right] J_{1,0}(R, x, r)\,dx\right\}}{2 - \int_0^\infty \left[\Phi_1^{(-)}(R, \xi, \tau, x) + \Phi_1^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\,dx}\,, \tag{20}$$

$$T_2(r, z, \tau) = T_0 - \frac{(T_0 - T_{fr}) \int_0^\infty \left[\Phi_2^{(-)}(R, z, \tau, x) + \Phi_2^{(+)}(R, z, \tau, x)\right] J_{1,0}(R, x, r)\,dx}{\int_0^\infty \left[\Phi_2^{(-)}(R, \xi, \tau, x) + \Phi_2^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\,dx}\,. \tag{21}$$

Substituting the found expressions (20) and (21) into the boundary conditions of freezing (11) and (12), we arrive at two integro-differential characteristic equations from which we determine the moving coordinates $\xi(\tau)$ and $\eta(\tau)$:

$$\lambda_1 (T_{fr} - T_m) \frac{I_1(R, \xi, \eta, \tau)}{2 - S_1(R, \xi, \eta, \tau)} - \lambda_2 (T_0 - T_{fr}) \frac{I_2(R, \xi, \eta, \tau)}{S_2(R, \xi, \eta, \tau)} = \rho^* W \gamma_2 \frac{d\xi}{d\tau}\,, \tag{22}$$

$$\lambda_1 (T_{fr} - T_m) \frac{-2\delta(R - \eta) + G_1(R, \xi, \eta, \tau)}{2 - S_1(R, \xi, \eta, \tau)} - \lambda_2 (T_0 - T_{fr}) \frac{G_2(R, \xi, \eta, \tau)}{S_2(R, \xi, \eta, \tau)} = \rho^* W \gamma_2 \frac{d\eta}{d\tau}\,, \tag{23}$$

where $\delta(x)$ is the Dirac function (symmetric unit impulse function);

$$I_i(R, \xi, \eta, \tau) = \int_0^\infty \left[\frac{x}{R}\left(\Phi_i^{(-)}(R, \xi, \tau, x) - \Phi_i^{(+)}(R, \xi, \tau, x)\right) + \frac{2}{\sqrt{\pi a_i \tau}} \exp\left(-\frac{\xi^2}{4 a_i \tau} - \frac{a_i \tau}{R^2} x^2\right)\right] J_{1,0}(R, x, \eta)\,dx\,,$$

$$G_i(R, \xi, \eta, \tau) = \int_0^\infty \frac{x}{R}\left[\Phi_i^{(-)}(R, \xi, \tau, x) + \Phi_i^{(+)}(R, \xi, \tau, x)\right] J_{1,1}(R, x, \eta)\,dx\,;$$

$$S_i(R, \xi, \eta, \tau) = \int_0^\infty \left[\Phi_i^{(-)}(R, \xi, \tau, x) + \Phi_i^{(+)}(R, \xi, \tau, x)\right] J_{1,0}(R, x, \eta)\,dx\,.$$

The formulated problem (1)–(9), (11), and (12) is actually solved. The main difficulty of practical use of the obtained equations (20) and (21) lies in identifying the moving boundaries $\xi(\tau)$ and $\eta(\tau)$ based on Eqs. (22) and (23). It should be noted that this difficulty holds for the simplest one-dimensional solutions of the problems with a moving boundary.

Let us consider the limiting case where the radius of a circle R on the surface of an isotropic half-space tends to infinity, i.e., where we have the case of a one-dimensional problem [2]. In this case, when $R \to \infty$, Eq. (17) takes the form

$$T_{\mathrm{m}} + B_1 \operatorname{erf}\left(\frac{\xi}{2\sqrt{a_1\tau}}\right) = T_0 - B_2 \operatorname{erfc}\left(\frac{\xi}{2\sqrt{a_2\tau}}\right) = T_{\mathrm{fr}} = \mathrm{const}\,, \tag{24}$$

and since in this one-dimensional case B_1 and B_2 are constant for any value $\tau > 0$, it is clear that the ratio $\xi/\sqrt{\tau}$ must also be a constant, i.e., we can write that $\xi = \beta\sqrt{\tau}$, where β is the proportionality factor characterizing the velocity of deepening of the freezing zone in the one-dimensional case.

Thus, from (24) we find that

$$B_1 = (T_{\mathrm{fr}} - T_{\mathrm{m}}) \Big/ \operatorname{erf}\left(\frac{\beta}{2\sqrt{a_1}}\right), \quad B_2 = (T_0 - T_{\mathrm{fr}}) \Big/ \operatorname{erfc}\left(\frac{\beta}{2\sqrt{a_2}}\right). \tag{25}$$

Consequently, the corresponding limiting solutions from (20) and (21) with account for the value of the improper integral $\int_0^\infty J_1(x)dx = 1$ (see, for example, [6]) will be written in the form

$$\lim_{R\to\infty} T_1(r, z, \tau) = T_1(z, \tau) = T_{\mathrm{m}} + (T_{\mathrm{fr}} - T_{\mathrm{m}}) \operatorname{erf}\left(\frac{z}{2\sqrt{a_1\tau}}\right) \Big/ \operatorname{erf}\left(\frac{\beta}{2\sqrt{a_1}}\right), \tag{26}$$

$$\lim_{R\to\infty} T_2(r, z, \tau) = T_2(z, \tau) = T_0 - (T_0 - T_{\mathrm{fr}}) \operatorname{erfc}\left(\frac{z}{2\sqrt{a_2\tau}}\right) \Big/ \operatorname{erfc}\left(\frac{\beta}{2\sqrt{a_2}}\right), \tag{27}$$

which is totally consistent with the solutions of the one-dimensional problem [2, p. 425].

The value of the coefficient β is determined from the formula

$$\lambda_1 \frac{T_{\mathrm{fr}} - T_{\mathrm{m}}}{\sqrt{a_1} \operatorname{erf}\left(\frac{\beta}{2\sqrt{a_1}}\right)} \exp\left(-\frac{\beta^2}{4a_1}\right) - \lambda_2 \frac{T_0 - T_{\mathrm{fr}}}{\sqrt{a_2} \operatorname{erfc}\left(\frac{\beta}{2\sqrt{a_2}}\right)} \exp\left(-\frac{\beta^2}{4a_2}\right) = \frac{1}{2}\sqrt{\pi}\, \rho^* W \gamma_2 \beta\,, \tag{28}$$

obtained from (22) for $R \to \infty$, while the method of determination of β from Eq. (28) can be found, for example, in [2, p. 425].

We note that when $T_{\mathrm{fr}} = 0$ and $W = 1$ the value of β in (28) is related to the process of formation of ice in stagnant water (this process is investigated in [3, p. 256]). The corresponding formulas (26)–(28) are simplified, especially in the case $T_0 = 0$.

In conclusion, we note that in investigating the process of local freezing of moist ground in the case of the presence of mixed discontinuous boundary conditions on its surface (for example, the temperature T_{m} is prescribed in the circle $0 < r < R$ and the condition of ideal heat insulation is prescribed beyond the circle $R < r < \infty$), one must solve the corresponding two-dimensional nonstationary problem for an isotropic half-space, which is associated with investigation of the so-called pair integral equations with the *L*-parameter (see, for example, the monograph [5, Section 2]).

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