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SHORT COMMUNICATIONS

Method for Solving Nonstationary Heat Problems with Mixed Discontinuous Boundary Conditions on the Boundary of a Half-Space

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Suppose that we must find a solution of the differential nonstationary ($\tau > 0$) heat equation

$$\theta_{rr}(r,z,\tau) + r^{-1}\theta_r(r,z,\tau) + \theta_{zz}(r,z,\tau) = a^{-1}\theta_\tau(r,z,\tau) \tag{1}$$

in the cylindrical coordinates ($r, z > 0$) for a half-space with the homogeneous initial condition

$$\theta(r,z,0) = 0 \tag{2}$$

and the mixed boundary conditions

$$\theta(r,0,\tau) = f_1(r)f_2(\tau), \qquad 0 < r < R, \qquad \theta_z(r,0,\tau) = 0, \qquad R < r < \infty, \tag{3}$$

on the surface $z = 0$ in the case of the axial symmetry $\theta_r(0,z,\tau) = 0$.

If to problem (1)–(3), we apply the Laplace integral transform (the L-transform)

$$\bar\theta(r,z,s) = L[\theta(r,z,\tau)] = \int\limits_0^\infty \theta(r,z,\tau)\exp(-s\tau)d\tau, \qquad \operatorname{Re} s > 0,$$

then the problem acquires the form

$$\bar\theta_{rr}(r,z,s) + r^{-1}\bar\theta_r(r,z,s) + \bar\theta_{zz}(r,z,s) = \sigma\bar\theta(r,z,s), \qquad \sigma = s/a, \tag{4}$$

$$\bar\theta(r,0,s) = f_1(r)\bar f_2(s), \qquad 0 < r < R, \qquad \bar\theta_z(r,0,s) = 0, \qquad R < r < \infty, \tag{5}$$

where

$$\bar f_2(s) = \int\limits_0^\infty f_2(\tau)\exp(-s\tau)d\tau. \tag{6}$$

Theorem. *Suppose that the integral* (6) *exists, the real part of the parameter s is positive* (*i.e.*, $\operatorname{Re} s > 0$), *and the temperature remains bounded as* $\sqrt{r^2+z^2} \to \infty$. *Then the solution of the problem can be represented via L-transforms in the form*

$$\begin{aligned}
\bar\theta(r,0,s) = {} & \frac{2\bar f_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\right)\int\limits_0^\infty \exp\left(-z\sqrt{p^2+\sigma}\right)\frac{pJ_0(pr)dp}{\sqrt{p^2+\sigma}} \\
& \times \int\limits_0^R \cos\left(t\sqrt{p^2+\sigma}\right)\left[\frac{d}{dt}\int\limits_0^t F_1(t,\mu)\right]dt \\
& + \frac{\bar f_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\right)\int\limits_0^\infty \exp\left(-z\sqrt{p^2+\sigma}\right)\frac{pJ_0(pr)dp}{\sqrt{p^2+\sigma}}\sum_{k=0}^\infty s^{(k+1)/2}\int\limits_0^R \cos\left(t\sqrt{p^2+\sigma}\right) \\
& \times \left[\frac{d}{dt}\int\limits_0^t D_{k+1}(R,t,\mu)F_1(t,\mu)d\mu + \sum_{m=0}^{k+1}\int\limits_0^R C_m(x,t)\varphi_{k-m+1}(x)dx\right]dt,
\end{aligned} \tag{7}$$

where

$$C_m(x,t) = (1/m!)\left(\sqrt{a}\,\right)^{-m} \sin(m\pi/2)\left[(x-t)^{m-1} + (x+t)^{m-1}\right], \tag{8}$$

$$D_n(R,t,\mu) = \frac{2}{n!}\left(\sqrt{a}\,\right)^{-n} \sum_{j=0}^{n} \binom{n}{j} R^{n-j} \cos\left(\frac{j\pi}{2}\right)\left(\sqrt{t^2-\mu^2}\,\right)^{j}, \tag{9}$$

$F_1(t,\mu) = f_1(\mu)\mu/\sqrt{t^2-\mu^2}$, *the* $\binom{n}{j}$ *are the binomial coefficients, and* $J_0(pr)$ *is the Bessel function of the first kind.*

Proof. The solution of Eq. (4) has the form [1]

$$\bar\theta(r,z,s) = \int_0^\infty \bar C(p,s)\exp\left(-z\sqrt{p^2+\sigma}\,\right) J_0(pr)dp, \tag{10}$$

where $\bar C(p,s)$ is the unknown function for which the following dual integral equations in terms of L-transforms can be obtained from (5):

$$\begin{aligned} &\int_0^\infty \bar C(p,s) J_0(pr)dp = f_1(r)\bar f_2(s), \qquad 0<r<R,\\ &\int_0^\infty \bar C(p,s)\sqrt{p^2+\sigma}\,J_0(pr)dp = 0, \qquad R<r<\infty. \end{aligned} \tag{11}$$

If we introduce a new unknown function $\bar\varphi(t,s)$ by the formula

$$\bar C(p,s) = \frac{p}{\sqrt{p^2+\sigma}} \int_0^R \bar\varphi(t,s)\cos\left(t\sqrt{p^2+\sigma}\,\right)dt, \tag{12}$$

then, using the substitution (12), we can readily show that the second dual integral equation in (11) is automatically valid with regard for the following well-known value of the discontinuous integral [2, p. 203]:

$$\int_0^\infty \frac{pJ_0(pr)}{\sqrt{p^2+\sigma}} \sin\left(x\sqrt{p^2+\sigma}\,\right)dp = \begin{cases} 0 & \text{if } x<r \\ \cos\left(\sqrt{(x^2-r^2)\,\sigma}\,\right)(x^2-r^2)^{-1/2} & \text{if } x>r. \end{cases}$$

The substitution of (12) into the first equation in (11) gives the integral equation

$$\int_0^r \frac{\bar\varphi(t,s)}{\sqrt{r^2-t^2}} \exp\left(-\sqrt{(r^2-t^2)\,\sigma}\,\right)dt - \int_r^R \frac{\bar\varphi(t,s)}{\sqrt{t^2-r^2}} \sin\left(\sqrt{(t^2-r^2)\,\sigma}\,\right)dt = f_1(r)\bar f_2(s), \quad 0<r<R, \tag{13}$$

with regard for the following well-known value of the discontinuous integral [2, p. 203]:

$$\int_0^\infty \frac{pJ_0(pr)}{\sqrt{p^2+\sigma}} \cos\left(t\sqrt{p^2+\sigma}\,\right)dp = \begin{cases} \exp\left(-\sqrt{(r^2-t^2)\,\sigma}\,\right)(r^2-t^2)^{-1/2} & \text{if } t<r \\ \sin\left(-\sqrt{(t^2-r^2)\,\sigma}\,\right)(t^2-r^2)^{-1/2} & \text{if } t>r. \end{cases}$$

The integral equation (13) in terms of L-transforms is a basis for finding the unknown function $\bar{\varphi}(r,s)$. We reduce it to a Fredholm integral equation of the second kind (but with the parameter s). To this end, we divide both sides of Eq. (13) by $\bar{f}_2(s) \neq 0$, replace r by μ, and multiply the resulting relation by the integrating factor $2\mu\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)(r^2-\mu^2)^{-1/2}$. Further, integrating both sides of the equation with respect to μ from 0 to r, we obtain

$$\int_0^r \frac{\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{(r^2-\mu^2)}}\mu\int_0^\mu \frac{\bar{\varphi}^*(t,s)}{\sqrt{\mu^2-t^2}}\exp\left(-\sqrt{(\mu^2-t^2)\,\sigma}\right)dt\,d\mu$$
$$-\int_0^r \frac{\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\int_\mu^R \frac{\bar{\varphi}^*(t,s)}{\sqrt{t^2-\mu^2}}\sin\left(\sqrt{(t^2-\mu^2)\,\sigma}\right)dt\,d\mu$$
$$=\int_0^r \frac{f_1(\mu)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\,d\mu,\qquad 0<r<R,$$

where

$$\bar{\varphi}^*(t,s)=\bar{\varphi}(t,s)/\left(s\bar{f}_2(s)\right). \tag{14}$$

Changing the order of integration on the left-hand side of the last relation and computing the resulting integrals, we arrive at the integral equation

$$\int_0^r \bar{\varphi}^*(t,s)dt-\frac{1}{\pi}\int_0^R \bar{\varphi}^*(t,s)\left[\operatorname{si}\left((t+r)\sqrt{\sigma}\,\right)-\operatorname{si}\left((t-r)\sqrt{\sigma}\,\right)\right]dt$$
$$=\frac{2}{\pi s}\int_0^r \frac{f_1(\mu)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\,d\mu,\qquad 0<r<R,$$

where $\operatorname{si}(x)$ is the sine integral [3, p. 59 of the Russian translation]. Hence, differentiating both sides with respect to r, we obtain the following integral equation in terms of L-transforms for the function $\bar{\varphi}^*(r,s)$:

$$\bar{\varphi}^*(r,s)-\frac{1}{\pi}\int_0^R \bar{\varphi}^*(t,s)\bar{K}(r,t,s)dt=\bar{F}(r,s),\qquad 0<r<R, \tag{15}$$

where

$$\bar{K}(r,t,s)=\frac{\sin\left((t-r)\sqrt{\sigma}\,\right)}{t-r}+\frac{\sin\left((t+r)\sqrt{\sigma}\,\right)}{t+r}, \tag{16}$$

$$\bar{F}(r,s)=\frac{2}{s\pi}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\,d\mu. \tag{17}$$

Note that the solution of problem (15)–(17) with $f_1(\mu)\equiv 1$ was found in [4]. Here we find the solution in the case of a general function $f_1(\mu)$.

We represent the unknown analytic function $\bar{\varphi}^*(r,s)$ as the function series

$$\bar{\varphi}^*(r,s)=\frac{1}{s}\exp\left(-R\sqrt{\sigma}\,\right)\sum_{n=0}^{\infty}\varphi_n(r)\left(\sqrt{s}\,\right)^n,\qquad 0<r<R, \tag{18}$$

where the $\varphi_n(r)$ are some auxiliary functions. The kernel $\bar{K}(r,t,s)$ given by (16) can be represented as

$$\bar{K}(r,t,s) = \sum_{k=0}^{\infty} C_k(t,r)\left(\sqrt{s}\right)^k,$$

and

$$\exp\left(R\sqrt{\sigma}\right)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right) = \frac{1}{2}\sum_{n=0}^{\infty} D_n(R,r,\mu)\left(\sqrt{s}\right)^n,$$

where $C_k(t,r)$ and $D_n(R,r,\mu)$ are given by (8) and (9), respectively.

Substituting these expressions into (15) and performing related multiplications of series, we arrive at the relation

$$\pi\sum_{n=0}^{\infty}\varphi_n(r)\left(\sqrt{s}\right)^{n-2} = \sum_{n=0}^{\infty}\left[\frac{d}{dr}\int_0^r D_n(R,r,\mu)f_1(\mu)\frac{\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \sum_{m=0}^{n}\int_0^R C_m(t,r)\varphi_{n-m}(t)\,dt\right]\left(\sqrt{s}\right)^{n-2},$$

$0<r<R$, which is not an integral equation for $\varphi_{n-m}(t)$, since, obviously, $C_0(t,r)\equiv 0$.

Therefore, for the unknown auxiliary functions $\varphi_n(r)$, we can readily write out the recursion formula

$$\varphi_n(r) = \frac{1}{\pi}\frac{d}{dr}\int_0^r D_n(R,r,\mu)\frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \frac{1}{\pi}\sum_{m=0}^{n}\int_0^R C_m(t,r)\varphi_{n-m}(t)\,dt, \qquad 0<r<R,$$

which, in particular, implies that

$$\varphi_0(r) = \frac{2}{\pi}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}}, \qquad \varphi_1(r) = \frac{2R}{\pi\sqrt{a}}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \frac{4}{\pi^2\sqrt{a}}\int_0^R \frac{f_1(\mu)\mu\,d\mu}{\sqrt{R^2-\mu^2}},$$

and so on.

Substituting the expression for $\varphi_n(r)$ into (18), we can find the transform $\bar{\varphi}^*(r,s)$, for which the inverse Laplace transform exists, since

$$L^{-1}\left[\frac{1}{s}\exp\left(-R\sqrt{\sigma}\right)\right] = \operatorname{erfc}\left(\frac{R}{2\sqrt{a\tau}}\right),$$

$$L^{-1}\left[s^{(k-1)/2}\exp\left(-R\sqrt{\sigma}\right)\right] = \frac{1}{2^k\sqrt{\pi\tau^{k+1}}}\exp\left(-\frac{R^2}{4a\tau}\right)H_k\left(\frac{R}{2\sqrt{a\tau}}\right),$$

where $\operatorname{erfc}(x) = (2/\sqrt{\pi}\,)\int_x^{\infty}\exp(-t^2)\,dt$ and $H_k(x)$ is the Hermite polynomial [3, p. 579 of the Russian translation]. Performing related substitutions, we obtain

$$\bar{\varphi}(r,s) = \frac{2\bar{f}_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\right)\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \frac{\bar{f}_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\right)\sum_{k=0}^{\infty}s^{(k+1)/2}\left[\frac{d}{dr}\int_0^r D_{k+1}(R,r,\mu)\frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \sum_{m=0}^{k+1}\int_0^R C_m(t,r)\varphi_{k-m+1}(t)dt\right].$$

Then, using (12) and (10), we can write out the solution of the original problem in terms of L-transforms in the form (7), which, together with the inversion formula for the Laplace integral [5, p. 205 of the Russian translation], gives $\theta(r, s, \tau)$. This completes the proof of the theorem.

Note that if $f_1(\mu) = T_c - T_0 = \text{const} \neq 0$, where T_0 is the initial temperature of the half-space and T_c is the temperature on the surface $z = 0$ in the disk $0 < r < R$, then the inner integrals in the solution (7) can readily be evaluated:

$$\int_0^R \cos\left(t\sqrt{p^2+\sigma}\right)\left[\frac{d}{dt}\int_0^t \frac{(T_c - T_0)\,\mu\,d\mu}{\sqrt{t^2-\mu^2}}\right]dt = (T_c - T_0)\,\frac{\sin\left(R\sqrt{p^2+\sigma}\right)}{\sqrt{p^2+\sigma}},$$

$$\int_0^R \cos\left(t\sqrt{p^2+\sigma}\right)\left[\frac{d}{dt}\int_0^t D_{k+1}(R,t,\mu)\frac{(T_c - T_0)\,\mu\,d\mu}{\sqrt{t^2-\mu^2}}\right]dt$$

$$= (T_c - T_0)\sum_{j=0}^{k+1}\sum_{n=0}^{j}\frac{(j+1)n!}{(k+1)!}\binom{k+1}{j}\binom{j}{n}B\left(1,\frac{j+1}{2}\right)\left(\frac{R}{\sqrt{a}}\right)^{k+1}$$

$$\times\cos\left(\frac{j\pi}{2}\right)\frac{\sin\left(R\sqrt{p^2+\sigma}+n\pi/2\right)}{R^n\left(\sqrt{p^2+\sigma}\right)^{n+1}},$$

where $B(\alpha, \beta)$ is the beta function [6, p. 24].

If $s \to 0$ ($\tau \to \infty$), then

$$\lim_{s\to 0}\left[s\bar{\varphi}(r,s)\right] = \frac{2}{\pi}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}}\lim_{s\to 0}\left[s\bar{f}_2(s)\right];$$

in particular, if $f_1(\mu) = T_c - T_0$, then we have $\lim_{s\to 0}\left[s\bar{\varphi}(r,s)\right] = (2/\pi)\,(T_c - T_0)$, which coincides with the similar solutions obtained in [7, 8] for the corresponding stationary Laplace equation with mixed boundary conditions.

REFERENCES

1. Kozlov, V.P., Yurchuk, N.I., and Mandrik, P.A., *Inzhenerno-Fiz. Zh.*, 1998, vol. 71, no. 4, pp. 734–743.
2. Prudnikov, A.P., Brychkov, Yu.A., and Marichev, O.I., *Integraly i ryady (spetsial'nye funktsii)* (Integrals and Series (Special Functions)), Moscow, 1983.
3. Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Washington, 1964. Translated under the title *Spravochnik po spetsial'nym funktsiyam* (Reference Book in Special Functions), Moscow, 1979.
4. Kozlov, V.P., Yurchuk, N.I., and Mandrik, P.A., *Vestn. Bel. Univ. Ser. 1*, 1999, no. 2, pp. 37–42.
5. Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F.G., *Tables of Integral Transforms*, vol. I, New York: McGraw-Hill, 1954. Translated under the title *Tablitsy integral'nykh preobrazovanii. T.1. Preobrazovaniya Fur'e, Laplasa, Mellina*, Moscow, 1969.
6. Kozlov, V.P., *Dvumernye osesimmetrichnye nestatsionarnye zadachi teploprovodnosti* (Two-Dimensional Axisymmetric Nonstationary Heat Problems), Minsk, 1986.
7. Uflyand, Ya.S., *Metod parnykh uravnenii v zadachakh matematicheskoi fiziki* (The Method of Dual Equations in Problems of Mathematical Physics), Leningrad, 1977.
8. Sneddon, I., *Mixed Boundary Value Problems in Potential Theory*, Amsterdam, 1966.