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SOLUTION OF MIXED CONTACT PROBLEMS IN THE THEORY OF NONSTATIONARY HEAT CONDUCTION BY THE METHOD OF SUMMATION-INTEGRAL EQUATIONS

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The laws governing the development of spatial nonstationary temperature fields in a bounded cylinder and a half-space where one of the end surfaces of the cylinder touches the surface of the half-space in a circular region are determined. A solution of a mixed axisymmetric nonstationary problem of heat conduction is obtained in the region of Laplace transforms. In solution of this problem, there appear summation-integral equations with the parameter of the integral Laplace transform (L-parameter) and the parameter of the finite integral Hankel transform (H-parameter).

The formulation of the problem is in the determination of the laws governing the development of spatial nonstationary temperature fields in a half-space and a bounded cylinder of radius R and height h where one of the end surfaces of the bounded cylinder touches the surface of the half-space. In this case, the thermophysical characteristics of the considered bodies and their initial temperatures are different and the side and nontouching end surfaces of the cylinder are maintained at a constant initial temperature. Ideal heat insulation exists on the half-space surface beyond the circular region of contact.

We introduce the following notation: r and z are the cylindrical coordinates, τ is the time; $T_1(r, z, \tau)$ is the temperature of the semibounded body ($r > 0$, $z > 0$, $\tau > 0$); $T_2(r, z, \tau)$ is the temperature of the cylinder ($0 < r < R$, $-h < z < 0$, $\tau > 0$); $\lambda_1 > 0$ and $a_2 > 0$ are the coefficients of thermal conductivity and thermal diffusivity of the semibounded body and the cylinder, respectively.

We consider the system of two heat-conduction equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1(r,z,\tau)}{\partial r}\right)+\frac{\partial^2 T_1(r,z,\tau)}{\partial z^2}=\frac{1}{a_1}\frac{\partial T_1(r,z,\tau)}{\partial \tau}, \quad r>0, \quad z>0, \quad \tau>0; \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2(r,z,\tau)}{\partial r}\right)+\frac{\partial^2 T_2(r,z,\tau)}{\partial z^2}=\frac{1}{a_2}\frac{\partial T_2(r,z,\tau)}{\partial \tau}, \quad 0<r<R, \quad -h<z<0, \quad \tau>0, \tag{2}$$

with the initial conditions

$$T_1(r,z,0)=T_{01}, \quad r>0, \quad z>0; \quad T_2(r,z,0)=T_{02}, \quad 0<r<R, \quad -h<z<0, \quad T_{01}\neq T_{02}, \tag{3}$$

and the boundary conditions (within the corresponding ranges of change of the coordinates)

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$$\frac{\partial T_1(r,\infty,\tau)}{\partial z}=\frac{\partial T_1(0,z,\tau)}{\partial r}=\frac{\partial T_1(\infty,z,\tau)}{\partial r}=\frac{\partial T_2(0,z,\tau)}{\partial r}=0\,; \tag{4}$$

$$T_2(R,z,\tau)=T_{02}\,,\quad T_2(r,-h,\tau)=T_{02}\,; \tag{5}$$

$$T_1(r,0,\tau)=T_2(r,0,\tau)\,,\quad 0<r<R\,; \tag{6}$$

$$K_\lambda\frac{\partial T_1(r,0,\tau)}{\partial z}=-\frac{\partial T_2(r,0,\tau)}{\partial z}\,,\quad 0<r<R\,; \tag{7}$$

$$\frac{\partial T_1(r,0,\tau)}{\partial z}=0\,,\quad R<r<\infty\,, \tag{8}$$

where $K_\lambda = \lambda_1/\lambda_2$.

We note that, according to [1], conditions (6) and (7) determine the boundary condition of the fourth kind in the region $z = 0$, $0 < r < R$, and the set of conditions (6)–(8) determines mixed boundary conditions on the surface $z = 0$ in the corresponding regions of change of the variable r.

The solution of Eq. (1) with conditions (3)–(5) in the region of the Laplace transform $\bar{T}_1(r, z, s) = \int_0^\infty T_1(r, z, \tau)\exp(-s\tau)d\tau$, $\mathrm{Re}\, s > 0$, can be written in the form [2]

$$\bar{T}_1(r,z,s)=\frac{T_{01}}{s}+\int_0^\infty \bar{C}(p,s)\exp\left(-z\sqrt{p^2+\frac{s}{a_1}}\right)J_0(pr)\,p\,dp\,,\quad r>0\,,\quad z>0\,, \tag{9}$$

where $J_0(pr)$ is the Bessel function of the first kind and zero order, $\bar{C}(p, s)$ is the unknown analytical function, and the restriction $\mathrm{Re}\, s > 0$ on the parameter of the Laplace transform here and below is omitted in our notation for brevity.

The solution of Eq. (2) with conditions (3)–(5) in the region of L-transforms using the finite Hankel transform $\bar{T}_{2\mathrm{H}}(p, z, s) = \int_0^R \bar{T}_2(r, z, s)J_0(pr)rdr$ can be found in the form [2]

$$\bar{T}_2(r,z,s)=\frac{T_{02}}{s}+\sum_{m=1}^{\infty}\frac{2J_0\left(\frac{\mu_m}{R}r\right)}{R^2J_1^2(\mu_m)}\bar{B}\left(\frac{\mu_m}{R},s\right)\sinh\left(|z|\sqrt{\frac{\mu_m^2}{R^2}+\frac{s}{a_2}}\right)\times$$

$$\times\left[\cot\left(|z|\sqrt{\frac{\mu_m^2}{R^2}+\frac{s}{a_2}}\right)-\cot\left(h\sqrt{\frac{\mu_m^2}{R^2}+\frac{s}{a_2}}\right)\right],\quad 0<r<R\,,\quad -h<z<0\,, \tag{10}$$

where $J_1(\mu_m)$ is the Bessel function of the first kind and first order, $\bar{B}(\mu_m/R, s)$ is an unknown analytical function, and μ_m are the roots of the equation

$$J_0(\mu)=0\,. \tag{11}$$

Taking into account the mixed boundary conditions (6)–(8) on the surface $z = 0$, we can explicitly obtain the following system of summation-integral equations with the L-parameter:

$$\frac{T_{01}}{s} + \int_0^\infty \overline{C}(p, s)\, J_0(pr)\, p dp = \frac{T_{02}}{s} + \sum_{m=1}^{\infty} \frac{2J_0\left(\frac{\mu_m}{R} r\right)}{R^2 J_1^2(\mu_m)} \overline{B}\left(\frac{\mu_m}{R}, s\right), \quad 0 < r < R\,; \tag{12}$$

$$\int_0^\infty \overline{C}(p, s) \sqrt{p^2 + \frac{s}{a_1}}\; J_0(pr)\, p dp =$$

$$= -\sum_{m=1}^{\infty} \frac{2J_0\left(\mu_m \frac{r}{R}\right)}{K_\lambda R^2 J_1^2(\mu_m)} \overline{B}\left(\frac{\mu_m}{R}, s\right) \sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\; \cot\left(h\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right), \quad 0 < r < R\,; \tag{13}$$

$$\int_0^\infty \overline{C}(p, s) \sqrt{p^2 + \frac{s}{a_1}}\; J_0(pr)\, p dp = 0\,, \quad R < r < \infty\,, \tag{14}$$

whence the unknown functions $\overline{B}(\mu_m/R, s)$ and $\overline{C}(p, s)$ must be determined.

We find the value of the function $\overline{B}(\mu_m/R, s)$ from Eq. (13), expanding the functions within the range $(0, R)$ into the Fourier–Bessel series in positive roots of Eq. (11) of the form [3]

$$\overline{f}(r, s) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\mu_m}{R} r\right), \quad A_m = \frac{2}{R^2 J_1^2(\mu_m)} \int_0^R r \overline{f}(r, s)\, J_0\left(\frac{\mu_m}{R} r\right) dr\,.$$

As a result, we have

$$\overline{B}\left(\frac{\mu_m}{R}, s\right) = -\frac{K_\lambda R^2 \tanh\left(h\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right)}{2\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}} J_1^2(\mu_m) \int_0^\infty \overline{C}(p,s) \sqrt{p^2 + \frac{s}{a_1}}\; J_0(pr)\, p dp\,, \quad 0 < r < R\,, \tag{15}$$

since (see, e.g., [1])

$$\int_0^R J_0\left(\frac{\mu_m}{R} r\right) J_0(pr)\, r dr = \begin{cases} 0\,, & \text{when } p \neq \frac{\mu_m}{R}\,, \\ \frac{R^2}{2} J_1^2(\mu_m)\,, & \text{when } p = \frac{\mu_m}{R}\,, \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{J_0\left(\frac{\mu_m}{R} r\right)}{\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}} \tanh\left(h\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right) = \frac{J_0(pr)}{\sqrt{p + \frac{s}{a_2}}} \tanh\left(h\sqrt{p^2 + \frac{s}{a_2}}\right), \quad 0 < r < R\,.$$

Substituting (15) into Eq. (12), we come to the paired integral equations with the L-parameter

$$\int_0^\infty \overline{C}(p,s)\left[1+K_\lambda\sqrt{\left(\frac{p^2+\frac{s}{a_1}}{p^2+\frac{s}{a_2}}\right)}\tanh\left(h\sqrt{p^2+\frac{s}{a_2}}\right)\right]J_0(pr)\,p\,dp=\frac{T_{02}}{s}-\frac{T_{01}}{s},\quad 0<r<R; \tag{16}$$

$$\int_0^\infty \overline{C}(p,s)\sqrt{p^2+\frac{s}{a_1}}\,J_0(pr)\,p\,dp=0,\quad R<r<\infty. \tag{17}$$

To solve the paired integral equations (16) and (17), i.e., to determine the unknown analytical function $\overline{C}(p, s)$, we use the substitution

$$\overline{C}(p,s)=\frac{1}{\sqrt{p^2+\frac{s}{a_1}}}\int_0^R \overline{\varphi}(t,s)\cos\left(t\sqrt{p^2+\frac{s}{a_1}}\right)dt, \tag{18}$$

which provides the fulfillment of Eq. (17) automatically due to the corresponding discontinuous integral within the range $R < r < \infty$ [4].

Substitution of (18) into (16) leads to the integral equation with the L-parameter for determination of the unknown analytical function $\overline{\varphi}(t, s)$:

$$\int_0^r \frac{\overline{\varphi}(t,s)}{\sqrt{r^2-t^2}}\exp\left(-\sqrt{\frac{s}{a_1}(r^2-t^2)}\right)dt-\int_r^R \frac{\overline{\varphi}(t,s)}{\sqrt{t^2-r^2}}\sin\left(\sqrt{\frac{s}{a_1}(t^2-r^2)}\right)dt+$$

$$+K_\lambda\int_0^R \overline{\varphi}(t,s)\,dt\int_0^\infty \frac{J_0(pr)\,p}{\sqrt{p^2+\frac{s}{a_2}}}\tanh\left(h\sqrt{p^2+\frac{s}{a_2}}\right)\cos\left(t\sqrt{p^2+\frac{s}{a_1}}\right)dp=\frac{T_{02}}{s}-\frac{T_{01}}{s},\quad 0<r<R. \tag{19}$$

We note that the inverse Laplace transform exists for the left-hand side of Eq. (19), since this is true for the right-hand side, viz.: $L^{-1}\left[\frac{T_{02}}{s}-\frac{T_{01}}{s}\right] = T_{02}-T_{01}\neq 0$.

Determination of $\overline{\varphi}(t, s)$ directly from Eq. (19) is a rather labor-consuming problem. Here we suggest a method of determination of $\overline{\varphi}(t, s)$ by reducing Eq. (19) to a simpler form. For this purpose, having replaced r by μ in advance, we multiply the left- and right-hand sides of Eq. (19) by the integrating factor $2\mu\,\dfrac{\cos\left(\sqrt{\frac{s}{a_1}(r^2-\mu^2)}\right)}{(r^2-\mu^2)^{1/2}}$ and integrate with respect to μ within the limits from zero to r. Then Eq. (19) is reduced to the form

$$\overline{\varphi}(r,s)-\frac{1}{\pi}\int_0^R \overline{\varphi}(t,s)\,\overline{K}(r,t,s)\,dt=\frac{2(T_{02}-T_{01})}{\pi s}\cos\left(r\sqrt{\frac{s}{a_1}}\right),\quad 0<r<R, \tag{20}$$

where

$$\bar{K}(r, t, s) = \frac{\sin\left[(t+r)\sqrt{\frac{s}{a_1}}\right]}{t+r} + \frac{\sin\left[(t-r)\sqrt{\frac{s}{a_1}}\right]}{t-r} -$$

$$- K_\lambda \int_0^\infty \frac{p}{\sqrt{p^2 + \frac{s}{a_2}}} \tanh\left(h\sqrt{p^2 + \frac{s}{a_2}}\right)\left[\cos\left((r-t)\sqrt{p^2 + \frac{s}{a_1}}\right) + \cos\left((r+t)\sqrt{p^2 + \frac{s}{a_1}}\right)\right] dp .$$

We note that in derivation of (20) the following values of the integrals were taken into account:

$$\int_0^r \frac{\cos\left(\sqrt{\frac{s}{a_1}(r^2 - \mu^2)}\right)}{\sqrt{r^2 - \mu^2}} J_0(p\mu)\, \mu d\mu = \frac{\sin\left(r\sqrt{p^2 + \frac{s}{a_1}}\right)}{\sqrt{p^2 + \frac{s}{a_1}}} ,$$

$$\int_t^r \frac{\cos\left(\sqrt{\frac{s}{a_1}(r^2 - \mu^2)}\right)\cosh\left(\sqrt{\frac{s}{a_1}(\mu^2 - t^2)}\right)}{\sqrt{(r^2 - \mu^2)(\mu^2 - t^2)}} \mu d\mu = \frac{\pi}{2} ,$$

$$2\int_0^r \frac{\sin\left(\sqrt{\frac{s}{a_1}(r^2 - \mu^2)}\right)\cos\left(\sqrt{\frac{s}{a_1}(r^2 - \mu^2)}\right)}{\sqrt{(t^2 - \mu^2)(r^2 - \mu^2)}} \mu d\mu = \mathrm{Si}\left((t+r)\sqrt{\frac{s}{a_1}}\right) - \mathrm{Si}\left((t-r)\sqrt{\frac{s}{a_1}}\right),$$

$$\bar{\varphi}^*(r, s) - \frac{1}{\pi}\int_0^R \bar{\varphi}^*(t, s)\, \bar{K}^*(r, t, s)\, dt = \frac{2(T_{02} - T_{01})}{\pi s} , \quad 0 < r < R ,$$

where $\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt$ is the sine integral function (integral sine).

The method for determining $\bar{\varphi}(r, s)$ from an equation of the type (20) is suggested, for example, in [5].

We note that if $R \to \infty$ and $h \to \infty$, then we have a one-dimensional nonstationary case of thermal contact of two semibounded bodies with different initial temperatures and different thermophysical properties [1]. In this case, the paired equations (16) and (17) do not appear.

Thus, having determined the value of the function $\bar{\varphi}(t, s)$ from Eqs. (19) or (20), we find the value of the function $\bar{C}(p, s)$ by formula (18) and then the value of the function $\bar{B}(\mu_m/R, s)$ by formula (15). Finally, using formulas (9) and (10), we find the temperature fields $\bar{T}_1(r, z, s)$ and $\bar{T}_2(r, z, s)$ in the region of L-transforms, and, having applied the inverse Laplace transformation, we determine the corresponding values of the inverse transforms $T_1(r, z, \tau)$ and $T_2(r, z, \tau)$.

In conclusion, we note that the existence of the continuously differentiable solution of the integral equation with the L-parameter (20) can be proved by writing it in the form

$$\bar{\varphi}^*(r,s) - \frac{1}{\pi}\int_0^R \bar{\varphi}^*(t,s)\,\bar{K}^*(r,t,s)\,dt = \frac{2\,(T_{02}-T_{01})}{\pi s}\,, \quad 0<r<R\,,$$

where

$$\bar{\varphi}^*(r,s) = \frac{\bar{\varphi}(r,s)}{\cos\left(r\sqrt{\frac{s}{a_1}}\right)}\,; \quad \bar{K}^*(r,t,s) = \frac{\cos\left(t\sqrt{\frac{s}{a_1}}\right)}{\cos\left(r\sqrt{\frac{s}{a_1}}\right)}\bar{K}(r,t,s)\,.$$

Then, since the inverse Laplace transform $L^{-1}\left[\frac{2(T_{02}-T_{01})}{\pi s}\right] = \frac{2(T_{02}-T_{01})}{\pi}$ exists, we can write the equation

$$\varphi^*(r,\tau) - \frac{1}{\pi}\int_0^R dt \int_0^\tau \varphi^*(t,\xi)\,K^*(r,t,\tau-\xi)\,d\xi = \frac{2\,(T_{02}-T_{01})}{\pi}\,, \quad 0<r<R\,, \quad \tau>0\,,$$

where $\varphi^*(r,\tau) = L^{-1}[\bar{\varphi}^*(r,s)]$ and $K^*(r,t,\tau) = L^{-1}[\bar{K}^*(r,t,s)]$, and according to the classical Fredholm theory [6] the convergence of the integral

$$\int_0^\infty \left| \frac{p}{\sqrt{p^2+\frac{s}{a_2}}}\tanh\left(h\sqrt{p^2+\frac{s}{a_1}}\right)\left[\cos\left((r-t)\sqrt{p^2+\frac{s}{a_1}}\right)+\cos\left((r+t)\sqrt{p^2+\frac{s}{a_1}}\right)\right]\right| dp\,, \quad 0<r\,,\,t<R\,,$$

and fulfillment of the inequality

$$\max\int_0^R\int_0^\tau \left|K^*(r,t,\xi)\right|\,d\xi dt < \pi\,, \quad 0<r<R\,, \quad \tau>0\,.$$

are the sufficient conditions for the existence of its solution.

REFERENCES

1. A. V. Luikov, *Theory of Heat Conduction* [in Russian], Moscow (1967).
2. V. P. Kozlov and P. A. Mandrik, *Systems of Integral and Differential Equations with the L-Parameter in Problems of Mathematical Physics and Methods for Identification of Thermal Characteristics* [in Russian], Minsk (2000).
3. V. P. Kozlov, *Two-Dimensional Axisymmetric Nonstationary Problems of Heat Conduction* [in Russian], Minsk (1986).
4. V. P. Kozlov, P. A. Mandrik, and N. I. Yurchuk, *Inzh.-Fiz. Zh.*, **71**, No. 4, 734–743 (1998).
5. V. P. Kozlov, P. A. Mandrik, and N. I. Yurchuk, *Vestn. Belor. Univer.*, Ser. 1, No. 2, 37–42 (1999).
6. Ya. S. Uflyand, *Method of Paired Equations in Problems of Mathematical Physics* [in Russian], Leningrad (1977).