Journal of Engineering Physics and Thermophysics, Vol. 74, No. 2, 2001

METHOD OF SUMMATION-INTEGRAL EQUATIONS FOR SOLVING THE MIXED PROBLEM OF NONSTATIONARY HEAT CONDUCTION

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UDC 517.968.536.24

The solution of a mixed axisymmetric nonstationary problem of heat conduction is obtained in the region of Laplace transforms. In solution of this problem, there occur summation–integral equations with the parameter of the integral Laplace transform (L-parameter) and the additional parameter of the finite integral Hankel transform (H-parameter). The laws governing the development of spatial nonstationary temperature fields in a bounded cylinder and a half-space when one end surface of the cylinder is in contact with the surface of the half-space in a circular region are determined.

We consider a system of a semibounded body and a bounded cylinder of height l and radius R that touch each other at one of the end surfaces of the cylinder. The initial temperatures of the bodies and their thermophysical characteristics are different; for example, zero temperature is maintained at the side surface and the nontouching end surface of the cylinder, and the initial temperature of the semibounded body is $T_0 > 0$. An ideal (in a thermal sense) heat insulation exists beyond the circular contact region on the surface of the semibounded body.

We denote the temperature of the semibounded body ($r > 0$, $z > 0$, $\tau > 0$) as $T_1(r, z, \tau)$, the temperature of the cylinder ($0 < r < R$, $-l < z < 0$, $\tau > 0$) as $T_2(r, z, \tau)$, and the coefficients of thermal conductivity and thermal diffusivity of the semibounded body and the cylinder as $\lambda_1 > 0$, $a_1 > 0$ and $\lambda_2 > 0$, $a_2 > 0$, respectively.

The above-formulated mixed axisymmetric nonstationary problem of heat conduction can be written as follows:

$$\frac{\partial^2 T_1(r,z,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial T_1(r,z,\tau)}{\partial r} + \frac{\partial^2 T_1(r,z,\tau)}{\partial z^2} = \frac{1}{a_1}\frac{\partial T_1(r,z,\tau)}{\partial \tau}, \quad r>0, \quad z>0, \quad \tau>0; \tag{1}$$

$$\frac{\partial^2 T_2(r,z,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial T_2(r,z,\tau)}{\partial r} + \frac{\partial^2 T_2(r,z,\tau)}{\partial z^2} = \frac{1}{a_2}\frac{\partial T_2(r,z,\tau)}{\partial \tau}, \quad -l<z<0, \quad 0<r<R, \quad \tau>0, \tag{2}$$

with the initial conditions

$$T_1(r,z,0) = T_0, \quad r>0, \quad z>0; \quad T_2(r,z,0) = 0, \quad 0<r<R, \quad -l<z<0, \tag{3}$$

and the boundary conditions

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1062-0125/01/7402-0477$25.00 ©2001 Plenum Publishing Corporation

$$\frac{\partial T_1(r,\infty,\tau)}{\partial z}=\frac{\partial T_1(0,z,\tau)}{\partial r}=\frac{\partial T_1(\infty,z,\tau)}{\partial r}=0\,, \quad r>0\,, \quad z>0\,, \quad \tau>0\,; \tag{4}$$

$$\frac{\partial T_2(0,z,\tau)}{\partial r}=T_2(R,z,\tau)=T_2(R,-l,\tau)=0\,, \quad 0<r<R\,, \quad \tau>0\,; \tag{5}$$

$$T_1(r,0,\tau)=T_2(r,0,\tau)\,, \quad 0<r<R\,, \quad \tau>0\,; \tag{6}$$

$$\frac{\partial T_1(r,0,\tau)}{\partial z}=-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2(r,0,\tau)}{\partial z}\,, \quad 0<r<R\,, \quad \tau>0\,; \tag{7}$$

$$\frac{\partial T_1(r,0,\tau)}{\partial z}=0\,, \quad R<r<\infty\,, \quad \tau>0\,. \tag{8}$$

To Eqs. (1) and (2) we apply the Laplace transformation of the form

$$\overline{T}_i=\overline{T}_i(r,z,s)=L\,[T_i(r,z,\tau)]=\int_0^\infty T_i(r,z,\tau)\exp(-s\tau)\,d\tau\,, \quad i=1,2\,,$$

where Re $s>0$; for brevity, here and in what follows it is omitted in the representation. Then, with account for the initial conditions (3), it is necessary to solve the following system of differential equations in the region of L-transforms:

$$(\overline{T}_1)_{rr}+r^{-1}(\overline{T}_1)_r+(\overline{T}_1)_{zz}-\frac{s}{a_1}\left(\overline{T}_1-\frac{T_0}{s}\right)=0\,, \quad r>0\,, \quad z>0\,; \tag{9}$$

$$(\overline{T}_2)_{rr}+r^{-1}(\overline{T}_2)_r+(\overline{T}_2)_{zz}-\frac{s}{a_2}\overline{T}_2=0\,, \quad 0<r<R\,, \quad -l<z<0\,. \tag{10}$$

The boundary conditions (4)–(8) for the function-transforms $\overline{T}_i(r,z,s)$ do not change.

The general solution of Eq. (9) by the Fourier method of separation of variables can be written in the form of an improper integral:

$$\overline{T}_1(r,z,s)=\frac{T_0}{s}+\int_0^\infty \overline{C}(p,s)\exp\left(-z\sqrt{p^2+\frac{s}{a_1}}\right)J_0(pr)\,p\,dp\,; \quad r>0\,, \quad z>0\,, \tag{11}$$

where $\overline{C}(p,s)$ is the unknown analytical function and $J_0(pr)$ is the Bessel function of the first kind of zero order.

Taking the finite Hankel transform of Eq. (10)

$$\overline{T}_{2H}=\overline{T}_{2H}(\overline{p},z,s)=\int_0^R \overline{T}_2(r,z,s)\,J_0(\overline{p}r)\,r\,dr\,, \quad -l<z<0\,,$$

with account for the homogeneous initial condition from (3) and the boundary conditions (5), we obtain the solution for the transform $\overline{T}_{2H}(\overline{p}, z, s)$ in the form

$$\overline{T}_{2H}(\overline{p}, z, s) = \overline{B}(\overline{p}, s)\left[\cosh\left(|z|\sqrt{\overline{p}^2 + \frac{s}{a_2}}\right) - \sinh\left(|z|\sqrt{\overline{p}^2 + \frac{s}{a_2}}\right)\cot\left(l\sqrt{\overline{p}^2 + \frac{s}{a_2}}\right)\right], \quad -l < z < 0, \tag{12}$$

where $\overline{B}(\overline{p}, s)$ is the unknown analytical function and the values of $\overline{p}$ are determined from the characteristic equation

$$J_0(\overline{p}r) = J_0(\mu) = 0. \tag{13}$$

We note that the Bessel function of the material argument of the first kind of zero order $J_0(\mu)$ has an uncountable set of roots μ_m, namely: $\mu_1 = 2.4048$, $\mu_2 = 5.5201$, $\mu_3 = 8.6537$, etc.

The inversion formula of the Hankel transform for (12) by the positive roots of Eq. (13) is known [1, 2] and has the form

$$\overline{T}_2(r, z, s) = \sum_{m=1}^{\infty} \frac{2J_0\left(\mu_m \frac{r}{R}\right)}{R^2 J_1^2(\mu_m)} \overline{T}_{2H}\left(\frac{\mu_m}{R}, z, s\right), \quad 0 < r < R, \quad -l < z < 0, \tag{14}$$

where $J_1(\mu)$ is the Bessel function of the first kind of first order.

From expressions (11), (12), and (14) at $z = 0$ we find

$$\overline{T}_1(r, 0, s) = \frac{T_0}{s} + \int_0^{\infty} \overline{C}(p, s) J_0(pr)\, p dp, \quad r > 0; \tag{15}$$

$$\frac{\partial \overline{T}_1(r, 0, s)}{\partial z} = -\int_0^{\infty} \overline{C}(p, s)\sqrt{p^2 + \frac{s}{a_1}}\, J_0(pr)\, p dp, \quad r > 0; \tag{16}$$

$$\overline{T}_2(r, 0, s) = \sum_{m=1}^{\infty} \frac{2\overline{B}\left(\frac{\mu_m}{R}, s\right)}{R^2 J_1^2(\mu_m)} J_0\left(\mu_m \frac{r}{R}\right), \quad 0 < r < R; \tag{17}$$

$$\frac{\partial \overline{T}_2(r, 0, s)}{\partial z} = -\sum_{m=1}^{\infty} \frac{2\overline{B}\left(\frac{\mu_m}{R}, s\right)}{R^2 J_1^2(\mu_m)} \sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\, J_0\left(\mu_m \frac{r}{R}\right) \cot\left(l\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right), \quad 0 < r < R. \tag{18}$$

The boundary conditions (6)–(8) in the region of L-transforms take the form

$$\overline{T}_1(r, 0, s) = \overline{T}_2(r, 0, s), \quad 0 < r < R; \tag{19}$$

$$\frac{\partial \overline{T}_1(r, 0, s)}{\partial z} = -\frac{\lambda_2}{\lambda_1} \frac{\partial \overline{T}_2(r, 0, s)}{\partial z}, \quad 0 < r < R; \tag{20}$$

$$\frac{\partial \bar{T}_1 (r, 0, s)}{\partial z} = 0 , \quad R < r < \infty , \tag{21}$$

and with account for (15)–(17) lead to the following system of summation-integral equations from which one must determine the values of the unknown functions $\bar{C}(p, s)$ and $\bar{B}\left(\frac{\mu_m}{R}, s\right)$:

$$\frac{T_0}{s} + \int_0^\infty \bar{C}(p, s) J_0(pr) pdp = \sum_{m=1}^\infty \frac{2\bar{B}\left(\frac{\mu_m}{R}, s\right)}{R^2 J_1^2(\mu_m)} J_0\left(\mu_m \frac{r}{R}\right), \quad 0 < r < R ; \tag{22}$$

$$\int_0^\infty \bar{C}(p, s) \sqrt{p^2 + \frac{s}{a_1}}\, J_0(pr) pdp =$$

$$= - \sum_{m=1}^\infty \frac{2B\left(\frac{\mu_m}{R}, s\right)}{K_\lambda R^2 J_1^2(\mu_m)} \sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\, J_0\left(\mu_m \frac{r}{R}\right) \cot\left(l \sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right), \quad 0 < r < R ; \tag{23}$$

$$\int_0^\infty \bar{C}(p, s) \sqrt{p^2 + \frac{s}{a_1}}\, J_0(pr) pdp = 0 , \quad R < r < \infty , \tag{24}$$

where $K_\lambda = \lambda_1/\lambda_2$ is the criterion characterizing the relative thermal conductivity of the bodies.

It is known from the theory of Bessel functions that if the function $\bar{f}(r, s)$ within the range $0 < r < R$ is represented by the Fourier–Bessel series

$$\bar{f}(r, s) = \sum_{m=1}^\infty A_m J_0\left(\mu_m \frac{r}{R}\right), \tag{25}$$

where μ_m are the roots of Eq. (13), then the coefficients of expansion A_m are calculated by the formula

$$A_m = \frac{2}{R^2 J_1^2(\mu_m)} \int_0^R r \bar{f}(r, s) J_0\left(\mu_m \frac{r}{R}\right) dr . \tag{26}$$

With account for (25) and (26), from Eq. (23) we can find the value

$$\overline{B}\left(\frac{\mu_m}{R}, s\right) = \frac{-K_\lambda \tanh\left(l\sqrt{\frac{\mu_m^2}{R^2}+\frac{s}{a_2}}\right)}{\sqrt{\frac{\mu_m^2}{R^2}+\frac{s}{a_2}}} \int_0^\infty \overline{C}(p, s)\sqrt{p^2+\frac{s}{a_1}}\; p\int_0^R J_0\left(\mu_m \frac{r}{R}\right) J_0(pr)\, rdr\, dp\,. \tag{27}$$

The following assertion holds:

$$\int_0^R J_0\left(\frac{\mu_m}{R} r\right) J_0(pr)\, rdr = \begin{cases} 0, & \text{if } p \neq \frac{\mu_m}{R}; \\ \frac{R^2}{2} J_1^2(\mu_m), & \text{if } p = \frac{\mu_m}{R}, \end{cases} \tag{28}$$

where μ_m are the roots of Eq. (13).

We prove the above. Let $y_1 = y_1(x) = J_0(\overline{p}, x)$ and $y_2 = y_2(x) = J_0(px)$. It is known that the Bessel function of the first kind of zero order satisfies the corresponding differential equation. Consequently, the equalities $(xy_1')' = -\overline{p}^2 xy_1$ and $(xy_2')' = -p^2 xy_2$ hold for y_1 and y_2, whence we can write

$$p^2 xy_2 y_1 - \overline{p}^2 xy_1 y_2 = y_2 (xy_1')' - y_1 (xy_2')' \quad \text{or} \quad (p^2 - \overline{p}^2)\, xy_1 y_2 = (y_2 xy_1' - y_1 xy_2')',$$

or, having integrated both sides of the latter equality from 0 to x:

$$(p^2 - \overline{p}^2)\int_0^x xy_1 y_2 dx = xy_2 y_1' - xy_1 xy_2'\,. \tag{29}$$

Since $y_1' = \overline{p}J_0'(\overline{p}x) = -\overline{p}J_1(\overline{p}x)$ and $y_2' = pJ_0'(px) = -pJ_1(px)$, passing in (29) to the initial notation of formula (28) we can write

$$\int_0^R J_0(\overline{p}r)\, J_0(pr)\, rdr = \frac{pRJ_0(\overline{p}R)\, J_1(pR) - \overline{p}RJ_0(pR)\, J_1(\overline{p}R)}{p^2 - \overline{p}^2} \tag{30}$$

or under the assumption that $\overline{p}R = \mu_m$ is the root of Eq. (13)

$$\int_0^R J_0\left(\frac{\mu_m}{R} r\right) J_0(pr)\, rdr = \frac{pRJ_0(\mu_m)\, J_1(pR) - \mu_m J_0(pR)\, J_1(\mu_m)}{p^2 - \frac{\mu_m^2}{R^2}}\,. \tag{31}$$

Applying the L'Hospital rule and taking into account that $J_0' = -J_1(x)$ and $J_1'(x) = J_0(x) - \frac{1}{x} J_1(x)$, we represent the expression on the right-hand side of equality (31) for $pR = \mu_m$ as $\frac{R^2}{2}\left(J_0^2(\mu_m) + J_1^2(\mu_m)\right) = \frac{R^2}{2} J_1^2(\mu_m)$, and for $pR \neq \mu_m$ it vanishes since in the case considered the values of pR are also the roots of Eq. (13).

Thus, assertion (28) is proved and from (27) we can write that

$$\overline{B}\left(\frac{\mu_m}{R}, s\right) = -\frac{R^2}{2} J_1^2(\mu_m) \frac{K_\lambda}{\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}} \tanh\left(l\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right) \int_0^\infty \overline{C}(p, s) \sqrt{p^2 + \frac{s}{a_1}}\, pdp, \quad 0 < r < R.$$

We substitute the obtained value of $\overline{B}\left(\frac{\mu_m}{R}, s\right)$ into the right-hand side of Eq. (22):

$$\frac{T_0}{s} + \int_0^\infty \overline{C}(p, s) J_0(pr)\, pdp = -\int_0^\infty \overline{C}(p, s) K_\lambda \sqrt{p^2 + \frac{s}{a_1}}\, pdp \sum_{m=1}^\infty \frac{J_0\left(\mu_m \frac{r}{R}\right)}{\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}} \tanh\left(l\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right), \quad 0 < r < R.$$

With allowance for the fact that

$$\sum_{m=1}^\infty \frac{J_0\left(\mu_m \frac{r}{R}\right)}{\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}} \tanh\left(l\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right) = \frac{J_0(pr)}{\sqrt{p^2 + \frac{s}{a_2}}} \tanh\left(l\sqrt{p^2 + \frac{s}{a_2}}\right), \quad 0 < r < R,$$

the first paired equation (22) takes the form

$$\int_0^\infty \overline{C}(p, s) \left[1 + K_\lambda \frac{\sqrt{\frac{s}{a_1} + p^2}}{\sqrt{\frac{s}{a_2} + p^2}} \tanh\left(l\sqrt{p^2 + \frac{s}{a_2}}\right)\right] J_0(pr)\, pdp = -\frac{T_0}{s}, \quad 0 < r < R. \tag{32}$$

The second paired equation (24) remains virtually unchanged.

To solve the paired integral equations (32) and (24), with the aim of determining the functions $\overline{C}(p, s)$, we use the substitution [3, 4]

$$\overline{C}(p, s) = \frac{1}{\sqrt{p^2 + \frac{s}{a_1}}} \int_0^R \overline{\varphi}(t, s) \cos\left(t\sqrt{p^2 + \frac{s}{a_1}}\right) dt, \tag{33}$$

which automatically ensures the fulfillment of the second paired equation (24), since [5]

$$\int_0^R \overline{\varphi}(t, s) \int_0^\infty \cos\left(t\sqrt{p^2 + \frac{s}{a_1}}\right) J_0(pr)\, pdp\, dt =$$

$$= \begin{cases} \sqrt{\frac{\pi}{2}} \left(\sqrt{\frac{s}{a_1}} \right)^{3/2} \int\limits_0^R \bar{\varphi}(t, s)\, (t^2 - r^2)^{-3/4} J_{-3/2} \left(\sqrt{\frac{s}{a_1} (t^2 - r^2)} \right) \sqrt{t}\, dt\,, & 0 < r < t < R\,; \\ 0\,, & 0 < t < R < r\,. \end{cases}$$

In order to obtain the equation for determining the analytical function $\bar{\varphi}(t, s)$, we substitute the value of $\bar{C}(p, s)$ from (33) into the first paired equation (32):

$$\int\limits_0^R \bar{\varphi}(t, s) \int\limits_0^\infty \frac{\cos\left(t \sqrt{p^2 + \frac{s}{a_1}} \right)}{\sqrt{p^2 + \frac{s}{a_1}}} J_0(pr)\, p dp\, dt +$$

$$+ K_\lambda \int\limits_0^R \bar{\varphi}(t, s) \int\limits_0^\infty \frac{\tanh\left(l \sqrt{p^2 + \frac{s}{a_2}} \right)}{\sqrt{p^2 + \frac{s}{a_2}}} \cos\left(t \sqrt{p^2 + \frac{s}{a_1}} \right) J_0(pr)\, p dp\, dt = -\frac{T_0}{s}\,, \quad 0 < r < R\,. \tag{34}$$

With account for the value of the discontinuous integral [6]

$$\int\limits_0^\infty \frac{\cos\left(t \sqrt{p^2 + \frac{s}{a_1}} \right)}{\sqrt{p^2 + \frac{s}{a_1}}} J_0(pr)\, p dp = \begin{cases} \dfrac{\exp\left(-\sqrt{\frac{s}{a_1}} \sqrt{r^2 - t^2} \right)}{\sqrt{r^2 - t^2}}\,, & t < r\,; \\ -\dfrac{\sin\left(\sqrt{\frac{s}{a_1}} \sqrt{t^2 - r^2} \right)}{\sqrt{t^2 - r^2}}\,, & r < t\,, \end{cases}$$

the integral equation with the L-parameter (34) takes the form

$$\int\limits_0^r \frac{\bar{\varphi}(t, s)}{\sqrt{r^2 - t^2}} \exp\left(-\sqrt{\frac{s}{a_1} (r^2 - t^2)} \right) dt - \int\limits_r^R \frac{\bar{\varphi}(t, s)}{\sqrt{t^2 - r^2}} \sin\left(\sqrt{\frac{s}{a_1} (t^2 - r^2)} \right) dt +$$

$$+ \frac{K_\lambda}{l} \int\limits_0^R \bar{\varphi}(t, s) \int\limits_{l\sqrt{s/a_2}}^\infty \tanh(x) \cos\left(t \sqrt{\frac{x^2}{l^2} - \left(\frac{a_1 - a_2}{a_1 a_2} \right) s} \right) J_0\left(r \sqrt{\frac{x^2}{l^2} - \frac{s}{a_2}} \right) dx\, dt = -\frac{T_0}{s}\,, \quad 0 < r < R\,. \tag{35}$$

Equation (35) is a basis one for determining the unknown analytical function $\bar{\varphi}(r, s) = \bar{\varphi}(-r, s)$. The method for solving these equations is formulated in [7] in detail. Having determined $\bar{\varphi}(r, s)$, we find the value of $\bar{C}(p, s)$ by formula (33). Substituting $\bar{C}(p, s)$ into (11), we find the L-transform of the temperature field $\overline{T}_1(r, z, s)$ which develops in the half-space with the mixed boundary conditions (6)–(8) in the region of L-transforms:

$$\overline{T}_1(r, z, s) = \frac{T_0}{s} + \int_0^R \overline{\varphi}(t, s) \int_0^\infty \frac{pJ_0(pr)}{\sqrt{p^2 + \frac{s}{a_1}}} \exp\left(-z\sqrt{p^2 + \frac{s}{a_1}}\right) \cos\left(t\sqrt{p^2 + \frac{s}{a_1}}\right) dp\, dt\,; \quad r > 0\,, \quad z > 0\,. \tag{36}$$

Using formulas (14), (27)–(29), and (33), we find the solution for the L-transform $\overline{T}_2(r, z, s)$ in the bounded cylinder

$$\overline{T}_2(r, z, s) = -\sum_{m=1}^{\infty} J_0\left(\mu_m \frac{r}{R}\right)\left[\cosh\left(|z|\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right) - \sinh\left(|z|\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right)\cot\left(l\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right)\right] \times$$

$$\times \frac{K_\lambda \tanh\left(l\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}\right)}{\sqrt{\frac{\mu_m^2}{R^2} + \frac{s}{a_2}}} \int_0^\infty p \int_0^R \overline{\varphi}(t, s) \cos\left(t\sqrt{p^2 + \frac{s}{a_1}}\right) dt\, dp\,, \quad 0 < r < R\,, \quad -l < z < 0\,, \tag{37}$$

where μ_m are the roots of Eq. (13).

We consider the limiting case of the solutions (36) and (37) for $R \to \infty$ that corresponds to the contact of an unbounded plate of thickness (height) l with a half-space. In this case, the second paired equation (24) actually disappears and the value of $\overline{C}(p, s)$ is found directly from (32) on the basis of the inversion formula for the Hankel transform:

$$\overline{C}(p, s) = -\frac{T_0}{s} \int_0^\infty J_0(px)\, x dx \left[1 + \frac{K_\lambda \sqrt{\frac{s}{a_1} + p^2}}{\sqrt{\frac{s}{a_2} + p^2}} \tanh\left(l\sqrt{p^2 + \frac{s}{a_2}}\right)\right]^{-1}. \tag{38}$$

Substituting (38) into the solution (10), we find

$$\overline{T}_1(r, z, s) = \frac{T_0}{s} - \int_0^\infty p \int_0^\infty \left[\frac{\frac{T_0}{s}\exp\left(-z\sqrt{p^2 + \frac{s}{a_1}}\right)}{1 + \frac{K_\lambda \sqrt{\frac{s}{a_1} + p^2}}{\sqrt{\frac{s}{a_2} + p^2}} \tanh\left(l\sqrt{p^2 + \frac{s}{a_2}}\right)}\right] J_0(pr)\, J_0(px)\, x dx\, dp =$$

$$= \frac{T_0}{s} - \frac{T_0}{s} \frac{\exp\left(-z\sqrt{r^2 + \frac{s}{a_1}}\right)}{1 + \frac{K_\lambda \sqrt{\frac{s}{a_1} + r^2}}{\sqrt{\frac{s}{a_2} + r^2}} \tanh\left(l\sqrt{p^2 + \frac{s}{a_2}}\right)} . \qquad (39)$$

If in (39) $r \to 0$, then we obtain the corresponding one-dimensional solution

$$\overline{T}_1 (z, s) = \overline{T}_1 (0, z, s) = \frac{T_0}{s} \left[1 - \frac{\exp\left(-z\sqrt{\frac{s}{a_1}}\right)}{1 + K_b \tanh\left(l\sqrt{\frac{s}{a_2}}\right)} \right], \quad z \geq 0 , \qquad (40)$$

where $K_b = K_\lambda \sqrt{a_1/a_2}$.

Thus, we obtained the solution in the region of L-transforms for the system, of contact at the instant $\tau > 0$, of an unbounded plate of height l with zero initial temperature and a semibounded body with the initial temperature T_0. All the inverse transforms of the temperature fields $\overline{T}_i(r, z, s)$ are found by the inversion formula of the Laplace integral in the complex plane Re $s > 0$.

If the height of the unbounded plate is $l \to \infty$, then tanh $(l\sqrt{s/a_2}) \to 1$ and we obtain the known solution of A. V. Luikov for two contiguous semibounded bodies [2].

It is more convenient to consider the limiting case $R \to \infty$ for (37) from Eqs. (12) and (40), since in this case the paired equations are absent and the finite Hankel transform changes over to infinite. In this case, we take into account the boundary condition (6) in the corresponding region, and proceeding from Eq. (40) we obtain at $z = 0$

$$\overline{T}_{1H} (0, 0, s) = \overline{T}_{2H} (p, 0, s) = \overline{B} (p, s) = \frac{T_0 K_b}{s\left[K_b + \cot\left(l\sqrt{\frac{s}{a_2}}\right)\right]} \int_0^\infty J_0 (px)\, xdx . \qquad (41)$$

Substituting (41) into (12) and using the inversion formula for the iterated Hankel integral [1], we obtain for $R \to \infty$

$$\overline{T}_2 (0, z, s) = \int_0^\infty p \int_0^\infty \left[\cosh\left(|z|\sqrt{p^2 + \frac{s}{a_2}}\right) - \sinh\left(|z|\sqrt{p^2 + \frac{s}{a_2}}\right)\cot\left(l\sqrt{p^2 + \frac{s}{a_2}}\right)\right] \times$$

$$\times \left[\frac{T_0 K_b}{s\left[K_b + \cot\left(l\sqrt{\frac{s}{a_2}}\right)\right]}\right] J_0 (pr)\, J_0 (px)\, xdx\, dp = - \frac{T_0 K_b}{s\left[K_b + \cot\left(l\sqrt{\frac{s}{a_2}}\right)\right]} \times$$

$$\times\left[\cosh\left(|z|\sqrt{\frac{s}{a_2}}\right)-\sinh\left(|z|\sqrt{\frac{s}{a_2}}\right)\cot\left(l\sqrt{\frac{s}{a_2}}\right)\right],\quad -l<z<0\,. \qquad (42)$$

When $l \to \infty$, cot $(l\sqrt{s/a_2}) \to 1$, and from (42) we obtain the one-dimensional equation for the case of contact of two semibounded rods whose side surfaces are heat insulated [2].

It is seen from the solutions (40) and (42) that the temperature $\overline{T}_1(r, 0, s) = \overline{T}_2(r, 0, s)$ in the region of contact ($0 \le r \le R$, $z = 0$) of the bounded cylinder with the semibounded body depends on the parameter s, and consequently, on time (for the inverse transform). It was assumed earlier [2] that a constant temperature is instantly established in the plane of contact of two half-spaces ($z = 0$) depending on the complex $1/(1 + K_b)$ and is held constant, including the steady state.

Thus, the solution of the given problem yields the following result: contact between a bounded body and a semibounded one does not lead to a constant temperature being held at the site of contact and depends on time and on the linear dimensions of the bounded cylinder.

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