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VALUE-AT-RISK PORTFOLIO OPTIMIZATION: A NOTE ON MULTIOBJECTIVE GENETIC ALGORITHM[*]

Рассматривается дальнейшее развитие общего подхода оптимизации риска в условиях рынка. Предлагается модель, ориентированная на различные меры риска, и эмпирическая методика расчета на основе многоцелевого генетического алгоритма.

In this paper we develop a general framework for market risk optimization. The model is valid for any given risk measure. Our empirical procedure is focused on VaR. We solve the problem using a multiobjective genetic algorithm (GA). The algorithm is very efficient and it can handle hundreds of assets in reasonable computer time. One of the advantages of this approach is that it is easily extendable.

1. Introduction

When VaR is considered as the risk measure to minimize, it leads to a non-convex and non-differential risk-return optimization problem. This problem is tackled in the literature in various ways. Using Arzac and Bawa (1977) framework, Jansen et al. (2000) and Campbell et al. (2001) use a safety-first theory approach to maximize expected return subject to a VaR constraint. In order to avoid the use of smoothing techniques, in this paper we propose a genetic algorithm (GA) approach to deal with the problem of minimizing VaR or any other measure that leads to non-convex and non-differential risk-return optimization problems. Also, we present a multiobjective evolutionary approach that optimizes simultaneously the return and the level of risk and evaluates the differences between mean-variance and mean-VaR efficient portfolios. One of the benefits of using GAs for multiobjective optimization is that GAs work with a population of individuals, which allows us to find several nondominated solution in a single run. Also, GAs are less susceptible than other techniques to the non-convexity of the search space.

The paper is organized as follows: section 2 describes the portfolio optimization problem solved. Section 3 describes how they have been applied to the portfolio optimization problem. Section 4 shows the results yielded by the GAs to the optimization problem, and the related conclusions are reported in section 5.

2. Portfolio optimization

In this work, we consider the Value-at-Risk (VaR) as an appropriate risk measure. VaR is defined as the maximum expected loss on an investment over a specified horizon given a confidence level 1-α. Usually α is fixed to be a 5 % or 1 %. In our study, we use the VaR definition given in Jorion (2001). That is,

$$\text{VaR}(R_p) = E(R_p) - q_\alpha(R_p), \tag{1}$$

where $q_\alpha(R_p)$ is the α-quantile of R_p.

Approaches to quantify VaR such as delta-normal, delta-gamma or Monte Carlo simulation method rely on the normality assumption or other prespecified distributions. These approaches have several drawbacks, such as the estimation of parameters and whether the distribution fit properly the data in the tail or not (Baixauli and Alvarez, 2004). In our analysis we computed the VaR by historical simulation using Equation (1). Hence, $q_\alpha(R_p)$ is the empirical α-quantile of the actual historical data, this specification is valid for any underlying distribution, discrete or continuous, fat or thin-tailed.

As we pointed out, mean-VaR problem becomes a non-convex and non-differential risk-return optimization problem. For this reason, we use a multiobjective GA approach to find VaR-efficient portfolios and σ-efficient portfolios by solving the following optimization problems:

Mean-VaR problem	Mean-variance problem
$\min_w E(R_p) - q_\alpha(R_p)$	$\min_w w'\Omega w$
$\max_w w'E(R)$	$\max_w w'E(R)$
$w'1 = 1$	$w'1 = 1$
$w \geq 0$	$w \geq 0$

3. Multiobjective genetic algorithm

The GA implementation is based on ECJ (\url{http://cs.gmu.edu/\~{}eclab/projects/ecj}), a research evolutionary computation system in Java developed at George Mason University's Evolutionary Computation Laboratory (ECLab). In this work the SPEA2 package of ECJ was used for the multiobjective aspect of the optimization (Zitzler et al., 2001). The reason for this choice was twofold. On the one hand, SPEA2 and NSGA-II have shown better performance than the others in various benchmark problems (Zitzler et al., 2002). On the other, the on-line avability of the package facilitates the reproducibility of the results presented in this paper.

The algorithm works as follows:

▪ In step 1 and 2 the archive, $A(g)$, where the nondominated solutions are stored and the population, $P(g)$, are initialized. $A(0)$ is an empty set and $P(0)$ is initialized at random.

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- In step 3 the generation counter g is set to 1 and then the evolution loop starts.
- In step 4 and 5 the individuals in the population and the archive are evaluated.
- According to this evaluation a new archive is created in step 6 containing all the nondominated individuals found in the union of the previous archive and the population.
- If the size of the resulting archive exceeds the archive size, in step 7 the archive is truncated. This truncation method removes those individuals which are at the minimum distance of another individual. This way the characteristics of the nondominated front are preserved and outer solutions are not lost.
- The termination criterion in step 8 stops the algorithm when the number of generations has been completed.
- In step 9 tournament selection with replacement is performed in the archive set in order to fill the mating pool, $M(g)$.
- The new population, $P(g)$, is created in step 10 by applying crossover and mutation to the mating pool.
- In step 11 the generation counter is increased.

The evaluation method works as follows:

- In step 1 the individual, ind, is normalized. ind is a vector of n integers $(w_1^{GA}, w_2^{GA}, ..., w_n^{GA})$, where n is the number of assets available in the portfolio.

- In step 2 the historical series of portfolio return is calculated as $\sum_{i=1}^{n} w_i R_{ij} \quad \forall j$, where w_i is the normalized weight assigned to asset i, n is the number of assets available in the portfolio and R_{ij} is the return of asset i at time j.

- In step 3 the expected return of the portfolio is calculated as: $E(R) = \frac{1}{T}\sum_{j=1}^{T} R_j$, where T is the number of observations per asset.

- In order to calculate the empirical VaR, the vector R is arranged from highest to lowest in step 4.
- The position the 0,05-quantile takes is calculated in step 5 as 0,05T (rounded if necessary).
- In step 6 the 0,05-quantile is set to the element in the position 0,05T in the returns vector.
- The VaR is calculated in step 7 as the expected return minus the 0,05-quantile of the historic return series.
- In step 8 and 9 the two objective values of ind are set to the expected return and the inverse of the VaR (since the GA implemented maximizes the objectives).

Table 1 shows the control parameters of the multiobjective GA used.

Table 1

GA control parameters

Parameter	Value
Replacement operator	Generational
Selection operator	Tournament selection
Tournament group size	7
Crossover rate	1
Mutation rate	0,05
Population size	1000
Archive size	100
Termination criterion	50 generations

4. Empirical results

The data used in this work were extracted from the Bloomberg database. It is a set composed of twelve composite returns indices from USA, Canada, Japan, UK, France, Germany, Spain, Holland and Sweden. We employed weekly data of these indices from January 1992 until December 2005.

Figure 1 shows both the σ-efficient frontier obtained with classical quadratic programming and the GA VaR-efficient frontier obtained with GAs in the VaR-return space for periods 92–01, 94–03 and 96–05. That is, we have plotted the VaR-efficient frontier obtained using GAs against the VaR values of the σ-optimal portfolios. The vertical axis shows the expected rate of return after a week, that is 5-trading days, in percentage points. The horizontal axis shows VaR values as a percentage of the original portfolio value.

In period 92–01 it can be observed that the differences between the efficient frontiers are not so significant (the gap is negligible). Although it should be noted that the vertical axis represents a larger range of expected returns than for other periods. Again the larger difference in expected returns appears for VaR values below 3,5 %. In period 94–03 the difference between both efficient frontiers is absolutely relevant for portfolios with VaR below 3,5 %. During the period 96–05 the differences between the efficient frontiers tend to disappear when portfolios with high expected return are compared. It must be highlighted that all this efficient frontiers have been obtained over ten year periods which included different conditions, high and small volatility periods and bullish and bearish markets. To sum up, we observe differences for all the periods considered. Such fact hints to use the GA algorithm in order to introduce new measures of risk.
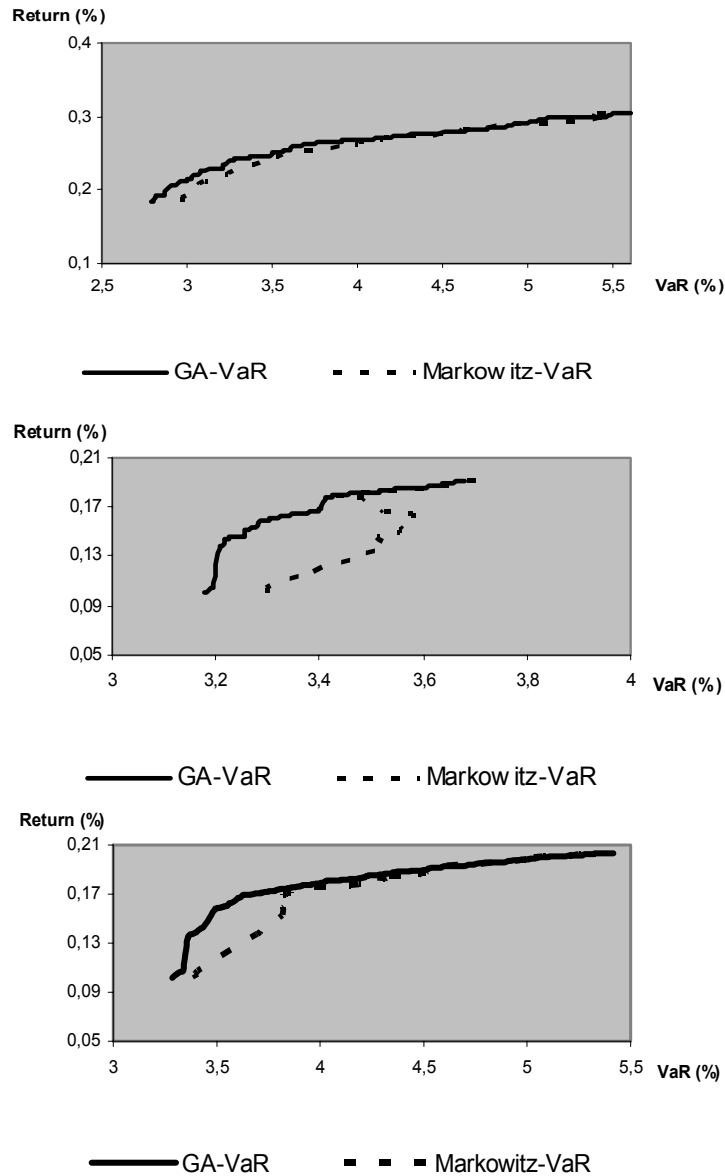
Figure 1. Efficient frontier for mean-VaR portfolios and frontier for mean-VaR with mean-variance efficient portfolios,
1992–2001, 1994–2003 and 1996–2005

Following Gaivoronski (2005), in order to quantify the differences between σ-optimal portfolios, w_σ^o, and VaR-optimal portfolios, w_{VaR}^o, we calculated the substitution error given by the largest value of VaR for some expected return. To measure this error, for each expected return value R^* we evaluated the VaR of the VaR-optimal portfolio, $VaR(w_{VaR}^o)$, and the VaR of the σ-optimal portfolios, $VaR(w_\sigma^o)$. That is, we computed,

$$E_{VaR}^o - E_\sigma^o = \left(\frac{R(w_{VaR}^o)}{VaR(w_{VaR}^o)} - \frac{R(w_\sigma^o)}{VaR(w_\sigma^o)} \right) 100. \tag{2}$$

This measure represents the relative improvement of VaR and expected return as percentage. We computed the mean of $E_{VaR}^o - E_\sigma^o$ and the percentage of cases in which the improvement exceed some threshold, given by θ.

When we compare the σ-efficient frontier and the VaR-efficient frontier in terms of expected return per percentage of VaR, table 2, we can observe that the majority of VaR-optimal portfolios are more efficient, if we measure efficiency in terms of Equation (2). Particularly, the percentage of VaR-optimal portfolios that are more efficient than σ-optimal portfolios, that is, $E_{VaR} - E_\sigma > 0$ %, goes from 81,21 % in 96–05 to 90,33 % in 94–03. The mean improvement goes from 0,134 % in 96–05 to 0,217 % in 92–01, which implies in annualized values 7 % to 11,3 %.

Comparison of results for efficient frontiers

	1992–2001	1994–2003	1996–2005
mean E_{VaR}–E_σ	0,2174	0,1354	0,1348
E_{VaR}–E_σ>–1 %	100	100	100
E_{VaR}–E_σ>–0,5 %	100	99,39	100
E_{VaR}–E_σ>0 %	82,47	90,33	81,21
E_{VaR}–E_σ>0,5 %	1,28	0	0
E_{VaR}–E_σ>1 %	0	0	0

Overall, the results point out the importance of solving the mean-VaR problem using an appropriate method in order to select an efficient portfolio when investors express their market risk in function of the VaR. Multiobjective GAs have proven to be able to solve the problem. Moreover, the time needed to compute around 300 points of the efficient frontier on a 2,8 GHz Celeron CPU with 1 GB RAM is of 60 seconds. This means that the algorithm could handle a massive amount of data (if available) in reasonable computing time.

5. Conclusions

We have developed a framework for portfolio selection that moves away from convex objective functions or standard mean-variance approach where non-differential restrictions can not be imposed. In our analysis, the risk measure minimized is VaR, which leads to non-convex objective functions. We have compared the mean-variance with mean-VaR approach to measure efficiency of the classical approach when investors are worried about portfolio's potential loss function, that is, the downside risk. We have evaluated optimal VaR-efficient portfolios and optimal σ-efficient portfolios for international stock indices observed weekly over the period 1992–2005. Results indicate reliability of VaR-efficient portfolios and significant improve over σ-efficient portfolios. Multiobjective GAs have demonstrated their adequacy for solving this problem in no-time.

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