Master Thesis - Active Management of Non-Granular Loan Portfolios

Martin Almqvist

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Abstract

This thesis considers quantitative tools for assessing concentration risks in credit portfolios that may underlie decision making in an active portfolio management setting. The study incorporates a literature review, which considers analytical and simulation based credit risk models in a Merton-type framework as well as aspects of credit portfolio management. The literature review is followed by a numerical analysis in which the credit risk models are evaluated with respect to accuracy and computational efficiency and the results suggests that the simulations based models are suitable for being incorporated into an active portfolio management framework in the setting tested.

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This work is dedicated to families Almqvist and Haubenwallner.

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1 Introduction

1.1 Problem Description

This thesis, written in co-operation with Danske Bank in Copenhagen, Denmark, aims to evaluate quantitative methods which may be used for actively managing the corporate and institutional banking portfolio of Danske Bank.

In contrast to retail banking portfolios, credit portfolios consisting of corporate and institutional clients tend to be characterized by large exposures to a relatively small set of counter-parties. This may give rise to effects of concentration with respect to single entities, industries and geographical regions and thus cause expenses in a risk/return perspective as well as with regards to the level of economic capital needed for supporting the portfolio.

Although Danske Bank is a global bank with presence in several northern European markets, possibilities of solely building a diversified corporate and institutional banking portfolio through origination oriented measures are restricted. Expanding the banks approach to active portfolio management may serve to enhance portfolio information, profitability and robustness with respect to the portfolio risk/return profile.

1.2 Research Question

How can active portfolio management be used given the limitations in being a Nordic bank with a non-granular corporate & institutional banking portfolio?

1.3 Methodology

The approach taken in this thesis is based on introduction of computationally effective methods for evaluating portfolio concentration risk and thus determining which exposures that are most favorable with respect to yield on capital at stake. This information may underlay decision making in transaction origination and capital market activities targeted to improve the structure of non-granular loan portfolios.

The study is initiated through a *literature review*, which is aimed to provide a theoretical presentation of the regarded credit risk models, with focus on the analytical multi-factor adjustment developed by Pykhin (2004) and the simulation based importance sampling Monte Carlo methods developed by Glasserman and Li (2005). This is followed by a study of portfolio management aspects and tools, which to a large extent are aligned with those in Hünseler (2013) and Bouteillé and Coogan-Pushner (2013). The purpose is here to give the reader insight in quantitative credit risk modeling and connect the considered risk models to the framework of active portfolio management.

In the proceeding *numerical analysis*, the risk models are evaluated through a Matlab implementation, with focus on accuracy and computational efficiency. The Pykhtin multi-factor adjustment is considered on portfolio level whereas the Monte Carlo methods are examined for portfolio, sector and obligor specific risk measures. The analysis is performed using the underlying multi-factor model and sample portfolio set outlined in section 4.1.1.

The approach of generated portfolio data is partly chosen for simplicity, as the portfolio structure outlined in section 4.1.1 is convenient for adjusting single properties in sensitivity analyses, and partly because of complications in using Danske Bank market data without disclosing details of the company's credit risk model. The setup chosen does however not limit generality, which is beneficial from an academical perspective.

Furthermore, the following computational setup is used for the numerical analysis:

Processor:	Intel(R) Core(TM) i5-2500 CPU @ 3.30GHz
Installed memory (RAM):	8.00 GB
Operating system:	Microsoft Windows Enterprise 7 64-bit
Computing software:	Matlab R2014a (8.3.0.532), 64-bit (win64)

1.4 Limitations

The credit risk models regarded in this study do not capture contagion risk, i.e. risk arising from direct business connections between counter-parties. This risk-type has proven to play a significant role in various financial crises, as stated in OECD (2012), but has here been left out in order to preserve simplicity of the considered factor risk models.

The models are further restricted to being default-mode only models, which are based on the foundational assumption that credit losses exclusively are stemming from counter-parties defaulting on their obligations. All exposures are additionally assumed to have a maturity of one year and we do not account for transitions in obligor credit qualities. These assumptions are however less restrictive from a perspective of generality, as the one year factor risk models considered in this thesis may be incorporated in a larger, multi-year credit model with little added complexity. Such a model could be adjusted to account for transition risk, but details on designing such structures will not be covered in this thesis.

The credit risk model derivations presented are general with respect to the distribution of the loss given default variables. Several distributions for modeling these variables have been suggested in academia and the appropriate choice of distribution depends on several factors connected to portfolio characteristics, and a detailed discussion of this subject lies outside the scope of this thesis for reasons of simplicity. Furthermore, the set of sample portfolios used in the numerical analysis is of limited size to make this study comprehensive.

A complete active portfolio management framework covers several aspects such as organizational structures, accounting standards, business strategies, trading strategies and credit risk models. Covering all aspects demands expertise within each separate field and although some of these fields are considered briefly in section 3, the core focus of this project lies in credit risk modeling, with focus on models which may be applied for portfolio analyses rather than regulatory purposes.

Finally, the computational hardware has set limits to the ability of reaching high levels of convergence for some of the considered Monte Carlo methods.

2 Credit Risk Modeling

2.1 An Introduction to the Multi-Factor Merton Model

The credit risk models regarded in this study are all *factor risk models*, a type of credit risk models originating from the pioneering work of Merton (1974). Factor risk models have gained popularity in the credit industry for effectively modeling asset correlations through a limited set of *risk factors*. The scope is further restricted to *default mode-only models*, meaning models where losses exclusively stem from obligors defaulting on their obligations.

2.1.1 The Multi-Factor Model Setup

This section aims to introduce the multi-factor model setup regarded in this study. Foundational definitions and assumptions are here followed by an introduction to portfolio loss rates. The framework presented here lays the foundation for the credit risk estimation techniques which will later be examined and evaluated.

2.1.1.1 Framework Definition

We define the properties of each loan in a portfolio analogous to (Bouteillé and Coogan-Pushner, 2013, p. 89).

Assumption 1
Each loan in a portfolio can be characterized by the following properties:
(i) Exposure at Default (EAD): The nominal amount of money at risk.
(ii) Probability of Default (PD): The probability that an obligor defaults on its obligations.
(iii) Loss Given Default (LGD): The relative amount of the exposure lost in case of default.
(iv) Tenor (T): The time during which the money is outstanding.

Following (Lütkebohmert, 2009, p. 5), we further assume that each obligor $i = 1...M$ in a portfolio correspond to one single loan and define the *exposure weight* of obligor i as:

$$w_i = \frac{EAD_i}{\sum\limits_{j=1}^{M} EAD_j}. \tag{1}$$

Assumption 2
An obligor defaults if its asset return X_i falls below a default threshold d_i (Lütkebohmert, 2009, p. 24).

We define the *default indicator* of obligor i as a Bernoulli distributed stochastic variable in accordance with (Lütkebohmert, 2009, p. 6), which takes the value 1 in the event of the obligor defaulting and zero else wise:

$$D_i = \mathbf{1}\{X_i < d_i\} \sim \text{Bern}\left[1, P(X_i < d_i)\right]. \tag{2}$$

Assumption 3
The logarithmic asset return X_i of obligor i in time period [0,T] is a linear function of N independent standard normal systematic risk factors $\{Z_k\}_{k=1...N}$ and an independent standard normal idiosyncratic shock ξ_i (Lütkebohmert, 2009, p. 24).

This assumption may be formulated mathematically in the following way:

$$X_i = r_i Y_i + \sqrt{1-r_i^2}\xi_i, \quad Y_i, \xi_i \sim N(0,1), \tag{3}$$

where r_i denotes the *factor loading* and Y_i the *composite risk factor* defined as:

$$Y_i = \sum_{k=1}^{N} \beta_{ik} Z_k, \quad Z_k \sim N(0,1), \tag{4}$$

in accordance to (Lütkebohmert, 2009, p 24-25). It must hold that $\sum_{k=1}^{N} \beta_{ik}^2 = 1$ in order for Y_i to satisfy unit variance and the asset return correlation in-between obligors i and j is thereby fully determined through the set of systematic risk factors:

$$corr(X_i, X_j) = r_i r_j \sum_{k=1}^{N} \beta_{ik}\beta_{jk}. \tag{5}$$

It follows from (3) that asset returns are assumed standard normal distributed under the assumptions of this model. Given normality, the property $PD_i = P(X_i < d_i)$ implies that $d_i = N^{-1}(PD_i)$ must hold. We may thus re-write (2) by using the probability of default:

$$D_i = \mathbf{1}\{X_i < N^{-1}(PD_i)\} \sim \text{Bern}\left[1, P(X_i < N^{-1}(PD_i))\right], \tag{6}$$

as is stated in (Lütkebohmert, 2009, p. 25).

We proceed with introducing a stochastic loss given default variable LGD_i with expected value $\mathbb{E}[LGD_i]$ and variance $\mathbb{V}[LGD_i]$, representing the loss given default of obligor i. The variables are assumed to be bounded in the interval[1] [-1,1] but their distribution is left unspecified at this stage[2].

Assumption 4
The default indicators $\{D_i\}_{i=1...M}$, the exposure at default variables $\{EAD_i\}_{i=1...M}$ and the loss given default variables $\{LGD_i\}_{i=1...M}$ are independent (Lütkebohmert, 2009, p. 7).

[1] Negative values of loss given default corresponds to short positions.

[2] Several approaches exist for modeling stochastic loss given default and we are therefore not restricting our derivation to a certain distribution. In the proceeding of this thesis, it will however be assumed that a distribution for modeling the loss given default variables has been chosen.

2.1.1.2 Portfolio Loss Rates

In this setup, the *loss rate* of a portfolio may be represented as a stochastic variable in the following way:

$$L = \sum_{i=1}^{M} w_i LGD_i D_i. \tag{7}$$

We may now proceed by seeking an expression for the expectation of the portfolio loss rate conditional on the composite risk factors $\{Y_i\}_{i=1...M}$, following (Lütkebohmert, 2009, p. 27). Given (3), the following expressions are equivalent:

$$X_i < N^{-1}(PD_i) \quad \iff \quad \xi_i < \frac{N^{-1}(PD_i) - r_i Y_i}{\sqrt{1 - r_i^2}}, \tag{8}$$

and the normality of ξ_i allows for the *conditional probability of default* of obligor i to be formulated as:

$$PD_i(Y_i) = N\left(\frac{N^{-1}(PD_i) - r_i Y_i}{\sqrt{1 - r_i^2}}\right). \tag{9}$$

The conditional probability of default may be interpreted as the likelihood of obligor i defaulting given a certain state of the economy[3], as stated in (Lütkebohmert, 2009, p. 26). We now obtain the conditional expectation of the portfolio loss rate by insertion of the conditional default probabilities in (7), as the exposure weights and loss given default variables are independent of the systematic risk factors (Lütkebohmert, 2009, p. 27):

$$\mathbb{E}[L \mid \{Y_i\}_{i=1...M}] = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] N\left(\frac{N^{-1}(PD_i) - r_i Y_i}{\sqrt{1 - r_i^2}}\right). \tag{10}$$

We further note that asset returns are conditionally independent given a realization of the systematic risk factors. This implies conditional independence of obligor losses, which is a foundational property for several of the models regarded in this study.

In this thesis we refer to all obligors with loadings larger than zero on the same set of systematic risk factors as a *sector*. This definition is intuitive as it is implied from (5) that the asset returns of obligors within a sector are closer linked than those of obligors in different sectors. In practice, a multi-factor model may be set up such that sectors and risk factors are related to risk drivers such as geographical regions and industries.

[3]This follows as the obligor specific composite risk factors $\{Y_i\}_{i=1...M}$ represents the impact market movements has on obligor asset returns.

2.2 Analytical Estimation Techniques

This section presents analytical estimation techniques for estimating portfolio Value at Risk and Expected Shortfall. These techniques allow for risk measures to be estimated without the corresponding portfolio loss distribution, in contrast to the simulation based methods considered later in this thesis.

2.2.1 The Asymptotic Single Risk Factor Framework

The Asymptotic Single Risk Factor framework, which we will from here on refer to as ASRF, was developed by Gordy (2003) and provides a method for approximating quantiles of the loss rate distribution of a portfolio under a few restricting assumptions.

2.2.1.1 Framework Definition

We begin with specifying one out of the three foundational assumptions in the framework.

Assumption 5
For a portfolio of interest, it holds that:
(i) The portfolio is infinitely fine-grained.
(ii) All obligors in the portfolio are influenced by the same single systematic risk factor.

This assumption is analog to that in (Lütkebohmert, 2009, p. 32). Assumption 5(i) refers to the negligibility of idiosyncratic risk that follows from the law of large numbers for large enough portfolios[4], meaning that the risk of a portfolio reduces to the uncertainty of the systematic risk factors. Assumption 5(ii) implies that the factor model in (3) reduces to:

$$X_i = r_i Y + \sqrt{1 - r_i^2}\xi_i, \quad Y, \xi_i \sim N(0,1), \tag{11}$$

where the Y denotes the standard normal distributed *single risk factor* of the portfolio.

Assumption 6
For the variables $V_i = LGD_i D_i$ $i = 1...M$, it holds that:
(i) The variables are bounded in the interval [-1,1].
(ii) The variables are mutually independent conditional on the systematic risk factor Y.

This assumption follows that in (Lütkebohmert, 2009, p. 32). Given assumptions 5(i) and 6 it holds almost surely, conditional on the systematic risk factor Y taking value y, that the conditional expectation of a portfolio loss rate converges to the true loss rate as the portfolio size tends to infinity:

$$\mathbb{P}\left(\lim_{M\to\infty} [L_M - \mathbb{E}[L_M \mid y]] = 0\right) = 1, \tag{12}$$

[4]The law of large numbers is described in detail in theorem 1, section 2.3.

where L_M denotes the loss rate of a portfolio with M obligors[5]. This follows as idiosyncratic risk is eliminated through diversification, making the loss rate fully driven by the systematic risk factors and the law of large numbers becomes applicable through assumption 6(*ii*).

A proof of (12) is to be found in (Hibbeln, 2010, p. 57). We proceed by making the following assumption, in line with (Lütkebohmert, 2009, p. 35)[6]:

Assumption 7
There exist an open interval ω that contains the quantile of the systematic risk factor $q_\alpha(Y)$ and a real number $m < \infty$ such that:
(i) for all M, $\mathbb{E}[V_M \,|\, y]$ is continuos in Y on ω,
(ii) $\mathbb{E}[L_M \,|\, y]$ is nonincreasing in y on ω for all $M > m$,
(iii) $\sup_{y\in\omega} \mathbb{E}[L_M \,|\, y] \leq \inf_{y \leq \inf \omega} \mathbb{E}[L_M \,|\, y]$ *for all $M > m$,*
(iv) $\inf_{y\in\omega} \mathbb{E}[L_M \,|\, y] \geq \inf_{y \geq \sup \omega} \mathbb{E}[L_M \,|\, y]$ *for all $M > m$.*

This assumption serves to ensure that a region around a quantile of the expected conditional loss rate $\mathbb{E}[L \,|\, Y]$ is related to a region of a quantile of the systematic single risk factor Y at the same confidence level. Under assumption 5(*ii*) and 7 it then holds for $M > m$:

$$q_\alpha\left(\mathbb{E}[L_M \,|\, Y]\right) = \mathbb{E}\left([L_M \,|\, y = q_{1-\alpha}(Y)]\right), \tag{13}$$

according to (Lütkebohmert, 2009, p. 34). The reader is here referred to (Hibbeln, 2010, p. 53)[7] for a proof of (13).

2.2.1.2 Estimation of Loss Rate Quantiles

Following assumptions 5, 6 and 7 and relations (12) and (13), it then holds that:

$$q_\alpha^{ASRF}(L) = \lim_{M\to\infty} q_\alpha(L_M) = \lim_{M\to\infty} q_\alpha(\mathbb{E}[L_M \,|\, Y]) = \lim_{M\to\infty} \mathbb{E}\left([L_M \,|\, y = q_{1-\alpha}(Y)]\right). \tag{14}$$

It further follows from (10) and (14) that an analytical formula for Value at Risk under the ASRF assumptions can be formulated as follows (Hibbeln, 2010, p. 38):

$$VaR_\alpha^{ASRF}(L) = q_\alpha^{ASRF}(L) = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] N\left(\frac{N^{-1}(PD_i) - r_i q_{1-\alpha}(Y)}{\sqrt{1-r_i^2}}\right). \tag{15}$$

[5]The loss rate is here indexed to simplify the formulation of assumption 7.
[6]Assumption 7 differs from (Gordy, 2003, p. 207, $\mathcal{A}$-4) since we in this thesis have defined the loss rate to be decreasing with Y, see equation (10).
[7]Gordy (2003) states a proof under the assumption that the expectation of the loss rate is monotonically increasing with Y. We choose to refer to the proof of Hibbeln (2010) as it is based on the assumption that the expectation is monotonically decreasing with Y.

2.2.2 The Pykhtin Multi-Factor Adjustment

Following the granularity adjustment technique for the ASRF developed by Gordy (2003), Pykhin (2004) presents an extension of the technique into a multi-factor framework.

This model aims to approximate Value at Risk and Expected Shortfall for a multi-factor portfolio loss rate distribution through an appropriately chosen single-factor model and a correction term, here referred to as a *multi-factor adjustment*, which accounts for effects of diversification and finite granularity. The reasonings and results presented here follows the original article by Pykhin (2004).

2.2.2.1 Taylor Expansion of Loss Rate Quantiles

For the loss rate L of a multi-factor model, we define a perturbed variable $L_\epsilon = \tilde{L} + \epsilon U$, where U denotes the perturbation $U = L - \tilde{L}$, and seek to estimate the α-quantile $q_\alpha(L)$ through a second order taylor expansion of $q_\alpha(L_\epsilon)$ around $q_\alpha(\tilde{L})$[8] evaluated at $\epsilon = 1$:

$$q_\alpha(L) = q_\alpha(\tilde{L}) + \left.\frac{dq_\alpha(L_\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \frac{1}{2}\left.\frac{d^2q_\alpha(L_\epsilon)}{d\epsilon^2}\right|_{\epsilon=0}, \tag{16}$$

as L_ϵ then reduces to L. The first and second order derivative of $q_\alpha(L_\epsilon)$ have been derived by Gourieroux et al. (2000) and can be obtained from:

$$\left.\frac{dq_\alpha(L_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[U \,\middle|\, \tilde{L} = l\right]\Big|_{l=q_\alpha(\tilde{L})}, \tag{17}$$

$$\left.\frac{d^2q_\alpha(L_\epsilon)}{d\epsilon^2}\right|_{\epsilon=0} = -\frac{1}{p_{\tilde{L}}(l)}\frac{d}{dl}\left(p_{\tilde{L}}(l)\mathbb{V}\left[U \,\middle|\, \tilde{L} = l\right]\right)\Big|_{l=q_\alpha(\tilde{L})}, \tag{18}$$

where $p_{\tilde{L}}(\cdot)$ denotes the probability density function of $\tilde{L}$.

In words, this approach is based on relating the quantile of a multi-factor loss distribution to that of a single factor limiting loss distribution through a second order taylor expansion, by letting the perturbed variable approach the multi-factor loss distribution. The correction terms in the taylor expansion may thus be interpreted as a multi-factor adjustment to the quantile of the single factor loss distribution.

2.2.2.2 Defining a Suitable Single-Factor Model

We now seek to define the variable $\tilde{L}$ in such a way that it in the best way possible captures the behavior of the multi-factor loss rate L. In doing so, we define $\tilde{L}$ through a single-factor

[8] This reasoning is based on the results of Martin and Wilde (2002), where it is shown that (16) hold if the confidence level α is set high enough.

limiting loss distribution in analogy with (10):

$$\tilde{L} = l(\tilde{Y}) = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] P\tilde{D}_i(\tilde{Y}), \tag{19}$$

$$P\tilde{D}_i(\tilde{Y}) = N\left(\frac{N^{-1}(PD_i) - \tilde{r}_i\tilde{Y}}{\sqrt{1-\tilde{r}_i^2}}\right), \tag{20}$$

where $P\tilde{D}_i(\tilde{Y})$ denotes the probability of default of obligor i conditional on the single systematic risk factor $\tilde{Y}$ defined under the assumptions of the ASRF, in accordance to (9). Analogous to (15), the α-quantile of $\tilde{L}$ can be obtained as:

$$q_\alpha(\tilde{L}) = l\left(N^{-1}(1-\alpha)\right). \tag{21}$$

Furthermore, conditioning on $\tilde{L} = q_\alpha(\tilde{L})$ is equivalent[9] to conditioning on $\tilde{Y} = N^{-1}(1-\alpha)$, which makes us able to re-formulate (17) and (18)[10] as:

$$\left.\frac{dq_\alpha(L_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[U \,\middle|\, \tilde{Y} = y\right]\Big|_{y=N^{-1}(1-\alpha)}, \tag{22}$$

$$\left.\frac{d^2q_\alpha(L_\epsilon)}{d\epsilon^2}\right|_{\epsilon=0} = -\frac{1}{\phi(y)}\frac{d}{dy}\left(\phi(y)\frac{v(y)}{l'(y)}\right)\Bigg|_{y=N^{-1}(1-\alpha)}, \tag{23}$$

where $v(y)$ denotes $\mathbb{V}\left[U \,\middle|\, \tilde{Y} = y\right]$ and $\phi(\cdot)$ refers to the standard normal probability density function.

In the article by Pykhin (2004), it is stated that choosing $\tilde{L}$ such that it equals $\mathbb{E}\left[L \,|\, \tilde{Y}\right]$ for any portfolio constellation is beneficial partly for intuitive reasons, as we are to approximate quantiles of L through $\tilde{L}$, and partly because of the fact that such a definition implies that (22) equals zero, as it then holds that:

$$\mathbb{E}\left[U \,\middle|\, \tilde{Y}\right] = \mathbb{E}\left[L - \tilde{L} \,\middle|\, \tilde{Y}\right] = \mathbb{E}\left[L \,\middle|\, \tilde{Y}\right] - \mathbb{E}\left[L \,\middle|\, \tilde{Y}\right] = 0. \tag{24}$$

This implies that we are now only interested in computing the second derivative in (16). We further assume that the single risk factor $\tilde{Y}$ can be described as a linear function of the systematic risk factors $\{Z_k\}_{k=1\ldots N}$ in the multi-factor model (4) such that:

$$\tilde{Y} = \sum_{k=1}^{N} \tilde{\beta}_k Z_k, \tag{25}$$

[9] This holds as $\tilde{L}$ is monotonically decreasing in $\tilde{Y}$.

[10] We here make use of the chain rule: $\frac{d}{dl} = \frac{d}{dy}\frac{dy}{dl} = \frac{d}{dy}\frac{1}{l'(y)}$.

and given that $\tilde{Y}$ is standard normal, $\sum\limits_{k=1}^{N} \tilde{\beta}_k^2 = 1$ must hold. Furthermore, Pykhin (2004) assumes that the obligor specific composite risk factor Y_i can be described as a function of the single risk factor $\tilde{Y}$ and a variable η_i which is standard normal distributed:

$$Y_i = \tilde{\rho}_i \tilde{Y} + \sqrt{1 - \tilde{\rho}_i^2} \eta_i. \tag{26}$$

The variables $\{\eta_i\}_{i=1...M}$ are assumed mutually dependent but independent of $\tilde{Y}$ and $\tilde{\rho}_i$ denotes the correlation between the composite risk factors in the single factor (25) and the multi-factor loss distribution (4), defined as:

$$\tilde{\rho}_i \equiv corr(Y_i, \tilde{Y}) = \sum_{k=1}^{N} \beta_{ik} \tilde{\beta}_k. \tag{27}$$

This enables us to formulate the following relation between the asset return of obligor i and the single risk factor:

$$X_i = r_i \tilde{\rho}_i \tilde{Y} + r_i \sqrt{1 - \tilde{\rho}_i{}^2} \eta_i + \sqrt{1 - r_i^2} \xi_i, \tag{28}$$

which, due to the mutual independence of η_i and ξ_i, we may simplify to:

$$X_i = r_i \tilde{\rho}_i \tilde{Y} + \sqrt{1 - (r_i \tilde{\rho}_i)^2} \tilde{\xi}_i, \tag{29}$$

where $\tilde{\xi}$ is standard normal distributed and independent of $\tilde{Y}$. In analogy with (10), we may write the expected loss rate conditional on the single risk factor as:

$$\mathbb{E}\left[L \middle| \tilde{Y}\right] = \sum_{i=1}^{M} w_i \mathbb{E}\left[LGD_i\right] N\left(\frac{N^{-1}(PD_i) - r_i \tilde{\rho}_i \tilde{Y}}{\sqrt{1 - (r_i \tilde{\rho}_i)^2}}\right). \tag{30}$$

The relation in (30), together with (19) and (20), enables us to determine the single risk factor loadings $\{\tilde{r}_i\}_{i=1...M}$, as they must satisfy the following condition:

$$\tilde{r}_i = r_i \tilde{\rho}_i = r_i \sum_{k=1}^{N} \beta_{ik} \tilde{\beta}_k. \tag{31}$$

Now remaining is to determine the parameters $\{\tilde{\beta}_k\}_{k=1...N}$, which constitutes the link between the single risk factor and the risk factors in the multi-factor model (25). In the article by Pykhin (2004), it is argued that a good choice of parameters is one that minimizes the difference between $q_\alpha(\tilde{L})$ and the sought after quantile $q_\alpha(L)$ and that an intuitive way of achieving this is to define the parameters such that the single risk factor $\tilde{Y}$ becomes as correlated with the composite risk factors $\{Y_i\}_{i=1\ldots M}$ as possible. We formulate this mathematically in the following optimization problem:

$$\begin{aligned} \underset{\{\tilde{\beta}_k\}}{\text{maximize}} \quad & \sum_{i=1}^{M} c_i corr(\tilde{Y}, Y_i), \\ \text{subject to} \quad & \sum_{k=1}^{N} \tilde{\beta}_k^2 = 1, \end{aligned} \tag{32}$$

and as we previously have defined the factor correlation in (27), the solution to (32) is[11]:

$$\tilde{\beta}_k = \sum_{i=1}^{M} \frac{c_i \beta_{ik}}{\lambda}, \tag{33}$$

where λ denotes a positive Lagrange multiplier defined such that the constraint in (32) is fulfilled and $\{c_i\}_{i=1\ldots M}$ is specified according to[12]:

$$c_i = w_i \mathbb{E}[LGD_i] N \left[\frac{N^{-1}(PD_i) + r_i N^{-1}(\alpha)}{\sqrt{1 - r_i^2}} \right]. \tag{34}$$

2.2.2.3 Multi-Factor Adjustment for Value at Risk

Having determined all parameters for mapping a multi-factor loss rate L to a single-factor loss rate $\tilde{L}$, we may now describe the taylor expansion in (16) as a single-factor model with a multi-factor adjustment $\Delta q_\alpha(L)$ in form of the correction term. Given (16) and (23) we may, by using the relation $\phi'(y) = -y\phi(y)$, express the multi-factor adjustment as:

$$\Delta q_\alpha(L) = q_\alpha(L) - q_\alpha(\tilde{L}) = -\frac{1}{2l'(y)} \left[v'(y) - v(y) \left(\frac{l''(y)}{l'(y)} + y \right) \right] \Bigg|_{y=N^{-1}(1-\alpha)}. \tag{35}$$

The derivatives $l'(y)$ and $l''(y)$ can easily be obtained from (19):

$$l'(y) = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] \tilde{PD}_i'(y), \tag{36}$$

$$l''(y) = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] \tilde{PD}_i''(y), \tag{37}$$

[11] We refer to Pykhin (2004) for the methodology used to obtain this result.

[12] This specification has been selected in Pykhin (2004) by empirically testing a number of specifications based on intuitive reasonings and assumptions. The reader is referred to the article for further details.

and the differentiated conditional probabilities analogously follow from (20):

$$\tilde{PD}_i'(y) = -\frac{\tilde{r}_i}{\sqrt{1-\tilde{r}_i^2}}\phi\left(\frac{N^{-1}(PD_i)-\tilde{r}_i y}{\sqrt{1-\tilde{r}_i^2}}\right), \tag{38}$$

$$\tilde{PD}_i''(y) = -\frac{\tilde{r}_i^2}{1-\tilde{r}_i^2}\frac{N^{-1}(PD_i)-\tilde{r}_i y}{\sqrt{1-\tilde{r}_i^2}}\phi\left(\frac{N^{-1}(PD_i)-\tilde{r}_i y}{\sqrt{1-\tilde{r}_i^2}}\right), \tag{39}$$

where N_2 denotes the bivariate standard normal distribution. As of now, all parameters in (35) are known except for the conditional variance $v(y)$ and its derivative $v'(y)$. We note that it must hold that $v(y) = \mathbb{V}(U \mid \tilde{Y} = y) = \mathbb{V}(L \mid \tilde{Y} = y)$, since $\tilde{L}$ is deterministic conditional on $\tilde{Y}$, and we will thus seek to determine the conditional variance of L. By using (26) and (27), we may rewrite the asset return for obligor i stated in (28) as:

$$X_i = \tilde{r}_i\tilde{Y} + \sum_{k=1}^{N}(r_i\beta_{ik} - \tilde{r}_i\tilde{\beta}_k)Z_k + \sqrt{1-r_i^2}\xi_i, \tag{40}$$

which shows that the asset returns are correlated through the systematic risk factors in the second term on the right hand side of (40). Given (40) and the previously stated constraints $\sum\limits_{k=1}^{N}\beta_{ik}^2 = 1$ and $\sum\limits_{k=1}^{N}\tilde{\beta}_k^2 = 1$, we may formulate the asset correlation for obligors i and j conditional on $\tilde{Y}$:

$$\rho_{ij}^{\tilde{Y}} = \begin{cases} \dfrac{r_i r_j \sum\limits_{k=1}^{N}\beta_{ik}\beta_{jk} - \tilde{r}_i\tilde{r}_j}{\sqrt{1-\tilde{r}_i^2}\sqrt{1-\tilde{r}_j^2}}, & i \neq j, \\ 1 & , \quad i = j. \end{cases} \tag{41}$$

In his article, Pykhin (2004) suggests that $v(y)$ can be divided in systematic and idiosyncratic components as the asset returns are independent conditional on the composite risk factors $\{Z_k\}_{k=1\ldots N}$ as follows:

$$\begin{aligned} v(y) = \mathbb{V}\left[L \mid \tilde{Y} = y\right] = \\ = \underbrace{\mathbb{V}\left[\mathbb{E}\left[L \mid \{Z_k\}_{k=1\ldots N}\right] \mid \tilde{Y} = y\right]}_{\text{Systematic}} + \underbrace{\mathbb{E}\left[\mathbb{V}\left[L \mid \{Z_k\}_{k=1\ldots N}\right] \mid \tilde{Y} = y\right]}_{\text{Idiosyncratic}}, \end{aligned} \tag{42}$$

where the systematic term accounts for the difference in variance in between the limiting loss distribution in a multi-factor setting (10) and the previously defined single-factor setting (19), and the idiosyncratic term adjusts for the granularity of the portfolio.

Pykhin (2004) further states that we may compute the systematic component of the conditional variance as:

$$v_{systematic}(y) = \sum_{i=1}^{M}\sum_{j=1}^{M} w_i w_j \mathbb{E}[LGD_i]\mathbb{E}[LGD_j] \\ \left[N_2\left(N^{-1}\left[\tilde{PD}_i(y)\right], N^{-1}\left[\tilde{PD}_j(y)\right], \rho_{ij}^{\tilde{Y}}\right), -\tilde{PD}_i(y)\tilde{PD}_j(y)\right], \tag{43}$$

and its derivative can be obtained from differentiation:

$$v'_{systematic}(y) = 2\sum_{i=1}^{M}\sum_{i=1}^{M} w_i w_j \mathbb{E}[LGD_i]\mathbb{E}[LGD_j]\tilde{PD}'_i(y) \\ \left[N\left(\frac{N^{-1}\left[\tilde{PD}_j(y)\right] - \rho_{ij}^{\tilde{Y}} N^{-1}\left[\tilde{PD}_i(y)\right]}{\sqrt{1-(\rho_{ij}^{\tilde{Y}})^2}}\right)\right]. \tag{44}$$

The corresponding relation for the idiosyncratic component is:

$$v_{idiosyncratic}(y) = \sum_{i=1}^{M} w_i^2 \\ \left(\mathbb{E}[LGD_i]^2\left[\tilde{PD}_i(y) - N_2\left(N^{-1}\left[\tilde{PD}_i(y)\right], N^{-1}\left[\tilde{PD}_i(y)\right], \rho_{ii}^{\tilde{Y}}\right)\right] + \mathbb{V}[LGD_i]^2\tilde{PD}_i(y)\right), \tag{45}$$

and its derivative can analogously be obtained as:

$$v'_{idiosyncratic}(y) = \sum_{i=1}^{M} w_i^2 \tilde{PD}'_i(y) \\ \left(\mathbb{E}[LGD_i]^2\left[1 - 2N\left(\frac{N^{-1}[\tilde{PD}_i(y)] - \rho_{ii}^{\tilde{Y}} N^{-1}[\tilde{PD}_i(y)]}{\sqrt{1-(\rho_{ii}^{\tilde{Y}})^2}}\right)\right] + \mathbb{V}[LGD_i]^2\right). \tag{46}$$

All terms in (35) have now been obtained and we are thus able to estimate a quantile $q_\alpha(L)$ through the multi-factor adjustment as follows:

$$q_\alpha(L) = q_\alpha(\tilde{L}) + \Delta q_\alpha(L). \tag{47}$$

As the multi-factor adjustment is a linear function of the conditional variance it is possible to divide it into a systematic and an idiosyncratic component analogous to the conditional variance, which we may write as: $\Delta q_\alpha(L) = \Delta q_\alpha^{systematic}(L) + \Delta q_\alpha^{idiosyncratic}(L)$. The quadratic weights (45) and (46) ensures that the idiosyncratic contribution goes to zero asymptotically as the granularity of a portfolio tends to infinity.

2.2.2.4 Multi-Factor Adjustment for Expected Shortfall

As we now are able to compute loss rate quantiles, we proceed by seeking a way to estimate Expected Shortfall through the multi-factor adjustment. In the following definition of Expected Shortfall:

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)\ du, \tag{48}$$

we substitute the quantile with the multi-factor adjustment $q_\alpha(L) = q_\alpha(\tilde{L}) + \Delta q_\alpha(L)$ and may thus re-formulate the expression as:

$$ES_\alpha(L) = ES_\alpha(\tilde{L}) + \frac{1}{1-\alpha}\int_\alpha^1 \Delta q_u(L)\ du, \tag{49}$$

where $ES_\alpha(\tilde{L})$ denotes the Expected Shortfall of our single-factor loss rate distribution. Following definition (103) in Appendix A.5, we may compute $ES_\alpha(\tilde{L})$ as:

$$ES_\alpha(\tilde{L}) = \mathbb{E}\left[l(\tilde{Y})\,\middle|\,\tilde{Y} \leq N^{-1}(1-\alpha)\right] = \frac{1}{1-\alpha}\int_{-\infty}^{N^{-1}(1-\alpha)} l(y)\phi(y)\ dy, \tag{50}$$

which, by insertion of $l(y)$ from (19) and use of the relation $\int_{-\infty}^{z}\phi(u)N(\frac{x-\tilde{r}y}{\sqrt{1-\tilde{r}^2}})du = N_2(x,z,\tilde{r})$, can be re-formulated as:

$$ES_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\sum_{i=1}^{M} w_i\mathbb{E}[LGD_i]N_2\left[N^{-1}(PD_i), N^{-1}(1-\alpha), \tilde{r}_i\right]. \tag{51}$$

In order to evaluate the integral term in (49), we make use of relations (16) and (23) to formulate an expression for $\Delta q_\alpha(L)$, which by insertion gives the following integral expression:

$$\Delta ES_\alpha(L) = -\frac{1}{2(1-\alpha)}\int_\alpha^1 \frac{1}{\phi(y)}\frac{d}{dy}\left(\phi(y)\frac{v(y)}{l'(y)}\right)\bigg|_{y=N^{-1}(1-u)} du. \tag{52}$$

According to (Pykhin, 2004, p. 88), this integral can be solved by exchanging variable u for $t = N^{-1}(1-u)$ in order to arrive at a complete expression for the multi-factor adjustment:

$$\Delta ES_\alpha(L) = -\frac{1}{2(1-\alpha)}\phi\left[N^{-1}(1-\alpha)\right]\frac{v\left[N{-}1(1-\alpha)\right]}{l'\left[N^{-1}(1-\alpha)\right]}. \tag{53}$$

In analogy with the Value at Risk multi-factor adjustment, (53) can be divided into systematic and idiosyncratic parts: $\Delta ES_\alpha(L) = \Delta ES_\alpha^{systematic}(L) + \Delta ES_\alpha^{idiosyncratic}(L)$.

Remark 1 *These results build on the assumption that the effective factor loadings $\{\tilde{r}_i\}_{i=1...M}$ are independent of the confidence level α, which is contradicted by the fact that the factor loadings are assumed to be a function of the quantile dependent parameters $\{\tilde{\beta}_k\}_{k=1...N}$, as can be seen in* (32) - (34)*. A solution to this problem would be to develop a new parameter estimation technique in order to make the parameters quantile independent (Pykhin, 2004, p. 88) but we will, however, not consider such a technique in this thesis.*

2.2.2.5 Enhancement of Computational Efficiency

This model's main determinant of computational efficiency is the number of obligors in a portfolio, following the double sums in (43) and (44), which are giving rise to a quadratic relationship between computational time and number of obligors.

A solution to this bottle-neck is suggested in (Hibbeln, 2010, p. 196), where it is stated that the model can be applied on sector level, conditional on a setup where the sectors are divided into sub-sectors after probability of default and all positions in each sub-sector are aggregated. The computational effort then reduces according to:

$$\text{Computational Effort} \propto (\text{Number of Sectors} \cdot \text{Number of Sub-Sectors})^2, \tag{54}$$

where the number of sub-sectors becomes a tradeoff between accuracy and computational efficiency. The idiosyncratic part of the granularity adjustment does, however, remain on obligor level in order to account for diversification effects across names, but this does not constitute a significant computational burden in comparison to computing (43) and (44).

2.3 Simulation Based Estimation Techniques

This section discusses the simulation based *Monte Carlo methods*, which is a family of estimation techniques which use independent draws from a set of random variables to empirically evaluate a function or a problem. The basic *Crude Monte Carlo method* is initially applied for estimating properties of portfolio loss rate distributions and we later advance to discussions about *importance sampling* based variance reduction techniques.

2.3.1 The Crude Monte Carlo Method

In this section we define the *Crude Monte Carlo estimator* and examine how it can be applied in order to estimate loss rate probabilities.

2.3.1.1 The Crude Monte Carlo Estimator

We here give an introduction to the Crude Monte Carlo estimator in line with (Pastel, 2012, p. 15-27).

Theorem 1 (Strong law of large numbers)
Consider a set $\{f_i\}_{i=1\ldots n}$ of independent identically distributed variables, which are distributed as f with finite expected value $\mathbb{E}[f] < \infty$. For such a set, it holds almost surely that the empirical average $\bar{f}_n$ of the set converges to the expected value (Pastel, 2012, p. 16). We describe this mathematically as:

$$\lim_{n\to\infty} \bar{f}_n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f_i = \mathbb{E}\left[f\right]. \tag{55}$$

The Crude Monte Carlo estimator is a direct application of theorem 1, as it uses a large number of independent realizations of a function or variable to evaluate its expected value. The estimator is unbiased following the strong law of large numbers and in order to evaluate its variance and rate of convergence, we state the central limit theorem.

Theorem 2 (Central limit theorem)
Consider a set $\{f_i\}_{i=1\ldots n}$ of independent identically distributed variables which are distributed as f with expected value $\mathbb{E}[f]$ and finite variance $\mathbb{V}[f] < \infty$. For such a set, it holds that the variable sequence $\sqrt{n}\left(\bar{f}_n - \mathbb{E}[f]\right)$ is asymptotically normal distributed with mean zero and variance $\mathbb{V}[f]$ (Pastel, 2012, p. 16). This implies the following relation:

$$\frac{\sum\limits_{i=1}^{n} f_i - n\mathbb{E}[f]}{\sqrt{\mathbb{V}[f]}\sqrt{n}} \xrightarrow{d} N(0,1) \quad as \quad n \longrightarrow \infty. \tag{56}$$

According to (Atzberger, p. 2), the central limit theorem implies that the estimation error of the Crude Monte Carlo estimator may be formulated as:

$$
\begin{aligned}
e_{\bar{f}} &= \left|\bar{f} - \mathbb{E}[f]\right| = \left|\frac{1}{n}\sum_{i=1}^{n} f_i - \mathbb{E}[f]\right| = \\
&= \frac{\sqrt{\mathbb{V}[f]}}{\sqrt{n}} \left|\frac{\sum\limits_{i=1}^{n} f_i - n\mathbb{E}[f]}{\sqrt{\mathbb{V}[x]}\sqrt{n}}\right| \approx \frac{\sqrt{\mathbb{V}[f]}}{\sqrt{n}} \left|N(0,1)\right|.
\end{aligned}
\tag{57}
$$

Following the strong law of large numbers and the central limit theorem, it holds that the Crude Monte Carlo estimator has expected value $\mathbb{E}[f]$ and variance is $\mathbb{V}[f]/n$. This implies that the Crude Monte Carlo estimator is an unbiased and consistent estimator with a rate of convergence of of $O(\sqrt{n})$, as stated in (Pastel, 2012, p. 16).

2.3.1.2 Estimation of Portfolio Tail Loss Probabilities

We here seek to estimate portfolio tail probabilities with the Crude Monte Carlo method presented in the previous section. This approach uses a number of scenarios S, for each of which a portfolio loss rate is computed and an indicator function $\mathbf{1}\{\mathrm{L} > \mathrm{x}\}$ is evaluated. The tail probabilities are estimated as the expected values of the indicator function at different thresholds x, analogous to (Pastel, 2012, p. 22). We describe this mathematically:

$$
P(L > x) = \mathbb{E}\left[\mathbf{1}\{L > x\}\right], \tag{58}
$$

and give a detailed pseudocode description in algorithm 1 of how the Crude Monte Carlo method can be applied for evaluation of the estimator in (58).

Algorithm 1 Crude Monte Carlo

```
for each scenario s = 1 to S do
    Draw systematic risk factors Z ~ N(0, I)
    Compute the composite risk factors Y
    Draw loss given default variables LGD
    Compute the conditional default probabilities PD(Y)
    Draw default indicators D_PD(Y)
    Compute scenario loss rate L_s and estimator 1{L_s > x}
end for
Estimate P(L > x) ≈ 1/S Σ_{s=1}^{S} 1{L_s > x}
```

Given the estimated tail probabilities, the loss distribution can be obtained from:

$$
F_L(x) = P(L \leq x) = 1 - P(L > x). \tag{59}
$$

It is stated in (Pastel, 2012, p. 22) that the empirical loss distribution converges almost surely to the true loss distribution as the number of scenarios tends to infinity, following from the law of large numbers. Risk measures Value at Risk and Expected Shortfall can be estimated from the empirical loss distribution in accordance to definitions (99) and (103) in Appendix A: Risk Measures.

2.3.2 The Importance Sampling Monte Carlo Method

Several methods exist for enhancing performance and improving accuracy by reducing the variance of Monte Carlo estimators. One such technique is the importance sampling Monte Carlo method, which uses a change of probability measure to increase precision in a region of interest. We here give an introduction to the importance sampling Monte Carlo estimator and examine how the technique can be applied to estimate loss rate probabilities.

2.3.2.1 The Importance Sampling Monte Carlo Estimator

We consider a function $f(u)$ of random variable u which has probability density $p(u)$, and introduce an (appropriately chosen) alternative density $q(u)$. As stated in (Pastel, 2012, p. 29), the importance sampling technique takes advantage of the definition of the expected value on integral form in order to make sampling from the alternative density function possible. We describe this mathematically as:

$$\mathbb{E}\left[f(u)\right] = \mathbb{E}_q\left[f(u)\frac{p(u)}{q(u)}\right] = \int_\Omega f(u)\frac{p(u)}{q(u)}q(u)\ du, \tag{60}$$

where the quote $p(u)/q(u)$ denotes the *likelihood ratio*. As can be seen in (60), it is possible to achieve an unbiased estimator for $\mathbb{E}[f(u)]$ by evaluating the integral expression under the alternative measure $q(u)$ and compensate for the shift of sampling distribution through the likelihood ratio. It follows from the relation between expectation and variance[13] and the equality between the expectations in (60) that variance reduction is obtained from importance sampling if the following relation holds:

$$\mathbb{E}_q\left[\left(f(u)\frac{p(u)}{q(u)}\right)^2\right] = \mathbb{E}\left[f(u)^2\frac{p(u)}{q(u)}\right] < \mathbb{E}\left[f(u)^2\right], \tag{61}$$

as stated in (Glasserman, 2003, p. 256). The efficiency of the method is thus dependent on the choice of $q(u)$, but the optimal choice does however depend on the problem at hand. We address this issue further when applying importance sampling to the simulation of portfolio tail loss probabilities in sections 2.3.2.2 - 2.3.2.3.

[13] $\mathbb{V}[f(x)] = \mathbb{E}[f(u)^2] - \mathbb{E}[f(u)]^2$.

2.3.2.2 Conditional Importance Sampling

The conditional importance sampling method of exponential twisting developed by Glasserman and Li (2005) applies a shift in default probabilities $\{Q_i\}_{i=1...M} \geq \{PD_i\}_{i=1...M}$ to increase the number of observations of large loss rates when estimating tail probabilities. We have previously introduced the concept of importance sampling in section 2.3.1 and will now seek a suitable way to shift the default probabilities in order to reduce variance in estimating loss rate probabilities.

In their article, Glasserman and Li (2005) recognize that a substitution of default probabilities gives rise to a convenient importance sampling estimator (60) under the condition that the obligors are independent. In this case, we obtain the likelihood ratio Λ from the following relation:

$$\Lambda = \prod_{i=1}^{M} \left(\frac{PD_i}{Q_i}\right)^{D_i} \left(\frac{1-PD_i}{1-Q_i}\right)^{1-D_i}, \tag{62}$$

and the importance sampling estimator for the tail probabilities of a loss rate distribution under the new set of default probabilities may be formulated as:

$$P(L > x) = \mathbb{E}_Q \left[\mathbf{1}\{L > x\}\Lambda\right]. \tag{63}$$

For the choice of default probabilities, Glasserman and Li (2005) further suggest that a new set of probabilities can be obtained from the following twist function:

$$Q_i(\theta) = \frac{PD_i\, e^{w_i LGD_i \theta}}{1 + PD_i\left(e^{w_i LGD_i \theta} - 1\right)}, \qquad \theta \geq 0, \tag{64}$$

where the parameter θ is to be chosen such that the estimator variance is minimized. Under this set of default probabilities, the likelihood ratio may be written as:

$$\Lambda(\theta) = \prod_{i=1}^{M} \left(\frac{PD_i}{Q_i(\theta)}\right)^{D_i} \left(\frac{1-PD_i}{1-Q_i(\theta)}\right)^{1-D_i} = \exp(-\theta L + \kappa(\theta)), \tag{65}$$

where $\kappa(\theta)$ denotes the cumulant generating function of the loss rate L, defined as:

$$\kappa(\theta) = \log\left(\mathbb{E}\left[e^{\theta L}\right]\right) = \sum_{i=1}^{M} \log(1 + PD_i(e^{w_i LGD_i \theta} - 1), \tag{66}$$

in accordance to (Glasserman and Li, 2005, p. 1645). It holds from (60) and (61) that the importance sampling estimator is unbiased and that all variance reduction is determined by the estimator's second moment, which is to be obtained from the moment generating function as:

$$\mathcal{M}_t(x,\theta) = \mathbb{E}_Q\left[(\mathbf{1}\{L > x\}\Lambda(\theta))^t\right]\Big|_{t=2} = \mathbb{E}_Q\left[\mathbf{1}\{L > x\}e^{-t\theta L + t\kappa(\theta)}\right]\Big|_{t=2}. \tag{67}$$

The problem at hand is thus to determine θ such that (67) is minimized. Glasserman and Li (2005) suggest to minimize an upper bound rather than the expectation in (67) in order to avoid a fairly cumbersome minimization problem. Given the following upper bound:

$$\mathbb{E}_Q\left[\mathbf{1}\{L > x\}e^{-2\theta L + 2\kappa(\theta)}\right] \leq e^{-2\theta x + 2\kappa(\theta)}, \tag{68}$$

our optimization problem reduces to:

$$\begin{aligned} \underset{\{\theta\}}{\text{maximize}} \quad & \theta x - \kappa(\theta), \\ \text{subject to} \quad & \theta \geq 0. \end{aligned} \tag{69}$$

As it holds that $\theta x - \kappa(\theta)$ passes through the origin and is strictly convex, according to (Glasserman and Li, 2005, p. 1645), the solution to (69) can be obtained as:

$$\theta_x = \begin{cases} \text{unique solution to } \kappa'(\theta) = x, & x > \kappa'(0), \\ \qquad\qquad 0 \qquad\qquad\qquad\quad , & x \leq \kappa'(0), \end{cases} \tag{70}$$

where the solution to $\kappa'(\theta) = x$ for $x > \kappa'(\theta)$ easily can be obtained using numerical methods. Glasserman and Li (2005) state that the following relation holds:

$$\mathbb{E}_Q[L] = \kappa'(\theta). \tag{71}$$

It follows from (64) that $\mathbb{E}_Q[L] = \mathbb{E}[L]$ for $\theta = 0$ and given (71) it holds that $\mathbb{E}[L] = \kappa'(0)$. Following the analysis in (Glasserman and Li, 2005, p. 1645), we may interpret the effect of exponential twisting in the following way:

$$\begin{gathered} \text{if } \mathbb{E}[L] < x \longrightarrow \text{set } \mathbb{E}[L] = x, \\ \text{if } \mathbb{E}[L] \geq x \longrightarrow \text{do nothing.} \end{gathered} \tag{72}$$

The method thus ensures that the expected loss rate is greater than or equal to the threshold x, which is intuitively appealing when estimating $P(L > x)$. It further holds, in analogy to (60), that (63) is an unbiased estimator and we refer to (Glasserman and Li, 2005, p. 1645-1646) for a proof of asymptotical optimality.

The assumption of independent obligors, on which (62) is based, rarely holds, but it follows from (3) and (4) that obligors are conditionally independent given a realization of the systematic risk factors $\{Z_k\}_{k=1\ldots N}$. It is thus possible to twist conditional default probabilities according to the pseudo code procedure described in algorithm 2.

Algorithm 2 Importance Sampling Monte Carlo - Exponential Twisting

for each scenario $s = 1$ to S **do**
  Draw systematic risk factors $Z \sim N(0, I)$
  Compute composite risk factors Y
  Draw loss given default variables LGD
  Compute conditional default probabilities $PD(Y)$
  Compute twist parameter θ_x
  Compute twisted conditional default probabilities $Q(\theta_x)$
  Draw default indicators $D_{Q(\theta_x)}$
  Compute scenario likelihood ratio $\Lambda_s = \Lambda(\theta_x)$
  Compute scenario loss rate L_s and estimator $\mathbf{1}\{L_s > x\}\Lambda_s$
end for
Estimate $P(L > x) \approx \frac{1}{S} \sum\limits_{s=1}^{S} \mathbf{1}\{L_s > x\}\Lambda_s$

Remark 2 *As this method serves to set the expected loss rate larger than or equal to the threshold of interest, it is suggested by Glasserman and Li (2005) that the same value for θ may be used for a variety of thresholds x. This enhances the computational efficiency of the model, as solving the* (70) *calls for external numerical methods and is the most computationally demanding part of each iteration in algorithm 2.*

2.3.2.3 Two-Step Importance Sampling

It is stated in (Glasserman and Li, 2005, p. 1647) that the magnitude of portfolio loss rates becomes increasingly dependent on the systematic risk factors as the obligor correlation increases. In order to find an importance sampling method that performs well independently of the degree of obligor correlation, we here seek to complement the method of exponential twisting with importance sampling on the systematic risk factors.

Analogous to (Glasserman and Li, 2005, p. 1647-1648), we decompose the variance of the exponential twisting estimator (63) in the following way:

$$\mathbb{V}_Q\left[\mathbf{1}\{L > x\}\Lambda\right] = \\ \mathbb{E}_Q\left[\mathbb{V}_Q\left[\mathbf{1}\{L > x\}\Lambda \,\middle|\, \{Z_k\}_{k=1\ldots N}\right]\right] + \mathbb{V}_Q\left[\mathbb{E}_Q\left[\mathbf{1}\{L > x\}\Lambda \,\middle|\, \{Z_k\}_{k=1\ldots N}\right]\right]. \tag{73}$$

The first term in decomposition (73) is small due to the effect of the exponential twisting and we thus focus on minimizing the second term in order to obtain further variance reduction. Given a realization of the systematic risk factors, the estimator is unbiased and it holds that:

$$\mathbb{E}_Q\left[\mathbf{1}\{L > x\}\Lambda \,\middle|\, \{Z_k\}_{k=1\ldots N}\right] = P\left(L > x \,|\, \{Z_k\}_{k=1\ldots N}\right). \tag{74}$$

This implies that an optimal choice of sampling distribution for the systematic risk factors is one that minimizes the variance of the integral estimator (60) for $P(L > x \,|\, \{Z_k\}_{k=1\ldots N})$. It is stated in (Glasserman and Li, 2005, p. 1648) that the optimal conditional sampling distribution for the problem at hand is:

$$\varphi(z) = \frac{P(L > x \,|\, z)e^{-z^T z/2}}{P(L > x)}. \tag{75}$$

This distribution is however of limited use considering the requirement for the tail probabilities in the denominator, but may serve as a starting point in the search of one that is useful. Glasserman and Li (2005) suggest that the systematic risk factors may be sampled from a normal distribution with the same mode as (75), which is found from maximization of the numerator. We set the mean vector μ of this normal distribution to:

$$\mu = \underset{\{z\}}{\arg\max}\; P(L > x \,|\, \{Z_k\}_{k=1\ldots N} = z)e^{-z^T z/2}. \tag{76}$$

Finding a solution to this maximization problem is however cumbersome and it is therefore suggested in (Glasserman and Li, 2005, p. 1648) to introduce the following upper bound for the conditional loss probability:

$$P(L > x \,|\, \{Z_k\}_{k=1\ldots N} = z) \leq e^{F_x(z)}, \tag{77}$$

where F_x denotes the natural logarithm of the likelihood ratio $\Lambda(\theta_x(z), z)$ at $L = x$:

$$F_x(z) = \ln(\Lambda(\theta_x(z), z)) = -\theta(z)x + \kappa(\theta_x(z), z). \tag{78}$$

If the upper bound in (77) is treated as an approximation of the conditional loss probability, the expression in (76) becomes $e^{F_x(z) - z^T z/2}$ and the maximization problem reduces to one of maximizing the exponent. We may thus formulate a convenient approximation of μ which only depends on known quantities as:

$$\mu^* = \underset{\{z\}}{\arg\max}\; \left\{F_x(z) - z^T z/2\right\}, \tag{79}$$

according to (Glasserman and Li, 2005, p. 1648). In further accordance with the original article, we state the likelihood ratio linking the probability density of the normal distribution $N(\mu, I)$ to that of a standard normal distribution $N(0, I)$ as:

$$\Psi(\mu, Z) = e^{-\mu^T Z + \mu^T \mu/2}. \tag{80}$$

We refer to (Glasserman and Li, 2005, p. 1649) for a proof of asymptotical optimality and formulate the two-step importance sampling procedure in pseudo code:

Algorithm 3 Importance Sampling Monte Carlo - Two-Step Importance Sampling

```
Estimate the risk factor mean μ*
for each scenario s = 1 to S do
    Draw systematic risk factors Z ~ N(μ*, I)
    Compute composite risk factors Y
    Draw loss given default variables LGD
    Compute conditional default probabilities PD(Y)
    Compute twist parameter θ_x
    Compute twisted conditional default probabilities Q(θ_x)
    Draw default indicators D_{Q(θ_x)}
    Compute scenario likelihood ratio Λ_s Ψ_s = Λ(θ_x) Ψ(μ*, Z)
    Compute scenario loss rate L_s and estimator 1{L_s > x} Λ_s Ψ_s
end for
Estimate P(L > x) ≈ (1/S) Σ_{s=1}^{S} 1{L_s > x} Λ_s Ψ_s
```

Remark 3 *The method of shifting the risk factor expected values can be of use without the exponential twisting technique in order to avoid the procedure of finding θ_x. This results in a procedure where each iteration is computationally faster than the procedure described in algorithm 3 but lacks the feature explained in* (72)*.*

2.3.3 Risk Decomposition using Monte Carlo Methods

We here consider methods from Burton et al. (2005) for decomposing risk contributions into effects of expected loss, systematic risk and idiosyncratic risk as well as for computing marginal contributions to loss rates of a certain magnitude of interest.

2.3.3.1 Estimation of Systematic and Idiosyncratic Risk

In their working paper (Burton et al., 2005, p. 11-13) suggest that the loss rate of a portfolio can be divided into components corresponding to effects of expected loss, systematic risk and idiosyncratic risk in the following way:

$$L = \underbrace{\sum_{i=1}^{M} \mathbb{E}[L_i]}_{\text{Expected Loss}} + \underbrace{\sum_{i=1}^{M} (\mathbb{E}[L_i \,|\, Y_i] - \mathbb{E}[L_i])}_{\text{Systematic Loss}} + \underbrace{\sum_{i=1}^{M} (L_i - \mathbb{E}[L_i \,|\, Y_i]),}_{\text{Idiosyncratic Loss}} \tag{81}$$

where the three components may be computed as:

$$\sum_{i=1}^{M} \mathbb{E}[L_i] = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] PD_i, \tag{82}$$

$$\sum_{i=1}^{M} (\mathbb{E}[L_i \,|\, Y_i] - \mathbb{E}[L_i]) = \sum_{i=1}^{M} w_i \mathbb{E}[LGD_i] \left(PD_i(Y_i) - PD_i\right), \tag{83}$$

$$\sum_{i=1}^{M} (L_i - \mathbb{E}[L_i \,|\, Y_i]) = \sum_{i=1}^{M} \left(L_i - w_i \mathbb{E}[LGD_i] PD_i(Y_i)\right). \tag{84}$$

The first term on the right hand side of (81) is deterministic and can be computed without simulation, in accordance to (105) in Appendix A: Risk Measures. The second term corresponds to risk arising from market movements whereas the last represents name concentration risk and approaches zero as portfolio granularity goes to infinity. Furthermore, Burton et al. (2005) states that leaving out the idiosyncratic term results in a loss distribution corresponding to a portfolio fully diversified with respect to entity specific risk.

2.3.3.2 Marginal, Component and Incremental Value at Risk

We here follow (Burton et al., 2005, p.16-19) in order to present a methodology for computing marginal risk contributions on obligor level. The results presented in the original article are formulated on sector level and can be obtained from the results presented here by summation over the obligors in each sector.

Denoting the set of systematic and idiosyncratic risk factors that governs the portfolio loss $H = \{Z_k, \xi_i\}_{k=1\ldots N, i=1\ldots M}$ and noting that the portfolio loss is fully deterministic given a realization of H, here referred to as h, allows us to express the set of risk factor realizations that renders a loss equal to the α^{th} quantile of the loss rate distribution as:

$$\mathcal{H}_\alpha^0 = \{h \,|\, \ell(h) = q_\alpha(L)\}, \tag{85}$$

where $\ell(h)$ denotes the portfolio loss conditional on a realization h. Following (Gourieroux et al., 2000, p. 242) it can be shown[14] that the marginal contribution of obligor i to portfolio loss rate quantile $q_\alpha(L)$ can be written as:

$$\frac{dq_\alpha(L)}{dw_i} = \mathbb{E}\left[LGD_i D_i \,|\, H \in \mathcal{H}_\alpha^0\right]. \tag{86}$$

[14] According to (Burton et al., 2005, p. 17).

When using Monte Carlo methods for estimating (86), the condition of the portfolio loss rate being precisely equal to the quantile $q_\alpha(L)$ is likely to result in a large amount of useless scenarios and we therefore expand (85) to include losses close to the quantile of interest:

$$\mathcal{H}_\alpha^\epsilon = \{h \,|\, |\ell(h) - q_\alpha(L)| < \epsilon\}, \tag{87}$$

where the choice of ϵ becomes a tradeoff between accuracy and computational efficiency. In order to incorporate (87), we re-formulate (86) to get an approximative formula for marginal loss rate quantiles as:

$$\frac{dq_{\alpha(L)}}{dw_i} \approx \mathbb{E}\left[LGD_i D_i \,|\, H \in \mathcal{H}_\alpha^\epsilon\right], \tag{88}$$

which converges to the true formula as $\epsilon \to 0$. It follows from (88) that we are now able to compute *marginal Value at Risk* according to the definition (100) in Appendix A.2:

$$MVaR_\alpha^i(L) = \frac{\partial VaR_\alpha^{portfolio}(L)}{\partial w_i}. \tag{89}$$

By multiplying marginal Value at Risk by the exposure weight of an obligor, we obtain the *component Value at Risk*, which is exactly additive with respect to portfolio Value at Risk and accounts for the Value at Risk contribution of an obligor to that of the full portfolio. In accordance with the definition in Appendix A.3, we describe this mathematically as:

$$CVaR_\alpha^i(L) = w_i \frac{\partial VaR_\alpha^{portfolio}(L)}{\partial w_i}. \tag{90}$$

Additional to being an exactly additive measure of diversified Value at Risk, component Value at Risk may serve as an approximation for *incremental Value at Risk*, which denotes the change in portfolio Value at Risk stemming from adding or removing a position[15]:

$$IVaR_\alpha^{\pm i}(L) = \left| VaR_\alpha^{portfolio \pm i}(L) - VaR_\alpha^{portfolio}(L) \right| \approx w_i \frac{\partial VaR_\alpha^{portfolio}(L)}{\partial w_i}. \tag{91}$$

Computing incremental Value at Risk for a large set of obligors using Monte Carlo methods demands for a new simulation to be carried out for each position of interest, which can be very computationally expensive. The approximation of component Value at Risk may thus serve as an indicator of the positions carrying the largest incremental Value at Risk and help avoiding extensive Monte Carlo simulations for less risky positions. A negative aspect is however that the accuracy of this approximation decreases as exposure weights increase, which is a consequence of the approximation being based on a first order derivative.

[15]Incremental Value at Risk from a adding and removing a position is (most often) not equal. Removing a position can however be seen as adding a negative position, which is why we included both cases in the same definition.

Furthermore, Glasserman (2005) suggest that (88) can be evaluated with a two-step importance sampling estimator in which θ_x is set to the unique solution to $\kappa'(\theta) = x$ regardless of the value of $\kappa'(0)$, in contrast to (70). Setting $x = VaR_\alpha$ then results in an unbiased estimator where $\mathbb{E}[L \mid Z] = VaR_\alpha$ in every scenario, in contrast to (72). This implies an increase of scenarios in the region of Value at Risk and a more precise estimation of (88) than the Crude Monte Carlo estimator achieves.

Remark 4 *A risk-type decomposition according to* (85) *is not possible using the importance sampling estimator suggested by Glasserman (2005), following the property of* $\mathbb{E}[L \mid Z] = VaR_\alpha$*, which implies that the idiosyncratic loss estimator in* (81) *asymptotically approaches zero (the law of large numbers) and all loss contributions become expected and systematic.*

3 Credit Portfolio Management

3.1 Managing and Rebalancing Credit Portfolios

This section considers methods for building and rebalancing credit portfolios, following the previously considered risk estimation techniques. We begin by discussing how the work of portfolio managers and originators may be structured for building robust and cost effective portfolios and proceed by presenting methods for limiting and mitigating credit exposures.

3.1.1 Active Credit Portfolio Management

Active credit portfolio management is here approched as a framework of methods and tools for building sound credit portfolios and resolving matters of unwanted credit exposures.

3.1.1.1 Roles and Responsibilities of the CPM Unit

Credit portfolios have traditionally been constructed solely by the works of origination units, following that transactions were evaluated in isolation, similar to the previous "picking the winners" approach of the stock market. As financial theory developed along with quantitative methods for risk assessment, a deeper understanding for the underlying processes forming the profit and loss distribution of a portfolio has been achieved, motivating a holistic approach to credit portfolio management. This has caused for the roles of credit portfolio managers to evolve from strictly being a monitoring and reporting function towards actively optimizing and rebalancing the credit portfolios of financial institutions.

Although responsibilities of CPM units vary across organizations, some attempts of classification have been made in academia. In (Hünseler, 2013, p. 66), roles and mandates of CPM units are divided into four categories, ranging from a passive advisory role as *risk controller* to an active management role as *value creator*, where each category inherits the activities of the previous. A tabular form of this categorization is displayed in table 1.

It is argued in Hünseler (2013) that the organizational placement of a CPM unit often is connected to its functionality, where units focusing on value creation tend to be placed within business lines whereas units with a risk protecting approach most often are placed in connection to risk or finance departments. A placement within business lines may serve to increase specialization and bring portfolio managers closer to the assets of the business lines, whereas a placement within risk or finance departments may give a holistic perspective of the portfolio under management.

No general best practices for engaging in credit portfolio management exist and financial institutions must therefore seek to tailor their structure with respect to the firms strategies, competences and targets. Efforts have been made by IACPM (2005) to outline sound practices for engaging in credit portfolio management, which may serve as a point of reference. Detailed descriptions of organizational aspects are however not in the scope of this thesis, as focus here lies in quantitative assessment and mitigation of credit risk.

Table 1: Activities of the CPM unit (Hünseler, 2013, p. 66)

Category	**Activity**	**Description**
Risk Controller	Monitoring & Reporting	· Intelligence on key risk and capital measures. · Monitoring credit risk strategies. · Portfolio risk adjusted performance measurement. · Limit reporting, loan pricing and stress tests.
Risk Protector	Risk Limiting	· Reduce absolute portfolio concentration. · Limit credit Value at Risk. · Definition of EL/LLP trigger levels and management. · Optimization of growth allocation by minimizing cost of risk.
Risk Optimizer	Portfolio Optimization	· Defining target portfolio, including structural limits given risk appetite and capital allocation. · Optimization of growth allocation by maximizing risk adjusted return. · Stress test derived feasibility check for target portfolio. · Stress test based contingency plans.
Value Creator	Portfolio Reallocation	· Implementing portfolio risk profile changing measures. · Target economic/regulatory capital improvements. · Profit and loss responsibility.

3.1.1.2 Portfolio Management Objectives and Tools

We display two portfolio loss distributions in figure 1 following (Bouteillé and Coogan-Pushner, 2013, p. 262). The darker bars represent a sound distribution, which has its probability mass centered at low loss levels, with low probabilities of losses large enough to endanger the future of the corporation. The loss distribution represented by the darker bars is cost effective with respect to economic capital and may serve as a target distribution for the holder of the portfolio represented by the lighter bars.

The distribution represented by the lighter bars displays significant probabilities of large losses and is ineffective from an economic capital perspective, given that it takes large funds to cover these dangerous potential outcomes. This distributional structure may stem from concentrations in correlated segments or large exposures towards single counter-parties. Alignment of the lighter distribution with the darker target distribution may reduce costs by deploying economic capital or generate new business given the same amount of held capital. In both scenarios, the firms return on risk adjusted capital is enhanced and its lending activities becomes more efficient (Bouteillé and Coogan-Pushner, 2013, p. 262).

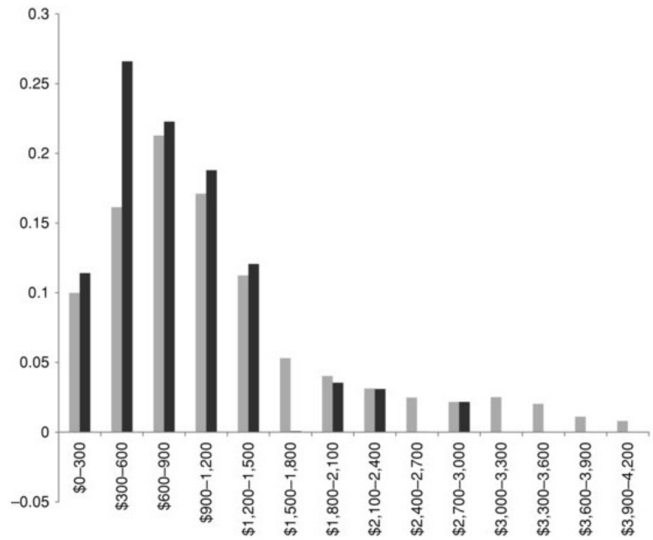


Figure 1: Example of portfolio loss distributions Bouteillé and Coogan-Pushner (2013)

In the survey *Principles and Practices in Credit Portfolio Management* conducted by the International Association of Credit Portfolio Managers, IACPM (2013), representatives from 66 financial institutions, based in 16 countries, answered to questions regarding the workings of their CPM units. A vast majority of the survey respondents (79%) stated revenue generation as one of their overall top priorities over the next 12 - 24 months, whereas the second most quoted priority (40%) was meeting capital targets. According to IACPM (2013), the relative importance of revenue generation has increased in the stable economic environment following the last financial crises, whereas firms were more concerned about meeting regulatory standards under conditions of financial stress.

On questions regarding specific key objectives, responses reflect a pro-active and cost-effective approach to portfolio management where diversification, portfolio information and support for origination processes constitute the highest priorities. The key objectives quoted in the survey are displayed in table 2, along with corresponding quote frequencies.

In a broad sense, the key objectives give an indication of the branch wide opinion of *what* to manage, whereas credit portfolio models, such as those previously regarded in this study, aims to point out *where* to target the efforts. In what follows, we will consider tools and methodologies for answering the question of *how* to manage a credit portfolio. The measures regarded in the proceeding sections were subject for evaluation in IACPM (2013), where the survey respondents rated management tools after perceived importance. Before engaging in discussions about risk mitigation methods, we display these results in figure 2.

Table 2: Key objectives of CPM unit, IACPM (2013)

Objective	**Quote Frequency**
Improve Portfolio Structure, Reduce Concentrations	89%
Provide Portfolio Information	86%
Help Guide Origination	74%
Optimize Risk and Return (Qualitative or Quantitative)	67%
Manage Maximum Risk Appetite Target	63%
Manage Regulatory Changes	56%
Manage Return on Equity, RAROC or Similar Target	49%
Manage P & L Volatility, Absolute P & L or Similar Target	44%
Scenario Analysis & Stress Testing	44%
Liquidity / Funding Risk Assessment	28%
Enterprise Risk Management	28%

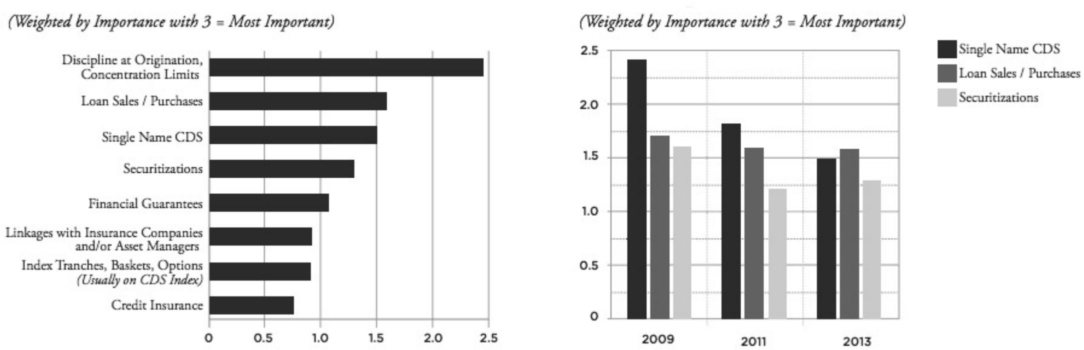


Figure 2: Relative importance of CPM tools, IACPM (2013)

3.1.1.3 Loan Pricing Methods

Adequately pricing transactions is crucial for the success of firms devoted to loan origination. An example of inconsistent pricing is given in (Hünseler, 2013, p. 43), where flat loan margins are considered, for which each counter-party are offered the same loan spread independently of their characteristics. This strategy implies beneficial offers to low credit quality borrowers, who would have to pay higher spreads in the market, whereas the opposite holds for high quality borrowers. This example points out that the lending institutes end up holding non-optimal credit portfolios by neglecting counter-party specific risk.

Through the use of *risk-adjusted pricing*, counter-parties are offered spreads based on how favorable a potential transaction is for the lender with respect to expected losses, costs of equity, funding and administration, required economic capital as well a *commercial adjustment*, which is a weight reflecting how aligned the transaction is with the firms overall strategy. This system aims to provide consistency in transaction pricing and encourage originators to target loan margins which are profitable net of costs (Hünseler, 2013, p. 84).

Transfer pricing may be applied in organizations where the CPM unit has profit and loss responsibility. In this setting, transactions are transferred to the CPM unit once originated and the originating unit is compensated according to a price at which the CPM unit is indifferent of holding the exposure on its balance sheet or transferring it to the market through sales or hedging (Hünseler, 2013, p. 85). Transactions originated above/below transfer price contributes positively/negatively to the balance sheet of the origination unit, whereas an increase/decline in value through out the holding period belongs to the balance sheet of the CPM unit. Even though transfer pricing may serve to enhance transparency and consistency in performance measurements across units, it is complex and must be implemented with care in order to avoid conflicts and effects of internal arbitrage.

3.1.1.4 Information and Transparency

Client relationships tend to give financial institutions routine-access to non-public financial statements of their counter-parties, as part of the credit evaluation process. Such information may serve to underlay negotiations of client terms and conditions as well as to monitor exposures and as long as there is no capital market involvement in forming and structuring the portfolio, financial institutions may act based on the disclosed non-public information.

Issues of insider trading may however arise if non-public information is used to hedge exposures through capital market activity, following the information bias between the institution and the market. Additional to moral aspects, severe penalties for such misconduct motivates firms to restrict the access to non-public information for departments involved in capital market activities. This is commonly referred to as building *chinese walls* and in the case of portfolio management, common practice is to divide the business into what banks refer to as *private side* and a *public side*. The former side makes credit assessments and evaluations based on private information, whereas the latter acts on public information to make decisions related to hedging exposures (Bouteillé and Coogan-Pushner, 2013, p. 230).

Although this organizational structure calls for some business activities to be conducted in parallel business units, which may raise organizational costs, it serves to protect financial institutions against penalties and reputation risk associated with insider trading.

3.1.2 Origination and Limit Setting

We here regard sound practices for origination of transactions, which may serve to build a strong foundational portfolio structure and help avoiding complications and costs associated with mitigating unwanted exposures.

3.1.2.1 Exposure and Concentration Limits

Setting a combination of qualitative and quantitative risk limits helps assuring that the exposures taken on by an institution are in line with its appetite for risk and serves as a first defense against unwanted portfolio structures.

As is stated in (Bouteillé and Coogan-Pushner, 2013, p. 218), limits should be set through the expertise and judgement of senior management. In this way, the knowledge of experienced professionals may serve to frame the institution's risk strategy and support junior management in their decision making. It is of importance that limits are clearly formulated and enforced in a strict manner, as the motivation for their existence is to protect the institution from hazardous financial developments. Limits should however be flexible enough to adapt to the development of the business and potential exceptions must be evaluated by a group which does not hold responsibility for portfolio performance.

Table 3: Examples of limit factors, Burmeister (2009)

Limit Factor	**Risk Driver**	**Description**
Value at Risk / Expected Shortfall	Overall Portfolio Credit Risk	The most important limit according to Burmeister (2009), as it incorporates several in-parameters reflecting both credit and market risk.
Country	Transfer Risk	Limits with regards to foreign countries. Burmeister (2009) suggest that countries within Organisation for Economic Co-operation and Development (OECD) and the European Union (EU) may be regarded as non-foreign following standardized regulatory frameworks and policies.
Sector	Correlational Risk	Due to high correlation and possible contagion effects, limiting exposures to certain sectors may serve to protect against clustering not captured by quantitative portfolio credit risk models.
Currency	FX Risk	Transactions in foreign currencies gives rise to foreign exchange risk, which could either be limited or hedged.
Asset Class	Operational and Liquidity Risk	Limiting exposures to complex products / derivatives may serve to avoid large exposures to financial asset classes which are hard to price. An example is the recent collapse of the subprime mortgage market, which initiated the latest global financial crisis.
Single Name Exposure	Maximum Loss (per Entity)	Restraining exposures to single entities may serve to decrease vulnerability against firm specific factors, which are hard to foresee.

In order to assure a holistic protection against negative portfolio structures, limits should be applied on a range of levels. Examples of possible limit factors and their corresponding risk drivers are illustrated in table 3, following (Burmeister, 2009, p. 204).

3.1.2.2 Sound Practices in Originating Transactions

Building a robust credit portfolio generally starts with originating appropriate transactions and avoiding engaging in those less favorable with respect to the overall strategy of the business. We here consider an overview of sound practices for selecting what transactions to originate, based on the checklist in (Bouteillé and Coogan-Pushner, 2013, p. 66-78).

When originating a transaction, it is crucial to make sure that the terms and conditions of the transaction is thoroughly understood. Financial products tend to evolve through minor changes rather than big leaps and unless the transaction details are approached with great care, a financial institution may end up engaging in transactions which affect the overall portfolio in ways they are not intended to.

Given that the terms and conditions have been thoroughly examined, it is of importance to consider the suitability of the transaction with respect to the overall strategy of the business. In the case of new transaction types, a holistic perspective may be achieved by consulting lawyers, accountants and senior management representatives in order to assure that no important aspects have been left out in the decision making. It is of great importance that the evaluation and approval processes are well documented in order to assure transparency.

Attention has to be payed to how the transaction conforms with exposure and concentration limits and in the event of a limit breach, aspects such as client relationships and other non-credit related value creators have to be taken into account in order to evaluate whether an exception shall be made or not. The capacity for taking on other transactions, which may contribute better to the portfolio risk/return profile, also have to be considered when evaluating the suitability of a credit with respect to the overall portfolio.

As credit risk is not static, originators must consider whether the institution has the possibility to surveillance the exposure throughout its life-time. Keeping a dialog with the departments responsible for quantitative modeling and credit risk monitoring may here serve to connect front and back end activities and help an institution to keep their monitoring activities up to date, in order to avoid declining favorable transactions due to lack of expertise. If appropriate surveillance is available, further attention should be paid to the availability of suitable mitigation tools and exit strategies, such that the institution is in power of reducing the exposure in the event of monitoring activities indicating increased risk for negative outcomes.

Bouteillé and Coogan-Pushner (2013) underline the importance of keeping possible information asymmetries between seller and buyer in mind, in order to avoid engaging into transaction with hidden flaws. Furthermore, it is suggested that the originator should keep informed about the transaction after conclusion of negotiations in order to make sure that the terms and conditions of the finalized deal reflects those agreed on at the negotiation table in a well structured and transparent manner.

3.1.3 Tools for Credit Risk Mitigation

We here regard financial instruments and methodologies commonly used for mitigating credit exposures and rebalancing credit portfolios. The information presented mainly follows the discussions of mitigation tools in Hünseler (2013), Saunders et al. (2007) and Bouteillé and Coogan-Pushner (2013).

3.1.3.1 Assignments

Assignments without *recursion*[16] refers to loan sales where loans are being permanently transferred from the balance sheet of the seller to that of the buyer. In this process, all obligations of the loan seller terminates and the claim on the counter-party is transferred to the buyer, which is favorable from a risk mitigation perspective. Assignments are dependent on the allowance of loan documentations, which may include clauses giving borrowers veto against loan sales or restricting the institution classes to which loans may be sold. Although effectively transferring exposures to the market, assignments may harm client relationships if not handled with care.

3.1.3.2 Participations

Participations are tools for sub-contracting loans to third party investors in an openly disclosed or silent manner. In *unfunded* participations, the sub-contractor guarantees to partly cover potential borrower defaults, whereas *funded* participations implies that the sub-contractor participates in the funding of the loan by transferring funds to the originator. In both cases, the sub-contractor is compensated by a share of revenue from the underlying loan proportional to its participation. The sub-contractor is hence exposed to the credit risk of both originator and borrower, although it only has a claim on the former.

Given that the sub-contractors have no contractual relation to the borrower, all loan administration remains conducted by the originator. It is argued in (Hünseler, 2013, p. 216) that this may cause an interest bias between the originator and the sub-contractor in the event of the borrower experiencing financial stress, following that the former is responsible for the client relationship and may have other incentives than purely economical influencing its decision making. Furthermore, the sub-contractor must rely on the originator to monitor the exposure and pass on relevant information in a timely manner[17]. Participations may however give sub-contractors access to credit market investments without implying an extensive administrative burden.

[16] Recourse refers to the ability of the buyer to put a loan back to the seller under certain conditions.

[17] In order to pass information to sub-contractors, a confidentiality agreements is generally established between the parties.

From a portfolio management perspective, participations are flexible with respect to eligible counter-parties and underlying assets while allowing for exposure mitigation without affecting client relationships. The method display low degree of standardization, which allows for tailored terms and conditions but also implies an extensive administrational burden and long time to market. Furthermore, originators may need to seek participants consent in order for changes in the underlying loan documentation to be carried out, which restricts flexibility for the risk manager once a participation has been established.

3.1.3.3 Credit Default Swaps

A *credit default swap* (CDS) is an over-the-counter derivative that allows for the credit exposure of a *reference entity* to be transferred from a *protection buyer* to a *protection seller* in exchange for fixed periodic payments, referred to as a *spread* or *premium*. In the event of the reference entity defaulting on its obligations, the protection seller compensates the protection buyer in accordance with the outcome of a *settlement process*, which we will consider briefly later in this section. The reference entity, which may be a corporation, a sovereign or an asset backed security, serves as an underlying to the CDS without being involved in (and possibly not aware of) the transactions implied by the contract[18].

Even though CDS may serve to insure an institution against unwanted credit exposure, it is not per definition a credit insurance. As opposed to insurances, a CDS is triggered by a publicly observable event such as a bankruptcy or failure to deliver payments in due time[19], rather than by a realized loss of the protection buyer. This property implies that CDS may compensate a protection buyer for a credit event to which it has no exposure, enabling the instrument class to be used for speculative purposes.

In the early development of the instruments, a *physical settlement* took place in the event of a CDS being triggered, meaning that the protection buyer would deliver the held security to the protection seller and for this be compensated at par value. This process has however often proved to be inconvenient as the nominal amount of outstanding CDS in many cases has come to exceed that of the outstanding deliverable securities, causing *bond squeezes*[20] to follow credit events. Given that bond squeezes marginalizes compensations, *cash settlements* have increasingly come to replace physical settlements as markets have developed.

In a cash settlement, the protection buyer is compensated by par value minus an expected recovery value, which is set based on the assumption of the protection buyers

[18]This does however not hold for CDS with physical settlements, addressed later in this section, where the claim on the reference entity is transferred to the protection seller.

[19]Further valid credit events may be debt restructuring, repudiation/moratorium, obligation acceleration and obligation default depending on the terms and conditions of the CDS.

[20]The term bond squeeze refers to an inflation in the price of deliverable bonds stemming from protection buyers rushing the market for a deliverable security following a credit event.

having a senior unsecured claim on the reference entity. The expected recovery values are determined through auctions supervised by the International Swaps and Derivatives Association (ISDA). As the structure of these auctions is of fairly high complexity, we do not engage in a detailed description in this thesis.

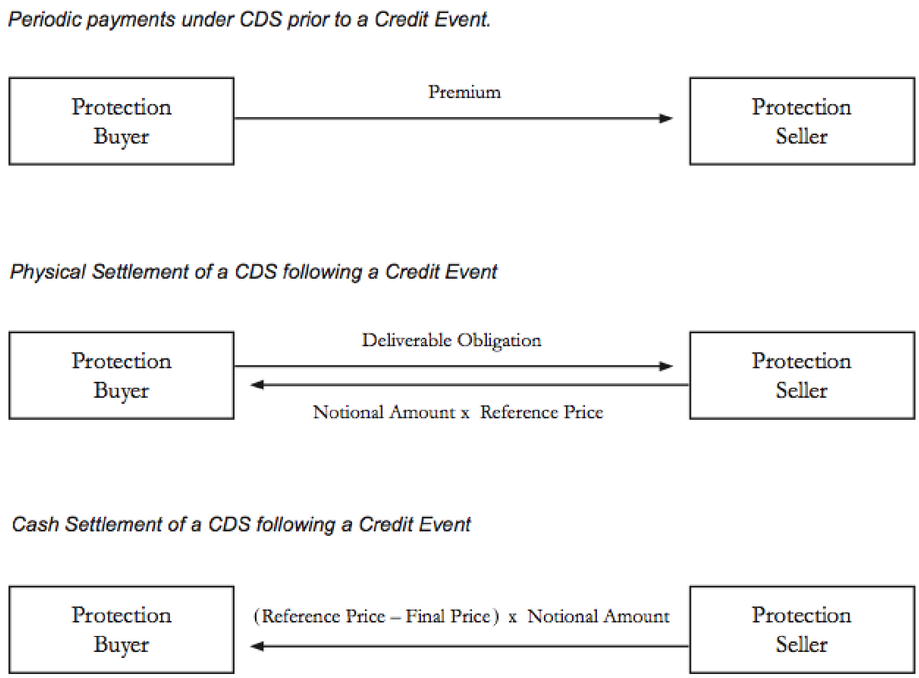


Figure 3: Schematic overview of fundamental CDS mechanisms. This figure is an edited version of that in (Bartlam and Artmann, 2006, p. 6)

The possibility of mitigating credit exposures without seeking the approval of a counter-party, as is the case for cash settlements, may help to preserve client relationships and spare business originators the trouble of telling clients that their credit is considered to have a negative influence on the overall portfolio. CDS provide flexibility to the risk managers toolbox and may be used for reducing concentrations and mitigating peak exposures in order to bring the portfolio risk/return ratio closer to the efficient frontier. The instruments may further serve as an insurance against deteriorating credit quality of a counter-party prior to default, as the value of CDS rise with increasing risk of a triggering credit event. Furthermore, CDS documentations have been standardized by ISDA to ensure liquidity and transparency, which reduces the administrative burden and time to market.

There is however no guarantee that the recovery rate covers the full exposure of a protection buyer, as it is set uniformly for all CDS on a reference entity in the cash settlements. This is referred to as *basis risk* and even though there is an upside of this coin, in the sense that a protection buyer may be overly compensated with respect to the realized loss, portfolio managers may be unwilling of baring the uncertainty. As derivatives are accounted for mark-to-market, they may induce volatility to profit and loss statements, stemming from market movements which are not necessarily linked to changes in the credit quality of the underlying. Furthermore, engaging in a derivative transaction exposes the institution to the counter-party risk of the protection seller, which must be taken into account when evaluating the protection.

Since the introduction of the single name CDS in the early 1990s, the instrument category has been expanded to include members with a variety of properties. We here consider three CDS family members and their areas of application for managing credit risk.

Basket CDS: The underlying is a basket of reference entities and the swap is triggered at the n-th credit event in the basket. This structure allows for protection against the credit risk of several counter-parties while taking reference entity correlation into account.

Loan-Only CDS: The reference obligations are syndicated secured loans and the need for separating loan-only CDS from regular single name CDS stems from the generally higher recovery value implied by this property. Furthermore, loan-only CDS terminate if the reference loan is re-paid or no deliverable at the required seniority is outstanding (FINCAD).

CDS Index: The most liquid and standardized credit derivative, which generally trades at lower spreads than regular single name CDS. The indices are mainly written on large, publicly traded firms and may therefore serve as a proxy hedge for such obligors, although caution has to be taken when estimating the correlation to the hedged exposure, due to the considerable basis risk proxy hedges implies.

3.1.3.4 Financial Guarantees

A *financial guarantee* is a financial contract through which a guarantor insures the holder against losses stemming from the failure of a debtor to make payments in due time. As opposed to CDS, these instruments are written on pre-specified payments and compensation is triggered by realized losses rather than credit events. It follows from these properties that the protection buyer must hold the security for which it is buying protection and can not be overly compensated with respect to the insured payments.

Another significant difference to CDS is the level of standardization. As previously mentioned, CDS are standardized to a high degree to ensure liquidity, whereas financial guarantees are tailored contracts between two parties, which implies a greater flexibility in formulating contract documentations. This flexibility does however come at the cost of illiquidity and high process complexity, although the latter problem often is partly resolved through reformulation of standardized CDS documentations.

Given that financial guarantees are perfect hedges with respect to the credit event 'failure to pay', they qualify as collateral under IAS 39. This property implies that they do not induce volatility on the profit and loss statement of the holder, as they do not have to be accounted for mark-to-market (Hünseler, 2013, p. 130).

3.1.3.5 Securitizations

From a risk management perspective, securitizations are tools for mitigating exposures by pooling and reselling assets of a certain class to investors. Practically, this is managed through a *special purpose vehicle* (SPV), which is a shell company whose only cause for existence is to issue securities. The SPV must be fully independent, such that no other entity than the investors in the issued securities can have a claim on the company, and management is outsourced to third party service providers.

The interest bearing debt securities issued by the SPV are typically structured into *tranches*. This securitization scheme reflects a waterfall structure where the tranches with the highest *seniority* experience the highest priority of being repaid in the event of default. Interest payments are made to investors from an *escrow account*, in which income from the pooled assets, as well as proceeds from security sales, is being held and invested. An overview of basic securitization mechanisms is presented in figure 4.

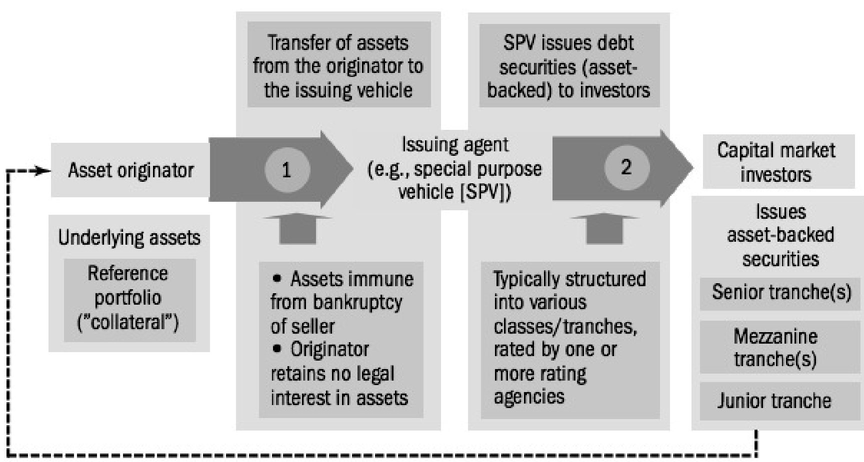


Figure 4: Schematic overview of basic securitization mechanisms (Jobst, 2008, p. 48)

Rather than engaging in a discussion spanning over all types of securitizations available, we here restrict our focus to *collateralized debt obligations* (CDO) and *collateralized loan obligations* (CLO) in particular. The term CDO refers to securitizations where the collateral posted consist of debt securities, whereas CLO constitute a subcategory in which the debt instruments are restricted to loans. The loan types posted as collateral are generally, but not formally, restricted to leveraged loans and loans to small and medium sized enterprises.

In the event of exposures being transferred from the balance sheet of the originating institution to the SPV through an assignment, the responsibilities of the originating institution terminates. This is referred to as *cash-flow* CDO and constitutes an ideal scenario for the originator, considering that it implies full mitigation of the unwanted exposures.

Assignments are in many situations infeasible due to factors such as contractual premises and customer relations, which has given rise to an alternative structure, *synthetic* CDO, in which risk transfer is achieved through CDS written on the reference portfolio of pooled assets. This implies that assets stays on the balance sheet of the originator, in contradiction to the process described in figure 4. The full nominal of the pooled assets is generally not covered, following that a simultaneous default on all assets is unlikely to occur under stable economical conditions. This structure implies high flexibility and low process complexity compared to cash-flow CDO, but also exposes the originator to credit and basis risk.

Table 4: Aspects of cash-flow and synthetic CDO Danske Bank (2014)

Property	**Cash-Flow**	**Synthetic**
Making Credit Capacity Available	Yes	Yes
Regulatory Equity Optimization	Yes	Yes
Refinancing	Yes	No
Portfolio Transfer	Yes	No
SPV Necessary	Yes	No
Balance Sheet Reduction	Yes	No
Cost	High	Average
Process Complexity	High	Average
Administrative Complexity	High	Average
Counter-Party Risk	No	Yes
Remaining Asset Ownership	No	Yes
Legal Matters - Documentation	High	Low
Requirements of Computer Systems	High	Low

4 Analysis

4.1 Numerical Analysis of Credit Risk Models

We here evaluate the credit risk models previously presented in this study, with focus on model accuracy and computational efficiency. The Pykhtin multi-factor adjustment is evaluated on portfolio level, whereas the evaluation of the Monte Carlo methods covers credit risk estimation on sector and obligor level as well.

The main purpose is to examine how the numerical estimation techniques considered in this study can be applied for gaining quantitative insight in credit portfolio risk profiles.

4.1.1 Numerical Analysis Setup

We here define a multi-factor model setup and a methodology for generating sample portfolios. This setup is later used for evaluation of the previously presented credit risk models.

4.1.1.1 Defining a Multi-Factor Model Setup

Throughout this numerical analysis, we will use an underlying multi-factor model with the following properties:

- Each sector is connected to a global Z_{N+1} and a sector specific $\{Z_k\}_{k=1...N}$ systematic risk factor,
- The systematic risk factor loadings $\{\beta_k\}_{k=1...N}$ are homogenous within sectors.

In analogy with (3) and (4), we describe this multi-factor model mathematically as:

$$X_i = r_i Y_i + \sqrt{1 - r_i^2}\xi_i, \quad Y_i, \xi_i \sim N(0,1), \tag{92}$$

$$Y_i = \beta_{k(i)} Z_{N+1} + \sqrt{1 - \beta_{k(i)}^2} Z_{k(i)}, \quad Z_{k(i)}, Z_{N+1} \sim N(0,1), \tag{93}$$

where the idiosyncratic shock, the composite risk factor and the systematic risk factors are standard normal distributed, following the specification in section 2.1. This model has been chosen as it gives rise to an intuitive correlational structure and simplifies generation of concentrated portfolios. All sample portfolios considered in this evaluation have further been divided into 5 sectors, without loss of generality.

The loss given default variables $\{LGD_i\}_{i=1...M}$ are set to be deterministic and the composite risk factor loadings $\{r_i\}_{i=1...M}$ are assumed to follow the Basel II IRB risk weight function for corporate assets, as defined in (Genest and Brie, 2013, p. 14):

$$r_i^2 = 0.12 \left(\frac{1 - e^{-50PD_i}}{1 - e^{-50}} \right) + 0.24 \left(1 - \frac{1 - e^{-50PD_i}}{1 - e^{-50}} \right). \tag{94}$$

All risk factor loadings, composite and systematic, are assumed positive and all Value at Risk and Expected Shortfall figures are based on confidence level $\alpha = 99.9\%$.

4.1.1.2 Homogenous Sample Portfolio Structure

We define a structure for homogenous sample portfolios in which the exposure weights of sectors $2-5$ are equal. This property can be described mathematically as:

$$w_k = \frac{1 - w_1}{4}, \quad \text{for } k = 2...5. \tag{95}$$

As can be seen in (95), this structure allows for concentrated portfolios to be created in a simple manner. It is implied from the sector homogeneity that the sector exposure weights are distributed equally across obligors. We describe this mathematically as:

$$w_i = w_{k(i)} \Big/ M_{k(i)} \ . \tag{96}$$

It further follows from the homogenous structure that all other obligor specific parameters equals a set of sector specific parameters, which are listed in table 5.

Table 5: Portfolio structure - Sector specific parameters

Property		**Sector 1**	**Sector 2**	**Sector 3**	**Sector 4**	**Sector 5**
Number of Obligors	M	M_1	M_2	M_3	M_4	M_5
Exposure Weight	w	w_1	$(1-w_1)/4$	$(1-w_1)/4$	$(1-w_1)/4$	$(1-w_1)/4$
Probability of Default	PD	0.02	0.01	0.02	0.01	0.02
Loss Given Default	$\mathbb{E}[LGD]$	0.3	0.2	0.3	0.2	0.3
	$\mathbb{V}[LGD]$	0	0	0	0	0
Syst. Factor Loading	β	$\sqrt{0.5}$	$\sqrt{0.5}$	$\sqrt{0.5}$	$\sqrt{0.5}$	$\sqrt{0.5}$

The composite risk factor loadings $\{r_k\}_{k=1...N}$ have been left out of the table since they are implied from (94) and the sector specific default probabilities $\{PD_k\}_{k=1...N}$.

4.1.1.3 Heterogenous Sample Portfolio Structure

We introduce a heterogenous portfolio structure with sector averages equal to those of the homogenous portfolio structure. In contrast to the homogenous case, the exposure weights are distributed randomly across obligors through normalized weights. Additionally, independent draws from uniformly distributed random variables are added to the expected loss given default and the probability of default for each obligor:

$$\begin{aligned} w_i &= w_{k(i)}\Delta w_i \Big/ \sum_{i=1}^{M} \Delta w_i, & \Delta w_i &\sim \mathcal{U}(0,1), \\ \mathbb{E}[LGD_i] &= E[LGD_{k(i)}] + \Delta LGD_i, & \Delta LGD_i &\sim \mathcal{U}(-0.1,0.1), \\ PD_i &= PD_{k(i)} + \Delta PD_i, & \Delta PD_i &\sim \mathcal{U}(-0.01,0.01). \end{aligned} \tag{97}$$

In generating sample portfolios, the stochastic parameters are drawn once and saved in order for all heterogenous portfolios to have the same basic set of obligor specific properties.

Furthermore, obligor specific composite risk factor loadings $\{r_i\}_{i=1\ldots M}$ given the generated default probabilities are computed according to (94) and the sector specific parameters are set equal to the homogenous case, which are to be seen in table 5.

4.1.2 The Pykhtin Multi-Factor Adjustment

We define 4 portfolios of each kind, homogenous and heterogenous, with $M = 100$ obligors in each sector. In order to examine how effects of exposure concentration are captured by the multi-factor adjustment, we let the exposure weight of sector 1 take values:

$$w_1^1 = 0.2, \quad w_1^2 = 0.4, \quad w_1^3 = 0.6, \quad w_1^4 = 0.8. \tag{98}$$

The standard Pykhtin model is referred to as Pykhtin I and we further derive a computationally cheaper Pykhtin II model by dividing each sector into two equally large homogenous groups after probability of default, where the properties of each group are obtained from exposure weighting[21]. In this setting, Value at Risk, Expected Shortfall and computational time for each combination of model and portfolio are presented in tables 6-7 along with close to exact estimates from a Monte Carlo routine[22].

For homogenous portfolios, Pykhtin II achieves very low computational time without compromising accuracy, as can be seen in table 6. This follows from the property of homogeneity, which implies that no information is discarded from grouping obligors after default probabilities. Table 7 shows that the approximation remains highly accurate for portfolio 1 (equally distributed exposures) in the heterogenous case, whereas estimates for concentrated portfolios display larger deviations from the "true" Monte Carlo estimates.

[21] This approach is close to the methodology suggested by Hibbeln (2010), described in section 2.2.2.5. The difference is that we do not apply a discrete scale for default probabilities in this thesis.

[22] A two-step importance sampling routine with 100 000 iterations denoted MC here serves as a benchmark. It is assumed that this number of iterations gives a sufficient level of convergence for the estimates to be regarded as "true values" for these portfolios.

Table 6: Homogenous portfolios: Value at Risk and Expected Shortfall

Measure	**Model**	**Portfolio 1**	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**
Value at Risk (%)	Pykhtin I	3.17	3.64	4.42	5.33
	Pykhtin II	3.17	3.64	4.42	5.33
	MC	3.18	3.65	4.42	5.33
Expected Shortfall (%)	Pykhtin I	3.71	4.27	5.21	6.30
	Pykhtin II	3.71	4.27	5.21	6.30
	MC	3.73	4.29	5.21	6.31
Computational Time (s)	Pykhtin I	108	108	107	111
	Pykhtin II	0.05	0.05	0.05	0.05
	MC	226	235	242	239

Table 7: Heterogenous portfolios: Value at Risk and Expected Shortfall

Measure	**Model**	**Portfolio 1**	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**
Value at Risk (%)	Pykhtin I	3.11	3.57	4.34	5.24
	Pykhtin II	3.10	3.53	4.27	5.13
	MC	3.11	3.58	4.35	5.24
Expected Shortfall (%)	Pykhtin I	3.64	4.19	5.12	6.22
	Pykhtin II	3.63	4.15	5.04	6.09
	MC	3.64	4.20	5.13	6.21
Computational Time (s)	Pykhtin I	108	109	108	111
	Pykhtin II	0.05	0.07	0.05	0.05
	MC	225	229	236	240

Value at Risk is further estimated with Pykhtin I & II at varying levels of systematic risk factor loadings in order to examine model sensitivty to composite factor correlation. Each homogenous and heterogenous portfolio 1-4 is here assigned the same β-values, ranging from 0 to 1, and the results are presented in figure 5.

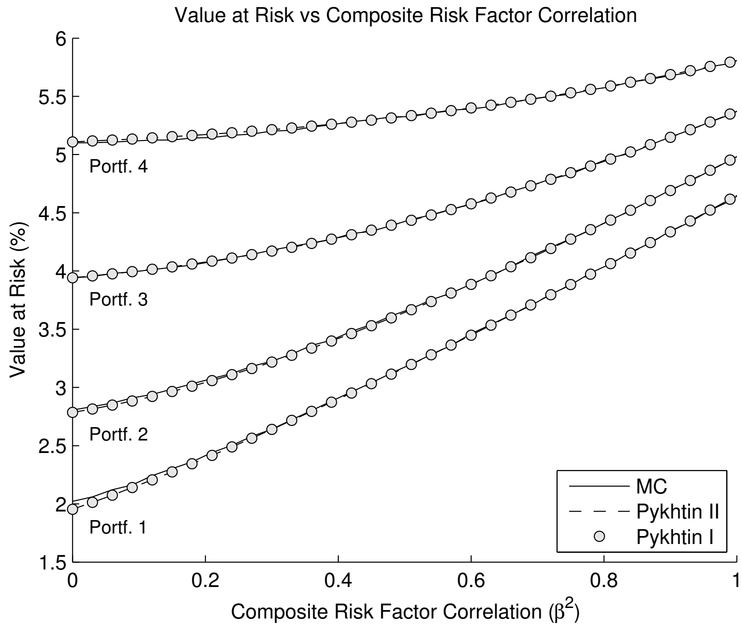
The Pykhtin I model show enhancements in accuracy with increasing composite factor correlation and exposure concentration, which can be explained by the asymptotic elimination of the systematic term in the multi-factor adjustment. As correlation and concentration increase, the underlying model converges to a single-factor model, reducing the estimation error of the multi-factor adjustment to that of the idiosyncratic term[23]. It can be seen in figure 5b that the Pykhtin I model sets an upper bound for the accuracy of Pykhtin II, which is expected given the crude approximation of averaging obligor parameters within sector groups. It is reasonable to think that the averaging process causes the excessive negative offset displayed in figure 5b for Pykhtin II and that a Monte Carlo routine would produce similar results for default probability weighted portfolios.

[23]This observation is in line with that in (Pykhin, 2004, p. 88).

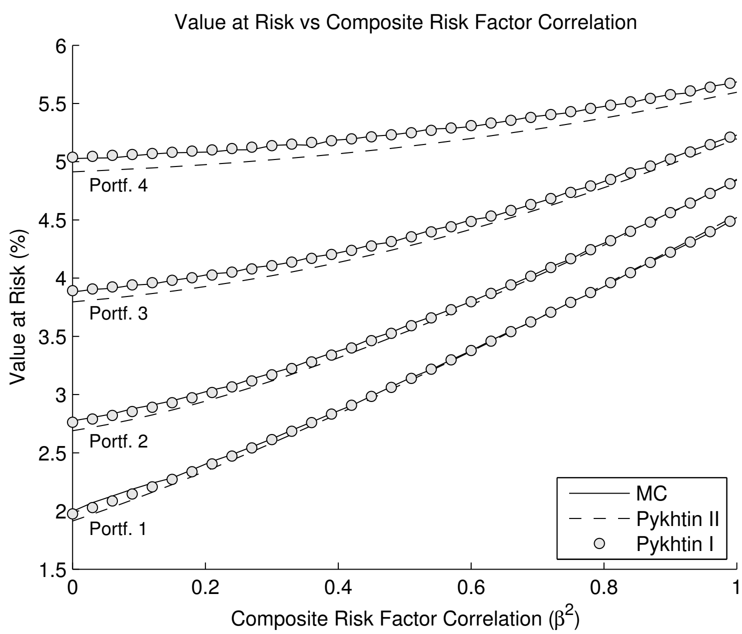
We further vary the sector sizes simultaneously in a range between 10 and 100 obligors for each of the portfolios and examine how such transitions affect the accuracy in computing Value at Risk. As can be seen in figure 6, the model performance is stable across sector sizes almost independent of sector exposure weights, where the only significant deviation from the Monte Carlo estimates occur for homogenous portfolio 4 with 10 obligors. This is an important result, which shows that the idiosyncratic part of the multi-factor adjustment is highly accurately describing the behavior it was designed to capture.

The accuracy of the Pykhtin II model is, as previously mentioned, highly dependent on the applied averaging method and even though it does not guarantee precise results, there are reasons for seeking very fast ways of determining the region of Value at Risk and Expected Shortfall. In section 4.1.3, we evaluate the importance sampling methods by Glasserman and Li (2005), which demand for an *accuracy threshold* to be set at the initiation of a simulation. In estimating Value at Risk, one may not know the region of interest in advance and the Pykhtin II model may thus serve as a time effective method for setting an approximate accuracy threshold.

Given the multi-factor model setting considered in this evaluation, the Pykhtin I model displays high accuracy and does not suffer from simulation noise, which is the case for Monte Carlo methods. As we have previously noted in section 2.2.2.5, the model computational time is approximately square proportional to the number of obligors in a portfolio, making it suitable for Value at Risk and Expected Shortfall computations for small portfolios. The model is however limited by restricting assumptions on the underlying model setting, which does not allow more complex multi-factor model structures including e.g. correlated systematic risk factors.

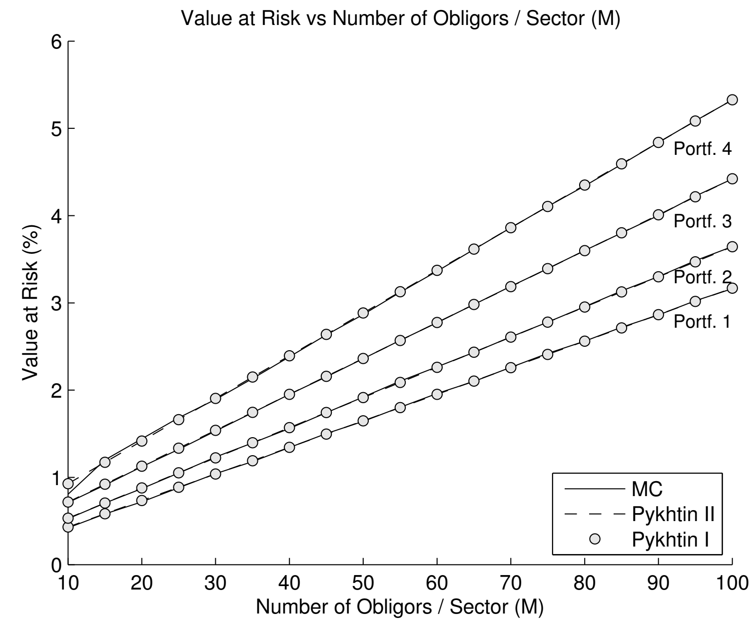


(a) Homogenous portfolios 1-4

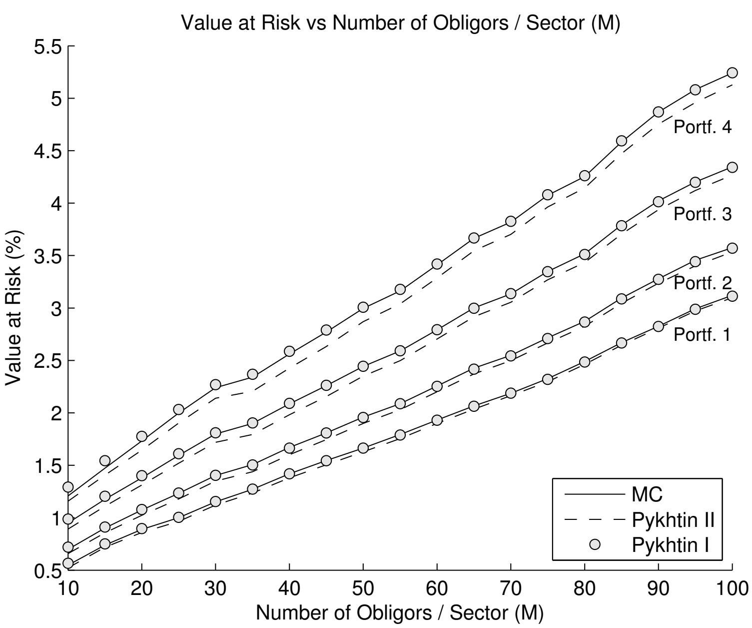


(b) Heterogenous portfolios 1-4

Figure 5: Value at Risk computed with Pykhtin I and Pykhtin II for varying levels of composite factor correlation and 100 obligors per sector. A two-step importance sampling routine with 200 000 iterations is used to compute the Monte Carlo figures.



(a) Homogenous portfolios 1-4



(b) Heterogenous portfolios 1-4

Figure 6: Value at Risk computed with Pykhtin I and Pykhtin II for varying sector sizes. A two-step importance sampling routine with 200 000 iterations is used to compute the Monte Carlo figures.

4.1.3 Monte Carlo Methods

We here consider heterogenous portfolio 1 from the evaluation of the Pykhtin multi-factor adjustment in section 4.1.2, with obligor specific properties as displayed in table (97). The portfolio loss distribution is simulated using the Monte Carlo estimators presented in section 2.3. For simplicity of notation, the following estimator descriptions are introduced:

CMC: The Crude Monte Carlo estimator

ISMC - MS: The risk factor mean-shift importance sampling Monte Carlo estimator

ISMC - ET: The exponential twisting importance sampling Monte Carlo estimator

ISMC - TS: The two-step importance sampling Monte Carlo estimator

The two-step importance sampling estimator here denotes the risk factor mean-shift and exponential twisting techniques combined, as described in algorithm 3. For each estimator, we simulate a range of loss probabilities $P(L > x)$, which are presented in figure 7 along with corresponding confidence intervals at 95% confidence level.

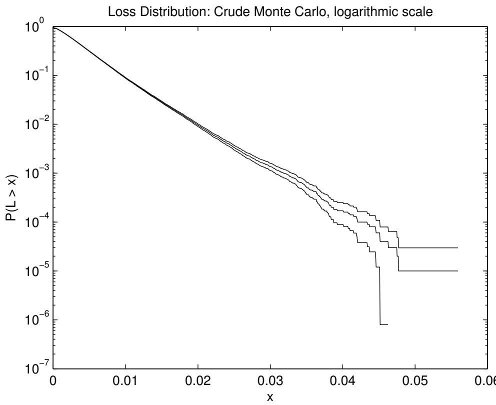
Rather than estimating parameters μ^* and θ_x for all loss levels x, we define an accuracy threshold x_a for which the parameter values are estimated[24]. Following (72), low losses become rare events when exponential twisting is applied, as can be seen in figure 7. We can however be certain, before initiating a simulation, that this region does not cover the loss level x_a for which μ^* and θ_x are estimated, which explains the term accuracy threshold.

Given that tail loss probabilities are of particular interest, we have here taken advantage of the Pykhtin II model's computational efficiency when setting the accuracy threshold, $x_a = VaR_{Pykhtin\ II} = 0.031$. This computation approximately takes 0.05 seconds and is thus a convenient way of assuring accuracy when estimating tail probabilities[25].

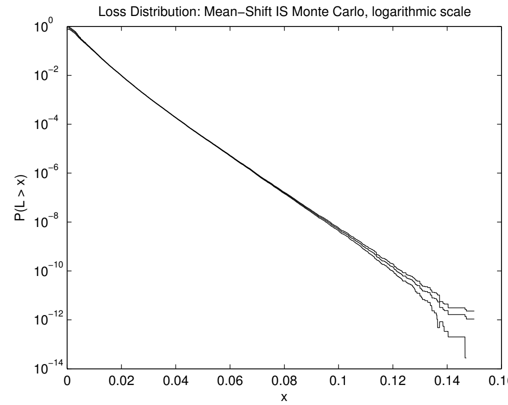
A significant variance reduction is obtained from shifting the risk factor expected values, which is intuitive considering that this method serves to reduce the probability of positive market scenarios. The confidence interval of the Crude Monte Carlo method strongly diverges for loss rates larger than 0.03, whereas methods including factor mean-shifts display stable confidence intervals until 0.12. This comes to a small (fixed) computational cost of 3 seconds for finding the risk factor expected value at the start of the simulation.

[24]This serves to significantly improve computational efficiency.

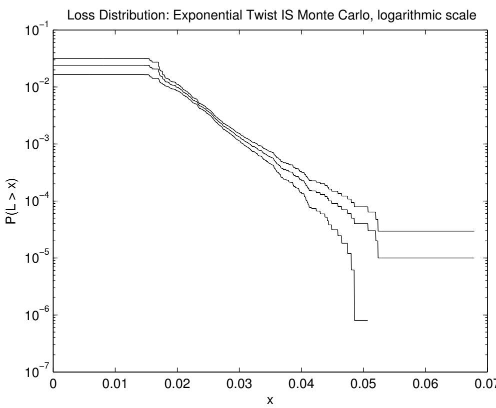
[25]As can be seen in figure 7c, higher accuracy in the region of Value at Risk ($x = 0.0311$) can be achieved by raising the accuracy threshold. We do however argue that the $VaR_{Pykhtin\,II}$ serves as a reasonable choice in case the loss distribution is unknown a priori simulation.



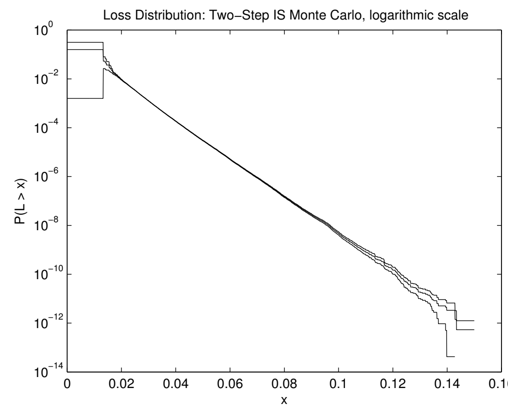
(a) CMC: 58 seconds



(b) ISMC - MS: 61 seconds



(c) ISMC - ET: 394 seconds



(d) ISMC - TS: 227 seconds

Figure 7: $P(L > x)$ computed with 4 different Monte Carlo estimators.
The number of iterations is set to 100 000 for each estimator.

The property of shifting the conditional expected loss rate described in (72) is explicitly reflected in figure 7c, where accuracy has been concentrated in a region close to the accuracy threshold. The most computationally expensive part of any scenario[26], all estimators considered, is the estimation of parameter θ_x. Given that default probabilities only are twisted in scenarios where the conditional expected loss rate is smaller than the accuracy threshold, shifting the risk factor expected values helps eliminating several optimization procedures for finding θ_x. This explains the significant decrease in computational time obtained from applying a risk factor mean-shift to the method of exponential twisting.

[26]The optimization procedure for finding a shifted risk factor mean μ^* is carried out before the first iteration and does thus not affect the average scenario computational time.

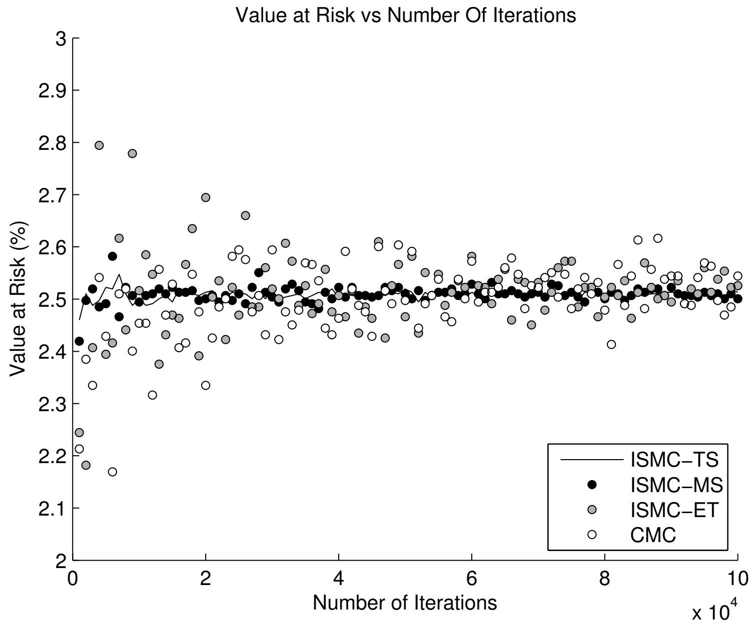
In order to examine convergence rates of the Monte Carlo estimators, we estimate Value at Risk for a varying number of iterations ranging from 1 000 to 100 000 for the previously considered portfolio. Two tests are performed in which the systematic risk factor correlation is set to 0.25 and 0.75, as β_k are assigned values $\sqrt{0.25}$ and $\sqrt{0.75}$ for all sectors. The results are displayed in figure 8.

All Monte Carlo methods display a faster rate of convergence for the highly correlated portfolio than for the less correlated. The exponential twisting estimator does not significantly outperform the Crude Monte Carlo estimator, which partly is a result of the choice of accuracy threshold. Raising the accuracy threshold would enhance the performance of the exponential twisting estimator, but we argue that this result serves a purpose in displaying the robustness of the estimators with respect to the initial guess of the threshold. In this sense, the factor mean-shift based estimators both display a robust performance with respect to the information (the Pykhtin II Value at Risk estimate) about the loss rate distribution that is fed into the estimators a priori simulation.

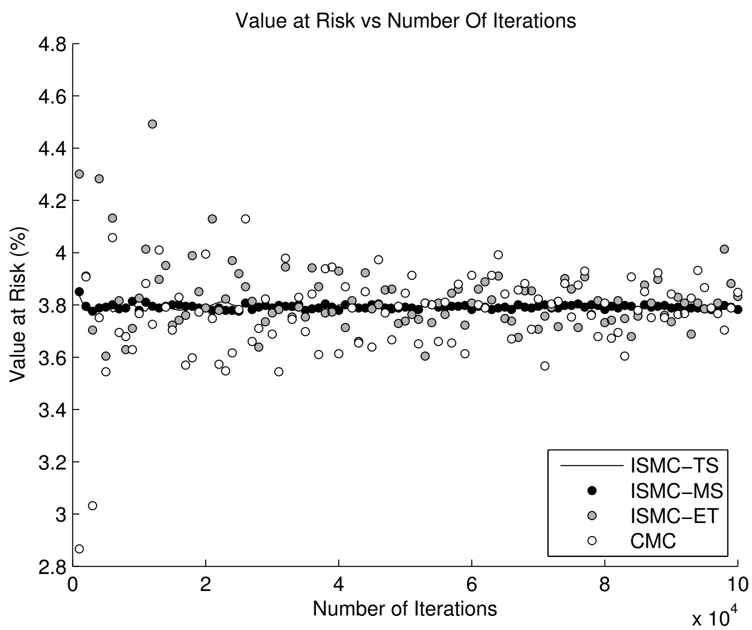
Furthermore, it can be seen that the two-step Monte Carlo estimator outperforms the factor mean-shift estimator at low risk factor correlation and low numbers of iterations. Given that estimators which do not apply exponential twisting are significantly faster than those that does, a tradeoff arises where this variance reduction could be achieved within the same computational time by raising the number of iterations for the factor mean-shift estimator.

Component Value at Risk is computed with 1 000 000 iterations for each obligor in the sample portfolio and the estimated values are presented with corresponding 95% confidence intervals in figure 9. Obligors have been sorted by magnitude of component Value at Risk in order to give the graph an increasing slope, which makes it simple to read and understand. The lower bounds in the confidence intervals are restricted to be larger than or equal to zero, as we here assume that no obligor has a hedging effect on the portfolio.

Estimating component Value at Risk is computationally demanding following the constraint in (88), which causes all scenarios with portfolio losses insufficiently close to Value at Risk to be discarded. This gives rise to a substantial estimator variance for the CMC estimator, following that valid scenarios are rare in this setting. Shifting the risk factor expected values does however drastically reduce estimator variance and figure 9 shows that the estimator ISMC - MS provides the highest accuracy/computational time ratio of all estimators. We further observe that computational time increase with number of valid scenarios, as this implies an increase in the average number of matrix-operations per iteration.

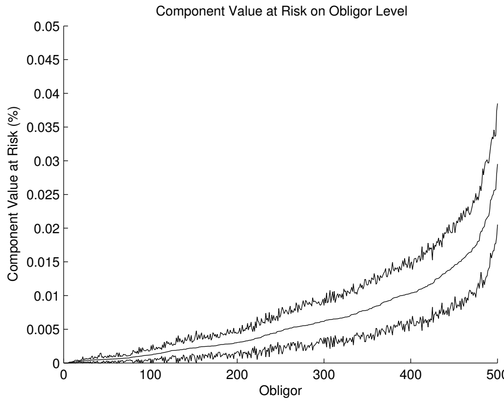


(a) Convergence test with risk factor correlation set to 0.25.

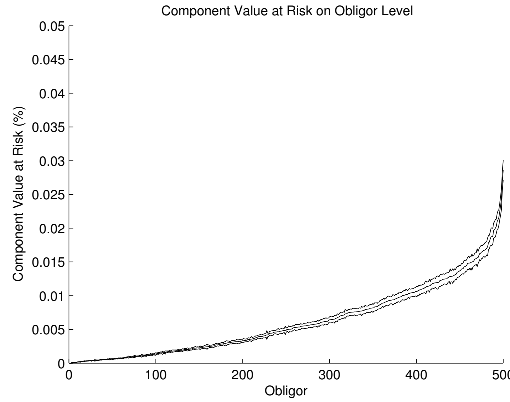


(b) Convergence test with risk factor correlation set to 0.75.

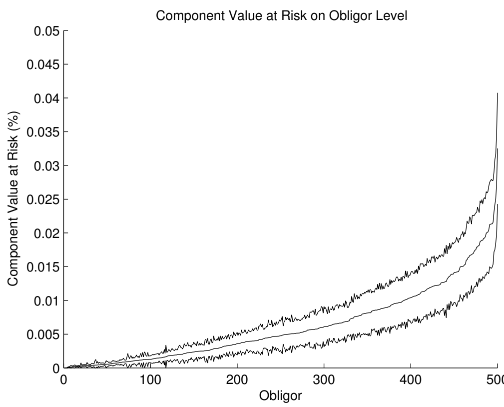
Figure 8: Convergence tests displaying Value at Risk for each Monte Carlo estimator. The number of iterations ranges from 1 000 to 100 000 with step size 1 000.



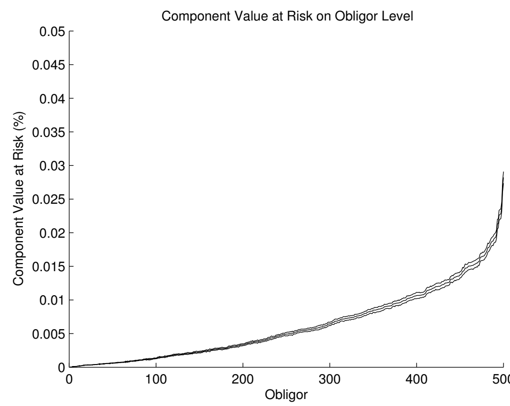
(a) CMC: 245 seconds



(b) ISMC - MS: 260 seconds



(c) ISMC - ET: 3601 seconds



(d) ISMC - TS: 3143 seconds

Figure 9: Component Value at Risk estimated for each obligor in the sample portfolio. The number of iterations is set to 1 000 000 and the tolerance ϵ in (87) is set to $VaR/100$ for each estimator.

The drastic offset in computational time for the exponential twisting estimators stems from the technique of shifting the default probabilities (and thus seeking θ_x) in every iteration, which is discussed in section 2.3.3.2. We do however note that shifting the risk factor expected values reduces the computational time of the exponential twisting estimator by eliminating part of the time allocated to the θ_x-solver.

We proceed by comparing the 50 largest values of component Value at Risk to incremental Value at Risk (the change in portfolio Value at Risk when removing a position) for the corresponding obligors. All figures are computed using the ISMC - TS estimator and the results are displayed in figure 10, along with a linear regression.

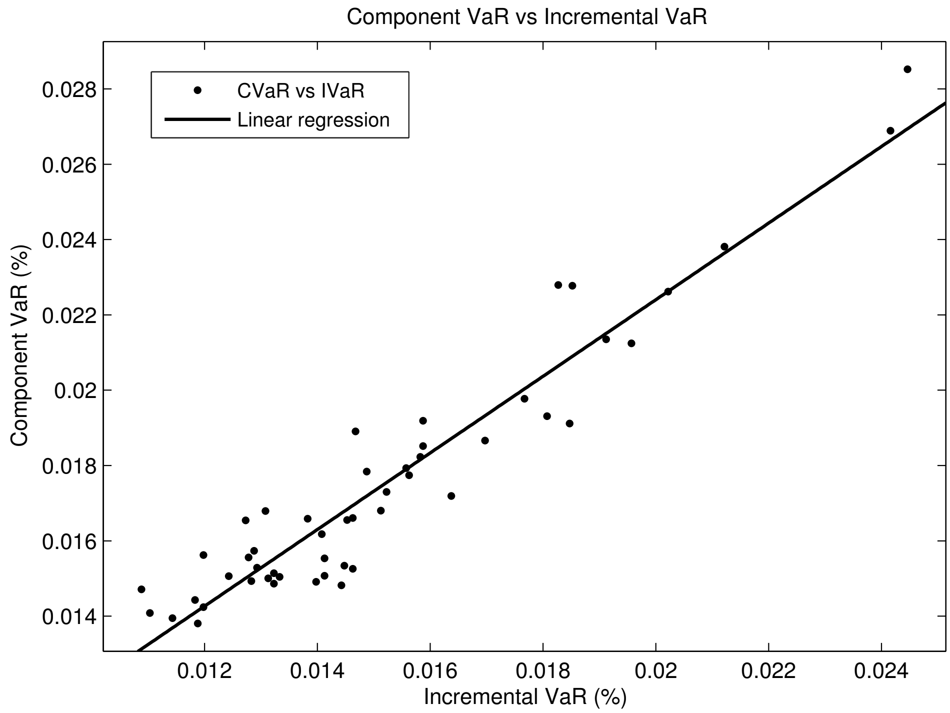


Figure 10: Component Value at Risk vs Incremental Value at Risk. Incremental Value at Risk is computed using the ISMC - TS estimator with 800 000 iterations and the component Value at Risk figures corresponds to those in figure 9d. Linear regression: $\mathrm{CVaR} = 1.017 \cdot \mathrm{IVaR} + 2.061 \cdot 10^{-3}$, $\mathrm{R}^2 = 0.9068$.

The approximation in (91) proves quite accurate, although the absolute value of component Value at Risk is consistently larger than or equal to that of incremental Value at Risk[27]. In the light of the computational effort needed to accurately compute incremental Value at Risk, it is reasonable to consider component Value at Risk an efficient indicator of a portfolios largest exposures. Furthermore, component Value at Risk can be interpreted as the contribution of an obligor or sector to the diversified portfolio Value at Risk and the properties of exact additivity and decomposability (81) makes it a convenient measure for assessing risk on obligor as well as sector level.

In conclusion, we demonstrate in tables 8 - 10 how the Monte Carlo methods regarded in this study can be applied for a comprehensive overview of a portfolios risk profile.

[27] This follows as the first order derivative neglects diversification/concentration effects stemming from adding/removing a position.

Table 8: Risk profile - Portfolio level

Measure	**Sector**	**Total**	**% Expected**	**% Systematic**	**% Idiosyncratic**
Value at Risk (%)	*All*	3.11	13.94	75.40	10.66
Expected Shortfall (%)	*All*	3.64	11.92	77.90	10.18

Table 9: Risk profile - Sector level

Measure	**Sector**	**Total**	**% Expected**	**% Systematic**	**% Idiosyncratic**
Component VaR (%)	*Sector* 1	0.81	13.84	75.48	10.68
	Sector 2	0.29	13.49	78.16	8.35
	Sector 3	0.79	14.00	74.64	11.36
	Sector 4	0.32	13.65	77.27	9.08
	Sector 5	0.90	14.21	74.45	11.34
Component ES (%)	*Sector* 1	0.95	11.76	77.75	10.49
	Sector 2	0.35	11.33	80.79	7.88
	Sector 3	0.91	12.14	77.28	10.58
	Sector 4	0.38	11.56	80.01	8.42
	Sector 5	1.05	12.21	76.84	10.95

Table 10: Risk profile - Obligor level (The 10 largest exposures)

Obligor ID	**Sector**	**Incremental VaR (%)**	**Component VaR (%)**	**Component ES (%)**
408	*Sector* 5	$2.45 \cdot 10^{-2}$	$2.85 \cdot 10^{-2}$	$3.17 \cdot 10^{-2}$
481	*Sector* 5	$2.42 \cdot 10^{-2}$	$2.69 \cdot 10^{-2}$	$3.05 \cdot 10^{-2}$
478	*Sector* 5	$2.21 \cdot 10^{-2}$	$2.38 \cdot 10^{-2}$	$2.73 \cdot 10^{-2}$
439	*Sector* 5	$1.82 \cdot 10^{-2}$	$2.28 \cdot 10^{-2}$	$2.60 \cdot 10^{-2}$
403	*Sector* 5	$1.85 \cdot 10^{-2}$	$2.28 \cdot 10^{-2}$	$2.56 \cdot 10^{-2}$
85	*Sector* 1	$2.02 \cdot 10^{-2}$	$2.26 \cdot 10^{-2}$	$2.56 \cdot 10^{-2}$
491	*Sector* 5	$1.91 \cdot 10^{-2}$	$2.14 \cdot 10^{-2}$	$2.39 \cdot 10^{-2}$
233	*Sector* 3	$1.96 \cdot 10^{-2}$	$2.13 \cdot 10^{-2}$	$2.49 \cdot 10^{-2}$
26	*Sector* 1	$1.77 \cdot 10^{-2}$	$1.98 \cdot 10^{-2}$	$2.23 \cdot 10^{-2}$
453	*Sector* 5	$1.80 \cdot 10^{-2}$	$1.93 \cdot 10^{-2}$	$2.30 \cdot 10^{-2}$

Remark 5 *The figures in table 10 show signs of estimator variance, which is a result of the limited computational power at hand when conducting this study. The estimates on sector level have however been confirmed accurate and it is reasonable to think that a financial institution would have the hardware needed to achieve accurate estimates on obligor level using these methods. This example serves a purpose in demonstrating how the models and risk type decompositions regarded in this study can be applied to examine risk-type contributions, sector concentrations and single name exposures in a holistic manner.*

5 Discussion

The Pykhtin models and the risk factor mean-shift based Monte Carlo techniques both display great performance throughout the numerical analysis in section 4.1. These results must however be interpreted in relation to the setting in which the models have been tested, considering the simplifying assumptions which have been made through out this thesis.

It is reasonable to think that the performance of the factor mean-shift Monte Carlo method would be negatively affected by the introduction of stochastic loss given default variables, following that the risk factor expected values are shifted a priori simulation. The estimator is therefore not able to account for scenario specific changes in the loss given default figures, which may cause the risk factor mean-shift to become sub-optimal with respect to the obligor specific properties in the scenarios. This should however not influence the performance of the exponential twisting estimator, as the default probabilities are twisted in each scenario, which allows for this estimator to account for scenario specific properties.

The Pykhtin model accounts for stochastic loss given default through the variable variance, but makes no further assumptions on the distribution of these variables. It is therefore reasonable to think that the performance of the model is dependent on the distribution chosen to model the loss given default variables. The computational time of the Pykhtin model is however independent of the choice of distribution, whereas the set of possible scenarios in the simulation based techniques is extended if stochastic loss given default is introduced. This may imply lower convergence rates for the simulation based methods and increase the relative computational efficiency of the Pykhtin model in such a setting.

Furthermore, the loss given default variables are assumed independent with regards to the systematic risk factors in the underlying multi-factor model. This is somewhat counter-intuitive, as it is reasonable to think that harsh market conditions implies low recovery rates through effects of contagion. In the Basel II framework, this simplification is compensated for through the use of loss given default estimates which "reflect economic downturn conditions where necessary to capture the relevant risks" (Hibbeln, 2010, p. 40). Following that the approach is in line with Basel II, we do not engage in further discussions on this topic.

The main limit factor in estimating component value at risk has been the RAM-memory, following the large matrices needed for storing scenario specific losses on obligor level. Given that scenario specific data only is necessary for computing the estimator variance, it is however possible to enhance the component Value at Risk estimator efficiency through addition of scenario losses, which reduces the matrix-size required for storage according to:

$$Number\, of\, Scenarios \cdot Number\, of\, Obligors \longrightarrow Number\, of\, Obligors.$$

As previously stated, this efficiency comes at the cost of neglecting the estimator variance. The factor mean-shift Monte Carlo estimator provides the highest accuracy/computational time ratio, whereas the two-step Monte Carlo estimator provides the highest accuracy with respect to allocated RAM-memory. This makes the latter a suitable tool in situations where internal memory, rather than computational time, constitutes the main constraint.

Fully covering active credit portfolio management is challenging and we have in this thesis taken a quantitative perspective. The idea was initially to provide a wide scope of portfolio management, ranging from computationally fast models to risk mitigation strategies, but as complexity grew in the credit risk modeling part of the project, a decision had to be taken on whether to equally balance these parts or put focus on the risk modeling part. The choice of the second option was motivated partly by the author's technical background and partly by a wish to find a model sophisticated enough to be practiced by a Nordic bank.

In conclusion, we note that the factor mean-shift based Monte Carlo methods are granular enough to cover all aspects of concentration risk regarded in the problem definition (in a setting where the systematic risk factors are connected to industries and geographical regions) with high accuracy and within reasonable computational time.

6 Conclusion

The Pykhtin models may serve to give a simplistic overview of portfolio risk profiles at low computational cost while taking granularity into account. They are further suitable for determining the accuracy threshold for the simulation based Monte Carlo models in the case of loss distribution tail regions being unknown a priori simulation.

The importance sampling Monte Carlo techniques allows for accurate estimation of concentration risk with respect to single entities, industrial sectors and geographical regions, which may serve to enhance awareness of concentration risk in loan portfolios.

Furthermore, the risk factor mean-shift estimator may, through a drastic decrease in computational time in comparison to the Crude Monte Carlo method, enable financial institutions to frequently update their risk estimates and better incorporate quantitative aspects into an active portfolio management setting.

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A Risk Measures

A.1 Value at Risk

Value at Risk is here defined as a quantile of a loss distribution, which may be expressed as the smallest value x such that the probability of encountering a loss of magnitude x or larger is smaller than $1-\alpha$. This can be formulated as:

$$VaR_\alpha(L) = \inf\{x : P(L > x) \leq 1 - \alpha\} = \inf\{x : F_L(x) \geq \alpha\}. \tag{99}$$

A.2 Marginal Value at Risk

Marginal Value at Risk is here defined as the change in portfolio Value at Risk following a marginal change in the exposure weight w_i of obligor i. This can be formulated mathematically in the following way:

$$MVaR_\alpha^i(L) = \frac{\partial VaR_\alpha^{portfolio}(L)}{\partial w_i}. \tag{100}$$

A.3 Component Value at Risk

Component Value at Risk is here defined as the contribution of obligor i to the diversified portfolio Value at Risk, which may be written as the product of the obligor exposure weight and marginal Value at Risk:

$$CVaR_\alpha^i(L) = w_i \frac{\partial VaR_\alpha^{portfolio}(L)}{\partial w_i}. \tag{101}$$

A.4 Incremental Value at Risk

Incremental Value at Risk is here defined as the absolute difference in portfolio Value at Risk which stems from adding or removing position i. We describe this mathematically as:

$$IVaR_\alpha^{\pm i}(L) = \left| VaR_\alpha^{portfolio \pm i}(L) - VaR_\alpha^{portfolio}(L) \right|. \tag{102}$$

A.5 Expected Shortfall

Expected shortfall is here defined as the expected loss conditional on the loss rate being larger than or equal to Value at Risk. The may be described mathematically as.

$$ES_\alpha(L) = \mathbb{E}[L \,|\, L \geq q_\alpha(L)] = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)\; du. \tag{103}$$

A.6 Expected Loss

The expected loss is defined as the unconditional expectation of the product of the exposure weight, loss given default and probability of default. We describe this mathematically in the following way:

$$EL = \mathbb{E}[w \cdot LGD \cdot PD]. \tag{104}$$

A.7 Economic Capital

Economic capital is defined as the unexpected part of Value at Risk. We describe this mathematically as:

$$EC_\alpha = VaR_\alpha - EL. \tag{105}$$