B.Sc. Thesis

The Heston Model

- Stochastic Volatility and Approximation -

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Abstract

The crude assumption on log normal stock returns and constant volatility in the Black-Scholes model is a big constraint which constructs smile and skew inconsistent prices. The Heston model and its suggested approximation built on stochastic volatility are introduced and faced against the Black-Scholes model in hope of producing option prices where the smile and skew are taken into account.. As one will observe later on is that numerical calculation and approximation of the Heston model will provide us with more accurate calculations.

Keywords

List of notation

cdf - Normal cumulative density function.

$E[X]$ - Expected value of s.v. X.

$f(\cdot)$ - Normal pdf.

$F(\cdot)$ - Normal cdf.

$\mathcal{F}_t^X$ - Filtration, all information about X until time t.

GBM - Geometric Brownian motion.

K - Strike price.

L - Likelihood function.

MC - Monte Carlo.

$N(0,1)$ - Normal distribution with mean 0 and variance 1.

$\mathbb{P}$ - Historical measure.

PDE - Partial differential equation.

pdf - Probability density function.

$\phi(\cdot)$ - Normal pdf.

Φ - Payoff.

$\Pi(S,t)$ - Derivative value, with underlying asset S at time t.

$\mathbb{Q}$ - Risk neutral martingale measure.

$\mathbb{R}^d$ - $(d \times 1)-$dimensional real value.

r_t - Interest rate at time t.

σ - Volatility.

SDE - Stochastic differential equation.

s.v. - Stochastic variable

S_t - Stock Value at time t.

T - Time to maturity.

θ- Parameter set.

Θ - Parameter space.

W - Wiener process, standard $N(0,1)$.

$\subseteq$ - Subset.

$\square$ - End of proof or derivation.

Contents

1 Introduction

1.1 Background

It was in the 1970s when Fischer Black, Myron Scholes and Robert Merton derived the Black-Scholes (sometimes Black-Scholes-Merton) which changed the way and impact the world of pricing derivatives using stocks as the underlying asset (Hull 2008: 277). Myron Scholes and Robert Merton where awarded the price of the Nobel Prize in economics in 1997 (Fischer Black died in the 1995) then one can understand the impact that this formula cased.

It was now possible to price derivatives by a very simple closed form solution. But with the crude assumption on constant volatility and in log normal returns really limits the model, and this is why the model only is used as a benchmark today. This thesis will study the first assumption of constant volatility and present a better proposal that can be used for pricing derivatives. One very simple model that is built on a stochastic volatility is the Heston model which is an extended version of the stochastic process which the Black-Scholes model is built on. The problem with this model is thus that there does not exist a closed for solution, but some approximations have been proposed. We will study one of these approximations of the closed form and compare this whit Monte Carlo simulation of the Heston stochastic process and with the Black-Scholes formula.

1.2 Purpose

The purpose of this thesis is to construct appropriate values for calculating options that are smile consistent by introducing stochastic volatility. The suggested closed form solution for the Heston model is faced against the Heston stochastic differential equation (SDE), and finally the Black-Scholes formula.

1.3 Outline

In section two are some basic fundamental mathematical and derivative theory introduced, the stochastic differential equation, Brownian motion, the geometric Brownian motion, Itô formula and then Black-Scholes model. Section three deduces Black-Scholes model, an arbitrage relationship and the Smile/Skew effect. The Heston stochastic differential equation and the suggested approximation presented are introduced in section four. In section five are some fundamental numerical methods presented, technique for approximating the stochastic differential equation, Monte Carlo techniques, a technique for solving equations numerically and at last a method for estimating parameters. Section six consists

of numerical implementation of the presented model. And in section seven are some empirical studies done, and finally will an European call option receive its arbitrage free price.

2 Stochastic Calculus and Derivatives

2.1 Mathematical Theory

We will throughout the thesis assume that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the sample space, $\mathcal{F}$ the σ-algebra generated by stochastic process $\omega = \{\omega_t : t \in \mathbb{R}\}$ and $\mathbb{P}$ the probability measure, $\mathbb{P} : \mathcal{F} \mapsto [0,1]$. We use the fundamental Brownian motion W_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to represent our important stochastic engine for modeling the randomness in the financial market.

Definition 2.1: (Brownian Motion)

The stochastic process $W = \{W_t : t \in \mathbb{R}\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Brownian motion if the following properties holds a.s.

1. *$W_0 = 0$.*

2. *The increments are independent and stationary, i.e. if $r < s \leq t < u$ then are $W_u - W_t$ and $W_s - W_r$ independent stochastic variables.*

3. *The increments of $W_{t+h} - W_t$ are normally distributed, $N\left(0, \sqrt{h}\right)$*

4. *W_t has continuous trajectories.*

A simple Matlab routine demonstrates the a simple simulation of the Brownian Motion with a step size $\Delta t = 1/100$, the simulation is done by using the built in Matlab function `randn` for representing a $N(0,1)$ stochastic variable:

Algorithm 1 Simulating a Brownian Motion in Matlab

```
k = 100; w = zeros(100,1); w(1) = 0;
for i=1:k-1
    w(i+1) = w(i) + sqrt(1/100)*randn;
end
```

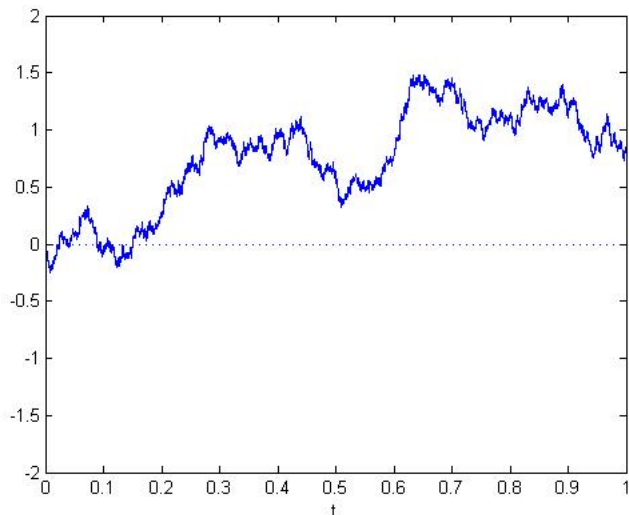The result by running is demonstrated in the following plot:

Figure 1: The Wiener process

Worth mentioning is that the process above is nowhere differentiable, i.e. the derivative of the process does not exist.

In introduction courses of mathematics one gets familiar with the deterministic ordinary differential equation, which consists of a unique solution. But in many cases for instance in financial economics this is not longer possible since these models seems to be random, a noise term is therefore added to the differential equation, hence the name Stochastic different equation denoted SDE. Most of the models used to simulate financial instruments can be described by these non deterministic models. Let $X(t)$ represent a stochastic processes, in our case a differential equation extended with a random part, hence the name stochastic differential equation (SDE) or the 1-dimensional Itô process given in the following definition,

Definition 2.2: (1-dimensional Itô Process)   *Let W_t be a Brownian motion, the Itô process (stochastic process) X_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is then given by*

$$X_t = X_0 + \int_0^t \mu(X_s, s) \, ds + \int_0^t \sigma(X_s, s) \, dW_s \qquad (1)$$

often written on a shorter form

$$dX_t = \mu dt + \sigma dW_s$$

such that the following conditions holds almost surely

$$P\left[\int_0^t \sigma\left(X_s, s\right)^2 ds < \infty \,, \forall t \geq 0\right] \;=\; 1$$

$$P\left[\int_0^t \left|\mu\left(X_s, s\right) ds\right| < \infty \,, \forall t \geq 0\right] \;=\; 1$$

SDE (1) consits of two terms, the first term μdt defined as the drift term, and the second term σdW_t which specifies the random part (the noise) of the process, named the diffusion part. For the existence and uniqueness of the solution of SDE given by (1) we need the following condition on μ and σ to be fulfilled

Theorem 2.3: (Existence and uniqueness) *Conditions that guarantees the existence and the uniqueness of the solution of SDE (1) is the growth condition, let μ and σ satisfying*

$$\left|\mu\left(x, t\right)\right| + \left|\sigma\left(x, t\right)\right| \leq C\left(1 + |x|\right), \qquad x \in \mathbb{R},\ t \in [0, T]$$

for some constant C, which guarantees global existence, and the Lipshitz condition

$$\left|\sigma\left(x, t\right) - \sigma\left(y, t\right)\right| + \left|\mu\left(x, t\right) - \mu\left(y, t\right)\right| \leq D\left|x - y\right|, \qquad x, y \in \mathbb{R},\ t \in [0, T]$$

for some constant D, which guarantees local uniqueness. And where $\mathcal{F}_t$ is the filtration generated by $W = \{W_t : t \in \mathbb{R}\}$, then the SDE

$$dX_t = \mu\left(X_t, t\right) dt + \sigma\left(X_t, t\right) dW_t$$

has a unique t-continuous solution $X\left(t\right)$ given by (1).

Proof. Omitted, See Oksendal (2000).

$\square$

One fundamental result to able to use and solve SDE:s is by applying the Itô formula. The formula is the stochastic analogue to the chain rule in ordinary mathematical analysis. The Itô formula transforms the Brownian motion given a function $Y\left(t\right) = f\left(t, X\left(t\right)\right)$, where $X\left(t\right)$ is defined by equation 1, the dynamics of $Y\left(t\right)$ is then given by applying the second order Taylor expansion.

Theorem 2.4: (Itô formula) *Let X_t be a stochastic process given by SDE (1) and let $g(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R})$[1] Then*

$$Y_t = g(t, X_t)$$

is an Itô process and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\, (dX_t)^2$$

and where the following rules has been used

$\times$	dt	dW_t
dt	0	0
dW_t	0	dt

Proof. Omitted

$\square$

We will throughout this thesis assume that the returns of each underlying asset will follow a log-normal distribution and can thereby be realized by the geometric Brownian motion (GBM) SDE.

Definition 2.5: (Geometric Brownian Motion) *A geometric Brownian motion is defined as*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2}$$

which is a short form of the following equation.

$$S_t = S_0 + \int_0^t \mu S_z dz + \int_0^t \sigma S_z dW_z$$

Let us assume that the daily asset returns follows a log normal distribution and this by introducing

$$Y_t = \ln\left(\frac{S_t}{S_0}\right)$$

By applying Itô's formula we receive the following expression

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[1]i.e. g is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$.

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$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

finding the primitive function

$$Y_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

and finally ending up with

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \tag{3}$$

$\square$

S_t is log normally distributed and there by does the following holds

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$$

The expected value of process (3) is given by

$$E[S_t] = S_0 e^{rt}$$

Let us demonstrate how (3) can be used to simulate a stock, assuming an initial spot price $s_0 = 10$, stock volatility $\sigma = 0.3$, return $\mu = 0.05$ and a year[2]. Stock will after 4 simulations have the following appearance

2.2 Derivatives

Derivatives can be seen as insurance and hedging contracts on the financial market in order to remove and avoid potential downside risk. A derivative derives its value from some underlying asset, hence the name derivative. Today derivatives can be derived by among different number of underlying assets: Stocks, Indexes, Interest rates, Commodities, Electricity etc. As one can see derivatives can be applied to almost any type of asset, and one of the simplest derivatives is the forward contract.

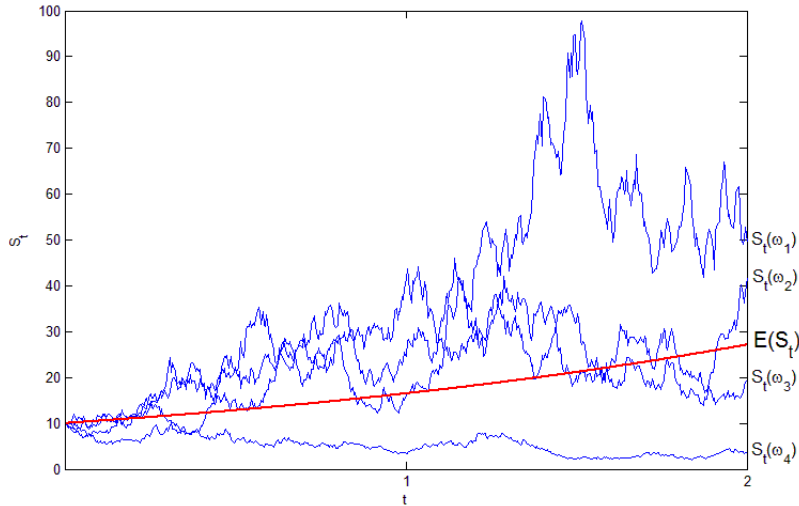[2]A year is assumed to have around 252 trading days

Figure 2: Simulated stock with 4 paths, and the mean $S_0 e^{rT}$.

Definition 2.6: (The Forward Contract) *The holder of a forward contract gives the obligation to buy/sell the underlying asset at some prespecified date for a prespecified price. The payers position on a asset S_t, at a specified time T and with strike price K have the following payoff function*

$$\Phi^{Payer}\left(S_T\right) = S_T - K$$

the contract for the seller position is defined in the analogue way

$$\Phi^{Seller}\left(S_T\right) = K - S_T$$

Just like forwards and futures are options derivatives contracts, but instead be forced to buy/sell the underlying asset at a specified date in the future, the option gives the holder an opportunity to buy/sell the underlying asset. The holder is thereby not forced to do something, and is only left with the positive outcome. The derivative market today is very big and can be build one a huge amount of different assets, in some cases have the derivatives market been dominating in size the market for the underlying asset. In the last 40 years there has been a huge development of the derivative market. (Byström, 2007). One of the reasons was the increased volatility and uncertainty after the OPEC oil crisis in the 1973, where the oil prices increased drastically and made a great impact on the global economy. This created a big demand to be able to insure and hedge not only commodities but all type of assets. The technical development is another aspect of the increased use in derivatives, by using the

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technique it have made it possible quicker price derivatives, trade and settle transactions than in earlier periods, but this area will lead to another crises in the late 80s. The third aspect which is the famous Black-Scholes model, this phenomenal model totally changed the world of pricing derivatives.

Before stating Black-Scholes we state some fundamental options fundamental for further studies regarding the subject of the thesis.

Definition 2.7: (The European Call/Put Option) *The holder of a call option have the option to exercise the option and thereby be able to buy the underlying asset S_t to specified price at the time to maturity T. The put option is defined in the analogue opposite way, where the holder of a put option have the option the sell the underlying asset S at a specified price at T. The payoff $\Phi^{Call}(S_T)$ for the European call option is given by*

$$\Phi^C(S_T) = (S_T - K)^+ = \begin{cases} S_T - K, & S_T \geq K \\ 0, & S_T < K \end{cases}$$

and for the European call option in the analogous way

$$\Phi^{Put}(S_T) = (K - S_T)^+ = \begin{cases} K - S_T, & K \geq S_T \\ 0, & K < S_T \end{cases}$$

The payoff of this both contracts are demonstrated in the following figure
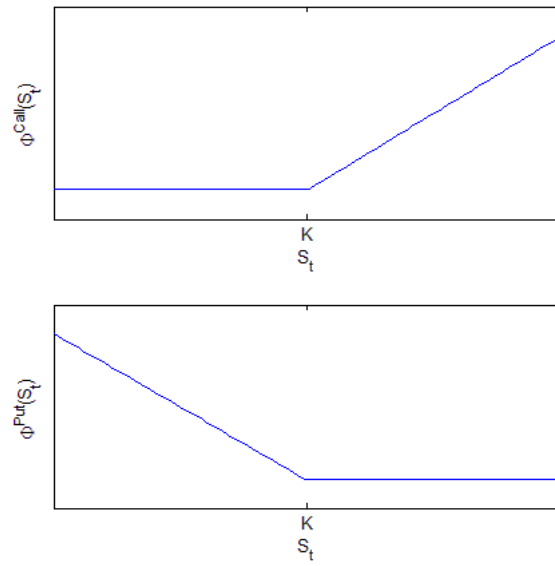
Figure 3: Upper: payoff for a Call option, Lower: payoff for a Put option, Both with $K = 50$.

3   The Black-Scholes Model

3.1   The Arbitrage Free Price

Before we state the Black-Scholes formula we need to introduce a fundamental formula which also will be used during the simulation later on. The formula named the risk neutral valuation formula, RNVF is stated next (Björk 2004: 99). The risk neutral valuation formula states that any asset risky and non risky will all have the same expected return as the risk-free rate of interest r (Hull 2008: 290), this means that in a risk neutral world all assets will all have the same expected return. One can familiar with financial mathematics can observe the connection between the risk neutral valuation formula and the solution proposed by the Feynman-Kač theorem.

Theorem 3.1: (The Risk Neutral Valuation Formula)   *Given the payoff function $\Phi(S_t)$ for a European type option, the arbitrage free price $\Pi(t, \Phi)$ of this claim is given by*

$$\Pi(t, \Phi) = e^{-rT} E^{\mathbb{Q}} \left[\Phi(S_t) \mid \mathcal{F}_t^S \right]$$

where $\mathbb{Q}$ denoted the risk-neutral martingale measure using the money market account (MMA) as a numeraire and $\mathcal{F}_t^S$ the filtration which contains all the information about S until time t.

One fundamental property of a martingale is that the expected value of a random variable X is always constant hence that the following condition holds with respect to the filtration $\mathcal{F}_s^X$ (Rasmusson, 2008:71)

Definition 3.2: (Martingale)   *A stochastic process $\{M_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a martingale w.r.t. the filtration $\{\mathcal{M}_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ if the following properties holds*

1. *M_t is $\mathcal{M}_t$-measurable, or M_t is $\mathcal{M}_t$ adapted[3] for all t,*

2. *$E[|X_t|] < \infty, \quad \forall t$*

3. *$E\left[X_t \mid \mathcal{F}_s^X\right] = X_s, \quad \forall\, 0 \leq s < t < \infty$*

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[3] *This means that the value of M_t is known given the information in $\mathcal{M}_t$.*

A martingale has no systematic drift and the notion above tells us that each asset in an arbitrage free world will be a martingale, thus that each asset discounted future expected value must be equal the present value, i.e. a fair game. In order to price process (2) correctly and thus that it does not cause any arbitrage possibilities we need perform a transformation of the Wiener process. We will only introduce the Girsanov theorem and the transformation of process (2) in the following theorem (Rasmusson 2008: 136). In order to change from the historical measure $\mathbb{P}$ into the risk neutral measure $\mathbb{Q}$ we use the following theorem

Theorem 3.3: (Girsanov theorem)

Assume that $W_t^{\mathbb{P}}$ is a standard $\mathbb{P}$-BM, The relationship between the historical measure $\mathbb{P}$ and the risk neutral measure $\mathbb{Q}$ is defined as

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t g\left(s, W_t^{\mathbb{P}}\right) ds$$

where $g\left(s, W_t^{\mathbb{P}}\right)$ is the unique Girsanov kernel letting a process defined by a $\mathbb{Q}$-measure to be arbitrage free.

The are some fundamental properties that are ignored above. By applying the Girsanov theorem to process (3) one can determine the unique pricing kernel and thus that in a arbitrage free world any type of asset will evolve with the interest free rate r and thus that we the drift will be replace by r

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{4}$$

We will throughout this thesis assume that the price process of the risk free asset, with the interest rate denoted $r \geq 0$ is defined according to the following process

$$dB\left(t\right) = rB\left(t\right) dt$$

3.2   Black-Scholes PDE

The relationship representing the arbitrage free price of the geometric Brownian motion will be derived in this section. This relationship on the form of a partial differential equation will be used later on when the option will be priced by using Black-Scholes formula. To determine the arbitrage free price we will find a replicating portfolio consisting of an option and the stock. One requirement for the portfolio is that it is self financing,

Definition 3.4: *a portfolio h is self-financed if the following conditions holds (Björk 2004: 92):*

$$
\begin{aligned}
V^h\left(0\right) &= 0 \\
P\left(V^h\left(T\right) \geq 0\right) &= 1 \\
P\left(V^h\left(T\right) > 0\right) &> 0
\end{aligned}
$$

Our portfolio will be composed by $\theta_c\left(t\right)$ fractions of call options and $\theta_s\left(t\right)$ fractions of stocks. The value V_t of our replicated portfolio can thereby be determined by

$$
V_t = \theta_c\left(t\right)F\left(t, S\left(t\right)\right) + \theta_s\left(t\right)S\left(t\right)
$$

and thus is the dynamics given as

$$
dV_t = \theta_c\left(t\right)dF\left(t, S\left(t\right)\right) + \theta_s\left(t\right)dS\left(t\right)
$$

Let $\omega_F\left(t\right)$ and $\omega_S\left(t\right)$ denote the relative proportions invested in each asset:

$$
\omega_F\left(t\right) = \frac{\theta_c\left(t\right)F\left(t, S\left(t\right)\right)}{V_t}; \ \omega_S\left(t\right) = \frac{\theta_S\left(t\right)S\left(t\right)}{V_t}
$$

the dynamics of the portfolio value can thus be written on the form

$$
dV_t = V\omega_F\left(t\right)\frac{dF\left(t, S\left(t\right)\right)}{F\left(t, S\left(t\right)\right)} + V\omega_S\left(t\right)\frac{dS\left(t\right)}{S\left(t\right)} \tag{5}
$$

We know that the dynamics of the stock is given by equation 2, the dynamics of the option value is given by $\Pi_t = F\left(t, S\left(t\right)\right)$

$$
dF_t = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}\sigma^2 S^2\right)dt + \frac{\partial F}{\partial S}\sigma S dW
$$

By inserting the option and stock dynamics into equation 4 we have

$$
dV_t = \frac{V\omega_F}{F}\left[\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}\sigma^2 S^2\right)dt + \frac{\partial F}{\partial S}\sigma S dW\right] +
$$

$$
+ \frac{V\omega_S}{S}\left[\mu S dt + \sigma S dW\right] =
$$

$$
= V\left[\frac{\omega_F}{F}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}\sigma^2 S^2\right) + \frac{\omega_S}{S}\mu\right]dt + V\sigma\left[\frac{\omega_F}{F}\frac{\partial F}{\partial S}S + \omega_S\right]dW
$$

The portfolio weights ω_S and ω_F are chosen such that

$$\begin{cases} \frac{\omega_F}{F} \frac{\partial F}{\partial S} S + \omega_S = 0 \\ \omega_S + \omega_F = 1 \end{cases}$$

which makes the diffusion part vanish which make our system look like

$$dV = V \left[\frac{\omega_F}{F} \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 F}{\partial t^2} \sigma^2 S^2 \right) + \frac{\omega_S}{S} \mu \right] dt$$

Knowing the fact that in a risk neutral world will all assets grow with the same rate, the interest free rate. The weights are thus given by

$$\omega_F = \frac{-F}{S \frac{\partial F}{\partial S} - F}$$

$$\omega_S = \frac{S \frac{\partial F}{\partial S}}{S \frac{\partial F}{\partial S} - F}$$

the following deterministic partial differential equations holds for the arbitrage free price of the option

$$\frac{\partial F}{\partial t} (t, S(t)) + r(t, S(t)) \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 (t, S(t)) \frac{\partial^2 F}{\partial S^2} (t, S(t)) - rF(t, S(t)) = 0$$

$$\square$$

The relationship above is more famous under the name Black-Scholes Equation (should not be confused with Black-Scholes formula, which is stated in next section).

The Black-Scholes equation stated above can be solved by using the Feynman-Kač theorem (Björk 2004: 70)

Theorem 3.5: (Feynman-Kač) *The solution the following PDE is given by* F

$$\frac{\partial F}{\partial t} (t, x) + r(t, x) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 (t, x) \frac{\partial^2 F}{\partial x^2} (t, x) - rF(t, x) = 0$$
$$F(T, x) = \Phi(x)$$

where X defined by equation 2, the solution to F is given by

$$F(t, x) = e^{-r(T-t)} E[\Phi(X_T)]$$

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3.3   Black-Scholes formula

By using formula 3 but replace the constant μ with the risk free rate of interest r will make our model risk neutral with a solution that is given by:

$$S_t = s_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

The derivation is done by using the payoff for the European Call option

$$\Phi^C(S_t) = (S_t - K)^+ = (S_t - K) \cdot 1_{\{S_t > K\}}$$

but this will also work by using the analogue European put option payoff. The price for the European put option can also be determined by using the put call parity which will be described later on. By using the RNVF we are receiving the following arbitrage free price.

$$
\begin{aligned}
\Pi_t^C &= e^{-rt}E^Q\left[\Phi(X_T)|\mathcal{F}_t^X\right] \\
&= e^{-rt}E^Q\left[(S_t - K)^+|\mathcal{F}_t^X\right] \\
&= e^{-rt}\underbrace{E^Q\left[S_t \cdot 1_{\{S_t > K\}}|\mathcal{F}_t^X\right]}_{E_1} - e^{-rt}K\underbrace{E^Q\left[1_{\{S_t > K\}}|\mathcal{F}_t^X\right]}_{E_2} \\
&= e^{-rt}E_1 - e^{-rt}KE_2
\end{aligned}
$$

The hard part is to calculate E_1, the calculation is done by assuming normality. Anyone with an introduction course in probability is familiar with the normal distribution $N(0,1)$ (Blom 2005: 143). The definition of the probability density function, pdf $f_X(x)$ is given by

$$f_X(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$$

thus the cumulative density function, cdf $F_X(x)$ is given by finding the primitive function of the pdf, and it is done by integration over the interval $-\infty \leq t \leq x$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2\pi}e^{-\frac{t^2}{2}}\,dt$$

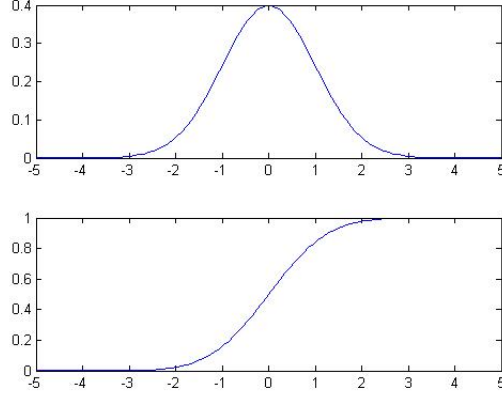Figure 4: Upper: density function f_X, Lower: Cumulative function F_X

$$
\begin{aligned}
E_1 &= E^Q\left[S_t \cdot 1_{\{S_t > K\}} | \mathcal{F}_t^X\right] \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} s_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z\right\} exp\left\{\frac{-z^2}{2}\right\} d \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} s_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z - \frac{z^2}{2}\right\} dz \\
&= \frac{1}{2\pi} s_0 e^{rt} \int_{-\infty}^{\infty} \exp\left\{\left(-\frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z - \frac{z^2}{2}\right\} dz \\
&= s_0 e^{rt} N\left[-z_0 + \sigma\sqrt{t}\right]
\end{aligned}
$$

since we require that the call option must be in-the-money, i.e. that the following condition is fulfilled $S_t > K$, the parameter d is the one that solves this

$$
s_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z_0\right\} = K \Leftrightarrow
$$

$$
z_0 = \frac{ln\frac{K}{s_0} - \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} \Rightarrow
$$

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$$
\begin{aligned}
E_1 &= s_0 e^{rt} N\left[-z_0 + \sigma\sqrt{t}\right] \\
&= s_0 e^{rt} N\left[-\frac{ln\frac{K}{s_0} - \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} + \sigma\sqrt{t}\right] \\
&= s_0 e^{rt} N\left[\frac{ln\frac{s_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right] \\
&= s_0 e^{rt} N\left[d\right]
\end{aligned}
$$

and for E_2

$$
\begin{aligned}
E_2 &= E^Q\left[1_{\{S_t > K\}} | \mathcal{F}_t^X\right] \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} exp\left\{\frac{-z^2}{2}\right\} dz \\
&= N\left[d - \sigma\sqrt{t}\right]
\end{aligned}
$$

$\square$

Theorem 3.6: (Black-Scholes formula for European Call Options)

$$
\Pi_t^C = s_0 N\left[d\right] - e^{-rt} N\left[d - \sigma\sqrt{t}\right]
$$

where d is given by

$$
d = \frac{ln\frac{s_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}
$$

The price of a European call option depends on

1. Time to maturity T

2. The current value of the underlying stock s

3. The strike price K

4. The interest rate r

5. The volatility σ

The following figure demonstrates how the Black-Scholes price changes when strike price K, volatility σ, and interest rate r is kept constant while time to maturity T and the stock price S varies. One can observe that the option price increases as the stock price and time to maturity increases
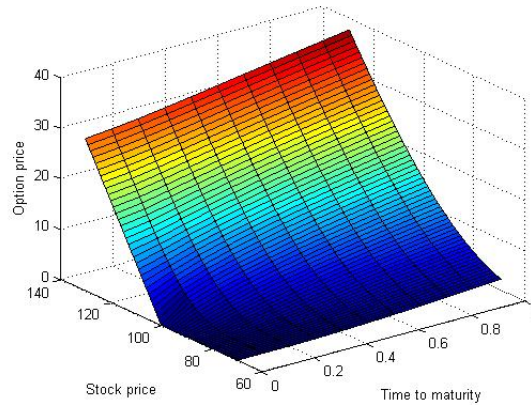

Figure 5: The Black-Scholes price for an European Call option with $K = 100$, $\sigma = 0.3$, $r = 0.05$.

The Black-Scholes formula is built on a few assumptions (Hull 2008: 286), some of them are very crude which limits the model

1. The stock follows geometric Brownian motion

2. Short selling is allowed

3. No market frictions, hence no transactions costs or taxes.

4. No dividend is paid out during the assets holding period.

5. There does not exist any arbitrage possibilities.

6. Constants volatilityσ, and constant interest r.

3.4   The Put-Call Parity

The put-call parity describes the important relationship between European call Π_t^C and put Π_t^P option in a arbitrage free world where both have the same strike price K, and time to maturity T.

$$\Pi_t^P = Ke^{-r(T-t)} + \Pi_t^C - S_t$$

If the relationship does not hold this means that there exists arbitrage opportunities and thus that we can make money from nothing, also known as "free lunches". The relationship above states that an European put option can be replicated by long position in the bank with amount $Ke^{-r(T-t)}$, a long position in a European call option and a short position i the stock.
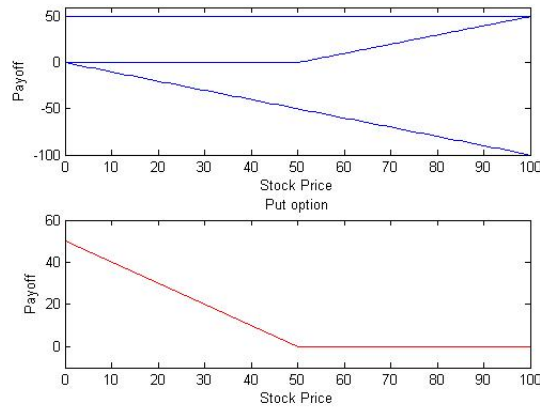

Figure 6: Put-Call parity

If you add the payoff of the assets in the upper plot it will result in the payoff given in the lower plot.

3.5   Volatility Smile

Assumption 6 with constant volatility in the Black-Scholes model is one of the drawbacks resulting in the phenomenon called the volatility smile. To demonstrate the volatility smile the implied volatility is introduced where we calculates the volatility of an option given the option price, stock price, strike price etc. The implied volatility is defined by finding the inverse of the Black-Scholes formula, the following function for the call option

$$\sigma_{imp} = C_{BS}^{-1}\left(\Pi_t, S_0, K, r, T\right)$$

But the problem is thus that there does not directly exists an inverse of the Black-Scholes formula, but it can be solved a numerical procedure. Solving the following relationship

$$C^M - C^{BS}\left(\sigma_{imp}\right) = 0$$

which can be solved by using the Bisection Method deduced in the section containing numerical methods.

The following picture demonstrates the smile effect by using a linear interpolating technique for plotting data collected for the currency mid pair EURUSD[4] quoted at 1.2714 the 10 March 2009. Strikes lower than the ATM strike are strikes of Put options and strikes higher are strikes of Call options, this method is more or less a standard way of describing the volatility in the FX market.
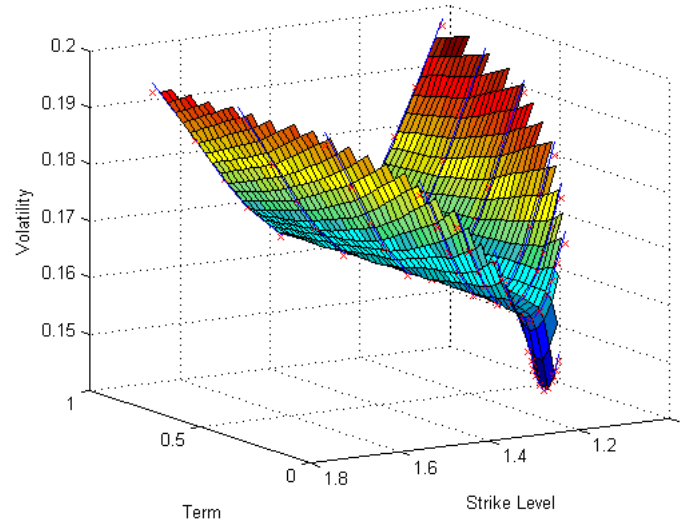

Figure 7: Volatility Smile in the FX market

_____

[4]One pay in USD and receive EUROs

4 The Heston model

4.1 Heston Stochastic Volatility Model

The crude assumption of constant volatility in the Black-Scholes formula causes problem. One model where the volatility is a stochastic process is the Heston Stochastic Volatility Model (Heston 1993: 328) which is an extended version of the Black-Scholes SDE with a volatility that follows a so called CIR-process (Rasmusson 2008: 115). The Heston Model takes the non-log normal distribution of the assets returns and the leverage effect into account, the correlation between the two Wiener processes.

Theorem 4.1: The Heston Model defined by following stochastic processes:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(S)} \tag{6}$$

$$dV_t = \kappa \left(\theta - V_t \right) dt + \sigma_V \sqrt{V_t} dW_t^{(V)} \tag{7}$$

and where $W_t^{(S)}$ and $W_t^{(V)}$ are correlated Wiener processes with ρ, i.e.

$$dW_t^{(S)} dW_t^{(V)} = \rho dt \tag{8}$$

parameters extended from the initial Black-Scholes model:

κ	Mean reversion rate
θ	Long run variance
V_0	Initial variance.
σ_V	Volatility of variance
ρ	Correlation parameter.

Steven L. Heston derived a closed form solution for the price of a European call option on an asset with stochastic volatility. By applying the Itô formula and some standard Black-Scholes arbitrage arguments one receives the Garman's partial differential equation (Heston 1993: 334) stated as:

$$
\begin{aligned}
&\frac{\partial C}{\partial t} + \frac{S^2 V}{2} \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC \\
&+ \left[\kappa \left(\theta - V \right) - \lambda V \right] \frac{\partial C}{\partial V} + \frac{\sigma^2 V}{2} \frac{\partial^2 C}{\partial V^2} + \rho \sigma SV \frac{\partial^2 C}{\partial S \partial V} = 0
\end{aligned} \tag{9}
$$

and λ is the market price of volatility risk.

4.2 The Heston Closed form approximation

The Heston approximation is build on Garman's PDE (9) stated in the previous part. Heston suggested a pricing function analogy to the Black-Scholes formula of the following form:

$$C\left(S_t, V_t, t, T\right) = S_t P_1 - K e^{-r(T-t)} P_2$$

P_1 and P_2 are defined by the inverse Fourier transformation

$$P_j\left(x, V_t, T, K\right) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-i\phi \ln(K)} f_j\left(x, V_t, T, \phi\right)}{i\phi}\right) d\phi, \quad j = 1, 2$$

and where

$$x = \ln\left(S_t\right)$$

$$f_j\left(x, V_t, T, \phi\right) = e^{C(T-t,\phi)+D(T-t,\phi)V_t+i\phi x}$$

$$C\left(T-t, \phi\right) = r\phi ir + \frac{a}{\sigma^2}\left[\left(b_j - \rho\sigma\phi i + d\right)\tau - 2\ln\left(\frac{1 - ge^{dr}}{1 - g}\right)\right]$$

$$D\left(T-t, \phi\right) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left(\frac{1 - e^{dr}}{1 - ge^{dr}}\right)$$

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}$$

$$d = \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}$$

for $j = 1, 2$, and where

$$u_1 = \frac{1}{2}, \quad u_1 = -\frac{1}{2}, \quad a = \kappa\theta \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa + \lambda$$

4.3 The Heston Smile and Skew

The motivation for using the Heston model is that as mentioned above the model creates consistent smile and skews. In this part will we simulate option prices from the suggested approximation and thereafter calculating the Black-Scholes implied volatility and this for different levels of the leverage effect ρ, where $\rho = \begin{bmatrix} -0.9 & 0 & 0.9 \end{bmatrix}$ and studying the outcome.
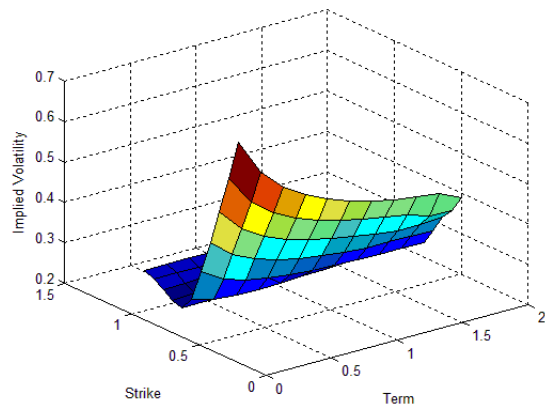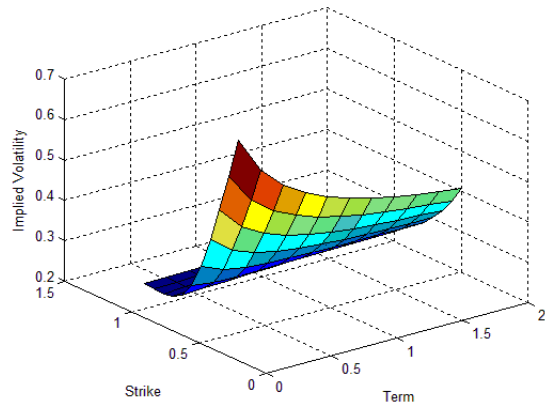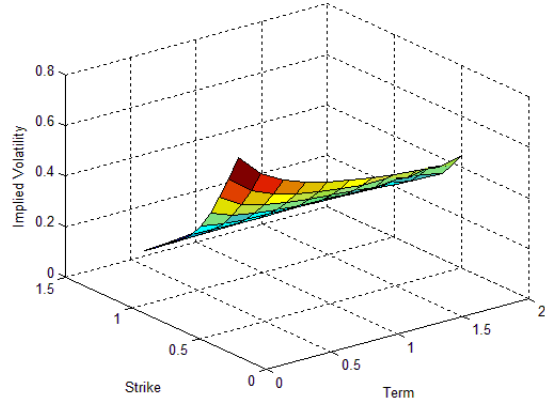
Figure 8: Implied volatility, Upper: $\rho = -0.9$, Middle: $\rho = 0$, Lower: $\rho = 0.9$

5   Numerical Methods

5.1   SDE Approximation

To be able to approximate the SDE's solution the process needs to be discretizised. To be able to do that the process is divided into small grids between an interval $[a, b]$

$$a = t_0 < t_1 < \cdots < t_n = b$$

The solution for the processes above can be approximated by using the Euler-Maruyama Method (Sauer 2005: 460), which is a Taylor approximation. The Euler-Maruyama Method is demonstrated on the following stochastic differential equation:

$$dY = \mu Y \, dt + \sigma Y \, dW_t$$

the Euler-Maruyama Method is defined as:

$$w_0 = y_0$$

$$w_{i+1} = w_i + \mu w_i \left(\Delta t_i \right) + \sigma w_i \left(\Delta W_t \right)$$

and ΔW_t is calculated as:

$$\Delta W_t = Z_i \sqrt{\Delta t_i}$$

and Z_i is a standard Gaussian random variable $N(0,1)$

Applying the Euler-Maruyama Method to the Heston Model above gives the following discrete relationship

$$S_t = S_{t-1} + \mu S_{t-1} dt + \sqrt{V_{t-1}} S_{t-1} \sqrt{dt} Z_t^{(S)}$$

$$V_t = V_{t-1} + \kappa \left(\xi - V_{t-1} \right) dt + \sigma_V \sqrt{V_{t-1} dt} Z_t^{(V)}$$

$$Z_t^{(S)} = G_t^{(S)}$$

$$Z_t^{(V)} = \rho G_t^{(S)} + \sqrt{1 - \rho^2} G_t^{(V)}$$

where $G_t^{(S)}$ and $G_t^{(V)}$ is chosen from $N(0,1)$ and are independent identically distributed.

$\square$

5.2 Monte Carlo

Our goal is to find a solution to the Heston process, by using the technique presented above together with Monte Carlo simulation the solution of the SDE will hopefully converge towards the real value. By simulating the process a large number of times its value will eventually converge towards the real value. Anyone familiar with Monte Carlo simulations knows that it is very time and computer consuming.

The Basic idea of MC is to approximate an integral by taking the average of some sequence of simulated paths. Say for instance that we want to evaluate the following integral

$$I = E\left[\phi\left(x\right)\right] = \int \phi\left(x\right) f\left(x\right) dx$$

where $X \in \mathbb{R}^d$, $\phi : \mathbb{R}^d \to \mathbb{R}$ and where f is the pdf. of X. $I = E\left[\phi\left(x\right)\right]$ can then be approximated in the following way

1. Draw N values $x_1, \ldots, x_N$ i.i.d from f.

2. The integral can then be evaluated as

$$I \approx \frac{1}{N}\sum_{i=1}^{N}\phi\left(x_i\right) \tag{10}$$

$\square$

Monte Carlo simulation is built on two famous theorems: the Law of Large Numbers and the Central Limit Theorem (Sköld 2006:28).

Theorem 5.1: (A Law of Large Numbers)

Assume $X_1, \ldots, X_n$ is a sequence of independent random variables with common means $E\left[X_i\right] = \tau$ and variance $Var\left[X_i\right] = \sigma^2$. If $T_n = \frac{1}{n}\sum_{i=1}^{n}X_i$, and such as the following condition holds almost surely

$$P\left(T_n \to \tau\right) = 1 \quad \text{as } n \to \infty$$

This means that our approximation will converge towards the real value as number of simulations tends to infinity. More precise information on the Monte-Carlo error $(T_n - \tau)$ is given by the Central Limit Theorem (CLT):

Theorem 5.2: (Central Limit Theorem)

Assume $X_1, \ldots, X_n$ is a sequence of i.i.d. random variables with common means $E[X_i] = \tau$ and variance $Var[X_i] = \sigma^2$. If $T_n = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$, we have:

$$P\left(\frac{\sqrt{n}\,(T_n - \tau)}{\sigma} \leq x\right) \to \Phi(x) \qquad as \ n \to \infty$$

where $\Phi(x)$ is the distribution function of the $N(0,1)$ distribution.

Slightly less formally, the CLT tells us that the difference $(T_n - \tau)$ has, at least for large n, approximately an $N\left(0, \frac{\sigma^2}{n}\right)$ distribution. With this information we can approximate probabilities like $P\left(|T_n - \tau| < \epsilon\right)$, and perhaps more importantly find ϵ such that $P\left(|T_n - \tau| < \epsilon\right) = 1 - \alpha$ for some specified confidence level α, and we have that the MC approximation converges with a rate of $\mathcal{O}\left(n^{-1/2}\right)$.

5.2.1 Antithetic Variates

There exists a couple of Monte Carlo simulation techniques, we will extend the crude MC technique by simulation using the variance reduction technique Antithetic Variates by introducing a negative dependence between each replication. The Antithetic Variates is defined in the following way (Rasmus 2008:160)

1. Sample n replicates of $z_i \in N(0,1)$

2. Set $s_i = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z_i\right\}$

3. Set $c_i = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}z_i\right\}$

4. The Antithetic Variate estimator is

$$\hat{\pi}_{av} = \frac{\sum\limits_{i=1}^{n}\left(\Phi(s_i) + \Phi(c_i)\right)}{2n}$$

The main idea with Antithetic Variates is that the outcome calculated by the first path will be balanced by the value calculated from the second path, or the Antithetic path, and thus that the variance is reduced. Let have a look why this work. Assume a random variable X and its antithetic variable $\tilde{X}$, the variance can be written as

$$
\begin{aligned}
Var\left[\frac{X + \tilde{X}}{2}\right] &= \frac{Var[X]}{4} + \frac{Var[\tilde{X}]}{4} + \frac{2\mathrm{Cov}[X, \tilde{X}]}{4} \\
&= \frac{Var[X]}{2}\left(1 + \mathrm{Corr}[X, \tilde{X}]\right) \\
&\leq Var[X]
\end{aligned}
$$

if $\text{Corr}\left[X, \tilde{X}\right] < 0$ the following relationship holds instead.

$$\text{Var}\left[\frac{X + \tilde{X}}{2}\right] < \frac{\text{Var}\left[X\right]}{2}$$

□

we know that X and $\tilde{X}$ have the same variance, in order to reduce the variance we need that the covariance between the both variables are negative $\text{Cov}\left[X, \tilde{X}\right] < 0$ and that is why we try to produce negative correlated pairs. As this technique can reduce the variance it can also increase it.

5.3   Bisection Method

To be able to solve equations numerically we will later on use the Bisection Method. The Bisection Method is an iterative method for finding root in some interval and is here described in the following pseudo code.

Algorithm 2 Bisection Method

Given an initial interval $[a, b]$ and a tolerance level TOL

```
while (b-a)/2 > TOL

    c = (a + b)/2
    if f(c) == 0 stop, end
    else if f(a)f(c) < 0
        b = c
    else
        a = c
    end

end
```

The final interval $[a, b]$ will contain the root and the approximate root is given by $(a + b)/2$

5.4   Estimating the parameters

Maximum-likelihood is a common technique for estimating unknown parameters from a specific distribution. Let $X_1, \ldots, X_n$ be a sequence of random variables from a distribution with unknown parameter θ from some parameter space Θ, often denoted as $\theta \subseteq \Theta$. We often assume that a sequence as the one given above is independent and that all random variables are from the same distribution.

Definition 5.3: (*Maximum Likelihood*)

$$L\left(\theta\right) = f_{X_1, X_2, \ldots, X_n}\left(x_1, x_2, \ldots, x_n; \theta\right)$$

since the sequence of random variables is assumed to be independent the likelihood function $L\left(\theta\right)$ above can be simplified by

$$L\left(\theta\right) = f_{X_1}\left(x_1; \theta\right) f_{X_2}\left(x_2; \theta\right) \cdots f_{X_n}\left(x_n; \theta\right) = \prod_{i=1}^{n} f_{X_i}\left(x_i; \theta\right)$$

It is often more convenient to maximize $\ln L\left(\theta\right)$ since they reach maximum in the same point, and the fact that by using the law of logarithmic function we receive a easier function to work with

$$l\left(\theta\right) = \ln L\left(\theta\right) = \ln\left(\prod_{i=1}^{n} f_{X_i}\left(x_i; \theta\right)\right) = \sum_{i=1}^{n} f_{X_i}\left(x_i; \theta\right)$$

The value θ^* which the function $L\left(\theta\right)$ receive the largest value in the parameter Θ is called the ML estimate of θ. (Blom (2005): page 255)

An extended version of the Maximum Likelihood is called the Simulated Maximum Likelihood, SMLE (Lindström 2008), which is a technique for estimating the parameters in stochastic processes.

Since the SMLE is built on the standard Maximum likelihood where the main goal is to find the following likelihood function:

$$L\left(\theta\right) = \prod_{n=1}^{N} p\left(y_i | y_{i-1}, \theta\right)$$

The model is not available in closed form and to be able to solve the likelihood function we need to approximation it which is done by the following equation:

$$L\left(\theta\right) = \prod_{n=1}^{N} p\left(y_i | y_{i+1}, \theta\right) \approx \prod_{n=1}^{N}\left(\frac{1}{K}\sum_{k=1}^{K} \phi\left(y_{n+1}, \mu_{n,k|\theta}, \sigma_{n,k|\theta}\right)\right)$$

where $\phi\left(y, \mu, \sigma\right)$ is the Normal distributed density function, the log likelihood is then given by:

$$l\left(\theta\right) = \log L\left(\theta\right) \approx \sum_{n=1}^{N}\left(\log\left(\frac{1}{K}\sum_{k=1}^{K} \phi\left(y_{n+1}, \mu_{n,k|\theta}, \sigma_{n,k|\theta}\right)\right)\right)$$

$\square$

6   Numerical Implementation

There are mainly three different techniques for approximating the prices stated by the Heston Stochastic Volatility model

- Numerical approximation of the stated pricing PDE.

- Monte Carlo methods, MCM.

- Different Fourier transformation techniques.

This thesis will use the Antithetic Variates Monte-Carlo method stated above for approximating the price of the contingent claim. The reason for it is that MCM is a more general method and very simple to use. The main drawback is that it is very computer intensive for higher price precisions. As stated above the Heston Stochastic Volatility model is discretizised by using the Euler-Maruyama Method, and are thereafter simulated by using MCM and compared with the closed form solution. The following values are used as initial parameter values:

r	0.05
κ	10
θ	0.16
V_0	0.16
σ_V	0.10
ρ	-0.80

Table 1: Parameters used when simulating the Heston model, approximate the Heston and calculating the BS value

The standard European call option price is calculated by Monte Carlo simulation, the Heston approximation and Black-Scholes formula. The price is calculated for different time to maturity T, moneyness MN defined in our case as the ratio between the strike price and the spot price of the underlying asset. The analogous European put option can be derived by applying the put-call-parity.

T	MN	MC	HA	(Error)	BS	(Error)
0.5	0.8	11.8364	12.014	(0.178)	11.987	(0.151)
	1.0	5.868	5.954	(0.086)	5.961	(0.093)
	1.2	2.533	2.572	(0.039)	2.615	(0.082)
1.0	0.8	13.998	13.989	(0.009)	13.968	(0.030)
	1.0	8.566	8.560	(0.006)	8.569	(0.003)
	1.2	5.028	5.027	(0.001)	5.069	(0.041)
1.5	0.8	15.892	15.647	(0.245)	15.631	(0.261)
	1.0	10.751	10.590	(0.161)	10.601	(0.150)
	1.2	7.175	7.064	(0.111)	7.104	(0.071)
2.0	0.8	17.615	17.095	(0.520)	17.083	(0.532)
	1.0	12.696	12.312	(0.384)	12.325	(0.371)
	1.2	9.109	8.836	(0.273)	8.875	(0.234)

Table 2: The Heston Approximation is compared to the Heston stochastic process which is simulated with a crude Monte Carlo technique with 10^7 trajectories. These values are compared to the Black-Scholes price. This is done for different levels of time to maturity and moneyness, moneyness defined as the strike price divided by the spot price of the underlying asset $K/S(0)$.

As one can observe from the table above is that the Heston approximation price European calls with time to maturity equally to one really close to the true value, and outperformance the Black-Scholes formula. The Heston loses the accuracy as the time to maturity increases, but this is the same for Black-Scholes. The Heston model faces biggest problem of pricing options with moneyness below 1. The overall view is that the Heston model beats the Black-Scholes since the assumption of log normal returns is ignored.
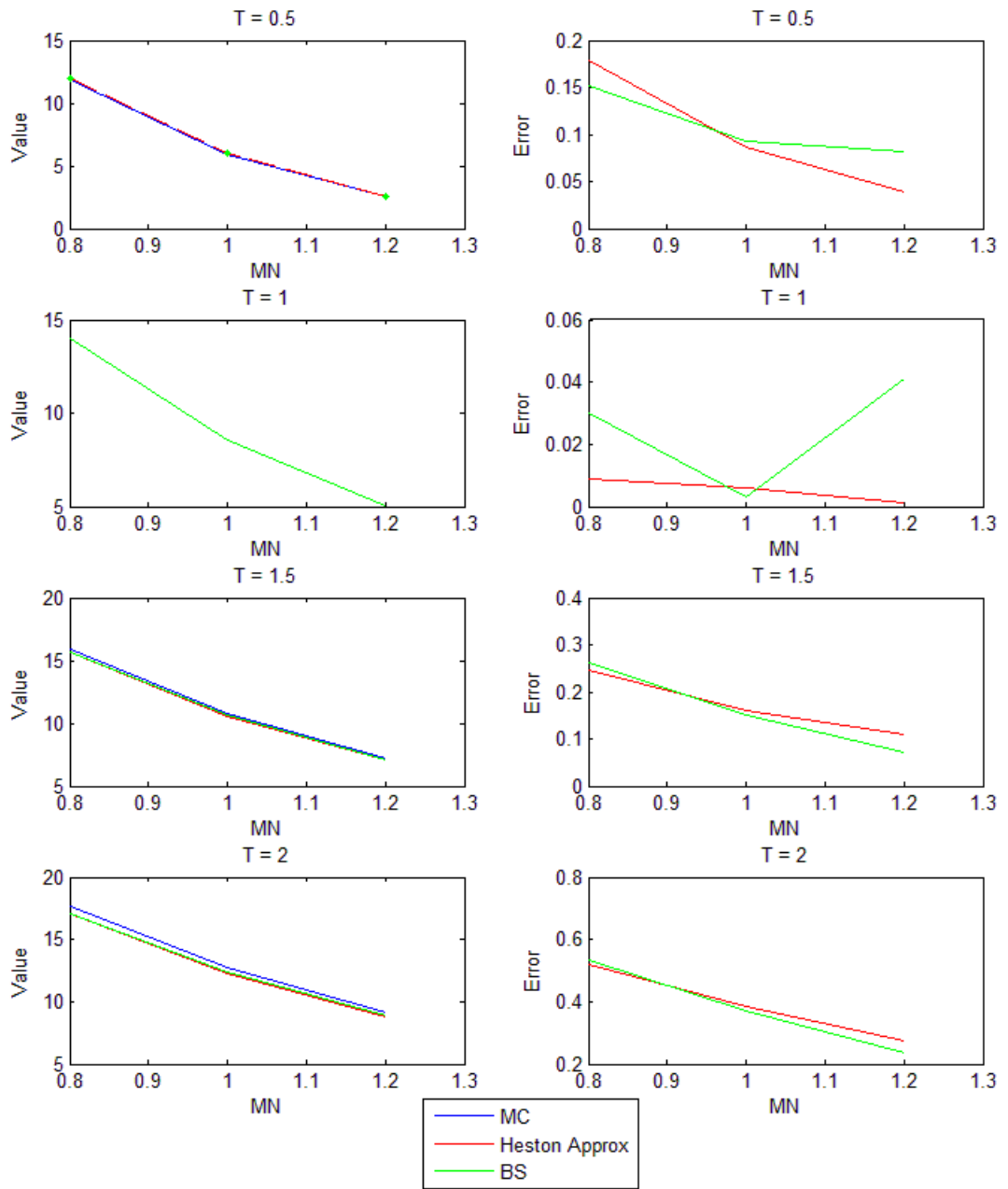
Figure 9: Results, comparing values derived with MC to those calculated by the Heston approximation and Black-Scholes formula.

7 Empirical Studies

The empirical studies are done by calibrating, in other words estimate the parameters for the Heston model and for the Black-Scholes formula. Thereafter are an European Call option priced by these two models. The used data are computer simulated. To be able to work with this data the prices the data needs to converted to asset returns and volatility. The problem for us is that the Black-Scholes formula assumes constant variance. So from our given data a constant variance is then calculated, to do so we use historical data of a period which has the same length as our time to maturity (Franke et. al 2008: 91) and hopefully resulting in an accurate value for the volatility. We are first calculating the returns for historical stock prices with the same horizon as our time to maturity:

$$R_t = \log S_t - \log S_{t-1}$$

The variance of the returns is then calculated by (Franke et. al 2008: 90):

$$\hat{v} = \frac{1}{n-1} \sum_{t=1}^{n} \left(R_t - \bar{R}_n \right)^2$$

where $\bar{R}_n = \sum_{t=1}^{n} R_t$, and an estimator for the volatility σ is then

$$\hat{\sigma} = \sqrt{\frac{\hat{v}}{\Delta t}}$$

A year is assumed to have 252 trading days resulting in $\Delta t = \frac{1}{252}$. Our volatility is then calculated to 0.0011. Inserting all the given data, $r = 0.03$, $S_0 = 120.6870$, $K = 108.1629$, $T = 0.667$ and $\sigma = 0.0011$ the price for the call options is then:

$$C_{BS} = 14.6668$$

The parameter estimation of the Heston model is done in the following way, we are first picking (done once) a large number of random number, referred as Common Random Numbers with goal to avoid Monte Carlo error and this by using the same sequence of random number each time we evaluate the likelihood function. We are thereafter estimating the parameters for the volatility process including the correlation term ρ. Then together with the estimated parameters for the volatility process the parameters for stock dynamics is estimated. The simulation gives us the following Heston parameters:

$$\mu = 0.0336,\ \kappa = 9.9636,\ \theta = 0.0406,\ \sigma_V = 0.1372,\ \rho = -0.9746$$

The correlations term ρ is negative and close to -1, perfect negative correlation. And why our estimated ρ is because of the leverage effect, the market is much more affected by "bad" news compared to "good" news. To see if the parameters makes any sense we are using them in the Heston Model with real volatility and stock values resulting in the following figure:
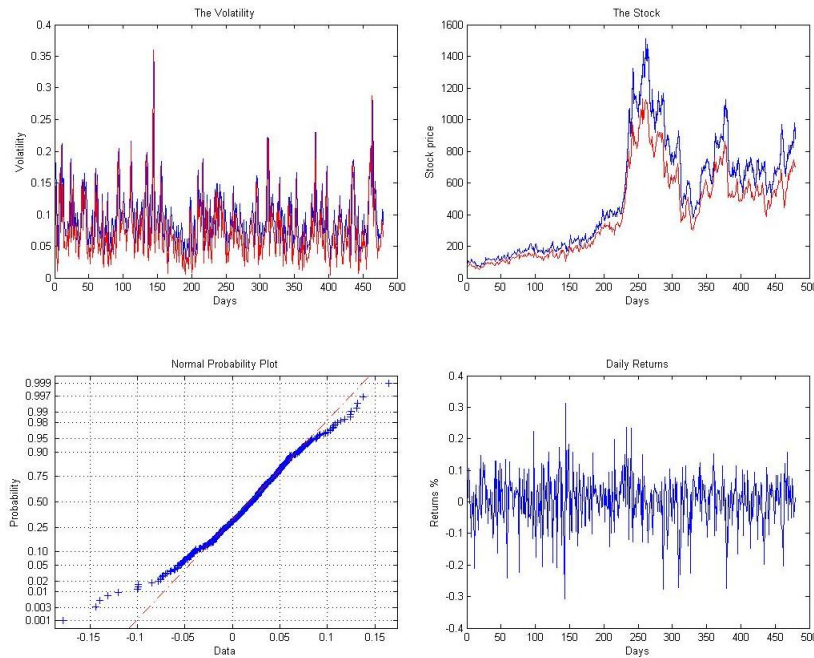

Figure 10: Upper left: The market volatility compared with simulated Heston volatility, upper right: the stock price compared with the simulated stock price. Lower left: The distribution of the returns compared with the normal distribution. Lower right: the daily returns.

The volatility from the Heston Model almost fit the real volatility; the simulated stock values are fairly the same as the real values but are over estimated at the peaks and this as a result from the Monte Carlo simulation. The third plot shows that the residuals is not normal distributed but have excess kurtosis and follows a student-t or a general error distribution. The forth plot is just the log-returns of the stock and we can see that the stock is more volatile where the returns are bigger (plus and minus).

With the estimated parameters we are now able to price the call option using the Risk-Neutral Valuation Formula and Monte Carlo simulation. Using

the time to maturity as the step size and simulate a large number of stock price trajectories with Heston model the generated stock price is then used in the pricing formula. There is improvement compared to Black-Scholes with the Heston model as the volatility is not assumed to be constant and more parameters to be calibrated better fit the option prices.

The crude Monte Carlo gives us the following value for the call option price

$$C_{Heston} = 14.7274$$

8   Epilouge

8.1   Conclusion

The study made in this thesis demonstrated a technique for constructing smile and skew consistent prices by violating one of the crude assumptions in the Black-Scholes model, constant volatility. The result shows that the Heston approximation works really well and only face big problems when options with high time to maturity are be priced. Another problem is that the approximation gives us incorrect prices when the moneyness is below one. To reduce this problem further studies of the volatility smile could be done and were the skew of options that are not in the money could be compare to options that are in the money and trying to repair this. As one could observe from the results above is that the Heston approximation loses its accuracy as the time to maturity increases, but Black-Scholes is also facing the same type of problem. Since the Heston model not is build on the assumption on non constant volatility showed an improvement of modeling stocks and receiving smile consistent option prices. Taking the leverage effect into account is another advantage why the model is and enhancement compared to Black-Scholes. The cons are thus that the integral (10) might not always converge.

8.2   Future Work

Further improvements in further studies could been done by introducing better variance reduction techniques for the Monte Carlo simulation resulting in even better option prices.

Introducing Local volatility models as introduced by Dupire (1994), a deterministic technique for determining the volatility σ_L from implied volatility σ_I as function of both term structure and current asset price $\sigma_L(S,t)$ is a technique that could be combined with these stochastic volatility models. Introducing Multi-Scale volatility processes since the market in some sense shows an effect of more than one volatility, e.g. volatility of volatility and thus being able to capture the market features even better. Additional improvement could be achieved by introducing so called jumps that are represented by Poisson processes.

"...only models that take into account local, jump and stochastic features of the volatility dynamics and mix them in the right proportion are adequate for pricing and risk management of forex options".

Lipton (2002)

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