Lund University

Bachelor's Thesis

An Introduction to Fair and Non-Manipulable Allocations of Indivisible Objects

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Abstract

This paper analyzes a way of allocating primarily three indivisible objects to the same number of individuals. We define an allocation rule that, given the preferences of the individuals, distributes an amount of money together with exactly one indivisible object to each of the individuals in a fair and optimal way. The monetary distributions are foremost interpreted as compensations and are regulated by an exogenously given upper limit. We examine some of the rule's properties, with the most important one being that the rule is coalitionally strategy-proof.

Keywords: indivisible objects, fairness, coalitionally strategy-proofness, auctions

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Contents

1 Introduction

Before presenting and discussing the topic of this paper, we will point out some problems and ideas in the field of Microeconomics upon which we will base the text. In an attempt to link the ideas to reality, we will combine them with a couple of examples.

In general Microeconomics little time is devoted to the problem of *indivisibility*, i.e. that all objects can not always be split or divided arbitrarily. We often get solutions of the form "maximize profits by producing $\frac{22}{7}$ units of output". However, in some cases we deal with indivisible objects - say cars, houses or positions in a company. We begin with an example of the mentioned problem.

Example King Solomon and the baby

> King Solomon is approached by two quarreling women - both claiming to be the mother of a baby boy. The king soon understands that he cannot decide who actually is telling the truth and calls on a guard: "I see no other fair solution than each of you being given half of the baby. Split the boy in two!". However, upon hearing this, one of the women cries out "Please, My Lord, give her the child - do not kill him!". The other potential mother agrees, stating "Yes, give him to me". But the king deduces that the first woman cares so much more for the baby, and instantly gives her the baby boy.

One branch of Microeconomics is called Game Theory. In short, it is a way of looking at "games" - situations where multiple individuals interact and their part of the outcome is not based solely on their own actions, but also on the other individuals' - and try to find solution concepts, which then can be used to anticipate what would happen once the game was played.

But this idea can also be reversed! In Mechanism Design you instead start out with an outcome you wish to achieve and then try to create a game and a mechanism, that, based on some game theoretic solution concept, will give you the desired outcome once the game is played.

Example Splitting the last of the Cola

> The two brothers are yet again fighting for the last drops of Cola. "I should have it! I'm so much bigger than he is and I need the extra energy!" screams the older of the two. "That's not fair! I need it to grow tall and strong", the younger replies. The mother, tired of the screaming, considers pouring it up herself, but is stopped by her husband, who tells the older brother to do it - and points out to the younger one that "When he is done, you get to decide which glass you want to have". The result? The older brother tries, with surgical precision, to fill the drinking glasses evenly.

The idea of Mechanism Design is examplified above. The big brother understands, that, if he fills the glasses unevenly, the little brother will take the drinking glass containing more Cola. Taking this into account, all other strategies for filling up the glasses are dominated (make him worse off) by the strategy corresponding to filling the glasses evenly.

But the example also illustrates another aspect frequently overlooked in Microeconomics - *fairness*. Sometimes achieving a fair outcome, in the sense that no one would be happier by switching his part of the outcome with someone else's, can be out of importance. Next is another example of this.

Example The sons of Abraham

> Abraham, on his deathbed, decides to draw up a will. His two sons, Isaac and Ishmael, are to inherit his two estates and giant pile of gold. However, one of the estates is substantially bigger than the other one. Abraham wants to distribute his wealth in a fair way - none of the sons is to be displeased in the sense that he would rather have what his brother got. Thus, Abraham has to compensate the son getting the smaller estate with a larger sum of gold. Furthermore, Abraham, as a loving father, wants to give away as much of the gold as possible to his sons. How is he to construct the will?

Another result that can be achieved by Mechanism Design is making people "tell the truth" - i.e. reveal some private information only known to them. This, for instance, might be their valuation of a painting.

Example The hunt for art

> Charles and Alex are walking down a central Stockholm street. Alex, who recently moved into a new apartment, needs a painting for the living room wall. They stumble upon a building where an auction is being held, and enter hoping of finding just what they are looking for.
>
> The auctioneer rambles on for a couple of minutes, until finally a beautiful painting is presented. "This painting stands out in oh so many ways, and it has been decided that it will be auctioned through something called a Vickrey auction. All of you who are interested are going to make sealed bids and the winner will be the one submitting the highest bid. But this type of auction has a twist - the winner will only have to pay the second highest bid!". Alex, trying to grasp what really is supposed to happen here, gives Charles a confused look. Charles immediately comes to the rescue though, and asks Alex what the painting is worth to him. "No more than sixty bucks I'd say". "Good, then sixty is your bid", Charles replies.
>
> Now, how can Charles be so sure? Alex basically has three options: (i) bidding his true valuation, (ii) bidding less than the valuation or (iii) bidding more than the valuation.
>
> First, consider Alex bidding less than 60, say 45. Now, if the highest other bid, call it b_m, is 55, he would have been better off bidding 60, which would have won him the object and "saved" him 5. In all other cases, that is b_m less than 45 or b_m above 60, he would achieve

the same result by bidding his true valuation. Thus, regardless of b_m, he is as least as well off by bidding 60 compared to bidding something less, making a bid of 60 dominate bids below 60.

On the other hand, say he bids 75, and b_m lands at 65. He will win the object but has to pay five more than what he actually thinks the painting is worth. Thus he would rather not have won the auction - which he wouldn't, had he bid 60. In all other cases, $b_m > 75$ and $b_m < 60$, he is indifferent between bidding 60 and a higher bid. Again, 60 becomes a strategy dominating, this time, bids above 60.

Combining the two above we see that bidding 60 is as least as good as something more or less, and in some cases actually better. Of course there is nothing special with the number 60; a more general approach would be a valuation of v, but basically the same observations would still have been made. The conclusion is that the Vickrey auction, named after and invented by Vickrey (1961), achieves to make (clever and rational) people tell their true feelings, since anything else is dominated.

1.1 Problems to engage

As has been proposed in the above examples, the usual Microeconomic models are not always fully applicable when considering indivisible objects. Furthermore, they mostly neither aim at reaching fair outcomes nor prevent people from manipulating the result by not telling the truth.

1.2 Purpose

We device an allocation rule that, given preferences reported by the participating individuals, achieves fair and optimal allocations of the indivisible objects together with some money. Furthermore, the allocation rule is proven to be coalitionally strategy-proof, in the sense that no group of individuals can manipulate the mechanism to their benefit.

1.3 Limitations

The model is limited especially in the areas of (i) the number of individuals and objects, (ii) the ratio between the number of individuals and objects and (iii) the utility functions used. Important to note is also that we look at *one* way of dealing with the problem, but this is by far not the *only* way. The rule could for instance be set up in a different way or we could refrain from using money as part of the solution entirely.

1.4 Outline

In section 2 a background to foremost set theory but also preferences is provided. In section 3 the model is discussed together with definitions of the key concepts. In the fourth section some of the results are both presented and proven. This continues on into the fifth section, where the proof of coalitionally strategy-proofness is presented. Section 6 describes how the model can be applied to situations like Ebay auctions, whilst related literature is discussed in the seventh section. Lastly, the text is summarized in section 8.

2 Background

This section is meant to provide a background including some useful mathematical methods and notations. It is recommended for everyone, even though it might be repetition for the more mathematically inclined. If not everything is crystal clear after you have read it through once, you can always use it as some kind of a glossary later on.

2.1 Set theory

A set is a collection of items. These items can be of any type, though most often words or numbers, and are called the set's elements. They are generally listed inside curly brackets. An example of a set is the menu at a pizzeria. If we denote the menu M we could, for instance, have

$$M = \{\text{Vesuvio, Capricciosa, Hawaii, Quattro Stagioni}\}.$$

This is an example of a finite set, but there also exist sets with an infinite number of elements, like the set of all integers.

Count The number of elements in a finite set S will henceforth be referred to as $\#S$. In the above case we have $\#M = 4$.

Equality The menu would perhaps not look the same, but the customers' set of choices would not have changed if we would rearrange the menu alphabetically. When comparing sets, we do not consider the ordering of the elements - two sets that contain the same elements (not necessary in the same order) are said to be equal.

Notation We will introduce a couple of elementary but important set theory symbols. We assume a set $S = \{3, 4, 5\}$ which will be used in the examples. A symbol can often be negated by adding a / to it, just like $=$ is read "equal to" whilst $\neq$ is read "not equal to".

First, we have $\in$, read "[is an element] in". We know for instance that $3 \in S$ but that $7 \notin S$. The colon should be read "such that", whilst $\exists$ is read "there exists". In our case

$$\exists\, x \in S : x > 4$$

is true, since there is an element (5) in S that is greater than 4.

Venn diagrams A graphical way of describing sets is using Venn diagrams. The diagrams' main purpose is to describe how different sets relate to each other. First we define some kind of a "universe" - the biggest possible set containing all relevant elements. Say we want to view our above menu in a diagram whilst considering all types of food. We let the universe be food and denote it X.

We start our diagram with a box, representing the set X (Fig. 1). Next, we mark the set of pizzas, P, in our just drawn box (Fig. 2). The shape of the figure representing the set is totally irrelevant. However, it is important that there be no elements in P that are not in X (otherwise there would be pizzas that are not regarded as food). When marking our little menu, we know that it only contains pizza, but by far not all types of pizzas. Thus M must be inside P (Fig. 3).

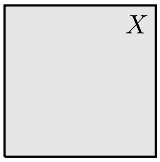


Fig. 1

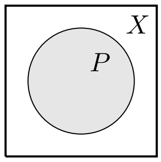


Fig. 2

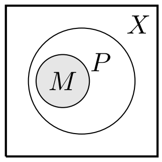
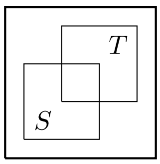


Fig. 3

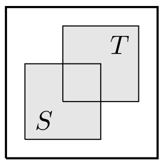
Subsets The fact that all elements in M are in P (and also in X) makes M a subset of P. We denote this $M \subseteq P$ and it is read "M is a subset of P". If the two compared sets are not the same, which is the case here, we have a proper subset, $M \subset P$. Furthermore, if there are elements in a set S that are not in T and vice versa, we have $S \not\subset T$.

Union The union of two sets S and T, $S \cup T$, contains all elements that are in *at least one* of S and T.

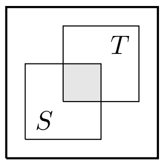
Intersection The intersection of two sets S and T, $S \cap T$, contains all elements that are in *both* S and T.



$S \not\subset T$



Union

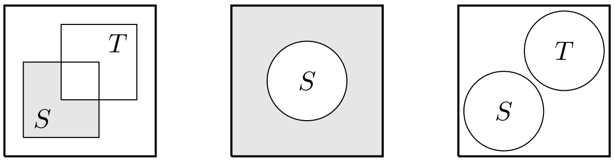


Intersection

Subtraction Subtracting a set T from a set S, $S - T$, creates a new set containing all elements that are in S but not in T.

Complement The complement of a set S, S^C, contains all elements in the regarded universe that are not in S.

Empty set A special set is the empty set, symbolized $\emptyset$, which contains no elements. Thus, if $\#S = 0$ then $S = \emptyset$. How can an empty set be out of any relevance? Consider the intersection of the sets of hamburgers and pizza - this is empty and shows that no pizzas are hamburgers and vice versa! Sets that do not share any elements are said to be disjoint.



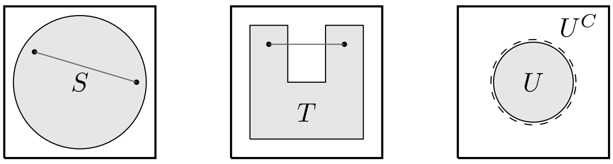
Subtraction Complement Disjoint sets

Convex sets A set S is said to be convex if we can take any two points in S and connect them with a line such that all points along the line also are in S. If this can not be done, the set is said to be non-convex. Mathematically, given elements x and y in S and $0 \leq \lambda \leq 1$,

$$\lambda x + (1 - \lambda) y \in S \text{ if } S \text{ is convex.}$$

Open sets A set S is said to be open if we, at a point in S, can move a small distance in any direction and still be within the set. Consider the set $S = \{x \in \mathbb{R} : x > 0\}$, i.e. the set of positive real numbers. We know that $0.01 \in S$ and we can move in any direction (to 0.001 or 0.1) and still be within S. The fact that, regardless of what value in S we pick, we always are able to find both a larger and a smaller value still in S, makes S an open set.

Closed sets If the complement of a set U is open, then U is said to be closed. In the Venn diagrams we mark closed set with solid lines, whilst using dashed ones for open sets.



Convex set Non-convex set Closed set

2.2 Vectors

A vector is, just like a set, a collection of items. The main difference is that the ordering of the items is of importance when it comes to vectors. Say we have a vector of burgers, $B =$ (Small, Medium, Large), and another vector for the burgers' prices, $P = (20, 30, 35)$, such that P_i, the i-th element in P, is the price of B_i. If we would shuffle the P-vector we would have a quite different situation, since it is connected to B.

Addition Assume a vector $x = (x_1, x_2, x_3)$. If we construct another vector, say $x' = x + (a, b, c)$, we have $x' = (x_1 + a, x_2 + b, x_3 + c)$. If we would add the same value to every component of the vector, we can use a shorter form:

$$x' = x + (t, t, t) = x + t = (x_1 + t, x_2 + t, x_3 + t).$$

Comparing vectors We will use three symbols when comparing vectors in this text. Given vectors x and $\bar{x}$, we have $x = \bar{x}$ if every element x_i in x is equal to $\bar{x}_i$. For $x < \bar{x}$ to hold, every element x_i in x must be less than $\bar{x}_i$. Lastly, $x \leq \bar{x}$ holds if there is some $x_i = \bar{x}_i$ and some $x_j < \bar{x}_j$. For $\bar{x} = (6, 8, 9)$,

$$\text{if } x = (6, 8, 9) \text{ then } x = \bar{x}$$
$$\text{if } x = (4, 7, 8) \text{ then } x < \bar{x}$$
$$\text{if } x = (6, 4, 8) \text{ then } x \leq \bar{x}.$$

2.3 Preferences and utility

For non-microeconomists the concept of utility, and in particular the idea of measuring a person's well-being with a mathematical function, might seem quite fictional. One way of getting around the confusion is by looking at the utility function as an ordinal function. That is, its only purpose is to represent preferences in the sense that, if I enjoy Cola more than Fanta, then my utility of Cola should be higher than that of Fanta.

However, some observations become quite meaningless when viewing the utility function in this way. For instance, $u(A) = 10$ and $u(B) = 5$ does not mean that I am twice as happy with A as with B - only that I am *happier* with A than with B. The fact that $u(A) = 10$ is totally irrelevant when doing anything else than comparing it to another object. The only objective for an ordinal utility function is to represent the person's preferences by ordering and ranking objects after his liking. The functions are not commonly known, they are private information - if I prefer listening to Jimi Hendrix or Westlife is not known to anyone other than myself, until I, in some way, reveal it.

3 Model and definitions

3.1 Basics

The model consists of three individuals gathered in the set $N = \{1, 2, 3\}$. These persons are all to be assigned both an object from the set $M = \{1, 2, 3\}$ and a sum of money. The objects in M are indivisible, whilst the money can be split in any way thinkable.

Consumption bundles A combination (j, α), consisting of both an object j and an amount $\alpha \in \mathbb{R}$ of money, is called a consumption bundle.

Allocations If we list all individuals' consumption bundles we get an allocation (a, x). It consists of two parts: (i) the assignment $a = (a_1, a_2, a_3)$ describing who gets what object in such a way that a_2 is the object individual 2 gets, and (ii) the distribution $x = (x_1, x_2, x_3)$, connected to the monetary compensations. Here, however, x_3 is the compensation that is bundled with *object* 3, thus is given the person that receives object 3. At an allocation (a, x) object j and monetary compensation x_j is given individual i if $a_i = j$.

Compensation limits We introduce maximum limits on each compensation in the distribution, using the vector $\bar{x} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$. This means that we can not compensate object j with more than $\bar{x}_j$, i.e. $x_j \leq \bar{x}_j$.

3.2 Preferences

We assume that the individuals have (quasilinear) preferences over the consumption bundles, known only to themselves, represented by a utility function $u_i(j, \alpha) = v_{ij} + \alpha$. What this means, is that the utility for individual i of a consumption bundle (j, α) is the sum of his valuation v_{ij} of the object and the monetary compensation α that he gets. One way of interpreting the valuation is that it is a monetary measurement of how much the individual would be willing to pay for the object. Furthermore, we put a quite reasonable restriction on the valuations - no object is infinitely better than another object. In other words, we can always alter the compensations in such a way that the individual's utilities of the objects are the same. This state we call *indifference*. Individual 1 is, for instance, indifferent between objects 2 and 3 at an allocation (a, x) if

$$u_1(2, x_2) = v_{12} + x_2 = v_{13} + x_3 = u_1(3, x_3).$$

If we make a list of all individuals' preferences we get a preference profile $u = (u_1, u_2, u_3)$, which is a vector of utility functions.

3.3 Definitions

Fair distributions A distribution is said to be fair if we can combine it with an assignment in such a way that every individual gets what he prefers. The combination is then called a fair allocation. A more technical way of describing this is the following. For (a, x) to be a fair allocation, we must have

$$u_i(a_i, x_{a_i}) = v_{ia_i} + x_{a_i} \geq v_{ik} + x_k = u_i(k, x_k)$$

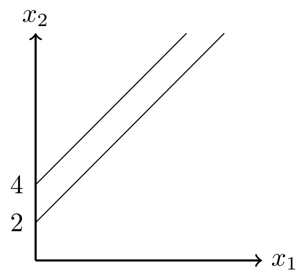
for all individuals $i \in N$ and objects $k \in M$. This might look rather complicated, but it is just a way of expressing that everyone gets the consumption bundle that gives them the highest utility.

Allowed distributions All distributions satisfying the restrictions in $\bar{x}$ are the so called allowed ones. This set is quite simple - it is something of a box including its borders, along which $x_j = \bar{x}_j$. This set is both closed and convex.

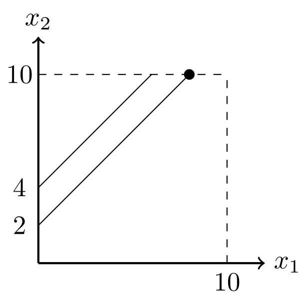
Optimal distributions A distribution that is fair, considering the preference profile u, and allowed with respect to $\bar{x}$ is said to be optimal if the sum of compensations, $x_1 + x_2 + x_3$, is maximized.

Let us shortly interrupt the listing of definitions, and try to apply what we have learned on a simple numerical version of the "The sons of Abraham" example. Say Isaac is individual 1 and Ishmael individual 2. Suppose Isaac's valuations of the two estates are $v_1 = (8, 4)$, whilst Ishamel values them at $v_2 = (5, 3)$. Furthermore, let us assume that no more than ten barrels of gold can be distributed together with an estate.

At what distributions $x = (x_1, x_2)$ is Isaac indifferent between the estates? Assuming preferences of the type described above, he will be indifferent if $8 + x_1 = 4 + x_2$. For Ishmael, we must have $5 + x_1 = 3 + x_2$. If we solve these equations for x_2, we get $x_2 = 4 + x_1$ and $x_2 = 2 + x_1$. Parts of these lines have been plotted in the graph to the right.



Along an indifference curve the individual is indifferent. If we increase x_2 we move upwards away from the curve. At such a point the individual must prefer estate 2. Thus, if we are at a point above both indifference curves, both Isaac and Ishmael prefer estate 2. For the distribution to be fair however, we require that everyone is assigned what they prefer - but since the estates are indivisible we can not assign estate 2 to both of the individuals. Thus a distribution either above or below both of the indifference curves can not be fair. The only ones remaining are the ones along or in between the indifference curves, which are all in the set of fair distributions.



What distributions are allowed then? Well, this is simply the set of all distributions x such that $x \leq \bar{x} = (10, 10)$. To the left we have plotted both the set of fair distributions and, the dashed box, the set of allowed distributions. Furthermore, the dot at $(8, 10)$ is our optimal distribution; it is both fair and allowed, and it maximizes $x_1 + x_2$. The optimal assignment will be $a = (1, 2)$.

Allocation rule The assignments and distributions will be handled using an allocation rule that selects the set of optimal allocations (a, x) given the preference profile u and $\bar{x}$. It is a correspondence rather than a function, since there may exist multiple optimal allocations for a single u. For instance, if all individuals report the same preferences, every possible assignment would be fair.

Manipulability An allocation rule is said to be manipulable by a single individual or a coalition of individuals, if they, by reporting other preferences than their true ones, are all made strictly better off. Described in a mathematical fashion: let $C \subseteq N$ be the set of individuals in the coalition. They all report preferences that differ from their true ones and we end up with an optimal allocation (b, y) instead of "the true" optimal allocation (a, x). The allocation rule is then manipulable if $u_i(b_i, y_{b_i}) > u_i(a_i, x_{a_i})$ for all individuals in the coalition.

Strategy-proof An allocation rule that is not manipulable by any single individual on his own, according to the definition above, is said to be individually strategy-proof. Furthermore, if it is not manipulable by any group of individuals (including groups of only one individual), the allocation rule is coalitionally strategy-proof.

Consider again the Vickrey auction. We know from before that no individual can be made better off by single-handedly bidding something else than his true valuation of the object. Thus the auction must be individually strategy-proof. Furthermore, we only have one object that is auctioned. If a group of individuals try to manipulate the auction together, still only one of the individuals will get the object, and the others will leave the auction empty handed. This would contradict them being made better off opposed to telling the truth, which, in its worst case, would yield the same result. Thus the one-object Vickrey auction must be coalitionally strategy-proof.

Observe that we do not take side payments into consideration. The case of one of the individuals bribing his friends in order to win the object cheaper and then distribute some of his saved money among his friends is outside the scope of this paper.

3.4 Procedure

The model's course of events can be varied in a few ways, but the inheritance example gives it a reasonably intuitive and realistic background. The example could continue with the father asking his sons what they feel about the objects, making them reveal their preferences. Of course, they could lie and report any preferences whatsoever, but, as will be proven later, this would not be to their benefit. Furthermore, no individual can gain by basing his decisions on the others' decisions. Thus, we do not need it to be "sealed bids" - the family could just as well get together in the living room and discuss it openly.

4 Results

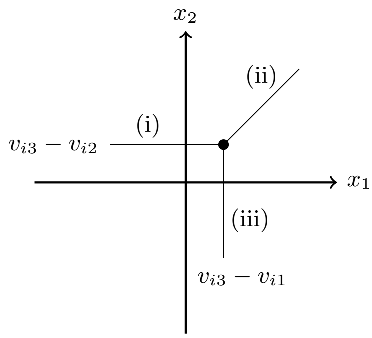
Proposition 1 The intersection of two joint convex sets is convex.

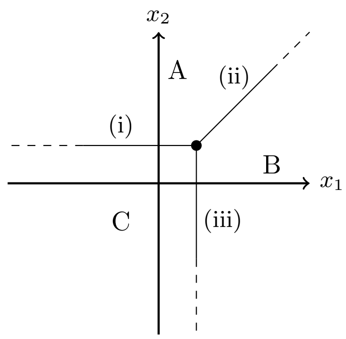
Proof Assume joint convex sets S and T, i.e. $S \cap T \neq \emptyset$. For two points $x, x' \in S \cap T$ we also, by the definition, have $x, x' \in S$ and T. Since S and T are convex, we must also have $\lambda x + (1-\lambda)x' \in S$ and T for $0 \leq \lambda \leq 1$ or equivalently $\lambda x + (1-\lambda)x' \in S \cap T$, proving that the intersection is convex.

4.1 Set of fair distributions

Individual i is indifferent between all objects at a distribution where $u_i(1, x_1) = u_i(2, x_2) = u_i(3, x_3)$, or more specifically $v_{i1} + x_1 = v_{i2} + x_2 = v_{i3} + x_3$. An example of such a distribution is $x = (-v_{i1}, -v_{i2}, -v_{i3})$. If the same amount is added to every compensation he will still be indifferent, thus, for any $t \in \mathbb{R}$, he will also be indfferent at $x' = (t - v_{i1}, t - v_{i2}, t - v_{i3})$. In particular, the distribution at $t = v_{i3}$, i.e., $\dot{x}' = (v_{i3} - v_{i1}, v_{i3} - v_{i2}, 0)$ is also one where he is indifferent. We can plot this point in an x_1/x_2-plane.

Since the individual is indifferent between all objects at the dot, he will still be indifferent between objects 2 and 3 if we decrease x_1. This, indifference between 2 and 3, is represented by the line (i). If we increase both x_1 and x_2 with the same amount, the individual will still be indifferent between the objects 1 and 2. This corresponds to the line (ii). Lastly, line (iii) is for objects 1 and 3.

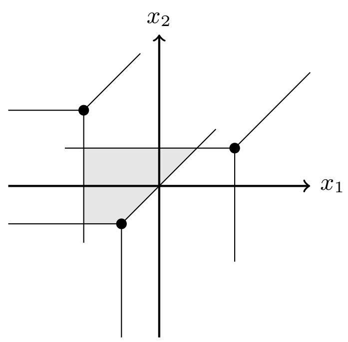




Consider the line (i) to the left. Along it the individual is indifferent between objects 2 and 3. Above it, with x_3 kept constant, he must prefer object 2. The same can be said about the area above (ii). Thus, in the combined area, here marked A, object 2 will be preferred. If we include the borders, we get a closed set with distributions such that no object is better than object 2.

Analogous, in the area B object 1 will be preferred and in the area C he will go for object 3, since x_3 is kept constant whilst both x_1 and x_2 are decreased. All of these sets (the area A and its borders, B and its borders ..) are closed and convex. We can do this for all three individuals, giving us three points in the x_1/x_2-plane - all only dependent on the individuals' reported valuations.

Here we have plotted points from each of the individuals' indifference curves such that $x_3 = 0$. The shaded area (including its borders) is, given the preferences and the restriction on x_3, the set of fair distributions. For all distributions outside this area there are at least two individuals strictly preferring the same object, which would contradict the distribution being fair.



This set is closed, since it is bounded by the indifference curves. The individuals have to get their preferred object - if they favour multiple objects they can be assigned any object amongst the preferred ones. Furthermore, the set, which is the intersection of convex sets, is convex according to Proposition 1.

It is quite obvious that we can construct a new fair distribution out of an old one by adding the same amount to each of the compensations - this would not change the individuals' internal ranking of the objects. For different values on x_3 we will get different values on x_1 and x_2 as well, but the shape of the area corresponding to the fair distributions will remain the same - it will however be moved further away or closer to the origin along the line $x_1 = x_2$. If we try to think of this in a cubic graph, we get some sort of edgy pipe with slope 1 in all three directions.

4.2 Properties

For every value on x_3 we will get the same shape - and it will always be a closed set. Since the borders always will be included, we can safely say that the entire set of fair distributions is closed.

Let us instead turn the attention to convexity. Take two fair distributions x and x'. Combine them into a third,

$$x'' = \lambda x + (1 - \lambda)x' \text{ for } 0 \leq \lambda \leq 1.$$

If we expand this, we have

$$x'' = \lambda(x_1, x_2, x_3) + (1 - \lambda)(x'_1, x'_2, x'_3).$$

From this we remove $\lambda x_3 + (1 - \lambda)x'_3$ and we get

$$\begin{aligned}\dot{x}'' &= x'' - \big(\lambda x_3 + (1 - \lambda)x'_3\big) \\ &= \lambda(x_1 - x_3, x_2 - x_3, 0) + (1 - \lambda)(x'_1 - x'_3, x'_2 - x'_3, 0).\end{aligned}$$

Thus, to prove that the entire set is convex, we only have to show that the set, given $x_3 = 0$, is convex. But this we know from above and Proposition 1 to be true, thus the set of fair distributions must be convex.

4.3 Optimal distributions

A reminder: at an allocation $(a, x), x_i$ is the compensation bundled with object i, whilst x_{a_i} is the compensation distributed to individual i.

Lemma 1 If $\bar{x}$ is fair, then $\bar{x}$ is the unique optimal distribution.

Proof Assume $\bar{x}$ is fair but that the optimal distribution is $x = (x_1, x_2, x_3)$ requiring that $x_1 + x_2 + x_3 > \bar{x}_1 + \bar{x}_2 + \bar{x}_3$. This would make some $x_j > \bar{x}_j$, contradicting x being allowed and, as a consequence, contradicting x being optimal.

Lemma 2A If $\bar{x}$ is not fair, then an optimal distribution x must lie on at least one indifference curve.

Proof Assume that $\bar{x}$ is not fair and that the optimal distribution consequently is $x \leq \bar{x}$. We must then have some $x_j < \bar{x}_j$, meaning that object j is not maximally compensated. Say this is object 1. Furthermore, assume x is located in the inner of the set of fair distributions, in the sense that it is not on the sets' borders (the indifference curves). At such a distribution everyone strictly prefers what they are assigned in comparison to the other objects. In this case we would be able to add a sufficiently small positive amount ε to x_1 and arrive at a new fair and allowed distribution x'. But the sums of distributions in $x', x_1+\varepsilon+x_2+x_3$ would be greater than the ones in $x, x_1+x_2+x_3$, contradicting x being optimal. Thus we can not have x in the inner of the set, and it must be on at least one indifference curve.

Lemma 2B If $\bar{x}$ is not fair, there must be someone who is indifferent between the object he is assigned and the object that is not maximally compensated.

Proof In the case that the optimal distribution x is positioned on multiple individuals' indifference curves, at least one of them is not assigned the object, and the statement is trivially true. Let us instead assume that x is on only one of the individuals' indifference curves. Assume object j is not maximally compensated and individual i is the person being indifferent. Consider assigning object j to i. Since x is optimal it must be fair, and furthermore, since x is not on any of the others' indifference curves, they must strictly be preferring the object they are assigned in comparison to j. But since $x_j < \bar{x}_j$ we could just as well increase x_j a little and still have a fair and allowed distribution. However, this contradicts x being optimal, since the new distribution would sum up to more than the original one. Thus we can not assign the object to the single individual being indifferent, and therefore there must be someone who is indifferent between the object he is assigned and the object that is not maximally compensated.

Theorem 1 The optimal distribution is unique for each preference profile u and $\bar{x}$.

Proof Assume that there are multiple different optimal distributions, of which x and y are two of them. These distributions are connected to optimal allocations (a, x) and (b, y) respectively. For them *both* to be optimal, we must have $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$. Moreover, for them to be allowed, we must have $x \leq \bar{x}$ and $y \leq \bar{x}$, and lastly, for them to be different, we must have $x \neq y$.

Using these three observations, we see that there must be some object i with

$$x_i < y_i \leq \bar{x}_i$$

since they are different, and some other object j with

$$y_j < x_j \leq \bar{x}_j$$

since they are both optimal - otherwise they would not be able to sum up to the same amount.

Since the allocations are optimal they must be fair, and $u_i(a_i, x_{a_i})$, the utility of the bundle that individual i is allocated at the allocation (a, x), must be greater than $u_i(j, x_j)$ for all other objects j. This we will refer to as *fairness*.

By our assumptions about the utility function, in specific the requirement that it is increasing in money, we know that $u_i(j, x_j) > u_i(j, y_j)$ if and only if $x_j > y_j$. This is rather intuitive; if he is assigned the same object but compensated with more money, he will be better off. This we will refer to as *monotonicity*.

Let us assume that $x_{a_i} < y_{a_i} \leq \bar{x}_{a_i}$ (since there had to be some $x_i < y_i$) and investigate the situation for individual i. We construct a list of inequalities:

$$u_i(b_i, y_{b_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(a_i, y_{a_i}) \underset{\substack{\uparrow \\ \text{monotonicity}}}{>} u_i(a_i, x_{a_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(b_i, x_{b_i}).$$

If we shorten this list, we see that

$$u_i(b_i, y_{b_i}) > u_i(b_i, x_{b_i}) \underset{\substack{\uparrow \\ \text{monotonicity}}}{\Leftrightarrow} y_{b_i} > x_{b_i}.$$

By Lemma 2B, we know that, whenever an object is not maximally compensated, there must be someone who is indifferent between this object (here: object b_i) and what he is assigned. Let us call him individual j. Again we set up a list of inequalities and equalities:

$$u_j(b_j, y_{b_j}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_j(b_i, y_{b_i}) \underset{\substack{\uparrow \\ \text{monotonicity}}}{>} u_j(b_i, x_{b_i}) \underset{\substack{\uparrow \\ \text{indifference}}}{=} u_j(a_j, x_{a_j}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_j(b_j, x_{b_j}).$$

Just like before, we can crop the list into a single interesting inequality: $u_j(b_j, y_{b_j}) > y_j(b_j, x_{b_j})$ making $y_{b_j} > x_{b_j}$.

Now we have $y_{b_i} > x_{b_i}$ and $y_{b_j} > x_{b_j}$. Remember that x and y both are said to be optimal, requiring that $x_1 + x_2 + x_3$ be equal to $y_1 + y_2 + y_3$. Thus, for the object assigned the remaining individual k, we must have $y_{b_k} < x_{b_k} \leq \bar{x}_{b_k}$.
If we rewrite the list of inequalities for individual i, we get the following:

$$u_i(b_i, y_{b_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(a_i, y_{a_i}) \underset{\substack{\uparrow \\ \text{monotonicity}}}{>} u_i(a_i, x_{a_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(b_k, x_{b_k}) \underset{\substack{\uparrow \\ \text{monotonicity}}}{>} u_i(b_k, y_{b_k}),$$

where $u_i(b_i, y_{b_i}) > u_i(b_k, y_{b_k})$ is particularly worth observing. In a similar fashion for individual j we can establish that $u_j(b_j, y_{b_j}) > u_j(b_k, y_{b_k})$. However, this contradicts y being optimal! Both individuals i and j strictly prefer what they are assigned, thus we could increase y_{b_k} with a sufficiently small positive ε and still have a fair distribution. Moreover, since $y_{b_k} < y_{b_k} + \varepsilon \leq \bar{x}_{b_k}$, the new distribution would still be allowed, but sum up to more than the original one.

The contradiction of y being optimal comes from the assumption that there are multiple optimal distributions, thus this can not be the case, and the optimal distribution, for each u and $\bar{x}$, must be unique.

5 Proof of coalitionally strategy-proofness

We will assume that the allocation (a, x) is the optimal allocation given that every individual reports his true preferences. Furthermore, we have another optimal allocation, (b, y), which is connected to the case when some coalition C of at least one individual tries to manipulate the allocation rule. Both distributions x and y are optimal with respect to the same $\bar{x}$.

An individual i is said to *favour* an object j if $u_i(j, x_j) \geq u_i(k, x_k)$ for all objects k. If he is indifferent between two objects, he will be favouring them both. We will also adopt the linguistic simplification of abbreviating "not maximally compensated" as *NMC*; if $x_j < \bar{x}_j$, then object j is said to be NMC.

Let us first consider the case of $\bar{x}$ being fair. We end up with an optimal allocation $(a, \bar{x})$ where everyone gets what they want, by fairness, and everyone is maximally compensated. No one can be made better off by trying to manipulate the outcome. Thus we will henceforth only have to consider distributions x such that $x \leq \bar{x}$, where some object is NMC.

Proposition 2 A distribution $x < \bar{x}$ can not be optimal.

Proof If $x < \bar{x}$ then no object is maximally compensated, and we could increase every compensation with the same small enough amount and reach a new fair and allowed distribution, contradicting x being optimal.

Lemma 3A If an object j is NMC, then there must be at least two individuals favouring it.

Proof Assume $x_j < \bar{x}_j$ and that only individual i favours object j. Then, by fairness, everyone else must strictly prefer what they are assigned in comparison to object j. But then we could increase x_j with a small enough amount and still have both a fair and allowed distribution, contradicting x being optimal.

Lemma 3B If two objects are NMC, then every individual must favour at least one of the two objects.

Proof Assume $x_i < \bar{x}_i$ and $x_j < \bar{x}_j$. From Lemma 3A, we know that there must be at least two persons favouring object i and the same goes for j. Assume some individual k does not favour any of the objects, i.e., he strictly prefers the remaining object. This would imply that we could increase x_i and x_j with the same small enough amount and still be at a fair and allowed distribution, which contradicts x being optimal.

Observation 1 Assuming that the allocation rule is manipulable, for all individuals i in the coalition C, we must have

$$u_i(b_i, y_{b_i}) \underset{\substack{\uparrow \\ \text{manipulability}}}{>} u_i(a_i, x_{a_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(b_i, x_{b_i})$$

or simply $u_i(b_i, y_{b_i}) > u_i(b_i, x_{b_i})$. By monotonicity, this implies $\bar{x}_{b_i} \geq y_{b_i} > x_{b_i}$.

Observation 2 If an individual i favours the object j at allocation (a, x) with $y_j > x_j$, we have

$$u_i(b_i, y_{b_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(j, y_j) \underset{\substack{\uparrow \\ \text{monotonicity}}}{>} u_i(j, x_j) \underset{\substack{\uparrow \\ \text{indifference}}}{=} u_i(a_i, x_{a_i}) \underset{\substack{\uparrow \\ \text{fairness}}}{\geq} u_i(b_i, x_{b_i})$$

or simply $u_i(b_i, y_{b_i}) > u_i(b_i, x_{b_i})$ and, by monotonicity, $\bar{x}_{b_i} \geq y_{b_i} > x_{b_i}$.

Proposition 3 If every individual is in the manipulating coalition, no object is maximally compensated and the distribution can not be optimal.

Proof From Observation 1 we know that object b_i is NMC for every individual i in C. If we assume that every individual is in the coalition, then no object is maximally compensated, which, by Lemma 1, contradicts x being optimal.

Theorem 2 The allocation rule is coalitionally strategy-proof.

Proof Assume that the allocation rule is manipulable by a coalition $C \neq \emptyset$ and thus not strategy-proof.

1. By Observation 1 we know that, for an individual i in C, we have an object b_i with $y_{b_i} > x_{b_i}$.
2. By Lemma 3A we know that there must be at least one other individual j favouring object b_i.
3. By Observation 2 we see that this leads to $y_{b_j} > x_{b_j}$.
4. By Lemma 3B we know that the last individual k must favour at least one of b_i and b_j.
5. By Observation 2 we see that this leads to $y_{b_k} > x_{b_k}$.
6. By Proposition 2 we know that $\bar{x} \geq y > x$ contradicts x being optimal.

Thus, the assumption that the allocation rule is manipulable must be wrong, and it must be coalitionally strategy-proof.

6 Applications

It is important to point out that the model can be interpreted in other ways than only distributing compensations and objects - for instance, if we make the compensations negative, they will in some sense act as prices instead.

As has been proven in Andersson & Svensson (2008), the results hold for any relation between the number of individuals and objects, in particular for the case of there being more objects than individuals. If we add a so called "no-object" for every individual, we can impose something called *individual rationality.* These no-objects can for instance be seen as the opportunity to not participate in an auction - once the prices are too high, you would rather refrain from buying the object than buying it, and instead favour the no-object. But now, with objects corresponding to the idea of not getting an object and compensations interpreted as prices, we can use our model to stage an auction instead.

An example of the model being used in reality is the auction procedure at sites like Ebay.com and its swedish branch Tradera.com. Generally a single object is auctioned out and everyone who has interest in it gets to bid on it. This can be seen as a special case of our model with n individuals, one object and $n-1$ no-objects. The allocation rule tries to find the lowest prices possible which make the outcome fair.

Not too long ago, a problem was that intense bidding started in the last minute of the auction, giving the casual, laid-back Ebayers a slight disadvantage compared to the more fanatic ones. The main problem was fairness - the bidder with the highest valuation of the object was perhaps not the one with the fastest browser or the most spare time, which could result in someone else winning the auction.

This has however now been dealt with! The auction sites have added the possibility of making a maximum bid rather than a single bid. The idea is presented in the following example.

Example The colourful t-shirt

> Tom has the current highest bid, \$6, on a colourful t-shirt on an Ebay auction, when Mark posts a maximum bid of \$16. The minimum increment is set to \$1, and Mark gets the current highest bid with \$7. A third participant, Travis, sees his chance to get his favourite t-shirt cheaply, and enters a maximum bid at \$10, unaware of the maximum bid of Mark. Now Travis gets the highest bid at \$8, increasing Mark's bid to \$9, followed by Travis' bid increasing to \$10 and Mark again getting the highest bid with \$11. But the required bid for Travis to outbit Mark is now \$12, which is more than what he has entered as his maximum bid, and his bid is not increased any further.
>
> The auction is ended with Mark winning the t-shirt at the price of \$11. Notice that this is a fair and optimal outcome: Tom's valuation of the t-shirt is \$6 and he rather gets a no-object than has to pay more than \$11 for the t-shirt. The situation is basically the same for

> Travis. For Mark, we know that his valuation is $16 but that he only has to pay $11, so he is also happy about the deal. The optimality comes from the price being the lowest possible (or highest negative compensation) given the level of the minimum increment.

As Ebay related trivia can be added that another technique called “sniping” has evolved, again by the more fanatic Ebayers. They wait until the last minute to enter their maximum bid, and hope thereby to avoid some sort of a “bidding war”. Anyway - the former disadvantages for the casual bidders has been dealt with, and the possibilities for a fair allocation have increased!

7 Related literature

The problem of fairly allocating a number of indivisible objects or positions together with some money has already been studied for many years. The idea of fairness as a state where no one envies anyone else was introduced by Foley (1967), but then applied to divisible objects.

The existence of *equilibrium* when dealing with indivisible objects, or what we call optimal allocations, was proven and investigated under various conditions by Svensson (1983), Alkan, Demange & Gale (1991) and Tadenuma & Thomson (1991) to mention a few.

Throughout the years of studying the indivisibles, a couple of different approaches and results have been established. Next we will describe these different alternatives, followed by a table where the elementary assumptions and results of some of the related papers easily can be compared.

n=m versus any The most general approach is to allow for any relation between the number of individuals and the number of objects. In the table this is called “any”. The most common assumption has however been $n = m$, i.e. as many individuals as objects, followed by forcing every individual to be assigned exactly one object.

Quasilinear versus general The idea of using quasilinear preferences sometimes simplifies the analysis. For instance, in our paper we can assume that everyone still feels the same about the objects if we compensate every object with an extra unit of money. Furthermore, the supposition is probably not too unrealistic in some cases - it is merely an assumption that everyone can estimate their monetary valuation of the object. A side note is that, throughout every considered paper, the utility function is always assumed to be increasing in money and continuous.

$\bar{\mathbf{x}}$ versus lump sum The limitations on the compensations or prices have basically been used in two ways. Either you put a lump sum limitation on the total of all compensations or, as we have done using $\bar{x} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$, you put a limit on every single compensation. The difference between the approaches is presented in the following example.

Consider a firm looking to hire personnel. They present a couple of positions (our objects) together with some salaries (our compensations) to a group of interested individuals. The target is to achieve fairness by altering the salaries. In the lump sum case, we suppose that the company has some budget for the *sum* of the entire staff's salary, whilst our approach is a way of regulating the *individual* salary for the CEO, his assistant, and every other employee.

IR versus not IR Individual rationality is, as described before, a way of allowing individuals the opportunity of rejecting every object if, for instance, the compensations are too low or the prices too high. We do this by adding no-objects for every individual, which requires the model to have more objects than individuals. As is shown in Andersson & Svensson (2008), individual rationality actually is a simplifying assumption once the results have been proven for the case of more objects than individuals. However, for instance in the case of an auction, it is arguably more realistic.

ISP versus CSP The rule is individually strategy-proof if it is not manipulable by a single individual on his own, whilst it is coalitionally strategy-proof if no group of individuals (including groups of one) can team up, manipulate the rule and improve the situation for everyone of them.

Matching versus allocation Our model is an allocation model, where we imagine one person holding all the objects, gathering every individual's valuations and then distributing the objects according to the allocation rule. In a matching model we instead look at multiple individuals holding an object each, and then try to match these with another group of individuals to achieve a fair outcome. This can be thought of as a case of supply and demand, where every supplier holds one object. On the market prices establish in such a way that every supplier's object is demanded by exactly one individual, achieving some sense of stability between supply and demand. The optimality in this sort of a model could for instance be to have prices as low as possible.

There are a few additional remarks to be made about some of the papers in the following table. When it comes to CSP, the paper by Demange & Gale (1985) presents a generalization (using a matching model) of the earlier mentioned Vickrey auction by Vickrey (1961). Vickrey only considered one-object auctions, whilst Demange & Gale allow for multiple objects but require individually rational outcomes. The model was then generalized even further by Andersson & Svensson (2008), by allowing for, however not requiring, individual rationality.

Another remark is on the paper by Svensson (2009), which consists, unlike the rest, of an "if and only if" part, proving that the allocation rule is coalitionally strategy-proof if and only if it is fair and optimal.

	$n = m = 3$	$n = m$	any	quasilinear	general	$\bar{x}$	lump sum	IR	not IR	ISP	CSP	Matching	Allocation
Svensson (1983)		✓			✓		✓		✓				✓
Demange & Gale (1985)			✓		✓		✓	✓			✓	✓	
Alkan, Demange & Gale (1991)			✓		✓		✓	✓					✓
Tadenuma & Thomson (1991)			✓	✓			✓		✓				✓
Sun & Yang (2003)		✓			✓	✓			✓	✓			✓
Andersson & Svensson (2008)			✓		✓	✓		✓	✓		✓		✓
Svensson (2009)		✓		✓		✓			✓		✓		✓
This paper	✓			✓		✓			✓		✓		✓

8 Summary

This paper has been based on three main issues: (i) how to allocate indivisible objects, (ii) how to do it in a fair way and (iii) how to make the allocation non-manipulable. We have investigated these problems in a quite restricted environment, with limitations on the number of individuals equaling the number of objects equaling three and the use of quasilinear utility functions. Seeing that every result established in this paper has already been shown in much more general cases, a larger weight has been put on providing the results and ideas in a simpler and clearer way.

The first results of importance were that the set of fair distributions is closed and convex and that the optimal distribution is unique for each preference profile and compensation limit. The most important proof has however been the one concerning coalitionally strategy-proofness, in which we showed that the allocation rule is not manipulable by neither one nor an entire group of individuals. In some sense this adds stability to our model, and, given that we know the preferences of the participating individuals, we can correctly forecast the optimal allocation of the objects.

Next we noted that, by allowing for individually rational outcomes, we could interpret the model as an auction. Throughout the text we have often referred to and given examples of the so called Vickrey (or second-price) auction, which has been generalized by, for instance, Andersson & Svensson (2008) using the same model as we do. Furthermore, we provided an extensive glance at other related literature on the subject, ending up with a table comparing a few of the earlier articles and putting this paper in some perspective.

In what way can we develop this particular model further in the future? The proof of coalitionally strategy-proofness for a more general case, where we allow for an arbitrary number of objects and the same number of individuals, is probably not a too difficult hurdle to climb. In our special case with three objects the results regarding the fair distributions can be shown using a nice graphical intuition. This possibility disappears if we consider more objects. Furthermore, if we would step away from the assumption of quasilinear preferences, we would still be able to get graphs, but not necessarily with the simplifying linearity.

To sum the text up, we again note that the main purpose of the paper, i.e. constructing a way of allocating three indivisible objects to three individuals with some money in a fair and non-manipulable way, has been proven using a, hopefully, untechnical and straight-forward method.

9 References

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