M.Sc. Thesis

FX BASKET OPTIONS

- Approximation and Smile Prices -

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Abstract

Pricing a Basket option for Foreign Exchange (FX) both with Monte Carlo (MC) techniques and built on different approximation techniques matching the moments of the Basket option. The thesis is built on the assumption that each underlying FX spot can be represented by a geometric Brownian motion (GBM) and thus have log normally distributed FX returns. The values derived from MC and approximation are thereafter priced in a such a way that the FX smile effect is taken into account and thus creating consistent prices. The smile effect is incorporated in MC by assuming that the risk neutral probability and the Local Volatility can be derived from market data, according to Dupire (1994). The approximations are corrected by creating a replicated portfolio in such a way that this replicated portfolio captures the FX smile effect.

Sammanfattning (Swedish)

Prissättning av en Korgoption för valutamarknaden (FX) med hjälp av både Monte Carlo-teknik (MC) och approximationer genom att ta hänsyn till Korgoptionens moment. Vi antar att varje underliggande FX-tillgång kan realiseras med hjälp av en geometrisk Brownsk rörelse (GBM) och därmed har lognormalfördelade FX-avkastningar. Värden beräknade mha. MC och approximationerna är därefter korrigerade på ett sådant sätt att volatilitetsleendet för FX-marknaden beaktas och därmed skapar konsistenta optionspriser. Effekten av volatilitetsleende överförs till MC-simuleringarna genom antagandet om att den risk neutral sannolikheten och den lokala volatilitet kan härledas ur aktuell marknadsdata, enligt Dupire (1994). Approximationerna korrigeras i sin tur genom att skapa en replikerande portfölj på ett sätt så att denna fångar upp FX-leendet.

Keywords

Basket Option, Black-Scholes, Derivative Pricing, Foreign Exchange, Hedging, Local Volatility, Monte Carlo, Volatility Smile.

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Patrik Karlsson, Lund, May 2009.

List of Notations

cdf - Cumulative density function.

$E[X]$ - Expected value of the stochastic variable X.

$\mathcal{F}_t^X$ - Filtration $\mathcal{F}$, contains all information about the s.v. X up until time t.

FX - Foreign Exchange.

GBM - Geometric Brownian motion.

K - strike price.

L - Likelihood function.

MC - Monte Carlo.

MG - Martingale.

$N(\mu, \sigma)$ - Normal distribution with mean μ and standard deviation σ.

$\mathbb{P}$ - Historical probability measure, $\mathbb{P}: \mathcal{F} \mapsto [0, 1]$.

PDE - Partial Differential Equation.

pdf - Probability density function.

ϕ - Normal probability density function.

Φ - Payoff function.

$\Pi(S_t, t)$- Derivative value with underlying asset S_t at time point t.

qMC - quasi Monte Carlo

$\mathbb{Q}$ - Risk neutral martingale measure.

r_d - Domestic interest rate.

r_f - Foreign interest rate.

$\mathbb{R}^d$ - $(d \times 1)$-dimensional real value.

σ - Volatility.

SDE - Stochastic Differential Equation.

S_t - Stock value at time t.

T - Time to maturity.

θ - Parameter set.

Θ - Parameter space.

W - Wiener process, $N(0,1)$.

$(\Omega, \mathcal{F}, \mathbb{P})$ - Probability space.

Ω - Sample space.

$\mathcal{F}$ - σ-algebra generated by $\omega = \{\omega_t : t \in \mathbb{R}\}$.

$\subseteq$ - Subset.

$\square$ - End of derivation/proof.

$\forall$ - for all.

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1 Introduction

1.1 Background

A Basket option is an asset similar to the Asian option, a multidimensional derivative whose payoff depends on the average price of the underlying assets. But instead of taking the average of one asset, the value of the Basket option depends on the weighted sum of a number of underlying assets. These types of path-dependent derivatives are one of the more complicated contracts to value and price. The Basket option protect against drops in all underlying assets at the same time, now anyone can understand its importunateness for risk reducing. Instead of buying plain options in the different assets as one can do, it is intuitively cheaper to buy a Basket of options which allows an investor to hedge its risk exposure by only using one derivative. And the fact that the total amount paid in transaction cost is lower when a single asset is purchased instead of several ones.

What makes it challenging when pricing averaging options is that traditional methods as finding numerical approximations for the partial differential equation (PDE) is not efficient since the number of underlying assets might be large. Also the fact the assets built on several assets in some sense are correlated is another aspect that need to be taken into account. The simple Black-Scholes model is built on the crude assumption that assets returns are log-normally distributed, well it can be shown that a finite summation of log normally distributed random variables are not log normally distributed anymore, and thus that there does not exist a closed form solution on Black-Scholes form. Then in order to be able to price these weighted summation of underlying assets like the Basket option we need some heavy computation, which can be done by Monte Carlo and quasi-Monte Carlo techniques. Anyone familiar with these techniques knows that they can be very computer intensive and time consuming in order to be able retrieve accurate values, and traders selling these types of assets needs these calculation to be done instantly since the foreign exchange (FX) market is very liquid and change continuously. That is why we need to find an accurate closed form approximation solution, several analytical approximations have been presented the last decades and this thesis will investigate some of them and determine their accuracy.

Another effect of being in a Black-Scholes world is the assumption on constant volatility, and thus that we get inconsistent prices since options with different time to maturity and strikes are valued with different implied volatilities, giving arise to the Smile effect. That is why the values derived from both Monte Carlo and the approximations need to be priced in such a way that the Smile is taken into account. The first fundamental strategy to overcome this problem is the deterministic local volatility model, and by constructing replicated portfolio that integrate the smile effect.

1.2 Purpose

This research arise from Nordea Market's (in Copenhagen) interest in increasing their FX financial instruments by offering a tailor made FX Basket option. The purpose of this thesis is to construct appropriate approximations for calculating FX Basket options instantly that are smile consistent and that not deviates to much compare to the values derived from Monte Carlo simulation.

1.3 Outline

In the second section will we present some fundamental mathematical and derivative theory, the Black-Scholes market, the purpose of the Greeks, introduction to the FX market, conventions and properties. Thereafter will the important Smile and Skew effect be deduced, how to approximate it, and finally is the Basket option introduced.

The third section is devoted to numerical method for calculating the option prices and where both Monte Carlo and quasi-Monte Carlo techniques are presented, techniques for reducing variance in Monte Carlo, techniques transforming the distribution of random variables, and some other fundamental numerical methods for the thesis.

In section four is the approximations introduced: Geometric Average, Log-Normal-, Reciprocal Gamma-, four moment method- and the Taylor approximation.

Section five contains different hedging strategies; strategies build on considering the Greeks and methods trying to find both upper and lower levels of replicating portfolios.

Pricing considering the volatility smile and skew are presented in section six, we present the Local volatility, techniques for creating replication of a portfolio that considers the smile.

And in section seven will the numerical results be presented, different scenarios for testing the approximations, creating FX smile prices and hedging values.

All calculations and simulations in this thesis are coded and performed in Matlab, the report is written in LyX (an interface between the user and LaTeX).

2 Stochastic Calculus and Arbitrage Pricing

2.1 Mathematical Theory

We will throughout the thesis assume that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the sample space, $\mathcal{F}$ the σ-algebra generated by stochastic process $\omega = \{\omega_t : t \in \mathbb{R}\}$ and $\mathbb{P}$ the probability measure, $\mathbb{P}: \mathcal{F} \mapsto [0, 1]$. The stochastic process is thus defined as

Definition 2.1: (Stochastic process) *A stochastic process is a collection of randomized variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$

$$\{X_t\}_{t \in T} \quad , T \in [0, \infty)$$

and for each $t \in T$ *we have that*

$$\omega \mapsto X_t(\omega), \quad \omega \in \Omega$$

or by fixing $\omega \in \Omega$ *we have the path of* X_t

$$t \mapsto X_t(\omega), \quad t \in T$$

We use the fundamental Brownian motion W_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to represent our important stochastic engine for modeling the randomness in the financial market.

Definition 2.2: (Brownian Motion) *The stochastic process* $W = \{W_t : t \in \mathbb{R}\}$ *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a Brownian motion if the following properties holds a.s.*

1. $W_0 = 0$.
2. *The increments are independent and stationary, i.e. if* $r < s \leq t < u$ *then are* $W_u - W_t$ *and* $W_s - W_r$ *independent stochastic variables.*
3. *The increments of* $W_{t+h} - W_t$ *are normally distributed,* $N\left(0, \sqrt{h}\right)$

4. W_t *has continuous trajectories.*

Let $X(t)$ represent a stochastic processes, in our case a differential equation extended with a random part, hence the name stochastic differential equation (SDE) or the 1-dimensional Itô process given in the following definition,

Definition 2.3: (1-dimensional Itô Process) *Let W_t be a Brownian motion, the Itô process (stochastic process) X_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is then given by*

$$X_t = X_0 + \int_0^t \mu(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dW_s \tag{1}$$

often written on a shorter form

$$dX_t = \mu dt + \sigma dW_s$$

such that the following conditions holds almost surely

$$\begin{aligned} P\left[\int_0^t \sigma(X_s, s)^2\, ds < \infty\ , \forall t \geq 0\right] &= 1 \\ P\left[\int_0^t |\mu(X_s, s)\, ds| < \infty\ , \forall t \geq 0\right] &= 1 \end{aligned}$$

SDE (1) consits of two terms, the first term μdt defined as the drift term, and the second term σdW_t which specifies the random part (the noise) of the process, named the diffusion part. For the existence and uniqueness of the solution of SDE given by (1) we need the following condition on μ and σ to be fulfilled

Theorem 2.3: (Existence and uniqueness) *Conditions that guarantees the existence and the uniqueness of the solution of SDE* (1) *is the growth condition, let μ and σ satisfying*

$$|\mu(x,t)| + |\sigma(x,t)| \leq C(1 + |x|)\,, \qquad x \in \mathbb{R},\ t \in [0, T]$$

for some constant C, which guarantees global existence, and the Lipshitz condition

$$|\sigma(x,t) - \sigma(y,t)| + |\mu(x,t) - \mu(y,t)| \leq D|x-y|, \qquad x,y \in \mathbb{R},\ t \in [0,T]$$

for some constant D, which guarantees local uniqueness. And where $\mathcal{F}_t$ is the filtration generated by $W = \{W_t : t \in \mathbb{R}\}$, then the SDE

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t$$

has a unique t-continuous solution $X(t)$ given by (1).

Proof. Omitted, See Oksendal (2000).

□

The stochastic analogue to the chain rule in ordinary calculus, the Itô formula that transforms the Brownian motion given the function $Y_t = g(t, X_t)$, where X_t is defined by (1). The dynamics of Y_t is then given by applying the second order Taylor expansion, are stated in the following theorem.

Theorem 2.4: (Itô formula) *Let X_t be a stochastic process given by SDE* (1) *and let $g(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R})$*[1] *Then*

$$Y_t = g(t, X_t)$$

is an Itô process and

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\,(dX_t)^2$$

and where the following rules has been used

$\times$	dt	dW_t
dt	0	0
dW_t	0	dt

Proof. Omitted

□

We will throughout this thesis assume that the returns of each underlying asset will follow a log-normal distribution and can thereby be realized by the geometric Brownian motion (GBM) SDE.

[1] i.e. g is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$.

Definition 2.5: (Geometric Brownian Motion) *A geometric Brownian motion is defined as*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2}$$

which is a short form of the following equation.

$$S_t = S_0 + \int_0^t \mu S_z dz + \int_0^t \sigma S_z dW_z$$

Let us assume that the daily asset returns follows a log normal distribution and this by introducing

$$Y_t = \ln\left(\frac{S_t}{S_0}\right)$$

By applying Itô's formula we receive the following expression

$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

finding the primitive function

$$Y_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

and finally ending up with

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \tag{3}$$

□

S_t is log normally distributed and there by does the following holds

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$$

The expected value of process (3) is given by

$$E[S_t] = S_0 e^{rt}$$

Let us demonstrate how (2) can be used to simulate a stock, assuming an initial spot price $S_0 = 10$, stock volatility $\sigma = 0.3$, return $\mu = 0.05$ and a year[2]. Stock will after 4 simulations have the following appearance

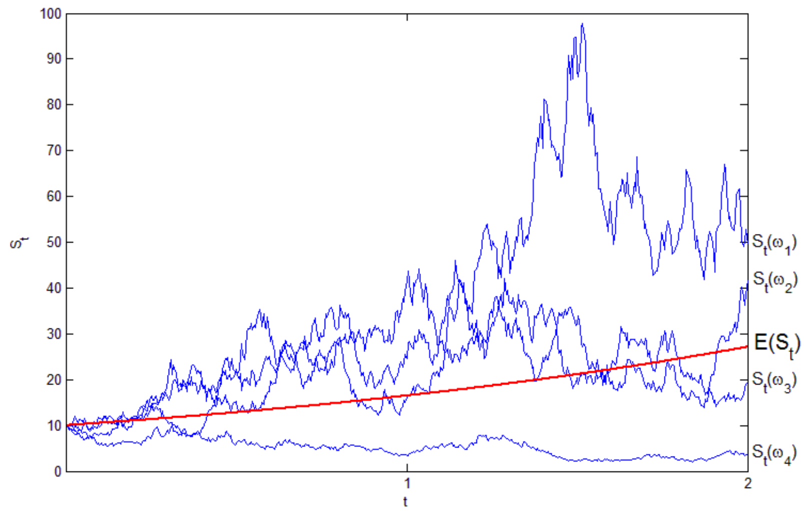

Figure 1: Simulated stock with 4 paths, and the mean $S_0 e^{rT}$.

Further assumptions in this thesis are that

1. There does not exists any arbitrage possibilities
2. There does not exists any transaction costs, hence the market is frictionless
3. We can enter any type of position at any time: short, long, arbitrary fraction and no constraints on liquidity.

The Normal Distributed Random Variable

We assume the each individual asset return will follow a log normal distributed and that it can be realized by GBM (3).

Definition 2.6: (The Normal Distributed r.v.) *A standard normal (Gaussian) random variable (r.v.)* $X \in N(0,1)$ *defined on the real axis, has the probability density function (pdf)* $f_X(x)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

[2]A year is assumed to have around 252 trading days

and the cumulative density function (cdf) $F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

Definition 2.7: (The log normal variable) *Assume $X \in N(\mu, \sigma)$, if $Y = \exp\{X\}$, then is Y called a log normal variable and thus defined as*

$$f_Y(y) = \phi\left(\frac{\ln y - \mu}{\sigma}\right)$$

where ϕ is the normal pdf.

Later on in the thesis will we use the k-th moment of the log normally r.v., which is defined as

Definition 2.8: (Moments of the log normal r.v.)

The k:th non centered moment $\mathbb{M}_k$ of a log normally distributed r.v. X is defined as (Lipton 2007: 32)

$$\begin{aligned} \mathbb{M}_k &= E\left[X^k\right] \\ &= e^{\mu k + k^2\sigma^2/2} \end{aligned}$$

Proof.

$$\begin{aligned} \mathbb{M}_k &= \int_0^{\infty} y^k \frac{e^{-(\ln y - \mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \frac{dy}{y} \\ &= \int_{-\infty}^{\infty} \frac{e^{-(x-\mu)^2/2\sigma^2 + kx}}{\sqrt{2\pi\sigma^2}} dx \\ &= e^{\mu k + k^2\sigma^2/2} \int_{-\infty}^{\infty} \frac{e^{-((x-\mu)/\sigma + k\sigma)^2/2}}{\sqrt{2\pi\sigma^2}} dx \\ &= e^{\mu k + k^2\sigma^2/2} \end{aligned}$$

The last equality holds since the third integral is one.

□

Let $\overline{\mathbb{M}}_k$ define the centered moment, of a random variable X, with expected value $E[X]$, variance $V[Y]$, skewness $\eta[Y]$ and kurtosis $\kappa[Y]$ defined according to

$$\begin{aligned} E[X] &= \mathbb{M}_1(X) \\ V[X] &= \overline{\mathbb{M}}_2(X) \\ \eta[X] &= \overline{\mathbb{M}}_3(X)/\overline{\mathbb{M}}_2(X)^{\frac{3}{2}} \\ \kappa[X] &= \overline{\mathbb{M}}_4(X)/\overline{\mathbb{M}}_2(X)^2 - 3 \end{aligned}$$

The first four moments of the log normally distributed r.v. Y with expected value μ and standard deviation σ are

$$\begin{aligned} E[Y] &= e^{\mu+\sigma^2/2} \\ \sigma[Y] = \sqrt{V[Y]} &= e^{\mu+\sigma^2/2}\sqrt{e^{\sigma^2}-1} \\ \eta[Y] &= \left(e^{\sigma^2}+2\right)\sqrt{e^{\sigma^2}-1} \\ \kappa[Y] &= e^{4\sigma^2}+2e^{3\sigma^2}+3e^{2\sigma^2}-6 \end{aligned}$$

Definition 2.9: (The Gamma Distribution)

The pdf g_Γ of a gamma distributed variable X is given by

$$g_\Gamma(x,\alpha,\beta) = \frac{e^{-x/\beta}(x/\beta)^{\alpha-1}}{\beta\Gamma(\alpha)},\ x \geq 0,\ \alpha,\beta \geq 0$$

the corresponding cdf G_Γ is defined as

$$G_\Gamma(x,\alpha,\beta) = \int_0^x g_\Gamma(u,\alpha,\beta)\,du = \frac{\int_0^x u^{\alpha-1}e^u du}{\Gamma(\alpha)} = \frac{\gamma\left(\alpha,\frac{x}{\beta}\right)}{\Gamma(\alpha)}$$

and where Γ is defined as the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

The i:th moment of the gamma distribution is given by

$$E\left[Y^i\right] = \frac{\beta^i\Gamma(i+\alpha)}{\Gamma(\alpha)}$$

The i-th moment of the inverse gamma distribution can be obtained for $-\alpha < i \leq 0$ using the same formula. For $i \leq -\alpha$ the moments are ∞.

2.2 Derivatives

Derivatives can be seen as insurance and hedging contracts on the financial market in order to remove and avoid potential downside risk. A derivative derives its value from some underlying asset, hence the name derivative. Today derivatives can be derived by among different number of underlying assets: Stocks, Indexes, Interest rates, Commodities, Electricity etc. As one can see derivatives can be applied to almost any type of asset, and one of the simplest derivatives is the forward contract.

Definition 2.10: (The Forward Contract) *The holder of a forward contract gives the obligation to buy/sell the underlying asset at some prespecified date for a prespecified price. The payers position on a asset S_t, at a specified time T and with strike price K have the following payoff function*

$$\Phi^{Payer}(S_T) = S_T - K$$

the contract for the seller position is defined in the analogue way

$$\Phi^{Seller}(S_T) = K - S_T$$

Just like forwards and futures are options derivatives contracts, but instead be forced to buy/sell the underlying asset at a specified date in the future, the option gives the holder an opportunity to buy/sell the underlying asset. The holder is thereby not forced to do something, and is only left with the positive outcome. The derivative market today is very big and can be build one a huge amount of different assets, in some cases have the derivatives market been dominating in size the market for the underlying asset. In the last 40 years there has been a huge development of the derivative market. (Byström, 2007). One of the reasons was the increased volatility and uncertainty after the OPEC oil crisis in the 1973, where the oil prices increased drastically and made a great impact on the global economy. This created a big demand to be able to insure and hedge not only commodities but all type of assets. The technical development is another aspect of the increased use in derivatives, by using the technique it have made it possible quicker price derivatives, trade and settle transactions than in earlier periods, but this area will lead to another crises in the late 80s. The third aspect which is the famous Black-Scholes model, this phenomenal model totally changed the world of pricing derivatives.

Before stating Black-Scholes we state some fundamental options fundamental for further studies regarding the subject of the thesis.

Definition 2.11: (The European Call/Put Option) *The holder of a call option have the option to exercise the option and thereby be able to buy the underlying asset S_t to specified price at the time to maturity T. The put option* *is defined in the analogue opposite way, where the holder of a put option have the option the sell the underlying asset S at a specified price at T. The payoff $\Phi^{Call}(S_T)$ for the European call option is given by*

$$\Phi^{C}(S_T) = (S_T - K)^{+} = \begin{cases} S_T - K, & S_T \geq K \\ 0, & S_T < K \end{cases}$$

and for the European call option in the analogous way

$$\Phi^{Put}(S_T) = (K - S_T)^{+} = \begin{cases} K - S_T, & K \geq S_T \\ 0, & K < S_T \end{cases}$$

The payoff of this both contracts are demonstrated in the following figure

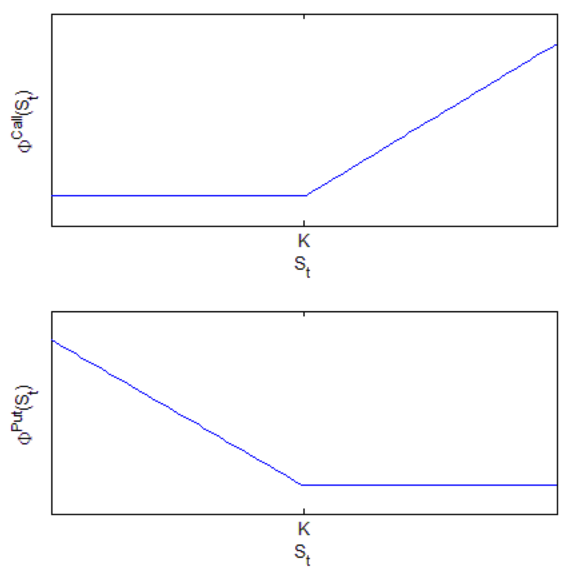

Figure 2: Upper: Payoff European call option, lower: European put option

Definition 2.12: (The Straddle) *A Straddle is a derivative consisting of a long call and put positions, with the same strike K and with the following payoff function*

$$\Phi^{S}(S_T) = \begin{cases} S_T - K, & \text{if } 0 \leq S_T \leq K \\ K - S_T, & \text{if } K \leq S_T \end{cases}$$

Definition 2.13: (Risk-Reversal) *The Risk-Reversal (RR) is a derivative consisting of a long call with strike K_2 and a short put with strike K_1*

$$\Phi^{RR}(S_T) = \begin{cases} -(K_1 - S_T), & \text{if } 0 \le S_T \le K_1 \\ 0, & K_1 \le S_T \le K_2 \\ S_T - K_2, & \text{if } K_2 \le S_T \end{cases}$$

such as the following condition holds $K_1 \le K_2$.

Definition 2.14: (Butterfly) *The Butterfly is a derivative consisting of a long call with strike K_1, two short call with strike K and finally a long call with strike K_2*

$$\Phi^{B}(S_T) = \begin{cases} 0, & \text{if } S_T \le K_1 \\ S_T - K_1, & K_1 \le S_T \le K \\ K - K_1 - S_T, & \text{if } K \le S_T \le K_2 \\ 0 & K_2 \le S_T \end{cases}$$

such as the following condition holds $K_1 \le K \le K_2$. It is also possible to construct a Butterfly consisting with the same setup as above but by just changing from calls to puts.

The three defined options payoff are realized in figure 3, the lowest plot the Butterfly demonstrates the residual constructed from the three positions in call options.

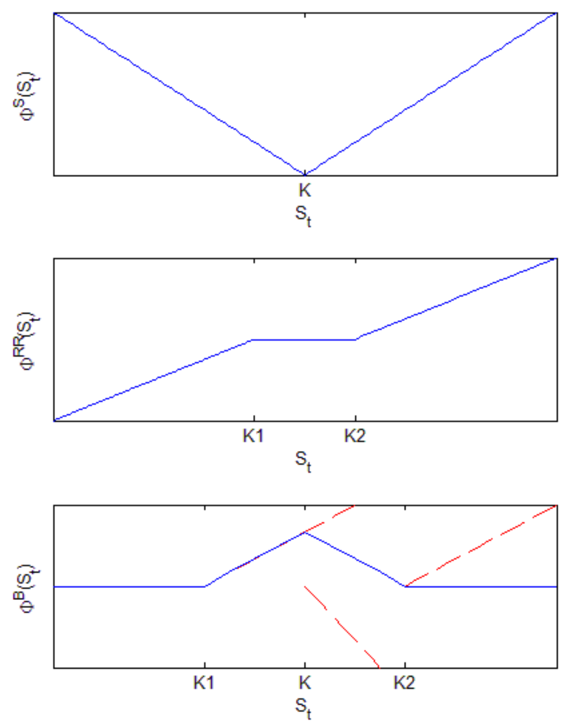

Figure 3: Upper: Straddle, Middle: Risk Reversal, Lower: Butterfly

A straddle can be seen as a simple volatility strategy since the delta for this type of assets are zero. These type of contracts are very commonly used in the industry for determining the ATM implied volatility.

2.3 The Arbitrage Free Price

Before we state the Black-Scholes formula we need to introduce a fundamental formula which also will be used during the simulation later on. The formula named the risk neutral valuation formula, RNVF is stated next (Björk 2004). The risk neutral valuation formula states that any asset risky and non risky will all have the same expected return as the risk-free rate of interest r (Hull 2008), this means that in a risk neutral world all assets will all have the same expected return. One can familiar with financial mathematics can observe the connection between the risk neutral valuation formula and the solution proposed by the Feynman-Kač theorem.

Theorem 2.15: (The Risk Neutral Valuation Formula) *Given the pay-off function $\Phi(S_t)$ for a European type option, the arbitrage free price $\Pi(t, \Phi)$ of this claim is given by*

$$\Pi(t,\Phi) = e^{-rT} E^{\mathbb{Q}}\left[\Phi(S_t) \mid \mathcal{F}_t^S\right]$$

where $\mathbb{Q}$ denotes the risk-neutral martingale measure using the money market account, MMA as a numeraire and $\mathcal{F}_t^S$ the filtration which contains all the information about S up until time t.

One fundamental property of a martingale (MG) is that the mean of a random variable X always is constant. The following condition must hold with respect to the filtration $\mathcal{F}_s^X$ for a r.v. to be a MG (Rasmus, 2008)

Definition 2.16: (Martingale) *A stochastic process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a martingale w.r.t. the filtration $\{\mathcal{M}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ if the following properties holds*

1. *M_t is $\mathcal{M}_t$-measurable, or M_t is $\mathcal{M}_t$ adapted[3] for all t,*
2. $E[|X_t|] < \infty, \quad \forall t$
3. $E\left[X_t \mid \mathcal{F}_s^X\right] = X_s, \quad \forall\, 0 \leq s < t < \infty$

A martingale has no systematic drift and the notion above tells us that each asset in an arbitrage free world will be a martingale, thus that each assets discounted future expected value must be equal the present value, i.e. a fair game. In order to price process (2) correctly and thus that it does not cause any arbitrage possibilities we need perform a transformation of the Wiener process. We will only introduce the Girsanov theorem and the transformation of process (2) in the following theorem (Rasmus 2008: 136). In order to change from the historical measure $\mathbb{P}$ into the risk neutral measure $\mathbb{Q}$ we use the following theorem

Theorem 2.17: (Girsanov theorem)

Assume that $W_t^{\mathbb{P}}$ is a standard $\mathbb{P}$-BM, The relationship between the historical measure $\mathbb{P}$ and the risk neutral measure $\mathbb{Q}$ is defined as

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t g\left(s, W_t^{\mathbb{P}}\right) ds$$

where $g\left(s, W_t^{\mathbb{P}}\right)$ is the unique Girsanov kernel letting a process defined by a $\mathbb{Q}$-measure to be arbitrage free.

[3] *This means that the value of M_t is known given the information in $\mathcal{M}_t$.*

There are some fundamental properties that are ignored above. By applying the Girsanov theorem to process (2) one can determine the unique pricing kernel and thus that in a arbitrage free world any type of asset will evolve with the interest free rate r and thus that we the drift will be replace by r

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{4}$$

We will throughout this thesis assume that the price process of the risk free asset, with the interest rate denoted $r \geq 0$ is defined according to the following process

$$dB(t) = rB(t)\, dt$$

for some constant r.

2.4 Black-Scholes

Black and Scholes (1973) proposed a closed form solution for options written on stock of the following form.

Theorem 2.18: (Black-Scholes formula, on a dividend paying asset) *The value of a European call option, given the initial stock price S_0, strike price K, volatility σ, interest rate r and dividend q the arbitrage free price Π_T^C is given by*

$$\Pi_T^C = S_0 e^{-qT} N[d] - Ke^{-rT} N\left[d - \sigma\sqrt{T}\right]$$

and where

$$d = \frac{\ln \frac{S_0}{K} + \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \tag{5}$$

The analogue European put option can be determined by applying the Put-Call parity, PCP

Proof. See Rasmus (2008)

□

Theorem 2.19: (Put-Call Parity) *In an arbitrage free world the following relationship must hold,*

$$\Pi_T^C - \Pi_T^P = S_0 e^{-qT} - K e^{-rT}$$

where Π_T^C *determines the call option,* Π_T^P *the put option,* S_0 *the spot price and* K *the strike price.*

The following figure demonstrates how the Black-Scholes price changes when strike price K, volatility σ, and interest rate r and dividend yield q are kept constant while time to maturity T and the initial stock price S_0 varies. One can observe that the option price increases as the stock price and time to maturity increases

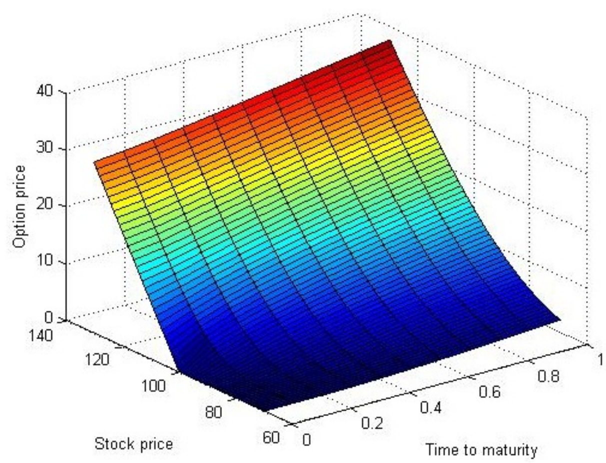

Figure 4: Black-Scholes price when varying T and stock spot price S and keeping all other parameters fixed.

The Black-Scholes formula is built on a few assumptions (Hull 2008: 286), some of them are very crude which limits the model

1. The stock follows a geometric Brownian motion.
2. Short selling is allowed.
3. No market frictions, hence no transactions costs or taxes.
4. There does not exist any arbitrage possibilities.
5. Constants volatility σ, and constant interest r.

2.5 The FX Greeks

The Greeks defines the risk exposure and how the option value Π changes when some parameter of the model changes. We will only present those FX Greeks that are importance for this thesis, anyone curious and want further information are recommend dig into relevant chapters in Hull (2008) or Wilmott (2006).

2.5.1 Delta

The delta Δ is defined as the partial derivative of the option value Π w.r.t the underlying asset S

$$\Delta = \frac{\partial \Pi}{\partial S}$$

$$\begin{aligned} \Delta_{\text{call}} &= \frac{\partial c}{\partial S} = e^{-r_f(T-t)} N(d) > 0 \\ \Delta_{\text{put}} &= \frac{\partial p}{\partial S} = -e^{-r_f(T-t)} N(-d) < 0 \end{aligned}$$

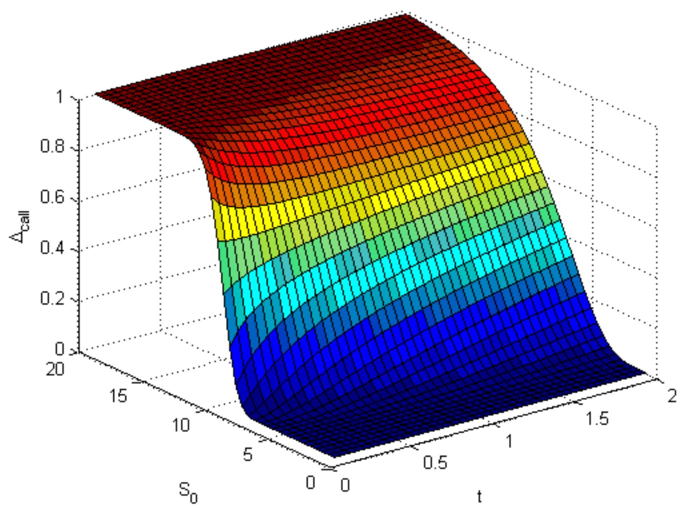

Figure 5: The spot Delta Δ_{call} of an European call option with $S_0 \in [1, 20]$, $T \in [0.1, 2]$, $K = 10$, $r_d = 0.05$, $r_f = 0$ and $\sigma = 0.3$.

For FX traders are delta one of the most important weapons in sense of hedging and minimize their risk exposure. As one can observe from Figure 5 is that the spot delta converges to one as we are deep-in-the-money, since $N(d)$ approaches one of the fact that the strike level K dominates the current spot S_0 level. The Δ value determines how many fractions of the underlying asset that we have to buy/sell in order to create a portfolio where the risk exposure in terms of change in the underlying asset will be eliminated.

The financial market consists of two different parties, the speculators and the hedgers. The speculators, the one that does not want to hedge at all, they believe on their implemented strategy hopefully generating them some form of profit. Hedgers on the other side is divided into two parts, the one holding a positions and who wants to eliminate some form of risk, and the one who is selling (buying) options that they believe have better values and hopefully can make some profit of by hedging away their risk exposure.

Say for instance that a bank is selling an option at $t = 0$ based on a underlying with a current $\Delta_{t=0}$. This mean that in order to eliminate the risk, we have to create a delta neutral hedge where we buy $\Delta_{t=0}$ amounts of the underlying asset. For all $t \in [0, T]$ we have to calculate a new Δ_t and rebalance or rehedge our portfolio. But we will face a set of problems, for instance since we in the theory need to rebalance our hedge continuously but in real life it is impossible, due to the existence of transaction costs.

2.5.2 Vega

Vega ν is defined in the analogous way but w.r.t changes in the underlying volatility σ

$$\upsilon = \frac{\partial \Pi}{\partial \sigma} = Se^{-r_f T} \phi\left(d\right) \sqrt{T} = Ke^{-r_d T} \phi\left(d - \sigma\sqrt{T}\right)$$

One can hedge its portfolio to decrease sensitivities to change of the volatility in the option, and the fact that we actually do not know the volatility (very precisely) can it be that useful[4]?

[4] This question is left to the reader

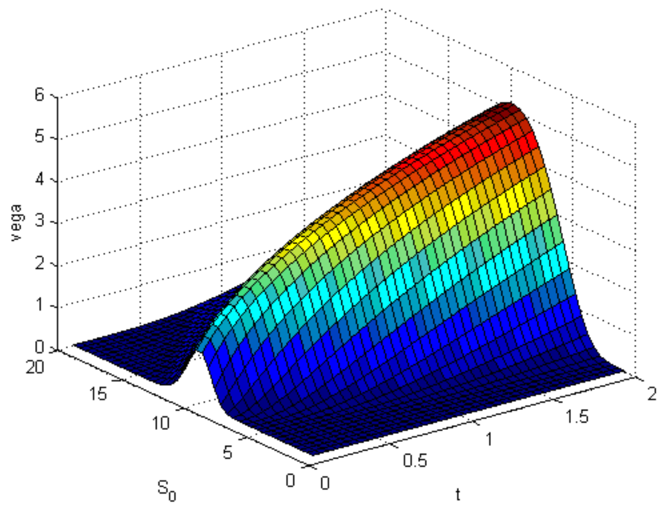

Figure 6: The spot Vega υ of an European call/put option with $S_0 \in [1, 20]$, $T \in [0.1, 2]$, $K = 10$, $r_d = 0.05$, $r_f = 0$ and $\sigma = 0.3$

2.5.3 Vanna

Vanna or also known as DdeltaDvol demonstrates how the change of delta Δ for small changes in volatility σ. The problem with Vanna is that in the Black-Scholes world, Vanna is not a function of a variable, but instead of a parameter. Vanna is very useful in the way that if it demonstrates high values it indicates that the volatility for calculating deltas becomes more significant. Vanna is defined as the partial derivate of the delta w.r.t. the underlying volatility σ

$$\text{Vanna} = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial^2 \Pi}{\partial S \partial \sigma} = e^{-r_f T} \phi(d) \frac{\left(d - \sigma\sqrt{T}\right)}{\sigma}$$

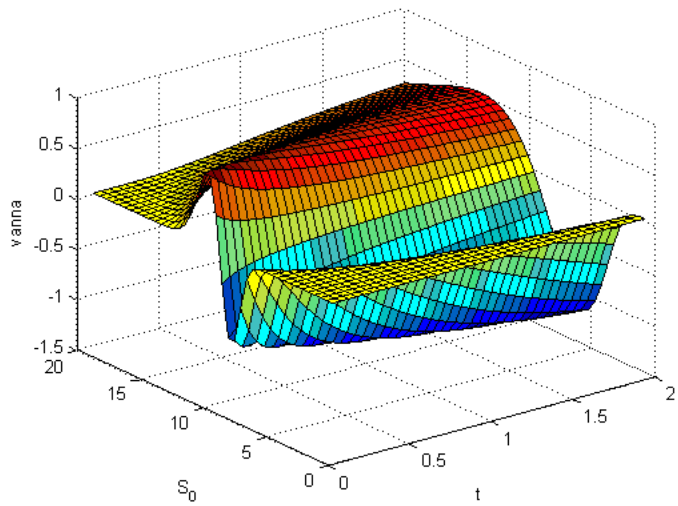

Figure 7: The spot Vanna of an European call/put option with $S_0 \in [1, 20]$, $T \in [0.1, 2]$, $K = 10$, $r_d = 0.05$, $r_f = 0$ and $\sigma = 0.3$

2.5.4 Volga/Vomma

Volga also known as Vomma is the partial derivative of Vega with respect to the underlying volatility, i.e. the second partial derivative of the option value Π w.r.t. the underlying volatility σ

$$\begin{aligned}\text{Volga} &= \frac{\partial \upsilon}{\partial \sigma} = \frac{\partial^2 \Pi}{\partial \sigma^2} \\ &= S e^{-r_f T} \phi(d) \sqrt{T} \frac{d\left(d - \sigma\sqrt{T}\right)}{\sigma} = \upsilon \frac{d\left(d - \sigma\sqrt{T}\right)}{\sigma}\end{aligned}$$

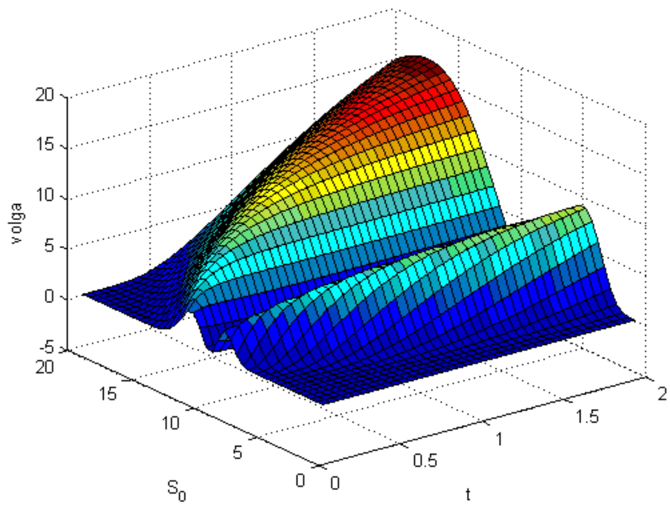

Figure 8: The spot Volga of an European call/put option with $S_0 \in [1, 20]$, $T \in [0.1, 2]$, $K = 10$, $r_d = 0.05$, $r_f = 0$ and $\sigma = 0.3$

2.6 The Foreign Exchange Market

The FX market is today one of the biggest and most liquid (up to about maturity of 2 years) markets, the bank for International Settlements reported in their finally year report of 2007 that the daily turnover on the FX market was approximately \$3.2 trillion[5]. The major of these transaction are done from the financial metropolises New York ($\sim 33\%$) and London ($\sim 20\%$).

The *spot exchange rate*, is the exchange between two currencies, i.e. the amount that ones have to pay in one currency to receive units in another currency. All these exchanges are accomplished through market makers, that is why we have the bid-ask spread, the difference between the rate at which the currency is purchased from and the rate that it is sold to these market makers. The FX market can be seen as a market that never sleeps, its open 24 hours from UTC 22:00 on Sunday and until 22:00 UTC Friday, and this compared to for instance the stock market, which closes for the day when the time hits the closing hour.

If we start with the assumption that we have a domestic currency (US dollars, \$) with the domestic interest rate r_d and a foreign currency (Euro, €) with interest rate r_f, and these interest rates will be assumed to be deterministic. The quotation or the exchange rate is defined how much one need to pay on the domestic currency to buy on unit of the foreign currency, and is according to the market convention quotation on the following form FOR-DOM (foreign-domestic) i.e. EUR/USD which means that one unit of EURO cost EUR/USD units of USD. Let S_t define the current spot exchange rate at time t, S_t is then

[5]1 trillion $= 10^{12}$

defined as

$$S_t = \frac{\text{units of the domestic currency}}{\text{units of the foreign curreny}}$$

On the 23 of April, 2009 the mid price of the EUR/USD was 1.3063, this means that one will have to pay \$1.3063 to receive 1€. The meaning of domestic and foreign should not be taken literally, instead it is only related to that the domestic currency is the currency that one will use as a numeraire or base currency.

There is a typically FX trading floor language and conventions, FX rates are usually quoted up to the first five relevant figures, e.g. the 23 April, 2009 is the EUR/JPY spot quoted as 128.66 and EUR/USD 1.2290. The last digit is called a *'pip'* and the middle digit *'big'*, since on trading floors the third often is displayed in a much bigger size compared to the other since it contains the most relevant information about the currency pair. A million is referred as a buck, and one billion as a yard. Some of the popular currency pairs have been given specific nicknames, for instance GBP/USD is named *cable*, since the FX information is sent between USA and Great Brittan through a cable in the Atlantic Ocean. EUR/JPY is called the *cross* since it is defined as the cross rate between the more traded USD/JPY and EUR/USD.

There exists typically 6 different ways of quoting vanilla options, often like in the Black-Scholes formula case are vanilla options quoted as d pips, and the other five ways are determined by the following relationship

$$d \text{ pips} \overset{\times \frac{1}{S_0}}{\rightarrow} \%f \overset{\times \frac{S_0}{K}}{\rightarrow} \%d \overset{\times \frac{1}{S_0}}{\rightarrow} f \text{ pips} \overset{\times S_0 K}{\rightarrow} d \text{ pips}$$

These 6 different standard quotation ways are listed the following example in Table 1 collected from Wystrup (2006)

Name	Symbol	Value in units of	Example
domestic cash	d	DOM	29,148 USD
foreign cash	f	FOR	24,290 EUR
% domestic	$\%d$	DOM/DOM	2.3318% USD
% foreign	$\%f$	FOR/FOR	2.4290% EUR
domestic pips	d pips	DOM/FOR	291.48 USD pips/EUR
foreign pips	f pips	FOR/DOM	194.32 EUR pips/USD

Table 1: Example: Quotation of option prices. FOR = EUR, DOM = USD, S0 = 1.2000, $r_d = 3\%$, $r_d = 2.5\%$, $\sigma = 0.10$, $K = 1.2500$,$T = 1$.

The payment date and expiry in FX are usually defined by 4 different dates: *trade date, spot date, expiry date* and *delivery date* (Lauritsen 2008b). The period between the trade date and the expiry date is the expiry term for options and the period between spot date and delivery date is the forward outright expiry term (deposit), illustrated in the following figure.

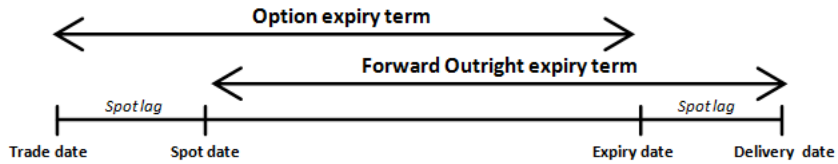

Figure 9: The Option Expiry term and Forward Outright expiry term.

The Forward Outright contract allows anyone to buy/sell a currency at a specified date for a specified rate in the future, the expiry date for this type of contracts will be referred as the delivery date. The *Spot date* for a currency cross is the first common day that is the second *good business day*[6]. For options with over the night and week expiries the expiry term is found as the first *open day*[7] after the trade date. For options with month and year expires the expiry date are determined from the Forward Outright contract by identifying the spot date, and the expiry date from the Forward Outright contract and finally by step back the length of the spot lag to determine the option expiry date for this contract. And the option expiry day is the first open day which is the second previous open day from the delivery date.

2.6.1 Black Scholes in FX

We will build our assumption that the spot exchange rate S_t follows a GBM.

Theorem 2.20: (The FX dynamics) Let S_t define the current spot FX rate, r_d and r_f the domestic and foreign interest rate and B_d and B_f the analogue MMA, we have the following FX dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_d &= r_d B_d dt \\ dB_f &= r_f B_f dt \end{aligned}$$

where μ, σ, r_d and r_f are deterministic constants and $W_t \in N(0,1)$.

We will also build all our models on the assumption that the market is frictionless and liquid. From a domestic point of view we can see r_f as being the same thing as the dividend for stocks, this since the holder of the foreign currency have the possibility to invest in the foreign MMA. If we return to the assumption on the FX dynamics and the view of a domestic investor which faces two types of market assets: the domestic MMA B_d and the value of the FX MMA given by $B_f S_t$, apply Itô's formula here the following relationship holds

[6]A day that not is a holiday of the two currencies, a US holiday or a weekend.
[7]A day that is not part of the weekend or 1st of January.

$$d\left(B_f S_t\right) = \left(r_f + \mu\right) B_f S_t dt + \sigma B_f S_t dW_t$$

using the domestic MMA as a numeraire and by introducing $\left(B_f S_t\right)^{'} = B_f S_t / B_d$ we have the following process

$$d\left(B_f S_t\right)^{'} = \left(r_f - r_d + \mu\right)\left(B_f S_t\right)^{'} dt + \sigma \left(B_f S_t\right)^{'} dW_t$$

By applying Girsanov's theorem with the unique Girsanov kernel $g = -\left(r_f - r_d + \mu\right)$ will then give us the unique equivalent martingale measure $\mathbb{Q}^d$ and thus making the process $\left(B_f S_t\right)^{'}$ a martingale and will we finally end up with the famous extended Black-Scholes formula known as the Garman and Kohlhagen formula (1983).

□

Theorem 2.21: (Garman and Kohlhagen FX Formula)

Given the risk neutral domestic measure $\mathbb{Q}^d$, using the domestic MMA B_d as numeraire the arbitrage free price $\Pi\left(t, \Phi\right)$ for the payoff $\Phi\left(S_t\right)$ and by $\Pi\left(t, \Phi\right) = F\left(t, S_t\right)$, where

$$F\left(t, s\right) = e^{-r_d (T-t)} E^{\mathbb{Q}^d}\left[\Phi\left(S_t\right) | \mathcal{F}_t^S\right] \tag{6}$$

our FX exchange rate $\mathbb{Q}$ dynamics will evolve according to

$$dS_t = \left(r_d - r_f\right) S_t dt + \sigma S_t dW_t^{\mathbb{Q}^d} \tag{7}$$

$F\left(t, S_t\right)$ can also be determined as the solution of the PDE by applying Feynman-Kač representation

$$\begin{aligned} \frac{\partial F}{\partial t} + s\left(r_d - r_f\right)\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2} - r_d F &= 0 \\ F\left(T, s\right) &= \Phi\left(s\right) \end{aligned}$$

or as a closed form solution on a modified Black-Scholes formula

$$\Pi = S_0 e^{-r_f (T-t)} N\left[d\right] - K e^{-r_d (T-t)} N\left[d - \sigma\sqrt{T-t}\right]$$

where

$$d = \frac{\ln \frac{S_0}{K} + \left(r_d - r_f + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Proof. Omitted, See Garman and Kohlhagen (1983)

The situation when $r_d > r_f$ is called *contango*, and *backwardation* when the opposite situation $r_d < r_f$ is true.

2.6.2 FX Correlations

To determine the FX correlation coefficients between each corresponding currency pair is not an easy process, it can either be done by observing historical data or by implied calibration. But since we in FX are trading currency pairs we can easily determine the correlation from these contracts. Let us illustrate a small FX market with only 3 currencies and thus 3 currency pairs.

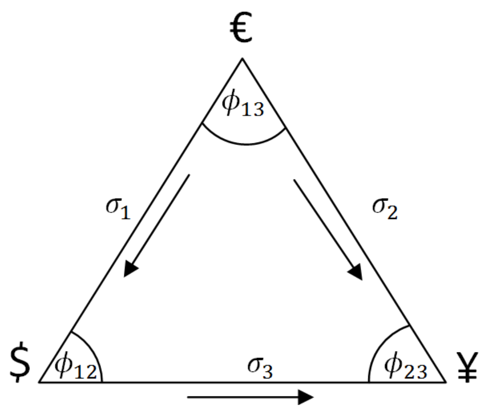

Figure 10: The traingle determines the relationship between the volatilities σ (edges) and the correlations ρ (cosinus of the angles) in a small FX market with only 3 currencies and 3 currency pairs, the arrows determines the standard quotation.

We will find an explicit formula for the correlation coefficient between the currency pairs, as usual let σ_i be the volatility of S_i, σ_{ij} the volatility of the spot FX rate between the currencies i and j, and ρ_{ij} the correlation between the currency pairs and we have the following relationship

$$\begin{aligned}
\mathrm{Cov}\left[\ln S_i(t), \ln S_j(t)\right] &= \mathrm{Corr}\left[\ln S_i(t), \ln S_j(t)\right] \sqrt{\mathrm{Var}\left[\ln S_i(t)\right]}\sqrt{\mathrm{Var}\left[\ln S_j(t)\right]} \\
\mathrm{Var}\left[\ln S_i(t)\right] &= \sigma_i^2 t \\
\rho_{ij} &= \frac{\mathrm{Cov}\left[\ln S_i(t), \ln S_j(t)\right]}{\sqrt{\mathrm{Var}\left[\ln S_i(t)\right]}\sqrt{\mathrm{Var}\left[\ln S_j(t)\right]}}
\end{aligned}$$

the second equality follows from the fact that we have a GBM. Consider figure 10 and let Euro € be the base currency and USD \$ and Yen ¥ be the two currencies in the basket. As usual let $S_{\$/€}$ determine the spot exchange rate between Euro and USD, which also can be expressed as

$$S_{\$/€} = S_{\$/¥} \cdot S_{¥/€}$$

then

$$\begin{aligned}
\frac{1}{S_{\$/€}} &= \frac{1}{S_{\$/¥}} \cdot \frac{1}{S_{¥/€}} \Rightarrow \\
S_{€/\$} &= S_{¥/\$} \cdot S_{€/¥} \Rightarrow \\
S_{¥/\$} &= \frac{S_{€/\$}}{S_{€/¥}} \Rightarrow \\
\ln S_{¥/\$} &= \ln S_{€/\$} - \ln S_{€/¥} \Rightarrow \\
\mathrm{Var}\left[\ln S_{¥/\$}\right] &= Var\left[\ln S_{€/\$}\right] + Var\left[\ln S_{€/¥}\right] - 2\mathrm{Cov}\left[\ln S_{€/\$}, \ln S_{€/¥}\right] \Rightarrow \\
\sigma_{¥/\$} t &= \sigma_{€/\$} t + \sigma_{€/¥} t - 2\mathrm{Corr}\left[\ln S_{€/\$}, \ln S_{€/¥}\right] \sigma_{€/\$}\sigma_{€/¥} t \Rightarrow \\
\mathrm{Corr}\left[\ln S_{€/\$}, \ln S_{€/¥}\right] &= \frac{\sigma_{€/\$} t + \sigma_{€/¥} t - \sigma_{¥/\$} t}{2\sigma_{€/\$}\sigma_{€/¥} t} = \frac{\sigma_{€/\$} + \sigma_{€/¥} - \sigma_{¥/\$}}{2\sigma_{€/\$}\sigma_{€/¥}}
\end{aligned}$$

□

we summarize the result in the following theorem

Theorem 2.22: (Correlation in FX) *Assume a market with 3 spot exchange rates, the correlation coefficient between two currencies are determined by*

$$\rho_{ij} = \frac{\sigma_i^2 + \sigma_j^2 - \sigma_{ij}^2}{2\sigma_i\sigma_j}$$

for $i \neq j$, where σ_i and σ_j determines the volatility and σ_{ij}^2 the covariance between two currency pairs. And the following properties must hold

$$|\sigma_i - \sigma_j| < \sigma_{ij} < \sigma_i + \sigma_j \text{ with } \sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j \Rightarrow$$

$$\rho_{ij} \in [0, 1]$$

and when $\rho_{ij} = 1$, we have that $\sigma_i = \sigma_j$.

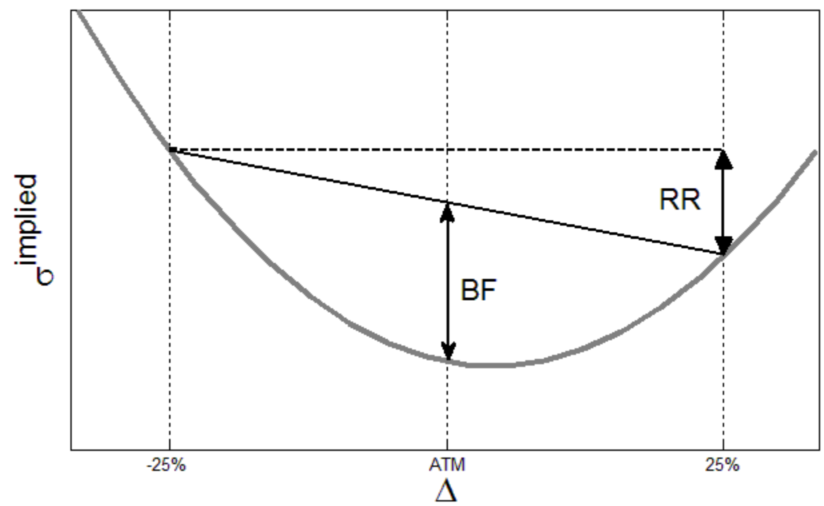

Figure 11: The Risk reversal and butterfly (or strangles) in terms of implied volatility for a FX option smile.

2.7 Volatility Smile

In the foreign exchange (FX) are options with strikes and maturity that differs from each other are priced with different implied volatilities, known as the smile/skew effect and these FX options are price according to their delta Δ^8. This means that each time the underlying asset S_t at time t changes, changing the delta a new implied volatility need to be considered. Figure 11 demonstrates how the Risk reversal and the Butterfly (or strangles) is used to determine the skewness from implied volatility in FX assets, which will used later on trying to recreate the Smile.

The assumption on constant volatility in the Black-Scholes model is one of the drawbacks resulting in the phenomenon called the volatility smile. To demonstrate the volatility smile the implied volatility is introduced where we calculates the volatility of an option given the option price, stock price, strike price etc. The implied volatility is defined by finding the inverse of the Black-Scholes formula, the following function for the call option

$$\sigma_{imp} = C_{BS}^{-1}\left(\Pi_t, S_0, K, r, T\right)$$

But the problem is thus that there does not directly exists an inverse of the Black-Scholes formula, but it can be solved a numerical procedure. Solving the following relationship

$$C^M - C^{BS}\left(\sigma_{imp}\right) = 0$$

[8]E.g. a 35Δ call is a call whose Delta is 0.35. Analogously, a 35△ put is one whose Delta is -0.35.

which can be solved by using the Bisection Method described in the section for numerical analysis.

	1W	1M	2M	3M	6M	9M	1Y
10ΔC	15.47	17.01	17.56	18.04	18.69	18.87	19.12
15ΔC	15.30	16.69	17.11	17.45	17.90	18.02	18.19
20ΔC	15.15	16.43	16.74	16.96	17.25	17.34	17.45
25ΔC	15.03	16.23	16.45	16.59	16.76	16.83	16.90
35ΔC	14.85	15.98	16.11	16.16	16.22	16.27	16.31
ATM	14.75	15.90	16.00	16.05	16.10	16.15	16.2
35ΔP	14.77	16.03	16.19	16.28	16.37	16.43	16.47
25ΔP	14.88	16.33	16.60	16.81	17.04	17.13	17.20
20ΔP	14.97	16.56	16.93	17.25	17.60	17.72	17.83
15ΔP	15.08	16.85	17.35	17.80	18.33	18.49	18.66
10ΔP	15.23	17.19	17.84	18.46	19.20	19.42	19.67

Table 2: Implied volatility for EURUSD at the 10 March, 2009.

The following picture demonstrates the smile effect by using a linear interpolating technique for plotting data collected for the currency mid pair EURUSD quoted at 1.2714 the 10 March 2009. Strikes lower than the ATM strike are strikes of Put options and strikes higher are strikes of Call options, this method is more or less a standard way of describing the volatility in the FX market.

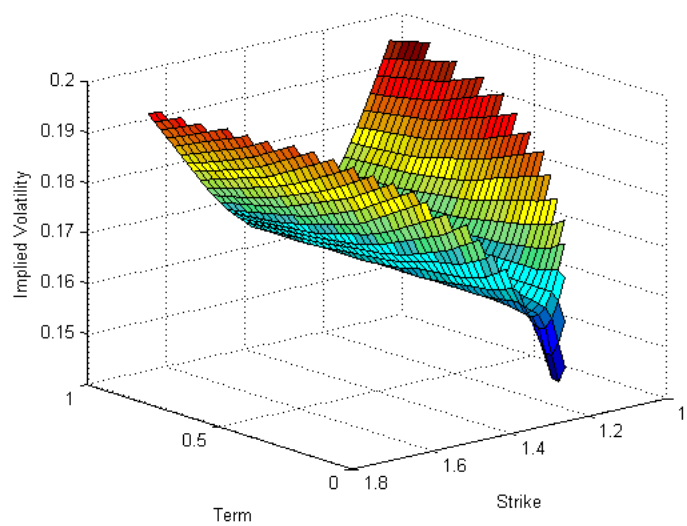

Figure 12: The implied volatility for the mid price of the currency pair EURUSD at the 10 March, 2009. Values has been converted from delta space inte strike space.

2.8 Strike From Delta

We will later on deduce how to determine the strike price from delta. There are two different scenarios, the first on when the option is quoted in the domestic currency (the right delta) and the second approach when the option is quoted in the foreign currency (the left delta) which makes the calculation a little bit heavier since we require some form of transformation.

2.8.1 Options Quoted in the Domestic Currency (Right Delta)

The spot delta will be defined as the absolute value of the partial derivative of the option value Π w.r.t. the underlying asset S

$$\Delta_s = \left| \frac{\partial \Pi \left(S, K \right)}{\partial S} \right|$$

we know that the modified Black-Scholes spot delta is including the foreign interest rate r_f is defined as

$$\Delta_s = e^{-r_f T} N \left(\alpha d \left(K \right) \right)$$

and we know that

$$d = \frac{\ln \frac{S_0}{K} + \left(r_d - r_f + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

solving for K we have that

$$K = F \exp \left\{ -\alpha N^{-1} \left(\Delta_s e^{r_f T} \right) \sigma \sqrt{T} + \frac{1}{2} \sigma^2 T \right\}$$

$\alpha = 1$ for call and $\alpha = -1$ for put, and where N^{-1} is the inverse of a standard normal cdf, and where the forward value is defined in the usual way

$$F = S e^{(r_d - r_f) T}$$

If the strike is quoted on the forward delta Δ_f is defined in the following way

$$\begin{aligned} \Delta_f &= \Delta_s e^{r_f T} \\ &= e^{r_f T} e^{-r_f T} N \left(\alpha d \left(K \right) \right) \\ &= N \left(\alpha d \left(K \right) \right) \end{aligned}$$

solving for K one receives

$$K = F \exp \left\{ -\alpha N^{-1} \left(\Delta_f \right) \sigma \sqrt{T} + \frac{1}{2} \sigma^2 T \right\}$$

□

The relationships are summarized below

$$K = \begin{cases} F \exp \left\{ -\alpha N^{-1} \left(\Delta_s e^{r_f T} \right) \sigma \sqrt{T} + \frac{1}{2} \sigma^2 T \right\}, & spot - \text{delta} \\ \\ F \exp \left\{ -\alpha N^{-1} \left(\Delta_f \right) \sigma \sqrt{T} + \frac{1}{2} \sigma^2 T \right\}, & forward - \text{delta} \end{cases}$$

2.8.2 Options Quoted in the Foreign Currency (Left Delta)

If the option is quoted in the foreign currency we need to find the delta in the foreign currency and thereafter transform the delta into the domestic currency, the right delta. The spot delta in the foreign currency is defined as

$$\Delta_s = \frac{1}{S} \left| \frac{\partial \frac{\Pi(S,K)}{S}}{\partial \left(\frac{1}{S} \right)} \right|$$

we have thus the following relationship between the spot delta Δ_s, and the forward delta Δ_f

$$\Delta_s = \frac{K}{S} \exp\left(-r_d T\right) N\left(\alpha\left(d\left(K\right) - \sigma\sqrt{T}\right)\right)$$
$$\Delta_f = \frac{K}{S} \exp\left(\left(r_f - r_d\right) T\right) N\left(\alpha\left(d\left(K\right) - \sigma\sqrt{T}\right)\right)$$

□

the systems of equations above can be solved by a numerical procedure.

To be able to solve the ATM strike, we can use several techniques as presented in Lauritsen (2008), UBS retrieves the ATM strike for the strike K that solves the 0-delta straddle[9]

2.8.3 The ATM Strike for options quoted om the domestic currency with spot delta

As mentioned above, UBS determines the ATM strike solving the 0-delta straddle

$$\frac{\partial}{\partial S}\left(C\left(S,K\right) + P\left(S,K\right)\right) = 0$$
$$d\left(K\right) = 0$$

receiving that K is defined as

$$K_{ATM} = Fe^{\frac{1}{2}\sigma^2 T}$$

□

2.8.4 The ATM strike for options quoted in the foreign currency with spot delta

When the option is quoted in the foreign currency instead we need once again transform the delta into the domestic currency

[9]An option that consists of both a Call and a Put with some strike K, defined as $C\left(S,K\right) + P\left(S,K\right)$.

$$0 = \frac{1}{S} \frac{\partial}{\partial \left(\frac{1}{S}\right)} \left(\frac{C(S,K) + P(S,K)}{S} \right)$$

$$\Leftrightarrow$$

$$0 = \frac{1}{S} \frac{\partial}{\partial \left(\frac{1}{S}\right)} \left(\frac{C(S,K)}{S} \right) + \frac{1}{S} \frac{\partial}{\partial \left(\frac{1}{S}\right)} \left(\frac{P(S,K)}{S} \right)$$

$$\Leftrightarrow$$

$$0 = -\frac{K}{S} \exp(-r_d T) N \left(\alpha \left(d(K) - \sigma\sqrt{T} \right) \right) + \frac{K}{S} \exp(-r_d T) N \left(\alpha \left(d(K) - \sigma\sqrt{T} \right) \right)$$

$$\Leftrightarrow$$

$$0 = d(K) - \sigma\sqrt{T}$$

solving for one receives the following relationship

$$K_{ATM} = F e^{-\frac{1}{2}\sigma^2 T}$$

□

The relationships are summarized below

$$K_{ATM} = \begin{cases} K_{ATM} = F e^{\frac{1}{2}\sigma^2 T}, & \text{foreign currency} \\ \\ K_{ATM} = F e^{-\frac{1}{2}\sigma^2 T}, & \text{domestic currency} \end{cases}$$

2.9 Approximating the Smile

Castagna and Marcurio (2006) demonstrated two simple approximations performing some great results, the approximation is built on three different implied volatilities, σ_{ATM}, $\sigma_{25\Delta C}$ and $\sigma_{25\Delta P}$. The ATM volatility are easy determined from the 0Δ straddle, the $\sigma_{25\Delta C}$ and $\sigma_{25\Delta P}$ are thus deduced from the risk reversal (RR) and the vega-weighted butterfly (VWB). The volatility of RR is typically quoted as the difference between the $\sigma_{25\Delta C}$ and $\sigma_{25\Delta P}$

$$\sigma_{RR} = \sigma_{25\Delta C} - \sigma_{25\Delta P}$$

The VWB volatility are defined as

$$\sigma_{VWB} = \frac{\sigma_{25\Delta C} + -\sigma_{25\Delta P}}{2} - \sigma_{ATM}$$

We need to retrieve $\sigma_{25\Delta C}$ and $\sigma_{25\Delta P}$ from the two system of equations above, which are easily solved and one receives

$$\sigma_{25\Delta C} = \sigma_{ATM} + \sigma_{VWB} + \frac{1}{2}\sigma_{RR}$$
$$\sigma_{25\Delta P} = \sigma_{ATM} + \sigma_{VWB} - \frac{1}{2}\sigma_{RR}$$

The different strikes are thereafter easily determined and the following condition holds

$$K_{25\Delta P} \leq K_{ATM} \leq K_{25\Delta C}$$

Castagna and Marcurio (2006) proposed the following second order approximation

Theorem 2.16: (Volatility Smile Approximation) *Knowing the following level of strikes $K_{25\Delta P}$, K_{ATM} and $K_{25\Delta C}$ the volatility smile can by approximated by the following formula for a given level of strike K and the volatility parameter σ*[10]

$$\sigma(K) = \sigma + \frac{-\sigma + \sqrt{\sigma^2 + d_1(K)\, d_2(K)\left(2\sigma D_1(K) + D_2(K)\right)}}{d_1(K)\, d_2(K)}$$

where

$$\begin{aligned}
D_1(K) =& \frac{\ln\frac{K_{ATM}}{K}\ln\frac{K_{25\Delta C}}{K}}{\ln\frac{K_{ATM}}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K_{25\Delta P}}}\sigma_{25\Delta P} + \frac{\ln\frac{K}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K}}{\ln\frac{K_{ATM}}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K_{ATM}}}\sigma_{ATM} \\
&+ \frac{\ln\frac{K}{K_{25\Delta P}}\ln\frac{K}{K_{ATM}}}{\ln\frac{K}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K_{ATM}}}\sigma_{25\Delta C} - \sigma \\
D_2(K) =& \frac{\ln\frac{K_{ATM}}{K}\ln\frac{K_{25\Delta C}}{K}}{\ln\frac{K_{ATM}}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K_{25\Delta P}}} d_1(K_{25\Delta P})\, d_2(K_{25\Delta P})\left(\sigma_{25\Delta P} - \sigma\right)^2 \\
&+ \frac{\ln\frac{K}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K}}{\ln\frac{K_{ATM}}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K_{ATM}}} d_1(K_{ATM})\, d_2(K_{ATM})\left(\sigma_{ATM} - \sigma\right)^2 \\
&+ \frac{\ln\frac{K}{K_{25\Delta P}}\ln\frac{K}{K_{ATM}}}{\ln\frac{K}{K_{25\Delta P}}\ln\frac{K_{25\Delta C}}{K_{ATM}}} d_1(K_{25\Delta C})\, d_2(K_{25\Delta C})\left(\sigma_{25\Delta C} - \sigma\right)^2
\end{aligned}$$

and

$$d_1(K) = \frac{\ln\frac{S_0}{K} + \left(r_d - r_f + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2(K) = d_1(K) - \sigma\sqrt{T}$$

[10] *Often chosen as* σ_{ATM}

Proof. See Castagna and Marcurio (2006)

These approximations are illustrated in the following figure, the crosses determines the observed implied volatilities and the lines are our approximations. We can conclude that our approximations fit the implied volatility really well.

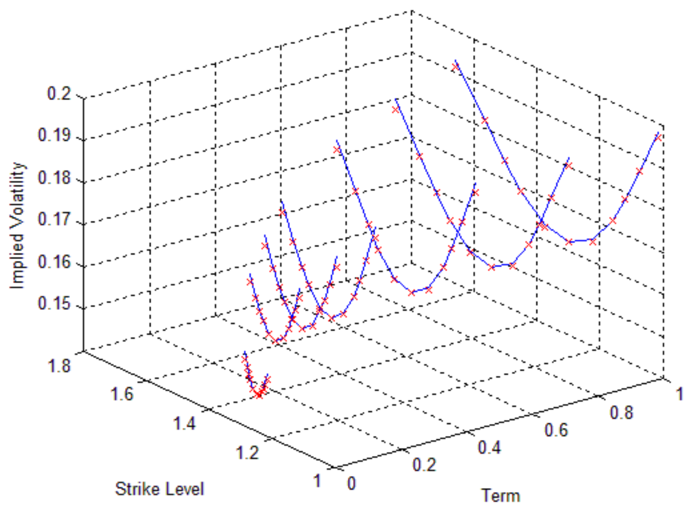

Figure 13: Implied Volatility approximation.

2.10 Basket Option

Throughout this thesis we will assume that the underlying assets of a Basket option follows a log-normal process and thus build our assumption by simulating each underlying asset by a GBM, where we have N correlated assets in our Basket, and thus each assets can be written as:

$$dS_i(t) = \mu_i S_i(t)\, dt + S_i(t) \sum_{j=1}^{N} \Omega_{ij} dW_j(t) \tag{8}$$

$$\text{Cov}\left[dW_i(t)\, dW_j(t)\right] = p_{ij} dt, \quad i \neq j$$

Since this thesis will mainly be concentrated on the FX market, μ_i will here denote the drift of the i-th currency pair, or the difference between the domestic and the foreign currency rate. The Ω_{ij} can be decomposed by applying the Cholesky decomposition, deduced later on which can be seen as a straight forward LU factorization of the covariance matrix $C_{ij} = \left(\Omega\Omega^T\right)_{ij} = \rho_{ij}\sigma_i\sigma_j$,

$$C = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_N \end{bmatrix} \begin{bmatrix} \rho_{11} & \cdots & \rho_{1N} \\ \vdots & \ddots & \vdots \\ \rho_{N1} & \cdots & \rho_{NN} \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_N \end{bmatrix}$$

where σ_i represents the volatility of asset i and ρ_{ij} the correlation between asset i and j. W_i and W_j are just correlated standard Brownian motions. Process (8) can also be applied to for instance a basket of only stocks, but μ_i will then under the $\mathbb{Q}$ measure be defined as the difference between the interest rate r and the dividend yield of the individual stock q_i, $\mu_i = r - q_i$.

The solution to the geometric Brownian motion is now given by

$$S_i(t) = S_i(0) \exp \left\{ \left(\mu_i - \frac{1}{2}\sigma_i^2 \right) t + \sum_{j=1}^{N} \Omega_{ij} W_j(t) \right\}$$

The payoff of the exotic European Basket call option is defined in the following way

$$\Phi(B(t)) = (B(t) - K)^+ = \left(\sum_{i=1}^{N} w_i S_i(t) - K \right)^+ \tag{9}$$

Thus can the value of the Basket option be determined by using the risk-neutral valuation formula and under the risk neutral measure $\mathbb{Q}$, the value is given by the stated expression below

$$\Pi_B(t) = e^{-rT} E^{\mathbb{Q}} \left[(B(tT) - K)^+ |\mathcal{F}_t \right] = e^{-rT} \left[\int_0^\infty (B(x) - K)^+ p(B(T)/B(0))\, dx \right]$$

where $\mathcal{F}_t$ denotes the information about the underlying assets up until time t, $p(x)$ the state-price density, SPD or the risk neutral pdf and $P(x)$ is the cumulative distribution function. Let N be denoting the number of assets, w_i the fraction of asset i satisfying $\sum_i w_i = 1$ for $i = 1, \ldots, N$, and K the strike price. The analogue European basket put can be derived by applying the put-call parity for Basket options as in Laurence and Wang (2003)

$$\left(K - \sum_{i=1}^{N} w_i S_i(t) \right)^+ = \left(\sum_{i=1}^{N} w_i S_i(t) - K \right)^+ + \left(K - \sum_{i=1}^{N} w_i S_i(t) \right)$$

The moments of a Basket option can be derived by matching the moments to a log-normal distribution. In a risk neutral world we are assuming that all assets are growing by the interest rate r, and in our case $\mu_i = r_d - r_{f_i}$, let us define F_i as the individual forward value

$$F_i = S_i\left(0\right)e^{\mu_i T}$$

and the Basket forward value F as

$$F = \sum_{i=1}^{N} w_i F_i$$

2.10.1 The Basket Moments

The non-normalized moments of the Basket option are defined in the following way.

$$\begin{aligned}
M_1 &= \sum_{i} w_i F_i \\
M_2 &= \sum_{i,j} w_i w_j F_i F_j e^{\rho_{i,j}\sigma_i\sigma_j T} \\
M_3 &= \sum_{i,j,k} w_i w_j w_k F_i F_j F_k e^{\rho_{i,j}\sigma_i\sigma_j T+\rho_{i,k}\sigma_i\sigma_k T+\rho_{j,k}\sigma_j\sigma_k T} \\
M_4 &= \sum_{i,j,k,l} w_i w_j w_k w_l F_i F_j F_k F_l e^{\rho_{i,j}\sigma_i\sigma_j T+\rho_{i,k}\sigma_i\sigma_k T+\rho_{i,l}\sigma_i\sigma_l T+\rho_{j,k}\sigma_j\sigma_k T+\rho_{j,l}\sigma_j\sigma_l T+\rho_{k,l}\sigma_k\sigma_l T}
\end{aligned} \tag{10}$$

2.10.2 Upper and Lower bound

For the Basket option the following condition holds, where the lower is the Geometric average.

$$\left(\prod_{i=1}^{N} S_i^{w_i} - K\right)^{+} \leq \left(\sum_{i=1}^{N} w_i S_i\left(t\right) - K\right)^{+} \leq \sum_{i=1}^{N} w_i\left(S_i\left(t\right) - k_i\right)^{+}$$

where

$$\sum_{i=1}^{N} w_i k_i = K$$

Proof. Omitted

□

3 Numerical Methods

3.1 Monte Carlo

To verify the stated approximations later on we are matching them against Monte Carlo (MC) simulations. By simulating the Basket process a large number of times its value will eventually converge towards the real value. Anyone familiar with Monte Carlo simulations knows that it is very time consuming and/or computer intensive.

The Basic idea of MC is to approximate an integral by taking the average of some sequence of simulated paths. Say for instance that we want to evaluate the following integral

$$I = E\left[\phi\left(x\right)\right] = \int \phi\left(x\right) f\left(x\right) dx$$

where $X \in \mathbb{R}^d$, $\phi : \mathbb{R}^d \to \mathbb{R}$ and where f is the pdf. of X. $I = E\left[\phi\left(x\right)\right]$ can then be approximated in the following way

1. Draw N values $x_{1,} \ldots, x_N$ i.i.d from f.
2. The integral can then be evaluated as

$$I \approx \frac{1}{N}\sum_{i=1}^{N}\phi\left(x_i\right) \tag{11}$$

□

Monte Carlo simulation is built on two famous theorems: the Law of Large Numbers and the Central Limit Theorem (Sköld 2006:28).

Theorem 3.1: (A Law of Large Numbers)

Assume $X_1, \ldots, X_n$ is a sequence of independent random variables with common means $E\left[X_i\right] = \tau$ and variance $Var\left[X_i\right] = \sigma^2$. If $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, and such as the following condition holds almost surely

$$P\left(T_n \to \tau\right) = 1 \quad \text{as } n \to \infty$$

This means that our approximation will converge towards the real value as number of simulations tends to infinity. More precise information on the Monte-Carlo error $\left(T_n - \tau\right)$ is given by the Central Limit Theorem (CLT):

Theorem 3.2: (Central Limit Theorem)

Assume $X_1, \ldots, X_n$ is a sequence of i.i.d. random variables with common means $E[X_i] = \tau$ and variance $Var[X_i] = \sigma^2$. If $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, we have:

$$P\left(\frac{\sqrt{n}(T_n - \tau)}{\sigma} \leq x\right) \to \Phi(x) \qquad as\ \ n \to \infty$$

where $\Phi(x)$ is the distribution function of the $N(0,1)$ distribution.

Slightly less formally, the CLT tells us that the difference $(T_n - \tau)$ has, at least for large n, approximately an $N\left(0, \frac{\sigma^2}{n}\right)$ distribution. With this information we can approximate probabilities like $P(|T_n - \tau| < \epsilon)$, and perhaps more importantly find ϵ such that $P(|T_n - \tau| < \epsilon) = 1 - \alpha$ for some specified confidence level α, and we have that the MC approximation converges with a rate of $\mathcal{O}\left(n^{-1/2}\right)$.

3.1.1 Antithetic Variates

There exists a couple of Monte Carlo simulation techniques, we will extend the crude MC technique by simulation using the variance reduction technique Antithetic Variates by introducing a negative dependence between each replication. The Antithetic Variates is defined in the following way (Rasmus 2008:160)

1. Sample n replicates of $z_i \in N(0,1)$
2. Set $s_i = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z_i\right\}$
3. Set $c_i = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}z_i\right\}$
4. The Antithetic Variate estimator is

$$\hat{\pi}_{av} = \frac{\sum_{i=1}^{n}\left(\Phi(s_i) + \Phi(c_i)\right)}{2n}$$

The main idea with Antithetic Variates is that the outcome calculated by the first path will be balanced by the value calculated from the second path, or the Antithetic path, and thus that the variance is reduced. Let have a look why this work. Assume a random variable X and its antithetic variable $\tilde{X}$, the variance can be written as

$$\begin{aligned} \text{Var}\left[\frac{X + \tilde{X}}{2}\right] &= \frac{\text{Var}[X]}{4} + \frac{\text{Var}\left[\tilde{X}\right]}{4} + \frac{2\text{Cov}\left[X, \tilde{X}\right]}{4} \\ &= \frac{\text{Var}[X]}{2}\left(1 + \text{Corr}\left[X, \tilde{X}\right]\right) \\ &\leq \text{Var}[X] \end{aligned}$$

if Corr $\left[X, \tilde{X}\right] < 0$ the following relationship holds instead.

$$\text{Var}\left[\frac{X + \tilde{X}}{2}\right] < \frac{\text{Var}\left[X\right]}{2}$$

□

we know that X and $\tilde{X}$ have the same variance, in order to reduce the variance we need that the covariance between the both variables are negative Cov $\left[X, \tilde{X}\right] < 0$ and that is why we try to produce negative correlated pairs. As this technique can reduce the variance it can also increase it.

3.2 Quasi-Monte Carlo

Quasi-Monte Carlo (qMC) is an alternative to MC based on low-discrepancy sequences, instead of randomness as for ordinary MC. The mainly motivation for qMC is that it hopefully will have faster convergence compared to MC from $\mathcal{O}\left(n^{-1/2}\right)$ to $\mathcal{O}\left(n^{-1}\right)$. This mean that increasing the number of simulated paths by a number of 100 will only increase the Monte Carlo accuracy by a factor of 10. Instead of be built on probability and pseudo random numbers, qMC is built on number theory and abstract algebra. As for the case of ordinary MC we are able to combine qMC with variance reduction techniques, but this must be done with a little bit of carefulness. The main goal with low-discrepancy methods is create draws x_i in (11) creating a small error as possible for a large number of draws. We will begin form the case of first construct uniformly low-discrepancy sequences (LDS) and since this is a thesis devoted to the financial area we need to transform those uniformly sequences into a Gaussian distributed sequence. We are using a LDS generated by Sobol which has been proved to be a very accurate method for generating LDS, compared to other methods. For further studies and introduction to LDS generator one can take a look at Glasserman (2003) who devotes a whole chapter on just qMC.

Discrepancy is defines how a d-dimensional vectors $\{x_i\}$ are distributed w.r.t. some subsets. Instead as for the random case where each point is chosen randomly we are now choosing next point in such a way that empty areas are filled up. This phenomenon by overcoming clustering can be observed from the following plot, where the first 512 (2^9) and 1024 (2^{10}) 2-dimensional Sobol numbers have been generated. The plot to the right are r.v. generated from the uniformed distribution Un $[0, 1]$ and for a relative small numbers of draws the empty areas are more and the points are not evenly distributed in $\mathbb{R}^2$.

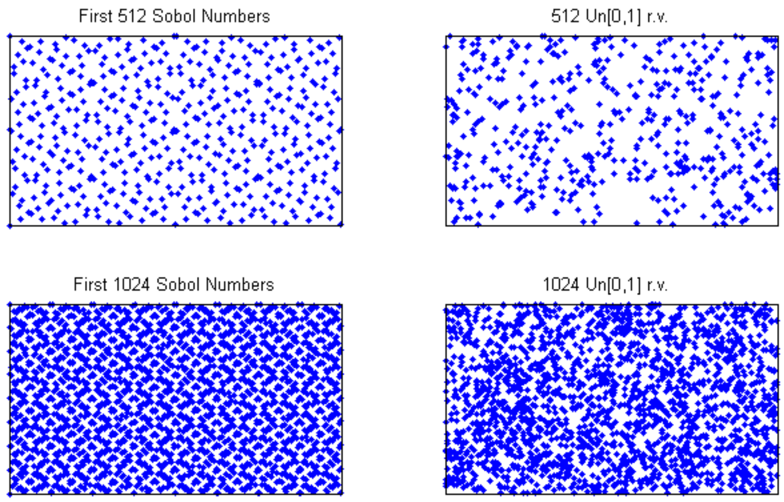

Figure 14: Left: the projection of the first 512 and 1024 points of the 2 dimensional Sobol numbers. Right: r.v. generated fromUn $[0, 1]$.

Definition 3.3: (Discrepancy) *Let $\mathcal{A}$ be a set of points in $[0,1]^d$, the discrepancy of set of points $\{x_1, \ldots, x_n\}$ relative $\mathcal{A}$ is*

$$D(x_1, \ldots, x_n; \mathcal{A}) = \sup_{A \in \mathcal{A}} \left| \frac{\#\{x_i \in A\}}{n} - vol(A) \right|$$

where $\#\{x_i \in A\}$ is the number of $x_i$in subset A and $vol(A)$ is the measure of A. And the discrepancy $D(x_1, \ldots, x_n; \mathcal{A})$ is thus the least upper bound or the supremum of error by integrating A using the vector $\{x_i\}$.

As $n \to \infty$ we will fill out our space totally and making the function $D(x_1, \ldots, x_n; \mathcal{A})$ converges towards 0.

$$\lim_{n \to \infty} D(x_1, \ldots, x_n; \mathcal{A}) = 0$$

3.3 Transforming Sequences

When we have generated the d-dimensional LDS with Sobol's sequence generator that are uniformly distributed on $[0,1]^d$ we need to transform them into a sequence that can be seen as a sequence generated from the normal distribution. A very simple transformation routine is the inverse method. Assume that we have a sequence $u_1, ..., u_n$ from $U(0,1)$, then can the sequence $x_1, ..., x_n$ be seen as a draw from F, if the inverse F^{-1} exists. We summerize the algorithm below

1. Generate $u_1, ..., u_n$ from $U(0, 1)$
2. Let $x_i = F^{-1}(u_i)$ for $i = 1, ..., n$

Proof. If U is $U(0, 1)$ and F is an arbitrary d.f. on $\mathbb{R}$ then $X = F^{-1}(U)$ has distribution function F. This follows by:

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) \\ &= F(x) \end{aligned}$$

□

3.4 Bisection Method

To be able to solve equations numerically we will later on use the Bisection Method. The Bisection Method is an iterative method for finding root in some interval is described in the following pseudo code.

Algorithm 1 Bisection Method

Given an initial interval $[a, b]$ and a tolerance level TOL

```
while (b-a)/2 > TOL
    c = (a + b)/2
    if f(c) == 0 stop, end
    else if f(a)f(c) < 0
        b = c
    else
        a = c
    end
end
```

The final interval $[a, b]$ will contain the root and the approximate root is given by $(a + b)/2$

3.5 Root Mean Square Error

To be able to measure the accuracy of each approximation V_i^{Approx} stated later on we will compare them against the values obtained by MC V_i^{MC}, one way of measuring the error is by Root Mean Square Error, RMSE defined as

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{V_i^{MC} - V_i^{Approx}}{V_i^{MC}}\right)^2}$$

3.6 Cholesky Decomposition

In process (8) we need to factorize $C = \Omega\Omega^T$ and where Ω is the lower triangular Cholesky factorization of the $d \times d$ covariance matrix C. C can only be Cholesky factorization if it is positive definite.

$$C = \begin{bmatrix} \Omega_{11} & & & \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & \vdots & \ddots & \\ \Omega_{d1} & \Omega_{d2} & \cdots & \Omega_{dd} \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{21} & \cdots & \Omega_{d1} \\ & \Omega_{22} & \cdots & \Omega_{d2} \\ & & \ddots & \vdots \\ & & & \Omega_{dd} \end{bmatrix}$$

This thesis follows the algorithm presented in Glasserman (2003), each element is visited in the covariance matrix C_{ij} by looping for $j = 1, \ldots, d$ and then $i = j, \ldots, d$ and producing the following equations

$$\begin{aligned} \Omega_{11} &= C_{11} \\ \Omega_{21}\Omega_{11} &= C_{21} \\ &\vdots \\ \Omega_{d1}\Omega_{11} &= C_{d1} \\ \Omega_{21}^2 + \Omega_{22}^2 &= C_{22} \\ &\vdots \\ \Omega_{d1}^2 + \cdots + \Omega_{dd}^2 &= C_{dd} \end{aligned}$$

We have that each A_{ij} is defined as

$$A_{ij} = \frac{\left(C_{ij} - \sum_{k=1}^{j-1} \Omega_{ik}\Omega_{jk}\right)}{A_j}, \qquad j < i$$

and

$$A_{ii} = \sqrt{C_{ii} - \sum_{k=1}^{i-1} \Omega_{ik}^2}$$

□

The Cholesky factorization is presented in the following pseudo algorithm

Algorithm 2 Cholesky Factorization

Given the correlation matrix C as input, the Cholesky factorization is given

```
A = zeros of d × d dimension
for j=1,...,d
    for i=,...,d
        v(i) = C(i,j)
        for k=1,...,j-1
            v(i) = v(i) - Ω(j,k)Ω(i,k)
        end
        Ω(i,j)= v(i)/sqrt(v(j))

    end
end
return Ω
```

4 Basket Option Approximations

4.1 Geometric Average

The geometric average of a basket option can be calculated in the following way.

$$\begin{aligned}
B^{GA}(t) &= \prod_{i=1}^{n} S_i(t)^{w^i} \\
&= \prod_{i=1}^{n} S_i(0)^{w^i} e^{w_i\left(r_i - \frac{\sigma_i^2}{2}\right)T + w^i \sigma_i W_i(T)} \\
&= B(0)\, e^{\sum\limits_{i=1}^{n} w_i\left(r_i - \frac{\sigma_i^2}{2}\right)T + \sum\limits_{i=1}^{n} w_i \sigma_i W_i(T)} \\
&= B(0)\, e^{\sum\limits_{i=1}^{n} w_i\left(r_i - \frac{\sigma_i^2}{2}\right)T + \left(\sum\limits_{i,j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j\right) W(T)}
\end{aligned}$$

□

A product of log-normally distributed variables are still log-normally distributed and a geometric average option can therefore be solved by a Black-Scholes type equation.

4.2 Log-Normal Approximation

Levy (1992) approximated the Basket option by assuming that the summation of correlated assets still is log-normally distributed. The approximation is done by matching the first two normalized moments, this means that the first moment equals 1 and the second moment M_2 is

$$M_2 = \frac{1}{F^2} \sum_{i,j} w_i w_j F_i F_j e^{\rho_{i,j}\sigma_i\sigma_j T}$$

and thus can the variance be matched according to

$$\sigma^2 = \ln(M_2)$$

$$B_{Call}^{logN}(T) = e^{-r_d T}\left[F \cdot N\left(\frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2}{\sigma}\right) - K \cdot N\left(\frac{\ln\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2}{\sigma}\right)\right]$$

where $N(\cdot)$ is the cumulative distribution of a standard normal random variable. This is a very simple model, its simplicity has made that it is well used in the industry.

4.3 Reciprocal Gamma Approximation

Milevsky and Posner (1998a) showed that a summation of correlated log-normally distributed stochastic variables will converges in distribution towards a reciprocal gamma distributions when $N \to \infty$. We will therefore approximate our finite summation log-normally distributed variables as a Reciprocal Gamma distribution. A random variable is reciprocal gamma distributed if the inverse is gamma distributed. The valuing of a Basket options with correlated underlying assets is done by moment matching technique, and in our case by just considering the first two moments as input to the closed form solution approximation.

If we let M_1be and M_2 be the first two normalized moments, which mean that $M_1 = 1$ and

$$M_2 = \frac{1}{F_2} \sum_{i,j} w_i w_j F_i F_j e^{\rho_{i,j} \sigma_i \sigma_j T}$$

where

$$F_i = S_i(0) \exp\{(r_d - r_{f_i}) T\}$$

and

$$F = \sum_i w_i F_i$$

By matching M_1 and M_2 with the moments given by the gamma distribution as presented above one receives the following system of equations

$$\begin{aligned} \beta =& \frac{1}{\alpha - 1} \\ M_2 =& \frac{1}{\beta^2 (\alpha - 1)(\alpha - 2)} \end{aligned} \tag{12}$$

solving the systems above the following result holds

$$\begin{aligned} \alpha =& \frac{2M_2 - 1}{M_2 - 1} \\ \beta =& 1 - \frac{1}{M_2} \end{aligned}$$

Let $B(t)$ define the arithmetic sum of the N underlying assets $S_i(t)$ with corresponding weights w_i

$$B(t) = \sum_{i=1}^{n} w_i S_i(t)$$

Let g_R be denoting the reciprocal gamma pdf, g_Γ the corresponding gamma pdf. The goal is to solve the following integral

$$\begin{aligned}
B_{Call}^{RG}(T) &= e^{-rT} \int_0^\infty (B(T) - K)^+ \, dP(B(T)/B(0)) \\
&= e^{-rT} \int_{K/F}^\infty \left(\frac{B(T)}{F} - \frac{K}{F} \right) g_R\left(\frac{B(T)}{F}, \alpha, \beta \right) dB\left(\frac{B(T)}{F \cdot B(0)} \right) \\
&= e^{-rT} \left[\int_{K/F}^\infty x g_R(x, \alpha, \beta)\, dx - K \int_{K/F}^\infty g_R(x, \alpha, \beta)\, dx \right] \\
&= e^{-rT} \left[\int_{K/F}^\infty \frac{g_\Gamma(1/x, \alpha, \beta)}{x} dx - K \int_{K/F}^\infty \frac{g_\Gamma(1/x, \alpha, \beta)}{x^2} dx \right] \\
&= e^{-rT} \left[\int_0^{F/K} \frac{g_\Gamma(u, \alpha, \beta)}{u} du - K \int_0^{F/K} g_\Gamma(u, \alpha, \beta)\, du \right] \\
&= e^{-rT} \left[\int_0^{F/K} g_\Gamma(u, \alpha - 1, \beta)\, du - K \int_0^{F/K} g_\Gamma(u, \alpha, \beta)\, du \right] \\
&= e^{-rT} \left[F \cdot G_\Gamma\left(\frac{F}{K}, \alpha - 1, \beta \right) - K \cdot G_\Gamma\left(\frac{F}{K}, \alpha, \beta \right) \right]
\end{aligned}$$

Where P is the risk neutral probability density function, G_Γ the gamma cdf. The fourth equality follows for the following relationship for the gamma pdf

$$g_R(x, \alpha, \beta) = \frac{g_\Gamma(1/x, \alpha, \beta)}{x^2}, \quad x \geq 0,\ \alpha, \beta > 0$$

and the second last equality holds since for the gamma distribution the following condition holds

$$
\begin{aligned}
\frac{g_\Gamma(x,\alpha,\beta)}{x} &= \frac{e^{-x/\beta}(x/\beta)^{\alpha-1}}{x\beta\Gamma(\alpha)} \\
&= \frac{e^{-x/\beta}(x/\beta)^{\alpha-2}}{\beta^2\Gamma(\alpha)} \\
&= \frac{e^{-x/\beta}(x/\beta)^{\alpha-2}}{\beta\Gamma(\alpha)/(\alpha-1)} \\
&= \frac{e^{-x/\beta}(x/\beta)^{\alpha-2}}{\beta\Gamma(\alpha-1)} \\
&= g(x,\alpha-1,\beta)
\end{aligned}
$$

□

Second last equality holds from (12). The analogue put basket price be derived by apply Put-Call-parity for basket for basket options.

4.4 4M Method Approximation

Milevsky and Posner (1998b) extends the approximation by taken the first four moments into account. The four moment (4M) approximation goal is to approximate the risk neutral pdf $p(x)$ with the Johnson family for matching the initial distribution with higher moments giving better accuracy. The Johnson (1949) family consists of four variables which transforms a standard normal distribution, Z in the following way by some general function φ

$$
X = c + d\varphi^{-1}\left(\frac{Z-a}{b}\right) \Leftrightarrow Z = a + b\varphi\left(\frac{X-c}{d}\right),
$$

and where the first four moments match the Johnson transformation in a perfect way. Milevsky and Posner (1998b) derived the arbitrage free value Π_t^{4M} as

$$
\begin{aligned}
\Pi_t^{4M} &= e^{-r_d T} E^{\mathbb{Q}} \left[(B(T) - K)^+ | \mathcal{F}_t \right] \\
&= e^{-r_d T} \int_K^\infty (x - K) p(x) \, dx \\
&= e^{-r_d T} \left(\int_0^\infty x p(x) \, dx - K \int_0^\infty p(x) \, dx - \int_0^K (x - K) p(x) \, dx \right) \\
&= e^{-r_d T} \left(M_1 - K - \int_0^K (x - K) p(x) \, dx \right) \\
&= e^{-r_d T} \left(M_1 - K - \left[(x - K) \int_0^x p(z) \, dz \right]_0^K + \int_0^K \left[\int_0^x p(z) \, dz \right] dx \right) \\
&= e^{-r_d T} \left(M_1 - K + \int_0^K P(x) \, dx \right)
\end{aligned}
$$

□

where $p(x)$ is the risk neutral pdf under the $\mathbb{Q}$ measure. We will mainly focus on two types of transformations, the logarithmic case called Type I, and the hyperbolic sinus case, Type II for unbounded systems. The parameters a, b, c and d are chosen such that the four moments M_1, M_2, M_3 and M_4 are matched according to 10.

Type I: The log normal system, S_L

$$
X = c + d \exp\left(\frac{Z - a}{b} \right) \Leftrightarrow Z = a + b \ln\left(\frac{X - c}{d} \right)
$$

knowing the skewness η and the kurtosis κ, Hill, Hill and Holder (1976) used the following moment matching technique. They defined a new variable ω according to

$$
(\omega - 1)(\omega + 2)^2 = \eta^2 \Leftrightarrow
$$

$$
\omega = \frac{1}{2} \sqrt[3]{8 + 4\eta^2 + 4\sqrt{4\eta^2 + \eta^4}} + \frac{2}{\frac{1}{2} \sqrt[3]{8 + 4\eta^2 + 4\sqrt{4\eta^2 + \eta^4}}} - 1
$$

if then $\kappa = \omega^4 + 2\omega^3 + 3\omega^2 - 3$ are approximately the same we know that the skewness forces the kurtosis, and the fact that we will apply Type I as our transformation, otherwise we apply transformation Type II. For the Type I, the parameters can be chosen according the following way

$$
\begin{aligned}
b &= (\ln\omega)^{-\frac{1}{2}} \\
a &= 0.5\ln\left(\omega\left(\omega-1\right)/\xi^2\right) \\
d &= \operatorname{sign}\left(\eta\right) \\
c &= dM_1 - e^{\left(\left(\frac{1}{2b}-a\right)/b\right)}
\end{aligned}
$$

where ξ is the variance.

Type II: The unbounded system, S_U

$$X = c + d\sinh\left(\frac{Z-a}{b}\right) \Leftrightarrow Z = a + b\sinh^{-1}\left(\frac{X-c}{d}\right)$$

For Type II, the parameters can be retrieved by numerically solving the following nonlinear equation system according to the determined first four moments.

$$
\begin{aligned}
M_1 = \quad & c - de^{\frac{1}{2b^2}}\sinh\tfrac{a}{b} \\
M_2 = \quad & c^2 - \tfrac{d^2}{2}\left(e^{\frac{2}{b^2}}\cosh\tfrac{2a}{b} - 1\right) - 2cde^{\frac{1}{2b^2}}\sinh\tfrac{a}{b} \\
M_3 = \quad & c^3 - 3c^2de^{\frac{1}{2b^2}}\sinh\tfrac{a}{b} + \tfrac{3}{2}cd^2\left(e^{\frac{2}{b^2}}\cosh\tfrac{2a}{b} - 1\right) \\
& + \tfrac{d^3}{4}\left(3e^{\frac{1}{2b^2}}\sinh\tfrac{a}{b} - e^{\frac{9}{2b^2}}\sinh\tfrac{3a}{b}\right) \\
M_4 = \quad & c^4 - 4c^3de^{\frac{1}{2b^2}}\sinh\tfrac{a}{b} + 3c^2d^2\left(e^{\frac{2}{b^2}}\cosh\tfrac{2a}{b} - 1\right) \\
& + cd^3\left(3e^{\frac{1}{2b^2}}\sinh\tfrac{a}{b} - e^{\frac{9}{2b^2}}\sinh\tfrac{3a}{b}\right) \\
& + \tfrac{d^4}{8}\left(e^{\frac{8}{b^2}}\cosh\tfrac{4a}{b} - 4e^{\frac{2}{b^2}}\cosh\tfrac{2a}{b} + 3\right)
\end{aligned}
$$

But this is very inefficient and since it is a system of 4 non-linear equations it might contains several false solutions. Tuenter (2001) proposed an algorithm to determine the S_U parameters by moment matching. Equation system (10) determines the four first non-centered moments. In order to be able to solve the parameters let us consider the following transformation instead

$$z = \gamma + \delta\sinh^{-1} y$$

And a simplification is done by a simple variable substitution $\omega = exp\left(\delta^{-2}\right)$ and $\Omega = \gamma/\delta$. Thus can the mean and variance be described by the following relationship

$$
\begin{aligned}
\mu &= -\,\omega^{\frac{1}{2}}\sinh\Omega \\
\sigma &= \frac{1}{2}\left(\omega-1\right)\left(\omega\cosh 2\Omega + 1\right)
\end{aligned}
$$

Let $\overline{M}_i$ determine the i:th centered moment, then one can calculate the skewness and kurtosis denoted $\beta_1 = \overline{M}_3^2/\overline{M}_2^3$ and $\beta_2 = \overline{M}_4/\overline{M}_2^2$, then the following relationship holds

$$\begin{aligned}\beta_1 =& \omega(\omega-1)\frac{(\omega(\omega+2)\sinh 3\Omega+3\sinh\Omega)^2}{2(\omega\cosh 2\Omega+1)^3}\\ \beta_2 =& \frac{\omega^2\left(\omega^4+2\omega^3+3\omega^2-3\right)\cosh 4\Omega+4\omega^2(\omega+2)\cosh 2\Omega+3(2\omega+1)}{2(\omega\cosh 2\Omega+1)^2}\end{aligned}$$

The hyperbolic cosinus and sinus can be eliminated by some substitution according to following, $t = \sinh\Omega$, $\cosh 2\Omega = 1 + 2t^2$, $\sinh 3\Omega = 4t^3 + 3t$ and $\cosh 4\Omega = 8t^4 + 8t^2 + 1$, and after some modifications one receives

$$\begin{aligned}\beta_1 =& \omega t^2(\omega-1)\frac{\left(4\omega(\omega+2)t^2+3(\omega+1)^2\right)^2}{2(\omega+2\omega t^2+1)^3}\\ \beta_2 =& \frac{\left(\omega^4+2\omega^2+3\right)(\omega+1)^2+8\omega^2(w+1)\left(\omega^3+\omega^2+2\omega-1\right)t^2+8\omega^2\left(\omega^4+2\omega^3+3\omega^3-3\right)t^4}{2(\omega+2\omega t^2+1)^2}\end{aligned}$$

Tuenter (2001) stated that the following conditions must hold $\omega > 1$, $0 \le \beta_1 \le (\omega-1)(w+2)^2$ and $\frac{1}{2}\left(w^4+2\omega^2+3\right) \le \beta_2 < w^4+2\omega^3+3\omega^2-3$. The parameter ω will first be defined by some upper and lower boundary in order to be able to solve it

$$\max\{\omega_0,\omega_1\} < \omega < \omega_2 = \sqrt{-1+\sqrt{2(\beta_2-1)}}$$

where ω_0 is determined by the positive root of $\beta_1 = (\omega-1)(\omega+2)^2$, and ω_1 the positive root of $\beta_2 = \left(\omega^4+2\omega^2+3\right)/2$. Tuenter (2001) follows Ferrari's method to be able to solve ω_1. When the upper and lower bound now are identified one need to solve the following equation $f(\omega) = \beta_1$, to receive ω

$$f(w) = \left(w+1-\sqrt{4+2\left(w^2-\frac{\beta_2+3}{\omega^2+2\omega+3}\right)}\right)\left(w+1+\frac{1}{2}\sqrt{4+2\left(w^2-\frac{\beta_2+3}{\omega^2+2\omega+3}\right)}\right)^2$$

we are using a the Bisection Method to find the value of ω. The performance of the algorithm can be enhanced by introducing better numerical solvers.

Having solved for ω we are more over only faced with straight forward calculations.

$$m = -2+\sqrt{4+2\left(\omega^2-\frac{\beta_2+3}{\omega^2+2\omega+3}\right)}$$

$$\Omega = -\text{sign}\overline{M}_3 \sinh^{-1} \sqrt{\frac{\omega+1}{2\omega}\left(\frac{\omega-1}{m}-1\right)}$$

the parameters for the shape value are thereafter calculated as

$$\delta = \frac{1}{\sqrt{\ln \omega}}, \quad \gamma = \frac{\Omega}{\ln \omega}$$

and

$$\lambda = \frac{\sqrt{\overline{M}_2}}{\omega-1}\sqrt{\frac{2m}{\omega+1}}, \quad \xi = \overline{M}_1 - \text{sign}\overline{M}_3 \frac{\sqrt{\overline{M}_2}}{\omega-1}\sqrt{\omega-1-m}$$

□

Once the parameters above are determined one can calculate the arbitrage free prices by the following formulas:

Type I:

$$\begin{aligned} B_{Call}^{4M-I}(T) &= e^{-r_d T}\left(M_1 - K + \int_0^K P(x)\,dx\right) \\ &= e^{-rT}\left(M_1 - K + (K-c)\,N(Q_1) - d\exp\left\{\frac{1-2ab}{2b^2}\right\} N\left(Q_1 - \frac{1}{b}\right)\right) \end{aligned}$$

where $Q_1 = a + b\ln\left(\frac{K-c}{d}\right)$.

Type II:

$$B_{Call}^{4M-II}(T) = e^{-r_d T}\left(M_1 - K + (K-c)\,N(Q_2) + \tfrac{1}{2} d e^{\frac{1}{2b^2}}\left[e^{\frac{a}{b}} N\left(Q_2 + \tfrac{1}{b}\right) - e^{-\frac{a}{b}} N\left(Q_2 - \tfrac{1}{b}\right)\right]\right)$$

where $Q_2 = a + b\sinh^{-1}\left(\frac{K-c}{d}\right)$ and $N(\cdot)$ is the standard normal cumulative distribution function.

4.5 Taylor Approximation

Ju (2002) derived the Basket value by a Taylor approximation. This by an expansion around zeros volatilities in order to be able to approximate the ratio of the characteristic function of the Basket value to the approximated log-normal random variable.

As usual let us consider that each underlying assets S_i follows the standard process

$$S_i(t) = S_i(0)\exp\left\{\left(\mu_i - \frac{1}{2}\sigma^2\right)t + \sigma_i W_i(t)\right\}$$

If we study the process it does not seem reasonable to perform a Taylor expansion around zero volatility since the volatility is different for each underlying asset, we can overcome this by introducing a scale parameter z, giving the process above the following look

$$S_i(z,t) = S_i(0)\exp\left\{\left(\mu_i - \frac{1}{2}\sigma^2\right)t + z\sigma_i W_i(t)\right\}$$

with $z = 1$ we are back to the standard process. Let the Basket value be defined as in (9) by updating it and introduce the parameter z we have

$$B(z) = \sum_{i=1}^{N} w_i S_i(z,t)$$

Let M_1 and $M_2\left(z^2\right)$ represent the first two moments of $B(z)$, let $Y(z)$ be a random variable with mean $m\left(z^2\right)$ and variance $v\left(z^2\right)$, and then match $\exp\{Y(z)\}$ to M_1 and M_2 in the following way

$$m\left(z^2\right) = 2\log M_1 - 0.5\log M_2\left(z^2\right)$$

$$v\left(z^2\right) = \log M_2\left(z^2\right) - 2\log M_1$$

Let the random variable $X(z)$ be defined as $X(z) = \log B(z)$ and the goal is to find the probability density function of this new introduced random variable. We consider the characteristic function given by

$$E\left[e^{i\phi X(z)}\right] = E\left[e^{i\phi Y(z)}\right]\frac{E\left[e^{i\phi X(z)}\right]}{E\left[e^{i\phi Y(z)}\right]} = E\left[e^{i\phi Y(z)}\right] f(z)$$

the characteristic function of the normal stochastic variable is

$$E\left[e^{i\phi Y(z)}\right] = \exp\left\{i\phi m\left(z^2\right) - \phi^2 v\left(z^2\right)/2\right\}$$

and where the ratio function is defined as

$$f(z) = \frac{E\left[e^{i\phi X(z)}\right]}{E\left[e^{i\phi Y(z)}\right]} = E\left[e^{i\phi X(z)}\right]\exp\left\{-i\phi m\left(z^2\right) + \phi^2 v\left(z^2\right)/2\right\} \tag{13}$$

We are thereafter expanding the $f(z)$ around zero volatility and $z = 0$ up to the third derivative. The expansion is done in two parts. Expression (13) is divided into two components, the first consisting of $\exp\left\{-i\phi m\left(z^2\right) + \phi^2 v\left(z^2\right)/2\right\}$

and the second one of $E\left[e^{i\phi X(z)}\right]$. Let us have a look at the expansion of the first component

$$\begin{aligned}
&\exp\left\{-i\phi m\left(z^2\right)+\phi^2 v\left(z^2\right)/2\right\}\approx\\
&e^{i\phi m(0)+\phi^2 v(0)/2-\left(i\phi+\phi^2\right)m'(0)z^2-\left(i\phi+\phi^2\right)m''(0)z^4/2-\left(i\phi+\phi^2\right)m^{(3)}(0)z^6/6}\approx\\
&e^{-i\phi m(0)+\phi^2 v(0)/2}\left(1-\left(i\phi+\phi^2\right)a_1+\left(\left(i\phi+\phi^2\right)^2 a_1^2-\left(i\phi+\phi^2\right)a_2\right)/2+\right.\\
&\left.\left(3\left(i\phi+\phi^2\right)^2 a_1 a_2-\left(i\phi+\phi^2\right)a_3-\left(i\phi+\phi^2\right)^3 a_1^3\right)/6\right)
\end{aligned} \tag{14}$$

where a_i is given by

$$a_1(z)=-\frac{z^2\sum_{i,j}^{N}w_i w_j F_i F_j\left(\rho_{ij}\sigma_i\sigma_j T\right)}{2\sum_{i,j}^{N}w_i w_j F_i F_j}$$

$$a_2(z)=2a_1^2-\frac{z^4\sum_{i,j}^{N}w_i w_j F_i F_j\left(\rho_{ij}\sigma_i\sigma_j T\right)^2}{2\sum_{i,j}^{N}w_i w_j F_i F_j}$$

$$a_3(z)=6a_1 a_2-4a_1^3-\frac{z^6\sum_{i,j}^{N}w_i w_j F_i F_j\left(\rho_{ij}\sigma_i\sigma_j T\right)^3}{2\sum_{i,j}^{N}w_i w_j F_i F_j}$$

Thereafter is $g(z)=E\left[e^{i\phi X(z)}\right]$ expanded as

$$g(z)\approx g(0)+\frac{z^2}{2}g''(0)+\frac{z^4}{24}g^{(4)}(0)+\frac{z^6}{720}g^{(6)}(0) \tag{15}$$

Let us identify the terms in the expression. By differentiating $g(z)$ twice, four and six times Ju (2002) showed that the following conditions holds

$$\begin{aligned}
\tfrac{z^2}{2}g^{(2)}(0)= \quad & e^{i\phi X(0)}\left(i\phi+\phi^2\right)a_1(z)\\
\tfrac{z^4}{24}g^{(4)}(z)= \quad & e^{i\phi X(z)}\left(-\left(i\phi-3\right)\left(i\phi-2\right)\left(i\phi+\phi^2\right)a_1^2(z)/2\right.\\
&\left.-\left(i\phi-2\right)\left(i\phi+\phi^2\right)b_1(z)-\left(i\phi+\phi^2\right)b_2(z)\right)\\
\tfrac{z^6}{720}g^{(6)}(z)= \quad & e^{i\phi X(z)}\left(-\left(i\phi-5\right)\left(i\phi-4\right)\left(i\phi-3\right)\left(i\phi-2\right)\left(i\phi+\phi^2\right)\left(-\tfrac{a_1^3(z)}{6}\right)\right.\\
&-\left(i\phi-4\right)\left(i\phi-3\right)\left(i\phi-2\right)\left(i\phi+\phi^2\right)c_1(z)\\
&-\left(i\phi-3\right)\left(i\phi-2\right)\left(i\phi+\phi^2\right)c_2(z)\\
&\left.-\left(i\phi-2\right)\left(i\phi+\phi^2\right)c_3(z)-\left(i\phi+\phi^2\right)c_4(z)\right)
\end{aligned}$$

where $b_i(z)$ and $c_i(z)$ are defined according to the following expression

$$
\begin{aligned}
b_1(z) = & \ \tfrac{z^4}{4B^3(0)} 2\sum_{ijk}^{N} w_i w_j w_k F_i F_j F_k \left(\rho_{ik}\sigma_i\sigma_k T\right)\left(\rho_{jk}\sigma_j\sigma_k T\right) \\
b_2(z) = & \ a_1^2(z) - \tfrac{1}{2} a_2(z) \\
c_1(z) = & \ -a_1(z)\, b_1(z) \\
c_2(z) = & \ \tfrac{z^6}{144B^4(0)} \Bigg(9 \cdot 8 \sum_{ijkl}^{N} w_i w_j w_k w_l F_i F_j F_k F_l \left(\rho_{il}\sigma_i\sigma_l T\right)\left(\rho_{jk}\sigma_j\sigma_k T\right)\left(\rho_{kl}\sigma_k\sigma_l T\right) \\
& + 2\sum_{ij}^{N} w_i w_j F_i F_j \left(\rho_{ij}\sigma_i\sigma_j T\right) \cdot \sum_{ij}^{N} w_i w_j F_i F_j \left(\rho_{ij}\sigma_i\sigma_j T\right)^2 \\
& + 4\sum_{i,j}^{N} w_i w_j F_i F_j \left(\rho_{ij}\sigma_i\sigma_j T\right) \sum_{i,j}^{N} w_i w_j F_i F_j \left(\rho_{ij}\sigma_i\sigma_j T\right)^2 \Bigg) \\
c_3(z) = & \ \tfrac{z^6}{48B^6(0)} \Bigg(4 \cdot 6 \sum_{ijk}^{N} w_i w_j w_k F_i F_j F_k \left(\rho_{ik}\sigma_i\sigma_k T\right)\left(\rho_{jk}\sigma_j\sigma_k T\right)^2 \\
& + 8\sum_{ijk}^{N} w_i w_j w_k F_i F_j F_k \left(\rho_{ij}\sigma_i\sigma_j T\right)\left(\rho_{ik}\sigma_i\sigma_k T\right)\left(\rho_{jk}\sigma_j\sigma_k T\right) \\
c_4(z) = & \ a_1(z)\, a_2(z) - \tfrac{2}{3} a_1^3(z) - \tfrac{1}{6} a_3(z)
\end{aligned}
$$

$f(z)$ is expressed as the product between $g(z)$ and $E\left[e^{i\phi X(z)}\right]$, if we multiply (10) with (15) the following expression holds

$$
f(z) \approx 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + i\phi^4 d_4(z)
$$

and where d_i is given by

$$
\begin{aligned}
d_1(z) = & \ \tfrac{1}{2}\left(6a_1^2(z) + a_2(z) - 4b_1(z) + 2b_2(z)\right) - \tfrac{1}{6}\left(1206a_1^3(z) - a_3(z)\right. \\
& \left. + 6\left(24c_1(z) - 6c_2(z) + 2c_3(z) - c_4(z)\right)\right) \\
d_2(z) = & \ \tfrac{1}{2}\left(10a_1^2(z) + a_2(z) - 6b_1(z) + 2b_2(z)\right) - \left(128a_1^3(z)/3 - a_3(z)/6\right. \\
& \left. + 2a_1(z)\, b_1(z) - a_1(z)\, b_2(z) + 50c_1(z) - 11c_2(z) + 3c_3(z) - c_4(z)\right) \\
d_3(z) = & \ \left(2a_1^2(z) - b_1(z)\right) - \tfrac{1}{3}\left(88a_1^3(z) + 3a_1(z)\left(5b_1(z) - 2b_2(z)\right)\right. \\
& \left. + 3\left(25c_1(z) - 6c_2(z) + c_3(z)\right)\right) \\
d_4(z) = & \ \left(-20a_1^3(z)/3 + a_1(z)\left(-4b_1(z)\right) + b_2(z) - 10c_1(z) + c_2(z)\right)
\end{aligned}
$$

$E\left[e^{i\phi X(1)}\right]$ is then approximated by

$$E\left[e^{i\phi X(z)}\right] \approx e^{-i\phi m(1)+\phi^2 v(1)/2}\left(1-i\phi d_1(1)-\phi^2 d_2(1)+\phi^3 d_3(1)+\phi^4 d_4(1)\right)$$

and thus are the pdf $h(x)$ of $X(1)$ given by

$$\begin{aligned} h(x) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\phi x} e^{-i\phi m(1)+\phi^2 v(1)/2}\left(1-i\phi d_1(1)-\phi^2 d_2(1)+\phi^3 d_3(1)+\phi^4 d_4(1)\right) d\phi \\ &= p(x)+\left(d_1(1)\frac{d}{dx}+d_2(1)\frac{d^2}{dx^2}+d_3(1)\frac{d^4}{dx^4}+d_4(1)\frac{d^4}{dx^4}\right)p(x) \end{aligned}$$

where $p(x)$ is the standard normal pdf with mean $m(1)$ and variance $v(1)$

$$\begin{aligned} p(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x+i\phi(1)+\phi^2 v(1)/2} d\phi \\ &= \frac{1}{\sqrt{2\pi v(1)}} e^{-\frac{(x-m(1))^2}{2v(1)}} \end{aligned}$$

The Basket call price on Black-Scholes style is then given by

$$B_{Call}^{Taylor}(T) = e^{-rt} E\left[\left(e^{X(1)}-K\right)^{+}\right] =$$

$$e^{-rT}\left[B(0)N(y_1)-KN(y_2)+K\left(z_1 p(y)+z_2\frac{d}{dy}p(y)+z_3\frac{d^2}{dy^2}p(y)\right)\right]$$

where

$$y=\log K,\ y_1=\frac{m(1)-y}{\sqrt{v(1)}}+\sqrt{v(1)},\ y_2=y_1-\sqrt{v(1)}$$

and

$$z_1=d_2(1)-d_3(1)+d_4(1),\ z_2=d_3(1)-d_4(1),\ z_3=d_4(1)$$

If we remove the last term in the Basket price we get exactly the same price as for the Levy log-normal approximation.

5 Hedging Strategies

5.1 Approximating the Greeks

For some of the approximations there does not exist an approximately closed form solution for calculating the Greeks, that is why we will use the derivative definition via difference quotients. In introduction to calculus one get familiar with the derivative $\partial f(x_0)/\partial x$ definition via difference quotients at a point x_0 as the limit function as h tends to zero

$$\frac{\partial f(x_0)}{\partial x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

which is an accurate approximation of the first derivative with an error of $\mathcal{O}(h^2)$ (Wilmott 2006: 2174)

Assume the basket option value $\Pi_t = B(S, T, K, r, q, \sigma)$ with underlying assets S_i, let us define the delta Δ_i and vega ν_i by using the definition of the derivative

$$\Delta_i = \frac{\partial \Pi}{\partial S_i} \approx \lim_{h_i \to 0} \frac{B(S + h_i, T, K, r, q, \sigma) - B(S - h_i, T, K, r, q, \sigma)}{2h_i}$$

and for vega

$$\nu_i = \frac{\partial \Pi}{\partial \sigma_i} \approx \lim_{h_i \to 0} \frac{B(S, T, K, r, q, \sigma + h_i) - B(S, T, K, r, q, \sigma - h_i)}{2h_i}$$

In our case can h not be to small, h at a level of 10^{-6} will give a great precision when calculating the two Greeks above. To be able to calculate the real Greeks using MC techniques we are using a technique called Common Random Numbers with goal to avoid Monte Carlo error and this by using the same sequence of random number each time we evaluate our function.

5.2 Hedging of the Reciprocal Gamma Approximation

Milevsky and Posner (1998) showed that a Basket option priced by assuming a summation of the underlying distribution converges towards a gamma distribution can be hedged by Δ_C^{RG} in the following way

$$\Delta_C^{\mathrm{RG}} = \begin{cases} \frac{e^{-r_f(T-t)} - e^{-r_d(T-t)}}{(r-q)T} G\left(\frac{T-t}{TK - tB(0)}, \alpha - 1, \beta\right), & B(0) < \frac{T}{t}K \\ \frac{e^{-r_f(T-t)} - e^{-r_d(T-t)}}{(r_d - r_f)T}, & B(0) \geq \frac{T}{t}K \end{cases}$$

5.3 Hedging of the Taylor Approximation

Ju (2002) simply showed that the hedging ratio of the Basket call B_C approximated priced by the Taylor approximation w.r.t the underlying asset S simply is

$$\Delta_C^{\text{Taylor}} = \frac{\partial B_C}{\partial S} = \frac{e^{-r_d T} U_1}{T} N\left(y_1\right) - \frac{e^{-r_d T} K}{S} \left(z_1 \frac{dp\left(y\right)}{dx} + z_2 \frac{d^2 p\left(y\right)}{dx^2} + z_3 \frac{d^3 p\left(y\right)}{dx^3} \right)$$

The analogous delta for the put option Δ_P^{Taylor} is defined as

$$\Delta_P^{\text{Taylor}} = \frac{\partial B_P}{\partial S} = \Delta_C^{\text{Taylor}} - e^{-rT} \frac{U_1}{T}$$

5.4 A Static Super-Hedging Strategy

Su (2006) proposed a hedging strategy for the basket option by replacing the basket with a number of individual plain vanilla options. As showed above this technique will give an upper boundary for the basket option according to

$$\begin{aligned} B\left(T\right) &= \left(\sum_{i=1}^{N} w_i S_i\left(T\right) - K \right)^+ \\ &= \left(\sum_{i=1}^{N} w_i \left(S_i\left(T\right) - k_i\right) \right)^+ \\ &\leq \sum_{i=1}^{N} w_i \left(S_i\left(T\right) - k_i\right)^+ \end{aligned}$$

and such that $\sum_{i=1}^{N} w_i k_i = K$. The last inequality follows from Jensen's inequality and the fact that N plain vanilla options never can be cheaper than the corresponding Basket options. The goal with the static super-hedging strategy is to optimize the optimal strike for each plain vanilla asset an according to

$$\begin{aligned} &\min_{k_i} e^{-r_d T} \sum_{i=1}^{N} w_i E^{\mathbb{Q}} \left[\left(S_i\left(T\right) - k_i\right)^+ \right] \\ &\text{s.t.} \qquad \sum_{i=1}^{N} w_i k_i = K \end{aligned}$$

and where $k_i \in [0, K]$.

Su (2006) proposed that each optimal k_i could be solved by the following theorem

Theorem 5.1: (Optimal k_i) *The optimal k_i for the inequality satisfying*

$$B(T) \leq e^{-r_d T} \sum_{i=1}^{N} w_i E^{\mathbb{Q}} \left[\left(S_i(T) - k_i \right)^+ \right]$$

is determined by solving the following sequence of equations

$$k_i = S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} e^{T\left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2} \sigma_i \sigma_1 \right) + \left(\frac{\sigma_i}{\sigma_1} q_1 - q_i \right) \right]}$$

and

$$\sum_{i=1}^{N} w_i k_i = K$$

Proof. Omitted, see appendix of Su (2006).

5.5 A Static Sub-Replicating Strategy

It can be proved that a lower bound of the Basket options can be determined by the forward contract with strike K denoted in the following theorem

Theorem 5.2: (A Lower Bound of the Basket Option) *A lower bound of the Basket Option with N assets with strike K, weights w_i, interest rate r is given by*

$$L = \sum_{i=1}^{N} w_i e^{-r_f^i T} S_i(t) - e^{-r_d T} K$$

6 Pricing with the Smile & Skew

6.1 Industry Model

We will here show how the volatility smile can be taken into account when basket options will be priced. The following procedure is as for the log-normal approximation a method that is well used the industry. The goal with this method is to find a replicated portfolio with plain vanilla options of the Basket option and where the volatility smile is taken into account. Let the approximated price Π_t of the basket option be defined by

$$B_t = B\left(F, T, \sigma^{ATM}, K\right)$$

where F denoted the forwards rates, T time to maturity, σ^{ATM} the at-the-money volatility and K the strike price. And the i:th vega ν_i defined in the traditional way

$$\nu_i = \frac{\partial B\left(F, T, \sigma^{ATM}, K\right)}{\partial \sigma_i}$$

As mentioned above, the goal is to create a portfolio of n plain vanilla options with a new strike K' such that the value of the replicated portfolio equals the Basket value. The new weights μ_i are constructed such that the exposure of the i:th vega ν_i will be zero. In order to get the new parameters the following systems of equations must be solved with a numerical procedure

$$\sum_{i=1}^{n} \mu_i C\left(K', T, r_i, q_i, \sigma_i^{ATM}\right) = B\left(F, T, \sigma^{ATM}, K\right)$$

$$\nu_i = \mu_i \frac{\partial C\left(K', T, r, q, \sigma^{ATM}\right)}{\partial \sigma_i}$$

The basket option price where the volatility smile is taking into account is then determined by

$$B_t^{smile} = \sum_{i=1}^{n} \mu_i C\left(K', T, r_i, q_i, \sigma_i^{smile}\right)$$

and where σ_i^{smile} is the volatility determined from the smile.

6.2 Replicated Portfolio

The goal is to find a replicated portfolio which eliminates the volatility risk satisfying the partial derivatives up to the second order. We want to find three time dependent weights $x_1(t,K)$, $x_2(t,K)$ and $x_3(t,K)$[11] in such a way that the replicated portfolio of three European calls with maturity T and strikes $K_{25\Delta P}$, K_{ATM} and $K_{25\Delta C}$ hedges up to the second order in the underlying assets volatility. Lets assume that our basket contains $j = 1, 2, \ldots, N$ currency pairs, to receive $x_i^j(t,K)$ we need to solve the following system of equations

$$
\begin{aligned}
\frac{\partial B}{\partial \sigma}(t,K) &= \sum_{i=1}^{3} x_i^j(t,K) \frac{\partial C_{BS}^j}{\partial \sigma}(t,K_i) \\
\frac{\partial^2 B}{\partial \sigma^2}(t,K) &= \sum_{i=1}^{3} x_i^j(t,K) \frac{\partial^2 C_{BS}^j}{\partial \sigma^2}(t,K_i) \\
\frac{\partial^2 B}{\partial \sigma \partial S_t}(t,K) &= \sum_{i=1}^{3} x_i^j(t,K) \frac{\partial^2 C_{BS}^j}{\partial \sigma \partial S_t}(t,K_i)
\end{aligned}
$$

The smile consistent price is now determined by

$$
B^{Smile}(K) = B(K) + \sum_{j=1}^{N} \left[\sum_{i=1}^{3} x_i^j(K) \left(C_{MKT}^j(K_i) - C_{i,BS}^j(K_i) \right) \right]
$$

6.3 Local Volatility

As motivation above that the implied volatility as solved by finding the inverse of Black-Scholes for a certain level and volatility, demonstrates a dependencies of both strike level K or underlying asset value S_t and time to maturity t instead of being constant, i.e. $\sigma = \sigma(S_t, t)$. The idea with Local Volatility as introduced by Dupire (1994) is to find a process that is consistent with the smile and skew[12] and still keeps the model complete and thus arbitrage free. We want the underlying volatility in eq (4) be dependent of the current asset value and time to maturity and thus letting it be on the following form

$$
dS_t = rS_t dt + \sigma(S_t, t) S_t dW_t \tag{16}
$$

Dupire (1994) showed how we can choose $\sigma(S_t, t)$ such that our requirements above are fulfilled and this by assuming that the risk neutral probability function $p(K, T; S_0)$ can be derived from the set of option prices $C(K, T : S_0)$ for different level of strike K and time to maturity T. Dupire's derivation of

[11] To simplify our notation let $i = 1, 2, 3$ determine $25\Delta P$, ATM and $25\Delta C$ respectively.

[12] From now on will we only use the term smile for representing the both the smile and skew effect.

local volatility is done by assuming that both domestic r_d and foreign r_f interest rates are equal to zero, but the can be easily be transformed into the common case. If we start from the ordinary relationship defining the option value $C(K, T : S_0)$, assuming that $r_d = r_f = 0$ and where $p(K, T; S_0)$ represents the risk neutral density.

$$\begin{aligned} C(K, T : S_0) &= E^{\mathbb{Q}}\left[(S - K)^{+} \mid \mathcal{F}_t^S\right] \\ &= \int_0^{\infty} (S_T - K)^{+} p(K, T; S_0)\, dS_T \\ &= \int_K^{\infty} (S_T - K)\, p(K, T; S_0)\, dS_T \end{aligned} \tag{17}$$

and where the risk neutral density $p(K, T; S_0)$ must satisfy the Fokker-Planck equation also known as Kolmogorov's forward formula.

Definition 6.1: (Fokker-Planck equation) *The Fokker-Planck equation describes the time evolution and the probability density function of a particle which evolves as*

$$\frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 p\right) - S\frac{\partial}{\partial S_T}\left(r S_T p\right) = \frac{\partial p}{\partial T}$$

with boundary condition $p = \delta(S - K)$, where $\delta(x)$ is the Dirac delta function and defined as

Definition 6.2: (Dirac delta function) *The Dirac delta function is a generalized function such that*

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$$

for all continuous and bounded f.

Taking the first and second partial derivative of (17) with respect to K following holds

$$\begin{aligned} \frac{\partial C}{\partial K} &= -\int_K^{\infty} p(K, T; S_0)\, dS_T \\ \frac{\partial^2 C}{\partial K^2} &= p(K, T; S_0) \end{aligned}$$

This as introduced by Breeden and Litzenberger (1978) and this indicates that the risk neutral transition density of S_T can be recovered directly from market option prices.

Differentiating with respect to T one receives

$$\begin{aligned}\frac{\partial C}{\partial T} &= \int_K^{\infty} (S_T - K) \frac{\partial}{\partial T} p(K, T; S_0)\, dS_T \\ &= \int_K^{\infty} \left[\frac{1}{2} \frac{\partial^2}{\partial S_T^2} \left(\sigma^2 S_T^2 p \right) - \frac{\partial}{\partial S_T} (r S_T p) \right] (S_T - K)\, dS_T\end{aligned}$$

solving the integral by integrating twice and by using the assumption that p and the first S_T derivative goes towards zero as S_T tends to infinity, and by assuming $r_d = r_f = 0$ one finally receives

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2} \frac{\partial^2 C}{\partial K^2}$$

and by rearranging we end up with Dupire's equation.

□

Theorem 6.3: (Dupire's equation) *The local volatility $\sigma(K,T)$ as introduced by Dupire (1994) assuming $r = 0$ is defined as*

$$\sigma^2 (K, T) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}$$

By voilating Dupire's assumption on non interest rate, and considering foreign and domestic interest rate one recives the local volatiliy function that is consistent in the FX market with the underlying asset S

Theorem 6.4: (Dupire's equation on FX options) *The local volatility $\sigma(K,T)$ in terms of FX option prices C and where r_d and r_f determines the domestic and foreign interest rate is*

$$\sigma^2 (K, T) = \frac{\frac{\partial C}{\partial T} + r_f C + K (r_d - r_f) \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}} \tag{18}$$

or in terms of implied volatility σ_I

$$\sigma^2 (K, T) = \frac{2 \frac{\partial \sigma_I}{\partial T} + \frac{\sigma_I}{T} + 2K (r_d - r_f) \frac{\partial \sigma_I}{\partial K}}{K^2 \left[\frac{\partial^2 \sigma_I}{\partial K^2} - d\sqrt{T} \left(\frac{\partial \sigma_I}{\partial K} \right)^2 + \frac{1}{\sigma_I} \left(\frac{1}{K\sqrt{T}} + d \frac{\partial \sigma_I}{\partial K} \right)^2 \right]}$$

where d is defined in (5).

Proof. Omitted, see Andersen (1997)

□

6.4 Discretization of Local Volatiliy

In order to create a nice local volatility surface of PDE (18) we need to discretisize it. First we need to divide the plane (K, T) into a grid with $N+2 \times M+2$ equal sized nodes. At node (K_i, T_i) for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$ we have the following value

$$
\begin{aligned}
\frac{\partial C}{\partial T} &\approx \frac{C\left(K_i, T_{j+1}\right) - C\left(K_i, T_j\right)}{\Delta_T} \\
\frac{\partial C}{\partial K} &\approx (1-\Theta)\frac{C\left(K_{i+1}, T_j\right) - C\left(K_{i-1}, T_j\right)}{2\Delta_K} \\
&\quad +\Theta\frac{C\left(K_{i+1}, T_{j+1}\right) - C\left(K_{i-1}, T_{j+1}\right)}{2\Delta_K} \\
\frac{\partial^2 C}{\partial K^2} &\approx (1-\Theta)\frac{C\left(K_{i+1}, T_j\right) - 2C\left(K_i, T_j\right) + C\left(K_{i-1}, T_j\right)}{2\Delta_K^2} \\
&\quad +\Theta\frac{C\left(K_{i+1}, T_{j+1}\right) - 2C\left(K_i, T_{j+1}\right) + C\left(K_{i-1}, T_{j+1}\right)}{2\Delta_K^2}
\end{aligned}
$$

where the parameter $\Theta \in [0, 1]$ determines at which time the partial derivative with respect to K are evaluated. The fully implicit finite difference method is when $\Theta = 0$, which mean that the K derivatives is evaluated at t_j. The case when $\Theta = 1$ the K derivatives are evaluated at t_{j+1}, also known as the explicit finite difference method. When $\Theta = 1/2$ giving rise to an average of the implicit and explicit method, named Crank-Nicholson scheme.

By continuing on the values used when calculating and interpolating the volatility smile we will use the Crank-Nicholson scheme discretization technique.

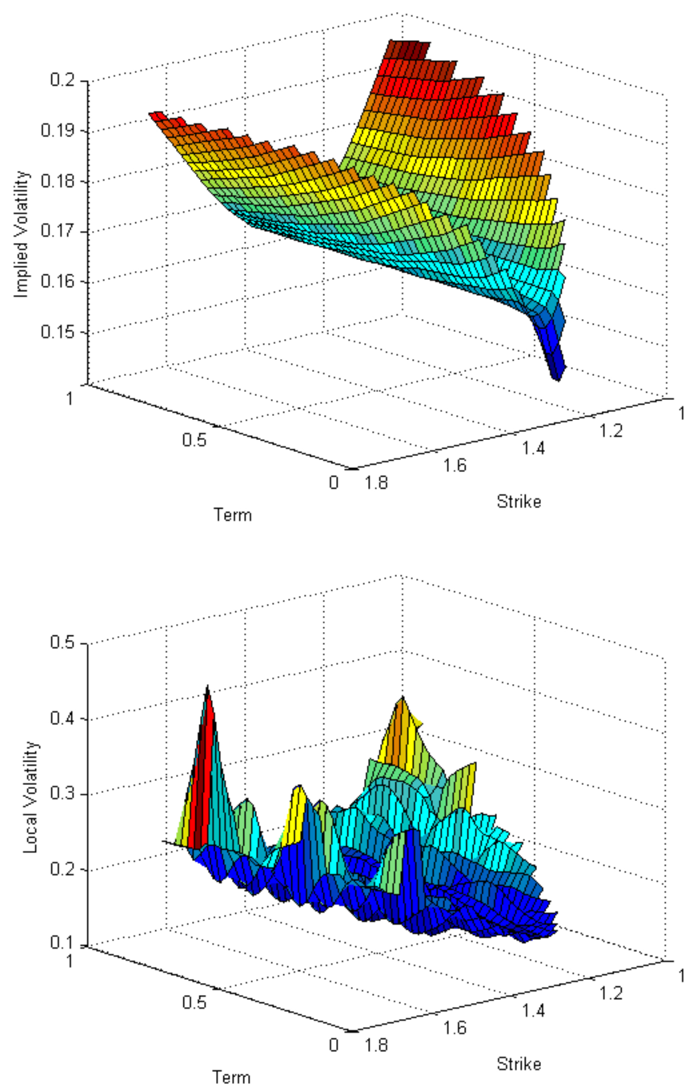

Figure 15: Upper: Implied Volatility. Lower: Local Volatility.

Or the lower plot from figure 16 as seen from above which we will use in our MC simulation, both figures generated from equation (18). We begin each simulated path at the spot value S_0 and will for each simulation step into the corresponding grid (square) and collect the current volatility (Local volatility) determined by the actual time t and current underlying asset value S_t[13] at the

[13]Note that we changed the Local volatility as function of K and t to be a function of S_t and t in order to get the correct current volatility, hence the name local volatility.

current t.

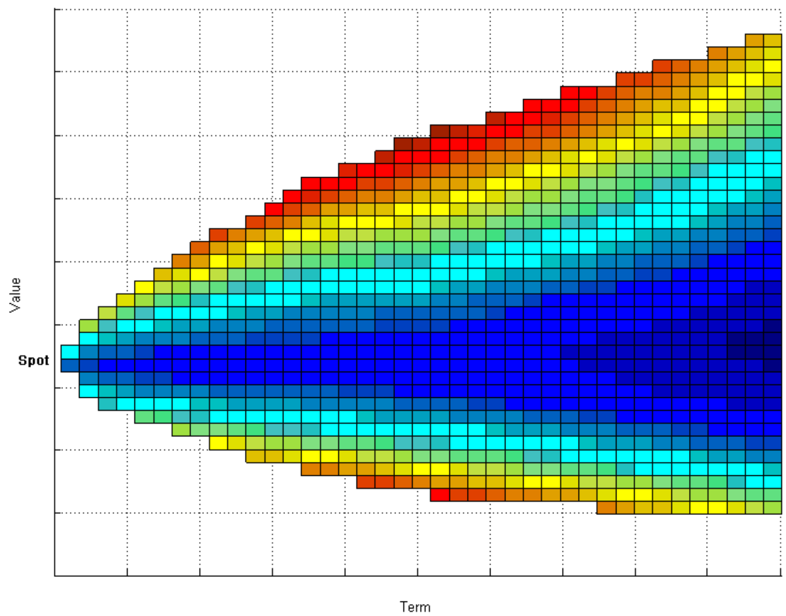

Figure 16: The Local volatility surface as a function S_t and t of seen from above.

7 Results

7.1 Basket Values

For our numerical test of the Basket option approximations are we using the following Basket setup were we are letting the number of FX assets to be $N = 4$, on a term structure of one year $T = 1$. For simplicity are both the domestic and foreign interest rate zero, $r_d = r_d = 0$. The initial FX spot values are equally sett to 100,

$$S_0 = \begin{bmatrix} 100 & 100 & 100 & 100 \end{bmatrix}$$

We price the Basket at the money with strike $K = 100$, and the volatility σ_i to

$$\sigma = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

the correlation ρ_{ij} where $i \neq j$ are arbitrary chosen to by equally weighted

$$\rho = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}$$

But it must be kept into mind that this is not possible for the FX market, since the relationship as presented in the section of FX correlation must fulfill a certain condition or else does there exists arbitrage possibilities, but this is ignored for now. We are letting the weight of each underlying asset in the Basket to equally weighted

$$w = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

The number of elements in our Sobol LDS sequence used for our qMC simulation is chosen to be $2^{24} - 1 = 16777216$, which keeps the standard deviation low, and we know that the mean from qMC is accurate to the last present digit. Valuing the Basket option without incorporating the smile effect is done by using different scenarios. The first one varying the moneyness (MN) at the spot, the ratio between the strike price K and the initial basket value $B(0)$. This case is done by letting MN vary in a range of 50% and 150%, but it should be kept into mind that the Basket option will at most scenarios be priced at-the-money.

The second case: varying the time-to-maturity, T in a range of 0.5 and 3 year. In the third case are the intial correlations ρ_{ij} where $i \neq j$ in the range $[0, 1]$. In the last case are the initial volatilities σ_i varied between 0.05 and 0.55. For each different scenario are we presenting the qMC value, the normalized standard deviation. As mentioned above about the bid-ask spread, if the approximated values (error subtracted and added) are within the bid-ask spread, we can also draw the conclusion that we have calculated accurate values.

7.1.1 Varying Moneyness

Let us begin our test by varying the Moneyness, $\mathrm{MN} = K/B(0)$ in a range of $[0.5, 1.5]$. We can see that 4M, LogN and the LogN produces the best overall RMSE value, but a closer look at the 4M values are that these are priced correct to the last *'pip'* for all different scenarios. It is only the RG approximation that does not produce so accurate values, and this due to that the summation of a log normal converges towards RG as the number of summations tends to infinity, and in our case are we only using 4 underlying assets. RG produces the worst values when pricing the Basket option around ATM.

MN	**qMC** (std)	**4M** (error)	**LogN** (error)	**RG** (error)	**Taylor** (error)	**UB**
0.5	50.0000 (0.0027)	50.0000 $(9.11 \cdot 10^{-6})$	50.0000 $(9.12 \cdot 10^{-6})$	50.0000 $(4.12 \cdot 10^{-6})$	50.0000 $(9.10 \cdot 10^{-6})$	50.0003
0.6	40.0020 (0.0027)	40.0020 $(9.29 \cdot 10^{-6})$	40.0020 $(1.05 \cdot 10^{-5})$	40.0005 $(1.50 \cdot 10^{-3})$	40.0020 $(9.05 \cdot 10^{-6})$	40.0263
0.7	30.0552 (0.0027)	30.0552 $(9.39 \cdot 10^{-6})$	30.0553 $(2.45 \cdot 10^{-5})$	30.0322 $(2.30 \cdot 10^{-2})$	30.0553 $(8.20 \cdot 10^{-6})$	30.2475
0.8	20.5062 (0.0026)	20.5062 $(1.27 \cdot 10^{-5})$	20.5063 $(6.60 \cdot 10^{-5})$	20.4185 $(8.77 \cdot 10^{-2})$	20.5063 $(1.19 \cdot 10^{-5})$	21.1840
0.9	12.2675 (0.0023)	12.2675 $(1.13 \cdot 10^{-5})$	12.2676 $(8.17 \cdot 10^{-5})$	12.1504 $(1.17 \cdot 10^{-1})$	12.2676 $(1.36 \cdot 10^{-5})$	13.5899
1.0	6.3059 (0.0018)	6.3059 $(9.52 \cdot 10^{-6})$	6.3060 $(3.63 \cdot 10^{-5})$	6.2604 $(4.56 \cdot 10^{-2})$	6.3060 $(1.27 \cdot 10^{-5})$	7.9668
1.1	2.7839 (0.0012)	2.7839 $(8.04 \cdot 10^{-6})$	2.7839 $(2.27 \cdot 10^{-5})$	2.8306 $(4.67 \cdot 10^{-2})$	2.7839 $(9.02 \cdot 10^{-6})$	4.2929
1.2	1.0694 (0.0007)	1.0694 $(4.63 \cdot 10^{-6})$	1.0694 $(4.77 \cdot 10^{-5})$	1.1517 $(8.22 \cdot 10^{-2})$	1.0694 $(3.57 \cdot 10^{-6})$	2.1475
1.3	0.3643 (0.0004)	0.3643 $(9.90 \cdot 10^{-6})$	0.3643 $(3.22 \cdot 10^{-5})$	0.4329 $(6.86 \cdot 10^{-2})$	0.3643 $(8.43 \cdot 10^{-6})$	1.0083
1.4	0.1123 (0.0002)	0.1123 $(6.87 \cdot 10^{-6})$	0.1123 $(1.73 \cdot 10^{-5})$	0.1539 $(4.16 \cdot 10^{-2})$	0.1123 $(5.88 \cdot 10^{-6})$	0.4494
1.5	0.0319 (0.0001)	0.0319 $(9.86 \cdot 10^{-6})$	0.0319 $(1.23 \cdot 10^{-6})$	0.0527 $(2.08 \cdot 10^{-2})$	0.0319 $(9.37 \cdot 10^{-6})$	0.1925
RMSE		$9.55 \cdot 10^{-5}$	$5.65 \cdot 10^{-5}$	$2.35 \cdot 10^{-1}$	$9.04 \cdot 10^{-5}$	

Table 3: Varying MN$\in [0.5, 1.5]$

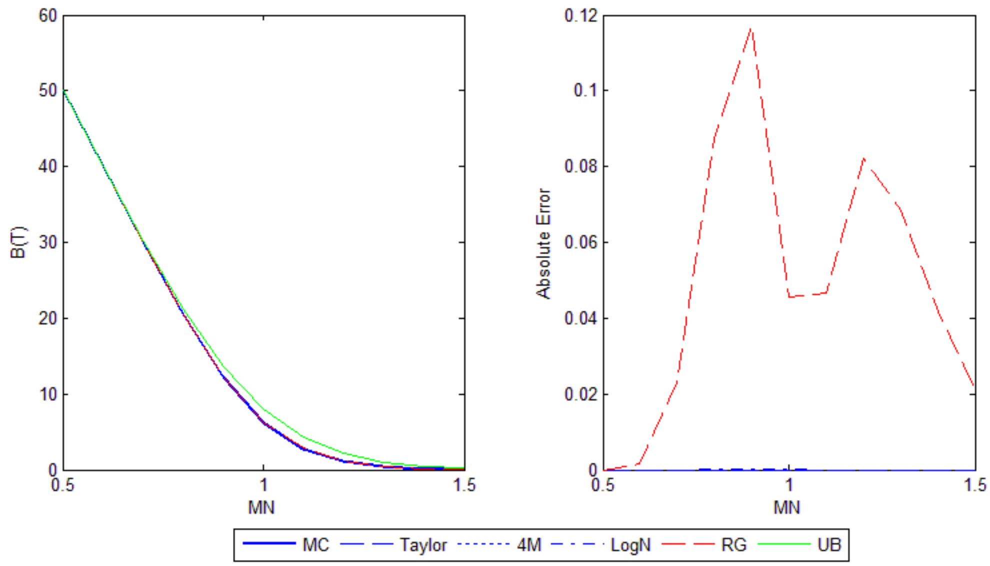

Figure 17: Varying the MN. Right: Basket option values derived from the different methods. Left: The absolute error of each approximation compared to values derived from qMC.

7.1.2 Varying Time-to-Maturity

For this scenario are we varying the time-to-maturity in the range $[0.5, 3]$. We have the same outcome as in the previous case, the 4M method is priced correct down to the last *'pip'*. The Taylor approximation is priced correct for values below $T = 1.75$ and after that just deviates by 1 *'pip'* . The absolute error of the log normal approximation increases as T increases and the values produced by RG can be neglected.

T	**qMC** (std)	**4M** (error)	**LogN** (error)	**RG** (error)	**Taylor** (error)	**UB** (diff)
0.50	4.4596 (0.0012)	4.4597 $(5.93 \cdot 10^{-6})$	4.4597 $(1.06 \cdot 10^{-5})$	4.4435 $(1.62 \cdot 10^{-2})$	4.4597 $(6.42 \cdot 10^{-6})$	5.6382
0.75	5.4615 (0.0015)	5.4615 $(7.36 \cdot 10^{-6})$	5.4615 $(2.03 \cdot 10^{-5})$	5.4318 $(2.97 \cdot 10^{-2})$	5.4615 $(8.82 \cdot 10^{-6})$	6.9005
1.00	6.3059 (0.0018)	6.30560 $(9.51 \cdot 10^{-6})$	6.30560 $(3.63 \cdot 10^{-5})$	6.2604 $(4.56 \cdot 10^{-2})$	6.30560 $(1.27 \cdot 10^{-5})$	7.9642
1.25	7.0497 (0.0020)	7.0498 $(1.10 \cdot 10^{-5})$	7.0498 $(5.81 \cdot 10^{-5})$	6.9861 $(6.36 \cdot 10^{-2})$	7.0498 $(1.70 \cdot 10^{-5})$	8.9034
1.50	7.7220 (0.0022)	7.7220 $(1.26 \cdot 10^{-5})$	7.7221 $(8.75 \cdot 10^{-5})$	7.6386 $(8.34 \cdot 10^{-2})$	7.7221 $(2.27 \cdot 10^{-5})$	9.7535
1.75	8.3401 (0.0024)	8.3401 $(1.33 \cdot 10^{-5})$	8.3402 $(1.24 \cdot 10^{-4})$	8.2352 $(1.05 \cdot 10^{-1})$	8.3401 $(2.90 \cdot 10^{-5})$	10.5235
2.00	8.9153 (0.0026)	8.9153 $(1.26 \cdot 10^{-5})$	8.9154 $(1.69 \cdot 10^{-4})$	8.7873 $(1.28 \cdot 10^{-1})$	8.9153 $(3.58 \cdot 10^{-5})$	11.2415
2.25	9.4555 (0.0028)	9.4554 $(1.25 \cdot 10^{-5})$	9.4556 $(2.23 \cdot 10^{-4})$	9.3030 $(1.52 \cdot 10^{-1})$	9.4554 $(4.53 \cdot 10^{-5})$	11.9241
2.50	9.9660 (0.0029)	9.9661 $(1.08 \cdot 10^{-5})$	9.9663 $(2.87 \cdot 10^{-4})$	9.7880 $(1.78 \cdot 10^{-1})$	9.9661 $(5.56 \cdot 10^{-5})$	12.5669
2.75	10.4516 (0.0031)	10.4517 $(7.44 \cdot 10^{-6})$	10.4520 $(3.61 \cdot 10^{-4})$	10.2467 $(2.05 \cdot 10^{-1})$	10.4517 $(6.71 \cdot 10^{-5})$	13.1696
3.00	10.9155 (0.0033)	10.9156 $(1.88 \cdot 10^{-6})$	10.9160 $(4.44 \cdot 10^{-4})$	10.6824 $(2.33 \cdot 10^{-1})$	10.9156 $(7.94 \cdot 10^{-5})$	13.7473
RMSE		$1.31 \cdot 10^{-6}$	$2.14 \cdot 10^{-5}$	$1.37 \cdot 10^{-2}$	$4.26 \cdot 10^{-6}$	

Table 4: Varying T in the range $[0.5, 3]$.

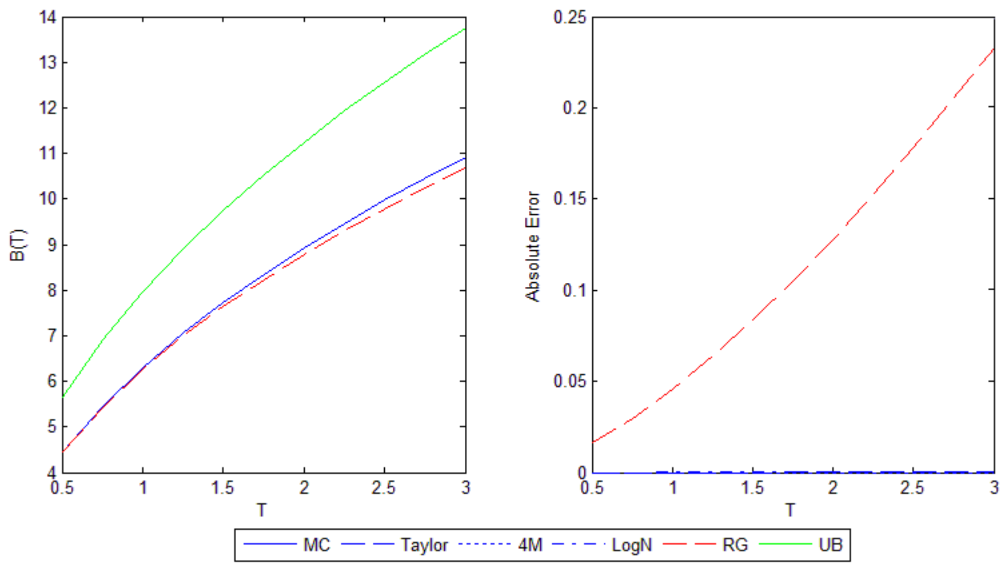

Figure 18: Varying the time-to-maturity, T. Right: Basket option values derived from the different methods. Left: The absolute error of each approximation compared to values derived from qMC.

7.1.3 Varying the correlation

Here are we varying the correlation coefficient ρ_{ij} in the rang $[0, 1]^{14}$, and where $i \neq j$. Four Moment method increases its absolute error significant when $\rho = 1$, but this is a scenario that could be neglected and the total RMSE of Taylor is the lowest.

[14] For $\rho = 1$ have we used the value $\rho = 0.999 \ldots 9$, in order to keep the covariance matrix positive definite.

ρ_{ij}	**qMC** (std)	**4M** (error)	**LogN** (error)	**RG** (error)	**Taylor** (error)	**UB** (diff)
0	4.0173 (0.0011)	4.0173 ($1.57 \cdot 10^{-5}$)	4.0177 ($4.49 \cdot 10^{-4}$)	4.0059 ($1.14 \cdot 10^{-2}$)	4.0174 ($1.53 \cdot 10^{-4}$)	7.9668
0.1	4.5672 (0.0012)	4.5672 ($3.51 \cdot 10^{-6}$)	4.5675 ($2.64 \cdot 10^{-4}$)	4.5501 ($1.71 \cdot 10^{-2}$)	4.5673 ($7.35 \cdot 10^{-5}$)	7.9656
0.2	5.0579 (0.0014)	5.0579 ($7.60 \cdot 10^{-6}$)	5.0581 ($1.64 \cdot 10^{-4}$)	5.0345 ($2.34 \cdot 10^{-2}$)	5.0579 ($4.39 \cdot 10^{-5}$)	7.9647
0.3	5.5052 (0.0015)	5.5052 ($1.15 \cdot 10^{-5}$)	5.5053 ($1.08 \cdot 10^{-4}$)	5.4749 ($3.03 \cdot 10^{-2}$)	5.5053 ($2.86 \cdot 10^{-5}$)	7.9658
0.4	5.9191 (0.0016)	5.9191 ($4.68 \cdot 10^{-6}$)	5.9191 ($5.58 \cdot 10^{-5}$)	5.8813 ($3.77 \cdot 10^{-2}$)	5.9191 ($1.24 \cdot 10^{-5}$)	7.9663
0.5	6.3059 (0.0018)	6.3060 ($9.52 \cdot 10^{-6}$)	6.3060 ($3.63 \cdot 10^{-5}$)	6.2604 ($4.56 \cdot 10^{-2}$)	6.3060 ($1.27 \cdot 10^{-5}$)	7.9658
0.6	6.6706 (0.0019)	6.6706 ($1.69 \cdot 10^{-5}$)	6.6707 ($2.95 \cdot 10^{-5}$)	6.6167 ($5.40 \cdot 10^{-2}$)	6.6706 ($1.81 \cdot 10^{-5}$)	7.9648
0.7	7.0165 (0.0020)	7.0166 ($1.60 \cdot 10^{-5}$)	7.0166 ($2.09 \cdot 10^{-5}$)	6.9538 ($6.28 \cdot 10^{-2}$)	7.0166 ($1.64 \cdot 10^{-5}$)	7.9637
0.8	7.3464 (0.0021)	7.3464 ($9.36 \cdot 10^{-6}$)	7.3464 ($1.07 \cdot 10^{-5}$)	7.2744 ($7.20 \cdot 10^{-2}$)	7.3464 ($9.44 \cdot 10^{-6}$)	7.9643
0.9	7.6621 (0.0022)	7.6621 ($1.02 \cdot 10^{-5}$)	7.6621 ($1.03 \cdot 10^{-5}$)	7.5805 ($8.16 \cdot 10^{-2}$)	7.6621 ($1.02 \cdot 10^{-5}$)	7.9688
1	7.9656 (0.0023)	7.9671 ($9.84 \cdot 10^{-2}$)	7.9656 ($5.54 \cdot 10^{-6}$)	7.8739 ($9.16 \cdot 10^{-2}$)	7.9656 ($5.54 \cdot 10^{-6}$)	7.9635
RMSE		$3.73 \cdot 10^{-3}$	$3.98 \cdot 10^{-5}$	$7.71 \cdot 10^{-3}$	$1.29 \cdot 10^{-5}$	

Table 5: Varying the correlation ρ_{ij} for $i \neq j$ between the underlying assets in the range of $[0, 1]$

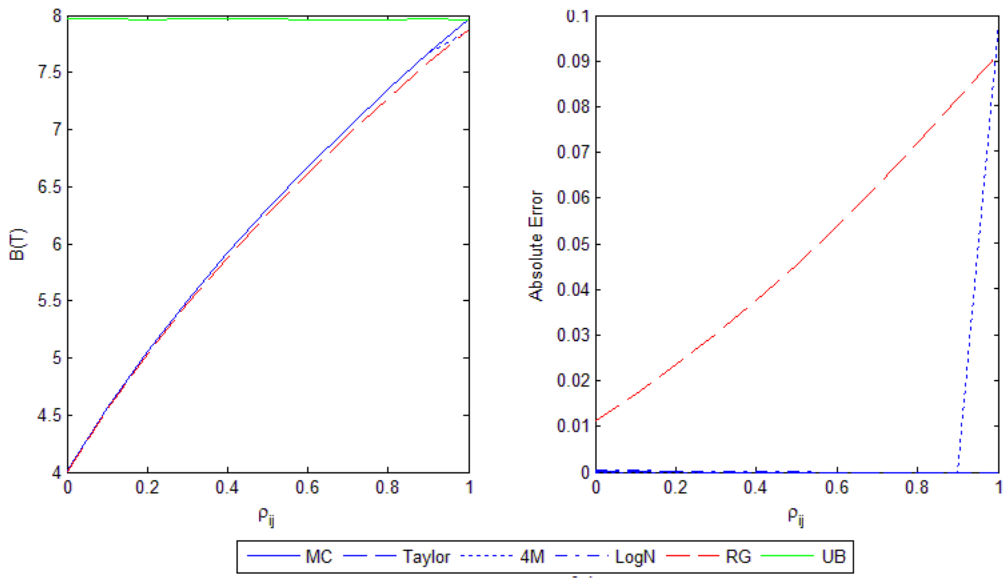

Figure 19: Varying the correlation between the underlying assets. Right: Basket option values derived from the different methods. Left: The absolute error of each approximation compared to values derived from qMC.

7.1.4 Varying the volatility

For the last numerical test are we varying the initial volatility σ_i of each underlying asset i in the range $[0.05, 0.55]$. Here is the 4M method the one that produces the best overall values, remarkable is that the Taylor approximation produces accurate values for large volatilities even thought the approximation is built on a Taylor approximation around just zero volatility. The absolute error of the values calculated by the RG approximation increases significant for increasing volatility.

σ_i	**qMC** (std)	**4M** (error)	**LogN** (error)	**RG** (error)	**Taylor** (error)	**UB** (diff)
0.05	1.5769 (0.0004)	1.5769 ($1.51 \cdot 10^{-6}$)	1.5769 ($1.53 \cdot 10^{-6}$)	1.5762 ($7.17 \cdot 10^{-4}$)	1.5769 ($1.51 \cdot 10^{-6}$)	1.9949
0.10	3.1537 (0.0008)	3.1537 ($3.57 \cdot 10^{-6}$)	3.1537 ($4.39 \cdot 10^{-6}$)	3.1479 ($5.73 \cdot 10^{-3}$)	3.1537 ($3.65 \cdot 10^{-6}$)	3.9873
0.15	4.7301 (0.0013)	4.7301 ($6.54 \cdot 10^{-6}$)	4.7301 ($1.28 \cdot 10^{-5}$)	4.7108 ($1.93 \cdot 10^{-2}$)	4.7301 ($7.22 \cdot 10^{-6}$)	5.9779
0.20	6.3059 (0.0018)	6.3060 ($9.52 \cdot 10^{-6}$)	6.3060 ($3.63 \cdot 10^{-5}$)	6.2604 ($4.56 \cdot 10^{-2}$)	6.3060 ($1.27 \cdot 10^{-5}$)	7.9650
0.25	7.8811 (0.0022)	7.8811 ($1.31 \cdot 10^{-5}$)	7.8812 ($9.61 \cdot 10^{-5}$)	7.7925 ($8.87 \cdot 10^{-1}$)	7.8811 ($2.44 \cdot 10^{-5}$)	9.9483
0.30	9.4553 (0.0028)	9.4554 ($1.25 \cdot 10^{-5}$)	9.4556 ($2.23 \cdot 10^{-4}$)	9.3030 ($1.52 \cdot 10^{-1}$)	9.4554 ($4.53 \cdot 10^{-5}$)	11.9220
0.35	11.0284 (0.0033)	11.0285 ($1.17 \cdot 10^{-7}$)	11.0289 ($4.66 \cdot 10^{-4}$)	10.7881 ($2.40 \cdot 10^{-1}$)	11.0285 ($8.24 \cdot 10^{-5}$)	13.8929
0.40	12.6002 (0.0039)	12.6002 ($5.40 \cdot 10^{-5}$)	12.6011 ($8.79 \cdot 10^{-4}$)	12.2442 ($3.56 \cdot 10^{-1}$)	12.6004 ($1.33 \cdot 10^{-4}$)	15.8506
0.45	14.1704 (0.0047)	14.1702 ($1.63 \cdot 10^{-4}$)	14.1720 ($1.57 \cdot 10^{-3}$)	13.6679 ($5.03 \cdot 10^{-1}$)	14.1706 ($2.29 \cdot 10^{-4}$)	17.7977
0.50	15.7387 (0.0051)	15.7383 ($4.02 \cdot 10^{-4}$)	15.7413 ($2.61 \cdot 10^{-3}$)	15.0560 ($6.84 \cdot 10^{-1}$)	15.7391 ($3.65 \cdot 10^{-4}$)	19.7341
0.55	17.3048 (0.0058)	17.3040 ($8.39 \cdot 10^{-4}$)	17.3090 ($4.19 \cdot 10^{-3}$)	16.4057 ($8.99 \cdot 10^{-1}$)	17.3054 ($5.7 \cdot 10^{-4}$)	21.6681
RMSE		$1.70 \cdot 10^{-5}$	$9.81 \cdot 10^{-5}$	$2.62 \cdot 10^{-2}$	$1.39 \cdot 10^{-5}$	

Table 6: Varying σ_i

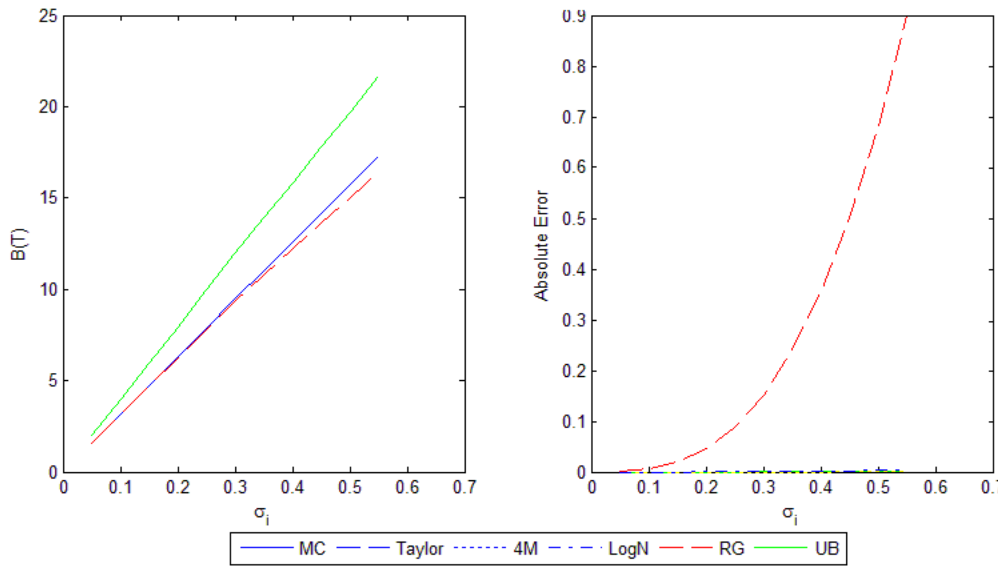

Figure 20: Varying the the intial variance σ_i between the underlying assets. Right: Basket option values derived from the different methods. Left: The absolute error of each approximation compared to values derived from qMC.

7.1.5 Real Market Data

Since the input data used when pricing the Basket option where chosen arbitrary and might be value that are “nice” to the approximations, we continue by using real market data collected the 24 March, 2009. The data used can be found in Appendix a. Each corresponding weights w_i in the Basket are one, $w_i = 1$. The Basket is valued at the money, $K = B(0)$, with a the time-to-maturity of one year, $T = 1$ and with number of simulation to $2^{23} - 1 = 8388607$.

	qMC (std)	**4M** (error)	**LogN** (error)	**RG** (error)	**Taylor** (error)	**UB**
$T = 0.5$	0.0572 $(2.21 \cdot 10^{-5})$	0.0572 $(1.53 \cdot 10^{-8})$	0.0572 $(4.41 \cdot 10^{-6})$	0.0571 $(7.43 \cdot 10^{-5})$	0.0572 $(1.53 \cdot 10^{-7})$	0.57886
$T = 1.0$	0.0786 $(3.16 \cdot 10^{-5})$	0.0786 $(3.33 \cdot 10^{-7})$	0.0786 $(1.29 \cdot 10^{-5})$	0.0969 $(2.08 \cdot 10^{-4})$	0.0786 $(5.48 \cdot 10^{-7})$	0.1667
$T = 2.0$	0.1064 $(1.03 \cdot 10^{-4})$	0.1064 $(2.63 \cdot 10^{-6})$	0.1064 $(3.89 \cdot 10^{-5})$	0.1058 $(5.776 \cdot 10^{-4})$	0.1064 $(2.61 \cdot 10^{-6})$	0.5064

Table 7: Values for each method when using real market data.

Even here does the 4M and Taylor show great accuracy.

7.1.6 Computation time

One interesting aspect and one of the purposes of this thesis is the saving in computational time when using approximation compared to valued from MC. To just get in idea of the saving we are timing each approximation for some arbitrary input data and letting the number of Basket option to be 4, these are compared to values derived from MC that are correct to the last *'pip'*.

	qMC	**4M**	**LogN**	**RG**	**Taylor**
Time (seconds)	88.639753	0.025989	0.009163	0.008426	0.042754

Table 8: The corresponding methods computational time.

Worth mentioning is that the current code for calculation both the MC value and the approximation are not optimized, for instance does the Taylor approximation are including some $\mathcal{O}\left(N^4\right)$ algorithms. Moving the code into higher level of programming language as C++ is one possibility for speeding up calculations.

7.2 FX Smile Prices

To value the FX Basket option consistent with the FX smile is done by using real market data, the implied volatility from each currency pair in delta space, the correlations between each currency pair. Each currency's deposit rate, each corresponding weight in the basket and finally each FX spot rate.

The numerical test is done by valuing a basket consisting of four currency pairs: EURUSD, EURGBP, EURJPY and EURSEK with euro as base currency, and letting all other currencies to represent the foreign currency. Data used for pricing the Basket with the smile can be found in appendix A.

For constructing approximation that are consistent with the FX smile, we are using the technique for construction a replicated portfolio as presented in section 6.2. We are using the same four currency pair EURUSD, EURGBP, EURJPY and EURSEK and summarize the relevant and used data for constructing the smile, all other data can be found in appendix A.

	EURUSD	**EURGBP**	**EURJPY**	**EURSEK**
S_0	1.3556	0.9238	133.17	10.9286
$\sigma_{25\Delta P}^{1M}$	17.690	15.830	22.770	16.630
σ_{ATM}^{1M}	17.750	15.675	21.125	17.200
$\sigma_{25\Delta C}^{1M}$	18.840	16.530	20.620	18.830

Table 9: Spot and implied volatility used when construction FX Basket smile prices.

In our case are we interesting of valueing the following expression function Φ

$$\text{Volume} \times \Phi\left(w_1 \cdot \frac{\text{USD}}{\text{EUR}} + w_2 \cdot \frac{\text{GBP}}{\text{EUR}} + w_3 \cdot \frac{\text{JPY}}{\text{EUR}} + w_4 \cdot \frac{\text{SEK}}{\text{EUR}}\right)$$

The values derived from both the Local volatility model and derived from the replicated portfolio is compared to each other. Each corresponding weights w_i in the Basket are set to one, $w_i = 1$. The strike value is set to that the moneyness is 0.8. The time-to-maturity is chosen to one month, $T = 30/365$. Number of grids used in the local volatility is set to 40×40, and the number of simulations are selected to 10^6 so that the value is correct to the last presented digit. The smile prices are compared to values calculated from the no smile consistent Taylor Approximation.

	Basket Value
Local Volatility (std)	0.3846 $(6.6804 \cdot 10^{-5})$
Replication	0.3838
Taylor	0.3835

Table 10: FX Smile prices

As one can observe from the results above is that differs a bit. Even though the Local volatility model incorporated in the MC simulation and the replication technique tries to correct the for the same problem one should remember that apples and pears are not the same, the two models should not be compared with each other. But it seems reasonable that the smile prices should be higher compared to the non smile prices. The Local Volatility in the example above want a greater adjustment compared to the replicated portfolio.

7.3 Hedging Values

We are using the same setup as in the first numerical test. These test are just demonstrating the calculated Greeks for the different approximations. All the calculation is done by using the definition of derivative. To be able to calculate qMC values we are using a common random numbers with the purpose of avoiding incorrect values due to Monte Carlo errors. Here is the 4M, LogN and the Taylor approximations the one that produces best delta and vega values. The delta values deviates by a few *'pips'*, but when calculating the vega values we have a deviation of a *'big'*.

Asset i	**qMC**	**4M**	**LogN**	**RG**	**Taylor**
Δ_1	0.1412	0.1407	0.1408	0.1352	0.1408
Δ_2	0.1416	0.1407	0.1408	0.1351	0.1408
Δ_3	0.1405	0.1407	0.1408	0.1352	0.1408
Δ_4	0.1416	0.1408	0.1408	0.1351	0.1408

Table 11: Delta values

Asset i	**qMC**	**4M**	**LogN**	**RG**	**Taylor**
ν_1	7.8618	7.8546	7.8577	7.2019	7.8554
ν_2	7.8709	7.8546	7.8577	7.2019	7.8554
ν_3	7.8505	7.8546	7.8577	7.2021	7.8554
ν_4	7.8526	7.8547	7.8577	7.2021	7.8554

Table 12: Vega values

8 Epilouge

8.1 Conclusion

This thesis has presented four closed form solution for pricing the multi currency Basket option. As observed from the numerical tests are that both the fourth moment method and the Taylor approximation produced the most accurate values which are acceptable and a computational time below a few hundred of a second. FX traders at Nordea Markets approved values that do not deviation from MC by more than a few *'pips'*. If we are about to choose a model, which of the four models should we then choose? If simplicity is of priority, we choose the Log Normal approximation. But if I were about to choose the method that produces the most accurate values, the Taylor approximation and 4M are the one to be chosen. This since the both approximations produces really accurate values, but this with a lead of the Taylor approximation due the fact that we have the opportunity to increase the accuracy by introducing higher orders in our Taylor approximation. The values calculated by the RG approximation are really bad, and this due to the fact that we only are using a small number of assets, and the summation of log-normally distributed random variables converges to a RG distribution when $N \to \infty$.

The second part was devoted for pricing the Basket option in such a way so that the FX smile is taken into account. Pricing method that we build on Local Volatility models and by creating a replicated portfolio in such a way that the smile is taken into account. The problem is thus when we are about to draw any conclusion about the derived prices since Basket options are not a very common traded and thus making it complicated trying to find out if the price is accurate or not. But since both Local Volatility model and the replicated portfolio in some since demonstrates the same significant trend, as long the derived price is within a traders bid-ask spread, and if we believe in the prices we can finally conclude that the new smile prices are correct.

8.2 Future Work

This thesis has opened the door for further research in very interesting and huge areas. The second part of this thesis covers just a very small part of a huge and a very importing area in risk management, pricing with the smile/skew. Further development and research could focus on just this subject, by introducing: volatility term structure, stochastic volatility models (as the Heston model) by violating the Black-Scholes assumption on constant volatility and with goal to capture the smile, also stochastic volatility models combined with Local volatility models. The market does often display more than one volatility and that is by further research could be done by introducing also multi-scale volatility processes (e.g. volatility of volatility) as the double-Heston Model, these models should be calibrated to market data in a clever way, and thus making us to construct even better Basket option prices. Further introduction and interesting reading on how to solve the smile problem can be found in Ayache et. al. (2004). Further research can also be done by introducing stochastic correlation models.

Hedging the models is a very important subject in risk management; no one is interesting in using an approximation that is not easy to hedge in the correct way. This thesis just cover the hedging in a not to deep way and something that have to be extended and finally determining which of the approximation that produces the best overall value. Hedging when using the Local volatility is nothing that has been considered and is not an easy task to solve.

Speeding up the calculation even further could be done by optimization the code and moving into higher level programming languages.

"...only models that take into account local, jump and stochastic features of the volatility dynamics and mix them in the right proportion are adequate for pricing and risk management of forex options".

Lipton (2002)

A Market Data

The following data are collected from the market on the 24 March, 2009, the implied volatility are collected from right delta.

	1W	1M	2M	3M	6M	9M	1Y	2Y
5ΔP	17.18	17.38	17.69	18.08	18.97	19.58	19.95	18.69
10ΔP	16.42	16.59	16.75	16.94	17.37	17.68	17.80	16.82
15ΔP	16.05	16.22	16.32	16.44	16.68	16.88	16.90	16.03
20ΔP	15.81	15.99	16.05	16.13	16.26	16.40	16.37	15.57
25ΔP	15.65	15.83	15.88	15.92	15.99	16.09	16.03	15.27
30ΔP	15.53	15.70	15.75	15.77	15.80	15.88	15.81	15.08
ATM	15.50	15.67	15.72	15.72	15.70	15.75	15.70	15.02
30ΔC	16.08	16.25	16.37	16.40	16.43	16.50	16.51	15.82
25ΔC	16.35	16.53	16.68	16.72	16.79	16.89	16.93	16.22
20ΔC	16.64	16.83	17.03	17.10	17.24	17.37	17.46	16.72
15ΔC	17.01	17.23	17.49	17.61	17.86	18.05	18.21	15.41
10ΔC	17.56	17.83	18.19	18.40	18.84	19.14	19.43	18.53
5ΔC	18.66	19.03	19.65	20.08	21.02	21.64	22.23	21.07

Table 13: The EURGBP implied volatiliy in delta space.

	1W	1M	2M	3M	6M	9M	1Y	2Y
5ΔP	25.23	26.41	27.10	27.99	29.80	30.80	31.29	30.95
10ΔP	23.95	24.73	25.14	25.65	26.76	27.46	27.89	28.09
15ΔP	23.29	23.86	24.13	24.41	25.10	25.58	25.90	26.24
20ΔP	22.84	23.25	24.42	23.55	23.96	24.28	24.50	24.86
25ΔP	22.48	22.77	22.87	22.87	23.07	23.26	23.39	23.73
30ΔP	22.15	22.31	22.33	22.27	22.32	22.40	22.44	22.67
ATM	21.37	21.12	20.92	20.65	20.32	20.15	20.02	19.95
30ΔC	21.21	20.63	20.26	19.81	19.19	18.85	18.61	18.29
25ΔC	21.28	20.62	20.22	19.72	19.02	18.66	18.39	18.03
20ΔC	21.41	20.70	20.27	19.71	18.94	18.55	18.25	17.83
15ΔC	21.61	20.85	20.42	19.78	18.96	18.55	18.20	17.68
10ΔC	21.95	21.15	20.72	19.99	19.14	18.71	18.30	17.62
5ΔC	22.67	21.82	21.43	20.56	19.74	19.33	18.82	17.82

Table 14: The EURJPY implied volatiliy in delta space.

	1W	1M	2M	3M	6M	9M	1Y	2Y
5ΔP	18.50	17.30	16.72	16.31	15.81	15.41	15.59	14.71
10ΔP	17.98	16.89	16.19	15.59	14.87	14.42	14.36	13.74
15ΔP	17.74	16.73	15.97	15.30	14.48	14.01	13.85	13.33
20ΔP	17.60	16.65	15.87	15.15	14.28	13.81	13.59	13.13
25ΔP	17.52	16.63	15.83	15.09	14.19	13.73	13.47	13.03
30ΔP	17.51	16.66	15.85	15.09	14.18	13.74	13.46	13.03
ATM	17.85	17.20	16.3750	15.60	14.70	14.30	14.00	13.50
30ΔC	18.80	18.38	17.61	16.88	16.02	15.64	15.35	14.66
25ΔC	19.17	18.83	18.08	17.39	16.54	16.18	15.92	15.13
20ΔC	19.55	19.32	18.61	17.96	17.15	16.80	16.57	15.67
15ΔC	20.01	19.93	19.28	18.70	17.96	17.63	17.44	16.39
10ΔC	20.67	20.80	20.27	19.80	19.20	18.91	18.80	17.51
5ΔC	21.92	22.43	22.16	21.98	21.69	21.48	21.60	19.79

Table 15: The EURSEK implied volatiliy in delta space.

	1W	1M	2M	3M	6M	9M	1Y	2Y
5ΔP	18.77	18.98	19.75	20.37	21.40	22.23	22.70	20.82
10ΔP	18.33	18.31	18.72	19.02	19.51	19.91	20.11	18.67
15ΔP	18.12	18.00	18.25	18.42	18.69	18.92	19.01	17.74
20ΔP	17.99	17.81	17.96	18.05	18.20	18.33	18.36	17.19
25ΔP	17.92	17.69	17.77	17.80	17.86	17.93	17.92	16.81
30ΔP	17.89	17.61	17.64	17.62	17.63	17.65	17.62	16.54
ATM	18.10	17.75	17.65	17.55	17.42	17.35	17.27	16.20
30ΔC	18.79	18.51	18.42	18.33	18.18	18.08	18.01	16.82
25ΔC	19.07	18.84	18.77	18.70	18.56	18.48	18.42	17.16
20ΔC	19.35	19.20	19.18	19.14	19.05	19.00	18.96	17.61
15ΔC	19.70	19.67	19.72	19.74	19.72	19.73	19.73	18.24
10ΔC	20.20	20.34	20.52	20.66	20.79	20.91	21. 00	19.29
5ΔC	21.12	21.67	22.19	22.63	23.19	23.63	23.96	21.68

Table 16: The EURUSD implied volatiliy in delta space.

	1W	1M	2M	3M	6M	9M	1Y	2Y
r_{EURO}	0.914	0.596	1.034	1.239	1.759	2.084	3.396	1.330
r_{GBP}	0.550	0.497	0.973	1.172	1.668	1.997	2.329	1.423
r_{USD}	0.525	0.475	0.975	1.225	1.825	2.179	2.525	1.493
r_{SEK}	1.067	0.768	0.861	0.889	1.290	1.629	1.960	1.164
r_{JPY}	0.08	-0.026	0.38	0.564	1.008	1.297	1.579	0.171

Table 17: The deposit rates.

EURUSD	EURGBP	EURJPY	EURSEK
1.3559	0.92391	133.27	10.9187

Table 18: Spot prices

	EURUSD	EURSEK	EURJPY	EURGBP
EURUSD	1	0.07	0.59	0.4
EURSEK	0.07	1	0.12	0.24
EURJPY	0.59	0.12	1	0.11
EURGBP	0.4	0.24	0.11	1

Table 19: Correlation between EURUSD, EURSEK, EURJPY and EURGBP.

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