Universal fractal scaling of self-organized networks

Paul J. Laurienti[1], Karen E. Joyce[2], Qawi K. Telesford[2], Jonathan H. Burdette[1], Satoru Hayasaka[3,1]

Departments of Radiology[1], Biomedical Engineering[2], and Biostatistical Sciences[3],
Wake Forest University School of Medicine
Medical Center Blvd,
Winston-Salem, North Carolina 27157
USA

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There is an abundance of literature on complex networks describing a variety of relationships among units in social, biological, and technological systems [1]. Such networks, consisting of interconnected nodes, are often self-organized, naturally emerging without any overarching designs on topological structure yet enabling efficient interactions among nodes [2, 3]. Here we show that the number of nodes and the density of connections in such self-organized networks exhibit a power law relationship. We examined the size and connection density of 46 self-organizing networks of various biological, social, and technological origins, and found that the size-density relationship follows a fractal relationship spanning over 6 orders of magnitude. This finding indicates that there is an optimal connection density in self-organized networks following fractal scaling regardless of their sizes.

There has been considerable interest in the organization of complex networks since the descriptions of small-world [3] and scale-free [2] networks at the end of the 1990's. Of particular interest are naturally occurring complex networks based on self-organizing principles [2]. In particular, self-organized processes have been shown to exhibit some scale-free and fractal behaviors [2, 4]. Demonstrating scale-free degree distributions in many self-organized networks, the work of Barabasi and colleagues [2, 5-7] has sparked great debate [8-10] on the actual existence of scale-free behavior in naturally occurring networks. Although the degree distributions of many networks were initially considered to follow power law distributions [10-14], severe truncation has been often observed. Nevertheless, it is intriguing that self-organized networks can exhibit scale-free degree distributions, and this has led scientists to the search for universality within self-organized systems.

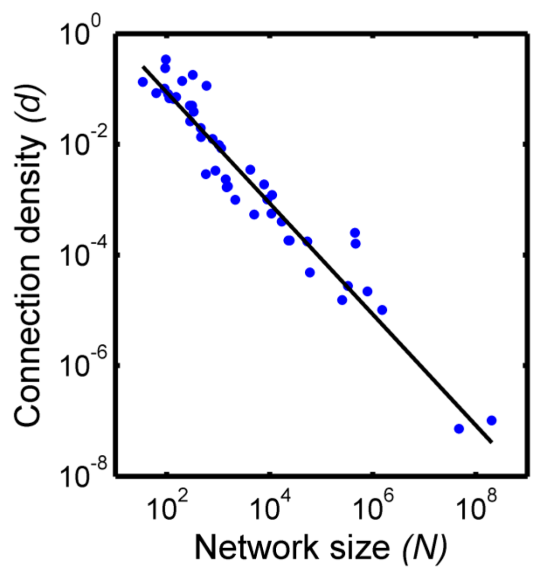

Figure 1. Log-log plot of the relationship between the number of nodes in a network (network size, *N*) and the density of connections (*d*). Each point represents a different network based on the previous literature. The fit shows a power law relationship that spans more than 6 orders of magnitude with an exponent of -0.989 consistent with a scale-free fractal behavior.

The literature on network organization encompasses a broad range of disciplines and disparate types of networks. The literature boasts networks that range from email communications to protein interactions to word frequencies in texts. The number of nodes and the density of connections in these networks span multiple orders of magnitude, complicating comparisons of metrics extracted from various studies. One common characteristic, however, is that the majority of them are self-organized--from social to technological to biological networks, the interactions between the nodes were not predetermined by a top-down blueprint design.

The work reported here describes a universal relationship between network size (the number of nodes, *N*) and connection density (the ratio of the number of existing edges to the number of all possible connections, *d*) across various types of systems. Network parameters from 46 unique networks were collected from the literature or publicly available databases as described in the supplemental methods. When the data are plotted on a log-log plot (Figure 1), there is an obvious linear relationship between the variables. The fit to the data ($d = 7.7227N^{-0.989}$) reveals a power law relationship between the size and density of the networks. The scaling exponent approaches negative one (-1), indicating that the relationship is fractal in nature with 1/f properties. Despite the wide variety of networks, there is a pronounced power law relationship between size and density covering more than 6 orders of magnitude for both node number and connection density. The fit to the data is very strong (Spearman's correlation $\rho = -0.956$, $p<0.0001$), and there is no indication of truncation at the very large network sizes. It can be seen from the figure that there are two extraordinarily large networks included in the analysis. These networks demonstrate that there is no truncation of the relationship at the extreme values. Even when these networks are removed, the correlation remains very strong (Spearman's correlation $\rho = -0.950$, $p<0.0001$) and the exponent is -0.979. Thus, these two points are not unduly influencing the analysis.

The findings reported here demonstrate a universal relationship in self-organized networks such that the network size dictates the density. The fractal behavior observed is of particular interest because it indicates that self-organized networks are critically organized. The number of connections within each network is scaled to the size of the network, and this universal behavior likely represents an optimal organization that ensures maximal capacity at a minimal cost. Furthermore, the critical organization would indicate that a density reduction would decrease the communication capabilities of the system. An increase in the density, on the other hand, would increase the wiring cost beyond the gain in capacity. It is true that one could artificially generate networks that do not exhibit this size-density relationship. In fact, the literature contains such artificially generated networks that do not lie near our plotted line. But such artificially created networks probably do not have real world relevance. We show here the scale-free relationship between network size and connection density in real networks from such diverse origins, supporting the notion of a universal law for network organization.

While replication of these findings from more networks will be important, there are a number of practical implications of these findings. First, the construction of networks is inherently limited by the sampling procedure used to identify nodes and links. If a self-organized network is found to disobey this relationship, one should seriously consider that there was a bias in the sampling of the network structure. Second, when building artificial networks to be compared to naturally occurring systems, the size-density relationship should roughly follow the 1/f relationship. For example, in studies of functional brain networks, cross-correlation matrices of nodal time series are often thresholded to identify links between nodes. The optimal threshold to be applied is not known, and the typical solution is to utilize multiple thresholds [15] producing networks with various densities. Based on the findings presented here, an optimal threshold can be easily determined, resulting in a network following the 1/f size-density relationship. Finally, engineered networks for practical applications may realize an optimal cost-benefit trade-off by ensuring that the density of connections is appropriate for the network size.

We show an important, apparently universal feature of self-organized networks: fractal scaling of size and density of connections. This fractal scaling is independent of network types, as the analysis spanned a wide gamut of networks, from biological (e.g., C. elegans neuronal architecture) to technological (e.g., the World Wide Web) to cultural (e.g., actors network) to social (e.g., student relationships). Thus, it appears that there is an underlying principle to organizing these self-emergent networks, a principle that probably ensures optimal network functioning.

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