The beauty of the mammalian vascular system

Julio Gil[1], Rafael Lahoz-Beltra[2, 3], Miguel Gimeno[1], Jesús Laborda[1], Javier Nubiala[1]

(1) Department of Anatomy, Embryology and Animal Genetics. Faculty of Veterinary Medicine, University of Zaragoza, c/ Miguel Servet 177, 50013 Zaragoza, Spain.

(2) Department of Applied Mathematics (Biomathematics). Faculty of Biological Sciences, Complutense University of Madrid, c/ Jose Antonio Novais 2, 28040 Madrid, Spain.

(3) **Corresponding author**: R. Lahoz-Beltra, Tel. +34 91 394 52 43, FAX +34 91 394 50 51, e-mail: lahozraf@bio.ucm.es

Beauty[1] is a characteristic of objects that provides a perceptual experience of pleasure. In nature, aesthetic appreciation thereof has given rise to the mathematical search for good series (e.g. the Fibonacci series) and proportions (e.g. the Golden proportion) as important elements of beauty. In 1928 the mathematician George David Birkhoff[2] introduced a formula for aesthetic measurement of an object. Birkhoff's equation defines the aesthetic value as the amount of order divided by the complexity of the product. These two features can be measured easily in poetry, music, painting, architecture, etc. In the fine arts, it is the artist who manipulates both these features, but how does nature manage order and complexity in living organisms or their parts? Here we show how Birkhoff's equation, applied to the mammalian vascular system of eight representative animals, results in new insights into the organization of the animal vascular system. We found that order and complexity are highly correlated in the mammalian vascular system (R^2=0.9511). Accordingly, in nature both features are not independently managed in the manner of artists. We found significant differences among the Birkhoff aesthetic values in the mammalian arterial system, whereas no such differences exist in the venous system. We anticipate our approach to be useful in the study of morphogenesis and evolution of tree-like structures, employing the Birkhoff aesthetic value as a simple tool for conducting such studies.

Aesthetic appreciation[1] of nature has always been a common attitude in poets, musicians, and artists, and this does not differ from the appreciation of beauty in naturalists and scientists. In the 19th century the American mathematician George David Birkhoff[2] introduced an equation to measure levels of aesthetics *M* based on a ratio of order *O* and complexity *C*:

$$M = \frac{O}{C} \quad (1)$$

The original equation had an artistic aim, assigning a high Birkhoff aesthetic value to orderliness and a low one to complexity. A good example of how to apply this equation can be found in music and poetry. Birkhoff assigned a value *O* for a piece of music[3], considering how much pattern underlies a piece, and *C* by how many notes its contains. In the case of poetry[3] the order *O* is given by $aa + 2r + 2m - 2ae - 2ce$, where *aa* stands for alliteration and assonance, *r* for rhyme, *n* for musical sounds, *ae* for alliterative excess, and *ce* for excess of consonant sounds. Birkhoff's equation appears to derive from St. Augustine's and Thomas Aquinas' definition of beauty as coherent complexity, or quality that "being perceived, pleases". For these philosophers beauty was difficult to directly define and measure, and they therefore used a trick: beauty was defined through other attributes which are easier to study. According to Birkhoff's equation, beauty increases as complexity decreases, a theory that can have a deep impact when applied to paintings, music, poetry, architecture or other fine arts. For example, a poem written by Alfred Lord Tennyson ranks nears the top on Birkhoff's aesthetic scale (0.77), whereas for a type of ancient Chinese vase, *M* was only 0.16. In the fine arts is the artist who manipulates both features of *M*, orderliness *O* and complexity *C*, but how does nature manage the order and complexity of a given structure? What is the aesthetic value of molecules, cells, tissues, organs, systems, or individuals? Here we show how Birkhoff's equation can be applied in the physiological study of living organisms, for example, to obtain new insights into the organization of the animal vascular system. For instance, how do we compare the embryonic circulatory system of two chicken eggs? By calculating Birkhoff's aesthetic value in 4 day-old (4.86) and 6 day-old (4.59) chicken embryos (Fig. 1), we found how the aesthetic value describes well changes in the egg during development[4].

An outstanding fact regarding many animals involves the presence of a circulatory system[5-7]. A circulatory system provides a blood distribution network that moves nutrients, gases, and wastes to and from cells. Furthermore, the circulatory system helps to stabilize body temperature and pH and maintains homeostasis. The more primitive animal phyla lack a circulatory system. Arthropods and most mollusks have an open circulatory system. In this type of system, there is neither a true heart nor capillaries, as found in humans and other animals. In contrast, the closed circulatory system of some mollusks and of all higher invertebrates and vertebrates is a much more efficient system. Compared with annelids such as the earthworm, the mammalian circulatory system is a highly evolved structure.

Limiting our study to mammals, we answered the following question: what is the aesthetic value of the circulatory system? To address this question, we selected the circulatory system of eight representative mammalians: dogs, cats, horses, pigs, cows, sheep, goats and humans. The mammalian circulatory system was represented as a Beck's map[8] (Fig. 2), and therefore like a map of the London Underground. Subsequently, the order *O* and complexity *C* of each vascular tree was calculated as a Shannon entropy *H* and a fractal dimension *D*, respectively. We then calculated the

Birkhoff aesthetic value M_1 and an alternative aesthetic measure M_2. Once these aesthetic measures were obtained, a statistical analysis was conducted.

To examine how mammalians manage the order and complexity of the vascular system, we obtained the regression line equation between entropy H and complexity D. The curve $H = -0.56 + 5.47\ D$ is shown in Fig. 3a. Our findings show that H and D are highly correlated in the mammalian vascular system. Indeed, this correlation suggests that order and complexity are not mutually independent. The beauty of the mammalian vascular system tallies with the theological interpretation of beauty provided by St. Augustine and Thomas Aquinas[9]: 'beauty consists of unit and order which emerge from complexity'. However, artists are able to manage both features independently. Thus, the beauty of paintings, music, poetry, etc. is in consonance with Pythagoras' and the Renaissance[1] view of beauty. Indeed, during the Renaissance, artists and architects used to create their works choosing determined proportions (e.g. 1:1.62 or the so-called golden section) considered to constitute important attributes of objects.

We next tested the correlation between the Birkhoff aesthetic value M_1 and the alternative aesthetic measure M_2. The regression line (Fig. 3b) $M_1 = 4.75 – 14.54\ M_2$ shows that both aesthetic measures are also correlated in the mammalian vascular system. We found significant differences between the Birkhoff aesthetic value M_1 in the arterial and venous systems (Birkhoff's aesthetic M_1 median value was 4.63, compared with 4.68; Kruskall-Wallis test, P=0.007). In addition, in the case of the arterial system in the eight selected mammalians (Fig. 4a), our study reveals significant differences among individuals. As a result, from a lower to higher M_1, mammalians are arranged as follows: humans (4.28) < dogs=cats=horses (4.60) < pigs (4.73) < cows=sheep=goats or ruminants (4.79) (Kruskall-Wallis test, P=0.006). However, when the value M_1 was studied in the venous systems of the eight selected mammals (Fig. 4b), we found no significant differences among these systems (Kruskall-Wallis test, P=0.2052). The likely explanation is as follows. Order O (or entropy H) is a measure of *relationship* among the nodes of blood vessel networks in the vascular tree. Complexity C (or the fractal dimension D) evaluates the *number* of nodes involved in blood vessel networks. Therefore, the high correlation between H and D responds to an optimised topology of the vascular tree. It should be noted that the vascular tree develops according to demand of nutrients by tissues. Likewise, the lines of the underground transport system grow according to demand by suburban dwellers. The classification of mammalians in accordance with the Birkhoff M_1 value in the arterial system provides similar groups of animals that can be obtained according to the left subclavian artery. This artery is one of the major arteries that supplies blood mainly to the head and front legs (or arms). Demand of nutrients by tissues and organs might be equivalent to the role played by population density during city formation[10] and the evolution of the urban layout. We anticipate our approach to be useful in the study of morphogenesis and evolution of tree-like structures[11], using Birkhoff aesthetic value as a simple tool for conducting such studies

METHODS

Beck representation. Beck's map is a schematic diagram, rather than a geographic map, showing the lines, stations and zones of the London underground. The map is based upon the relative positions of stations along the lines, as well as the connective relations of one station with another. One of the main features of Beck's map is that lines are drawn only horizontally, vertically or at 45 degrees. The result is a topological map that emphasizes connections bearing a resemblance with electrical circuit diagrams. We represented a circulatory system showing the connective relationships among blood vessels, and therefore, arteries, capillaries and veins, as a schematic Beck's diagram (Fig. 2). A similar schematic approach has been used for illustrating protein molecules[12] and to plot the molecular circuitry of cancer[13].

Spatial codification. Although the circulatory system is represented as a 2D Beck's map, each vessel has a label with information (coded with a single letter) regarding its geographical or real spatial position.

Vascular tree quantification. First, using vascular tree data[14-16], we build a standard model of a 3D vascular tree preserving the relationships among nodes. Data were stored in a customized database named NAVI_NA written in Borland Turbo C++ by the first author[17]. We then constructed a sample of vascular trees, changing the branches angles at random. Given a branch angle, its value is modified by adding a random value. These random values were set empirically. The former operation introduces some variability, which can be observed in real vascular trees.

In the present study we obtained a sample with N=30 vascular trees, where the fractal dimension D was measured. The results obtained were registered. The method described was conducted with Fractal3D, a program also written by the first author[17]. The program draws any of the reconstructed vascular trees, showing the box-counting step sequence. Secondly, the Shannon entropy H was calculated:

$$H = \sum_{i=1}^{n} -p_i \, ln(p_i) \qquad (2)$$

In the equation, the probability of a given symbol p_i is replaced by the node probability in a vascular tree. Thus, given the arborisation level b_i (b_i = 1, 2, 3 …), p_i is calculated as the number of nodes or cardinal $C(i)$, being ln (p_i) the neperian logarithm of the number of branches arising from node i (i = 1, 2, 3 …). Assuming that we are measuring entropy macroscopically in a vascular tree, n was set up equal to 14. In addition, we calculated another measure labelled as H^*

$$H^* = \sum_{i=1}^{n} C(i) \, . \, b_i \qquad (3)$$

Note how H^* is a measure of the number of nodes $C(i)$ in a vascular tree, but weighted with the arborisation level b_i. In third place, for each of the reconstructed vascular trees, the fractal dimension D is calculated by applying the box-counting method[18]. Thus, D was obtained by counting the boxes (d_n) filled with a section of the vascular tree at iteration n. The fractal dimension value is the average of the number of sample trees:

$$D = \frac{\sum_{i=1}^{N} K_i}{N} \qquad (4)$$

calculating the Kolmogorov entropy value, one per vascular tree:

$$K_i = \frac{d_n - d_{n-1}}{\log\left(\frac{1}{2^n}\right) - \log\left(\frac{1}{2^{n-1}}\right)} \qquad (5)$$

The results obtained were registered in an output file. The program[17] draws any of the reconstructed vascular trees, showing the box-counting step sequence. Finally, and in fourth place, the beauty or aesthetic measure of the vascular trees was obtained by means of the following expressions:

$$M_1 = \frac{H}{D} \qquad (6)$$

being (6) a Birkhoff aesthetic expression, and:

$$M_2 = \frac{H}{H^*} \qquad (7)$$

an aesthetic measure introduced by us in this study. As with D, the value of H^* is a measure of complexity, but without the spatial information included in D.

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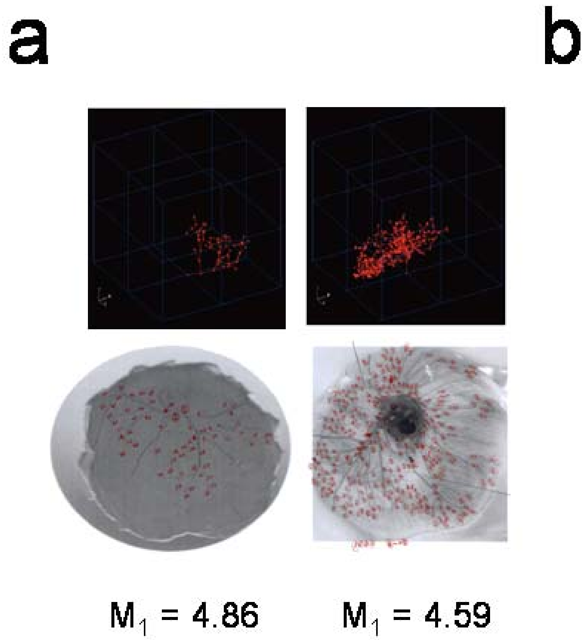

Fig. 1

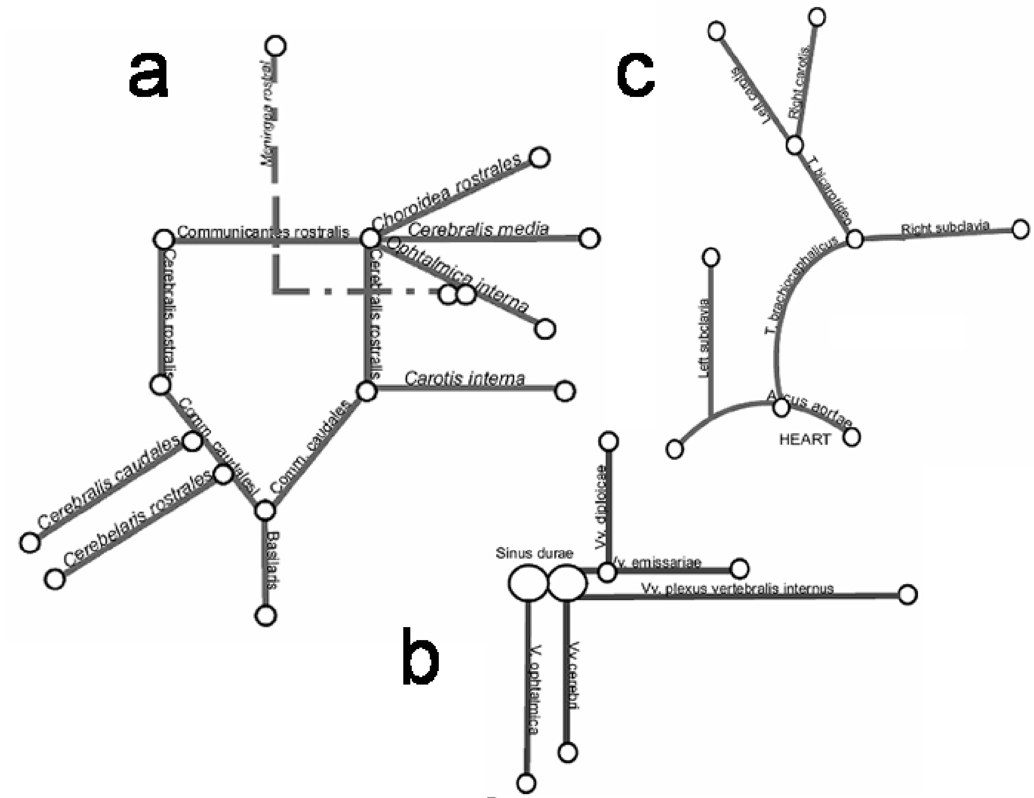
a
Meningea rostral
Choroidea rostrales
Communicantes rostralis
Cerebralis media
Ophtalmica interna
Cerebralis rostralis
Cerebralis rostralis
Carotis interna
Comm. caudales
Comm. caudales
Cerebralis caudales
Cerebelaris rostrales
Basilaris
b
Vv. diploicae
Sinus durae
Vv. emissariae
Vv. plexus vertebralis internus
V. ophtalmica
Vv. cerebri
c
Left carotis
Right carotis
T. bicarotideo
Right subclavia
T. brachiocephalicus
Left subclavia
Arcus aortae
HEART

Fig. 2

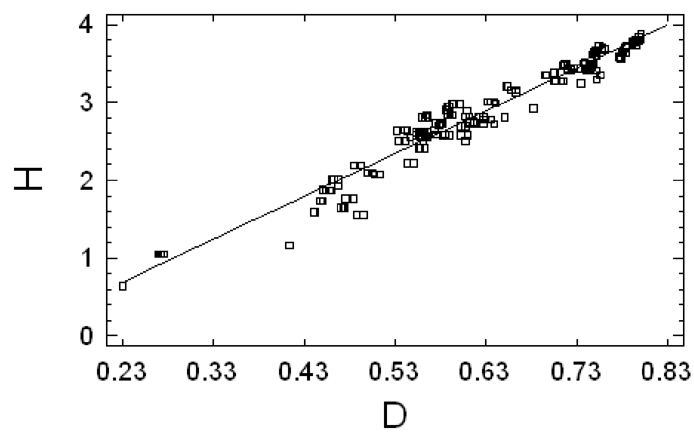
H
D
0
1
2
3
4
0.23
0.33
0.43
0.53
0.63
0.73
0.83

Fig. 3a

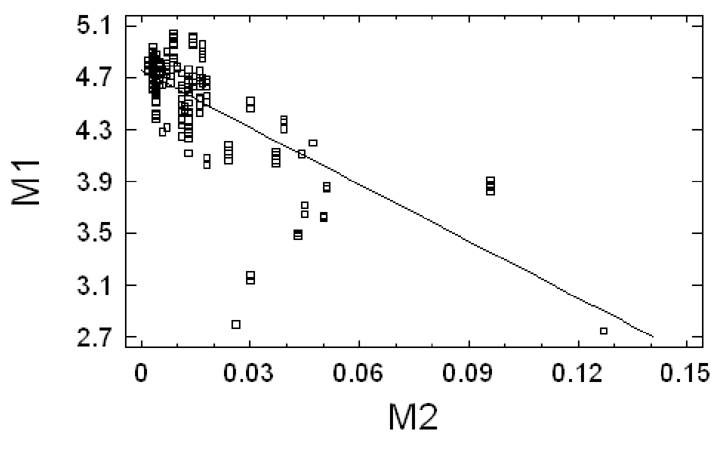
M1
M2
5.1
4.7
4.3
3.9
3.5
3.1
2.7
0
0.03
0.06
0.09
0.12
0.15

Fig. 3b

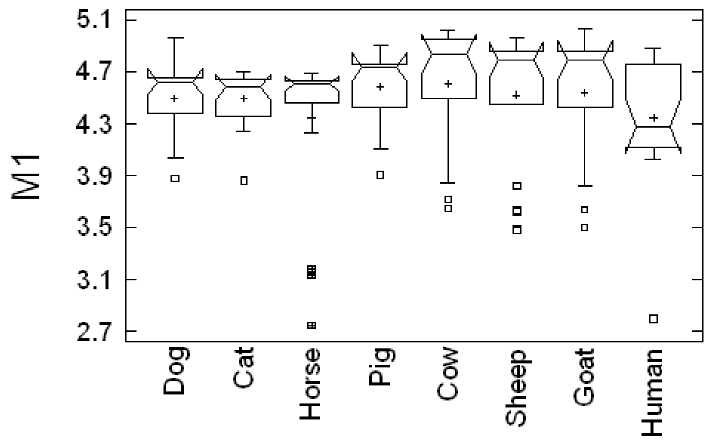
M1
5.1
4.7
4.3
3.9
3.5
3.1
2.7
Dog
Cat
Horse
Pig
Cow
Sheep
Goat
Human

Fig. 4a

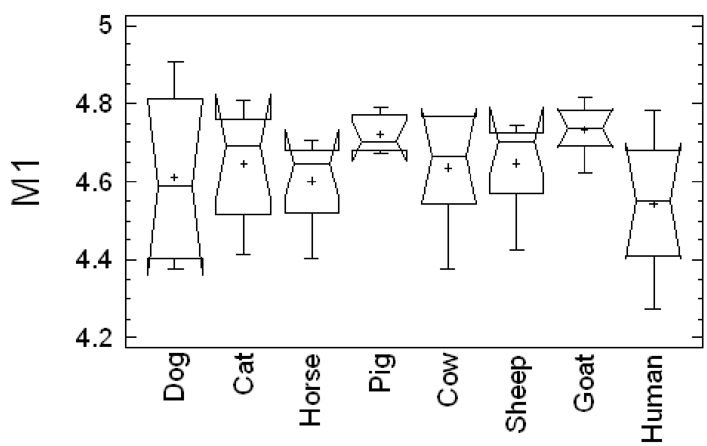
M1
5
4.8
4.6
4.4
4.2
Dog
Cat
Horse
Pig
Cow
Sheep
Goat
Human

Fig. 4b

Figure 1. Birkhoff aesthetic value M_1 in chicken embryo. a, 4 day-old egg (M_1=4.86, H=2.63, D=0.54). **b,** 6 day-old egg (M_1=4.59, H=3.43, D=0.74).

Figure 2. Schematic diagrams representing the mammalian vascular networks in a manner similar to Beck's Underground map. a, Arterial circle of the brain. **b,** Subclavian arterial circuit. **c,** Craneal venous sinus system.

Figure 3. Multicollinearity in the vascular system of mammalians. a, Order (entropy H) versus complexity (fractal dimension D). For standard linear regression, R^2 = 0.9511, P=0.0000. **b,** Birkhoff aesthetic value (M_1) versus M_2. For standard linear regression, R^2 = 0.5241, P=0.0000.

Figure 4. Birkhoff's aesthetic value M_1 of the mammalian vascular system. Distribution of variation according animals, shown as a notched-box-and-whisker plot. The edges of the box correspond to quartiles; the notches to the standard error of the median; crosses inside boxes are means; crosses outside boxes are outliers; and the vertical whiskers correspond to range. **a,** Box plot in the arterial system. **b,** Box plot in the venous system.