ON THE EXOTIC FISHES GIVEN TO ... GEOMETRY

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Polyhedral forms are extremely widespread both in animate and inanimate nature. Thus, crystals occur as polyhedra only. Besides, these forms are quite common with various primitive organisms, i.e. icosahedral viruses, radiolaria and algae. Here we discuss the cases of exotic Boxfish and Porcupinefish. The specific morphology of the Boxfish reveals in polygonal osseous blades, covering its body. As for the Porcupinefish, its polyhedral approximation was observed via certain geometrical techniques applied. Namely, their spine bases were considered the Delaunay point (R, r)-systems. Consequently, the respective Dirichlet tiling proved to be quasifullerenes and analogous to the Boxfish morphology. This unexpected geometrical dualism of the two families corroborates their taxonomic affinity within the *Tetradontiformes* order. The above biometrical method is highly recommended as a means of characterization of the *Tetraodontiformes* specimens in terms of the Delaunay (R, r)-systems.

It is striking that the universal principles of purposefulness manifest similarly both in animate and inanimate nature. Presumably, the same mathematical theories can be applied to characterize biological and mineral individuals. Specifically, in both cases polyhedral forms are widespread. It is plane-facet polyhedra that crystals occur as, their free growth provided. Furthermore, even mineral grains in rocks may be approximately considered adjacent polyhedra. Generally, polyhedral forms are a norm in mineral nature. As for the animate one, polyhedra are most common with primitive organisms, say, icosahedral viruses, green algae (*Pandorina morum, Volvox aureus, Volvox globator*) and some species of radiolaria (*Circogonia icosahedra, Circogonia dodecahedra*)[1]. Typically, most of these have the form of a fullerene, i.e. a polyhedron that has only penta- and hexagons on its surface. By the way, an elementary fullerene of a dodecahedron found its reflection in the name of an above mentioned radiolaria; icosahedron is geometrically dual to dodecahedron and reflected in the name of the other. However, polyhedral forms can be found among high-organization organisms as well. Thus, all but the whole body of the Boxfish (the *Ostraciontidae* family) is covered with polygonal osseous blades that make them quasipolyhedra (Fig. 1).

Fig. 1. *Acanthostracion quadricornis* (Scrawled cowfish, above) and *Tetrosomus gibbosus* (Humpback turretfish, below)[2]. See also pictures of the Boxfish in the Field Guide by Carpenter *et al.*[3] and on the Internet site[4].

Then, I turned to the Porcupinefish (the *Diodontidae* family) that belong to the same *Tetraodontiformes* order. From the first sight, their morphology has nothing to do with that of the Boxfish. The Delaunay (R, r)-systems presented in the current research, however, justify their affinity. But before we put forth the results of our analysis, let us have an insight into the Delaunay (R, r)-system theory, which is a pillar of the contemporary crystallography.

Studying the issue of what geometrical principles make atoms nest in a growing crystal at the very place they should, B. N. Delaunay[5] introduced a notion of the point (R, r)-systems dependent on the two axioms. The "discreteness" axiom states that the distance between any two points of the system is more than a certain constant *r*. The "overlapping" axiom requires that the distance from any point in space to the nearest point of the (R, r)-system is less than a certain constant *R*. The two axioms prevent points getting either too close or too far from each other, providing thus more or less uniform distribution of points in space. When studying (R, r)-systems, B. N. Delaunay implied an original "vacuous balloon" method. Supposing, there is an (R, r)-system, say, on a plane. Let us put rather a small circle in it lest there should be a single point inside the circle, and then begin to "blow" it like a balloon. When the circle touches the first point of the system, let us have it repulsed from this point. When it touches the second one, let us have it repulsed from both of them. Finally, the circle shall touch the third point. Provided that the (R, r)-system is occasional on a plane, here is the end of the process since it becomes impossible to enlarge the circle without capturing a point. If we connect the three points with intervals, we get a triangle, and having implied the technique several times, we get the Delaunay plane triangulation. Having drawn then middle perpendiculars to all the intervals till their intersection, we get the Dirichlet polygonal tiling of the plane (Fig. 2).

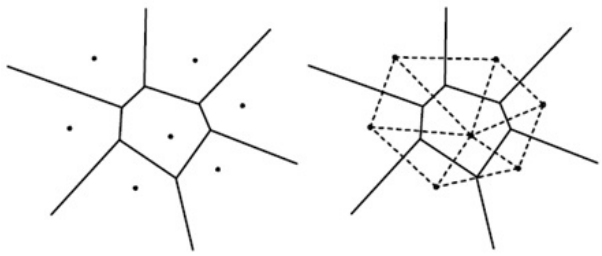

Fig. 2. The Dirichlet tiling (left) and the dual Delaunay triangulation (right) of the plane.

Each polygon contains the only point of the initial (R, r)-system. The main property of any polygon is its every inner point being closer to this only point than to any other. So far the same (R, r)-system is concerned, the Delaunay and Dirichlet tilings are dual, i.e. they determine each other. According to Voronoy, two (R, r)-systems are equivalent, if the respective Dirichlet tilings (or the related Delaunay ones) are combinatorially identical. The latter implies their one-to-one correspondence, where the respective polygons are surrounded by the same number of neighbours, and the same number of polygons meets at the respective vertexes. The notion of the (R, r)-system grounds the contemporary crystallographic geometry. Its prior achievement is local conditions (i.e. those for bounded domains surrounding each point of the system) that inevitably provide crystallographic ordering of the whole (R, r)-system. For example, it is proved that the Delaunay (R, r)-system will get the crystallographical order on a plane, if its every point is identically surrounded in a circle of 4R radius. There is a hypothesis that this "local equality" theorem is correct for 3D space as well. Below we apply all these concepts to the Delaunay tiling not on a plane, but on a quasispherical surface of the fish body. We cannot but fail to see that they are closely related to the Boxfish morphology (Fig. 1).

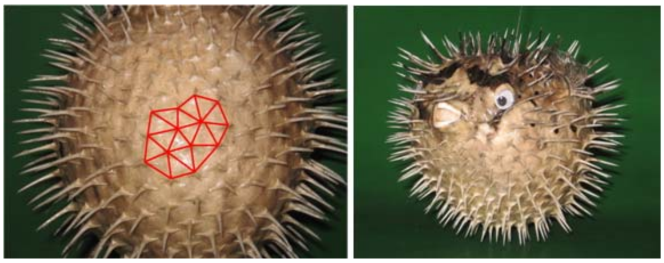

Fig. 3. The Delaunay triangulation of a *Diodon holocanthus*'s belly. All the three studied specimens have the diameter of 10-12 cm.

In the current research three specimens of the *Diodon holocanthus* were studied. Geometrically, the Porcupinefish spine bases form the (R, r)-system on the quasispherical surfaces of their bodies. Accordingly, the idea to study the Dirichlet tilings on several Porcupinefish specimens occurred. Thus, they are nothing if not polyhedral approximations of the Delaunay system. First, the Delaunay triangulations and the dual Dirichlet tilings were drawn on the surface of each specimen (Fig. 3). Second, considering the Dirichlet tiling, the number of various polygons occurring on the bodies of each was calculated (Table 1). In all cases pentagons and hexagons dominate, whereas 4- and 7-gons are very few. Logically, here the question rises – are the polyhedral approximations of the *Diodon holocanthus* fullerenes? Any fullerene, however, has 12 pentagons precisely and any number of hexagons[6]. In our case the number of pentagons is tens as many, no errors of geometrical drawing possible. Therefore, their polyhedral approximations are but quasifullerenes, and their having 4- and 7-gons is quite a pattern, whatever rare. So far the Dirichlet tiling is concerned, the penta- *vs* hexagons ratio is approximately the same with all the three specimens. In other words, statistically, these systems appear to be equivalent.

TABLE 1 The number of n-gons in the Dirichlet tiling of the *Diodon holocanthus's* surfaces

n	4	5	6	7	6 *vs* 5 *
# 1	1	48	69	0	1.437
# 2	0	58	76	1	1.310
# 3	1	65	84	2	1.292

* The hexagons *vs* pentagons ratio.

Furthermore, it is easy to see that the Dirichlet tilings on the studied Porcupinefish surfaces are analogous to the osseous blades of the Boxfish (Fig. 1). The latter also has the dominating number of penta- and hexagons on its body, with 4- and 7-gons being extremely rare. Thus, studying the Delaunay (R, r)-systems revealed the *Ostraciontidae* and *Diodontidae* families morphological dualism. Unexpected as it is, this is a conceptually new geometrical argument in support of their taxonomic affinity within the *Tetraodontiformes* order. With the above technique providing new biometrical characteristics, I recommend to apply it for studying the *Tetraodontiformes* species. Other research issues can also be studied with the Delaunay (R, r)-systems technique applied. Say, whether the Porcupinefish are mutually equivalent in the sense of Voronoy's definition given above is a prospective field of further research. By the way, the Boxfish and Porcupinefish morphology is not strictly speaking crystallographic since the “local equality” theorem is obviously not correct for it.

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