FACTORIZATION BY ELEMENTARY MATRICES, NULL-HOMOTOPY AND PRODUCTS OF EXPONENTIALS FOR INVERTIBLE MATRICES OVER RINGS

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Abstract. Let $R$ be a commutative unital ring. A well-known factorization problem is whether any matrix in $\text{SL}_n(R)$ is a product of elementary matrices with entries in $R$. To solve the problem, we use two approaches based on the notion of the Bass stable rank and on construction of a null-homotopy. Special attention is given to the case, where $R$ is a ring or Banach algebra of holomorphic functions. Also, we consider a related problem on representation of a matrix in $\text{GL}_n(R)$ as a product of exponentials.

1. Introduction

Let $R$ be an associative, commutative, unital ring. A well-known factorization problem is whether any matrix in $\text{SL}_n(R)$ is a product of elementary (equivalently, unipotent) matrices with entries in $R$. Here the elementary matrices are those which have units on the diagonal and zeros outside the diagonal, except one non-zero entry. In particular, for $n = 3, 4, \ldots$, Suslin [20] proved that the problem is solvable for the polynomials rings $\mathbb{C}[\mathbb{C}^m], \ m \geq 1$. For $n = 2$, the required factorization for $R = \mathbb{C}[\mathbb{C}^m]$ does not always exist; the first counterexample was constructed by Cohn [4].

In the present paper, we primarily consider the case, where $R$ is a functional Banach algebra. So, let $O(\mathbb{D})$ denote the space of holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Recall that the disk-algebra $A(\mathbb{D})$ consists of $f \in O(\mathbb{D})$ extendable up to continuous functions on the closed disk $\overline{\mathbb{D}}$. The disk-algebra $A(\mathbb{D})$ and the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$ may serve as good working examples for the algebras under consideration.

In fact, we propose two approaches to the factorization problem. The first one is based on construction of a null-homotopy; see Section 2. This method applies to the disk-algebra and similar algebras. The second approach is applicable to rings whose Bass stable rank is equal to one; see Section 3. This methods applies, in particular, to $H^\infty(\mathbb{D})$.

Also, the factorization problem is closely related to the following natural question: whether a matrix $F \in \text{GL}_n(R)$ is representable as a product of exponentials, that is, $F = \exp G_1 \ldots \exp G_k$ with $G_j \in M_n(R)$. For $n = 2$ and matrices with entries in a Banach algebra, this question was recently considered in [15]. In Section 4 we obtain results related to this question with emphasis on the case, where $R = O(\Omega)$ and $\Omega$ is an open Riemann surface.

2010 Mathematics Subject Classification. Primary 15A54; Secondary 15A16, 30H50, 32A38, 32E10, 46E25.

Frank Kutzschebauch was supported by Schweizerische Nationalfonds Grant 200021-178730.
2. Factorization and null-homotopy

Given \( n \geq 2 \) and an associative, commutative, unital ring \( R \), let \( E_n(R) \) denote the set of those \( n \times n \) matrices which are representable as products of elementary matrices with entries in \( R \).

For a unital commutative Banach algebra \( R \), an element \( X \in \text{SL}_n(R) \) is said to be null-homotopic if \( X \) is homotopic to the unity matrix, that is, there exists a homotopy \( X_t: [0, 1] \to \text{SL}_n(R) \) such that \( X_1 = X \) and \( X_0 \) is the unity matrix.

We will use the following theorem:

**Theorem 1** ([13, §7]). Let \( A \) be a unital commutative Banach algebra and let \( X \in \text{SL}_n(A) \). The following properties are equivalent:

(i) \( X \in E_n(A) \);

(ii) \( X \) is null-homotopic.

To give an illustration of Theorem 1 consider the disk-algebra \( A(\mathbb{D}) \).

**Corollary 1.** For \( n = 2, 3, \ldots \), \( E_n(A(\mathbb{D})) = \text{SL}_n(A(\mathbb{D})) \).

**Proof.** We have to show that \( E_n(A(\mathbb{D})) \supset \text{SL}_n(A(\mathbb{D})) \). So, assume that

\[
F = F(z) = \begin{pmatrix} f_{11}(z) & f_{1n}(z) \\ f_{n1}(z) & f_{nn}(z) \end{pmatrix} \in \text{SL}_n(A(\mathbb{D})).
\]

Define

\[
F_t(z) = F(tz) \in \text{SL}_n(A(\mathbb{D})), \quad 0 \leq t \leq 1, \ z \in \mathbb{D}.
\]

Given an \( f \in A(\mathbb{D}) \), let \( f_t(z) = f(tz), 0 \leq t \leq 1, z \in \mathbb{D} \). Observe that \( \| f_t - f \|_{A(\mathbb{D})} \to 0 \) as \( t \to 1^- \). Applying this observation to the entries of \( F_t \), we conclude that \( F \) is homotopic to the constant matrix \( F(0) \). Since \( \text{SL}_n(\mathbb{C}) \) is path-connected, the constant matrix \( F(0) \) is homotopic to the unity matrix. So, it remains to apply Theorem 1. \( \square \)

3. Factorization and Bass stable rank

**Definitions.** Let \( R \) be a commutative unital ring. An element \( (x_1, \ldots, x_k) \in R^k \) is called **unimodular** if

\[
\sum_{j=1}^k x_j R = R.
\]

Let \( U_k(R) \) the set of all unimodular elements in \( R^k \).

An element \( x = (x_1, \ldots, x_{k+1}) \in U_{k+1}(R) \) is called **reducible** if there exists \( (y_1, \ldots, y_k) \in R^k \) such that

\[
(x_1 + y_1 x_{k+1}, \ldots, x_k + y_k x_{k+1}) \in U_k(R).
\]

The **Bass stable rank** of \( R \), denoted by \( \text{bsr}(R) \) and introduced in [1], is the least \( k \in \mathbb{N} \) such that every \( x \in U_{k+1}(R) \) is reducible. If there is no such \( k \in \mathbb{N} \), then we set \( \text{bsr}(R) = \infty \).

**Remark 1.** The identity \( \text{bsr}(R) = 1 \) is equivalent to the following property: For any \( x_1, x_2 \in R \) such that \( x_1 R + x_2 R = R \), there exists \( y \in R \) such that \( x_1 + y x_2 \in R^* \).
3.2. A sufficient condition for factorization.

**Theorem 2.** Let $R$ be a unital commutative ring and $n \geq 2$. If $bsr(R) = 1$, then $E_n(R) = SL_n(R)$.

**Proof.** First, assume that $n = 2$. Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in SL_2(R).$$

Since $\det X = 1$, we have

$$x_{21}R + x_{11}R = R. $$

Hence, using the assumption $bsr(X) = 1$ and Remark 1, we conclude that there exists $y \in R$ such that

$$x_{21} + yx_{11} \in R^*. \quad (3.1)$$

Now, we have

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} X = \begin{pmatrix} x_{11} & x_{12} \\ \alpha & * \end{pmatrix}. $$

Next, using (3.1) we obtain

$$\begin{pmatrix} 1 & (1-x_{11})^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \alpha & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ \alpha & * \end{pmatrix}. $$

Finally, we have

$$\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ \alpha & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & x_0 \end{pmatrix}. $$

Since the determinant of the last matrix is equal to one, we conclude that $x_0 = 1$. Therefore, the $X$ is representable as a product of four multipliers.

For $n \geq 3$, let

$$X = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \in SL_n(R).$$

Since $\det X = 1$, there exist $\alpha_1, \ldots, \alpha_n \in R$ such that $\alpha_1 x_{11} + \cdots + \alpha_{n-1} x_{n-11} + \alpha_n x_{n1} = 1$. Therefore,

$$x_{n1}R + \left( \sum_{i=1}^{n-1} \alpha_i x_{i1} \right) R = R. $$

Applying the property $bsrR = 1$, we obtain $y \in R$ such that

$$x_{n1} + y \left( \sum_{i=1}^{n-1} \alpha_i x_{i1} \right) := \alpha \in R^*. $$

Put

$$L = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \ddots \\ \alpha_1 y & \cdots & \alpha_{n-1} y & 1 \end{pmatrix}. $$
Then

\[ LX = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n-1} \\ \alpha \end{pmatrix}. \]

Multiplying by the upper triangular matrix

\[ U_1 = \begin{pmatrix} 1 & 0 & \cdots & (1 - x_{11})\alpha^{-1} \\ 1 & 0 & \cdots & -x_{21}\alpha^{-1} \\ 0 & \cdots & 1 & -x_{n-11}\alpha^{-1} \\ 0 & \cdots & \cdots & 1 \end{pmatrix}, \]

we obtain

\[ U_1 LX = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \alpha \end{pmatrix}. \]

Now, put

\[ \tilde{L} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \cdots \\ -\alpha & 0 & 1 \end{pmatrix}. \]

We have

\[ \tilde{L}U_1 LX = \begin{pmatrix} 1 & * & * & * \\ 0 & \vdots & Y_1 \\ 0 & \cdots & \cdots \end{pmatrix}. \]

Observe that \( Y_1 \in \text{SL}_{n-1}(R) \). So, arguing by induction, we obtain

\[ \left( \prod_{i=1}^{n-1} \tilde{L}_i U_i L_i \right) X = \begin{pmatrix} 1 & * \\ 0 & \cdots & \cdots \\ & & \cdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} := U \]

or, equivalently,

\[ \left( \prod_{i=1}^{n-1} \mathcal{L}_i U_i \right) L_{n-1} X = U, \]

where \( \mathcal{L}_i \) are lower triangular matrices. So, we conclude that every \( X \in \text{SL}_n(R) \) is a product of \( 2n \) unipotent upper and lower triangular matrices. \( \square \)

**Corollary 2.** Let \( A \) be a unital commutative Banach algebra such that \( \text{bsr}(A) = 1 \). If \( X \in \text{SL}_n(A) \), then \( X \) is null-homotopic.

**Proof.** It suffices to combine Theorems 1 and 2. \( \square \)
3.3. Examples of algebras $A$ with $\text{bsr}(A) = 1$.

3.3.1. Disk-algebra $A(\mathbb{D})$. By Corollary 1, $E_n(A(\mathbb{D})) = \text{SL}_n(A(\mathbb{D}))$. Theorem 2 provides a different proof of this property. Indeed, Jones, Marshall and Wolff [12] and Corach and Suárez [5] proved that $\text{bsr}(A(\mathbb{D})) = 1$, so Theorem 2 applies.

3.3.2. Algebra $H^\infty(\mathbb{D})$. Let $f \in H^\infty(\mathbb{D})$. If $\|f_t - f\|_\infty \to 0$ as $r \to 1^-$, then clearly $f \in A(\mathbb{D})$. So the homotopy argument used for $A(\mathbb{D})$ is not applicable to $H^\infty(\mathbb{D})$. However, Treil [22] proved that $\text{bsr}(H^\infty(\mathbb{D})) = 1$, hence, Theorem 2 holds for $R = H^\infty(\mathbb{D})$. Also, Corollary 2 guarantees that any $F \in \text{SL}_n(H^\infty(\mathbb{D}))$ is null-homotopic.

3.3.3. Generalizations of $H^\infty(\mathbb{D})$. Tolokonnikov [21] proved that $\text{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface $G$ and for certain infinitely connected planar domains $G$ (Behrens domains). In particular, any $F \in \text{SL}_n(H^\infty(G))$ is null-homotopic. However, even in the case $G = \mathbb{D}$ the homotopy in question is not explicit. So, probably it would be interesting to give a more explicit construction of the required homotopy.

Let $T = \partial \mathbb{D}$ denote the unit circle. Given a function $f \in H^\infty(\mathbb{D})$, it is well-known that the radial limit $\lim_{r \to 1^-} f(r\zeta)$ exists for almost all $\zeta \in T$ with respect to Lebesgue measure on $T$. So, let $H^\infty(T)$ denote the space of the corresponding radial values. It is known that $H^\infty(T) + C(T)$ is an algebra, moreover, $\text{bsr}(H^\infty(T) + C(T)) = 1$; see [13].

Now, let $B$ denote a Blaschke product in $\mathbb{D}$. Then $C + BH^\infty(\mathbb{D})$ is an algebra. It is proved in [16] that $\text{bsr}(C + BH^\infty(\mathbb{D})) = 1$.

3.4. Examples of algebras $A$ with $\text{bsr}(A) > 1$.

3.4.1. Algebra $A_R(\mathbb{D})$. Each element $f$ of the disk-algebra $A(\mathbb{D})$ has a unique representation

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in \mathbb{D}.$$  

The space $A_R(\mathbb{D})$ consists of those $f \in A(\mathbb{D})$ for which $a_j \in \mathbb{R}$ for all $j = 0, 1, \ldots$ in (3.2). As shown in [17], $\text{bsr}(A_R(\mathbb{D})) = 2$. Nevertheless, the following result holds.

Proposition 1. For $n = 2, 3, \ldots$, $E_n(A_R(\mathbb{D})) = \text{SL}_n(A_R(\mathbb{D}))$.

Proof. For a function $f \in A_R(\mathbb{D})$, we have $f_t \in A_R(\mathbb{D})$ or all $0 \leq t < 1$. Hence, given a matrix $F \in \text{SL}_n(A_R(\mathbb{D}))$, we have $F_t \in \text{SL}_n(A_R(\mathbb{D}))$, where $F_t$ is defined by (2.4). Since $\|f_t - f\|_{A_R(\mathbb{D})} \to 0$ as $t \to 1^-$, $F$ is homotopic to the constant matrix $F_0 \in \text{SL}_n(\mathbb{C})$. Hence, $F$ is homotopic to the unity matrix. Therefore, $F \in E_n(A_R(\mathbb{D}))$ by Theorem 1. \hfill \Box

3.4.2. Ball algebra $A(B^m)$, polydisk algebra $A(\mathbb{D}^m)$, $m \geq 2$, and infinite polydisk algebra $A(\mathbb{D}^\infty)$. Let $B^m$ denote the unit ball of $\mathbb{C}^m$, $m \geq 2$. The ball algebra $A(B^m)$ and the polydisk algebra $A(\mathbb{D}^m)$ are defined analogously to the disk-algebra $A(\mathbb{D})$. By [6] Corollary 3.13,

$$\text{bsr}(A(B^m)) = \text{bsr}(A(\mathbb{D}^m)) = \left\lceil \frac{m}{2} \right\rceil + 1, \quad m \geq 2.$$  

The infinite polydisk algebra $A(\mathbb{D}^\infty)$ is the uniform closure of the algebra generated by the coordinate functions $z_1, z_2, \ldots$ on the countably infinite closed polydisk.
$D^\infty = D \times D \ldots$. Proposition 1 from [14] guarantees that $\text{bsr}(A(D^\infty)) = \infty$. Large or infinite Bass stable rank of the algebras under consideration is compatible with the following result.

**Proposition 2.** Let $n = 2, 3, \ldots$. Then

$$E_n(A(B^m)) = \text{SL}_n(A(B^m)), \quad m = 2, 3, \ldots, \infty,$$

$$E_n(A(D^m)) = \text{SL}_n(A(D^m)), \quad m = 2, 3, \ldots, \infty.$$

**Proof.** It suffices to repeat the argument used in the proof of Corollary 1 or Proposition 1.

3.4.3. **Algebra $H^\infty_\mathbb{R}(D)$.** It is proved in [17] that $\text{bsr}(H^\infty_\mathbb{R}(D)) = 2$. We have not been able to determine the connected component of the identity in $\text{SL}_n(H^\infty_\mathbb{R}(D))$.

**Problem 1.** Is any element in $\text{SL}_n(H^\infty_\mathbb{R}(D))$ null-homotopic?

4. **Invertible matrices as products of exponentials**

Let $R$ be a commutative unital ring. In the present section, we address the following problem: whether a matrix $F \in \text{GL}_n(R)$ is representable as a product of exponentials, that is, $F = \exp G_1 \ldots \exp G_k$ with $G_j \in M_n(R)$. For $n = 2$ and matrices with entries in a Banach algebra, this problem was recently studied in [15].

4.1. **Basic results.** There is a direct relation between the problem under consideration and factorization of matrices in $\text{GL}_n(R)$.

**Lemma 1.** Let $X \in \text{SL}_n(R)$ be a unipotent upper or lower triangular matrix. Then $X$ is an exponential.

**Proof.** For $n = 2$, we have

$$\exp \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Let $n \geq 3$. Given $\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots; \gamma_1, \gamma_2, \ldots$, we will find $a_1, a_2, \ldots; b_1, b_2, \ldots; c_1, c_2, \ldots$ such that

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \ldots \\ 1 & \beta_1 & \beta_2 & \ldots \\ 1 & \gamma_1 & \ldots \\ 0 & 1 & \ldots \end{pmatrix} = \exp \begin{pmatrix} 0 & a_1 & a_2 & a_3 & \ldots \\ 0 & b_1 & b_2 & \ldots \\ 0 & c_1 & \ldots \\ 0 & 0 & \ldots \end{pmatrix}.$$

Put $a_1 = \alpha_1, b_1 = \beta_1, \ldots$. Next, we have $a_2 = \alpha_2 - f(a_1, b_1) = \alpha_2 - f(\alpha_1, \beta_1)$. Analogously, we find $b_2, c_2, \ldots$. To find $a_3$, observe that $a_3 = \alpha_3 - f(a_1, a_2, b_1, c_2)$. Since $f$ depends on $a_i, b_i, c_i$ with $i < 3$, we obtain $a_3 = \alpha_3 - f(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$, and the procedure continues. So, the equation under consideration is solvable for any $\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots$.

**Corollary 3.** Assume that $\text{SL}_n(R) = E_n(R)$ and every element in $E_n(R)$ is a product of $N(R)$ unipotent upper or lower triangular matrices. Then every element in $\text{SL}_n(R)$ is a product of $N(R)$ exponentials.
Corollary 4. Let the assumptions of Corollary 3 hold. Suppose in addition that every invertible element in $R$ admits a logarithm. Then every $X \in \text{GL}_n(R)$ is a product of $N(R)$ exponentials.

Proof. Let $X \in \text{GL}_n(R)$. So, $\det X \in R^*$ and $\ln \det X$ is defined. Therefore, $\det X = f^n$ for appropriate $f \in R^*$ and

$$
\begin{pmatrix}
    f^{-1} & 0 \\
    & \ddots \\
    0 & & f^{-1}
\end{pmatrix}
X \in \text{SL}_n(R).
$$

Applying Corollary 3 we obtain

$$
X = \begin{pmatrix}
    f & 0 \\
    \vdots & \ddots \\
    0 & & f
\end{pmatrix}
\exp Y_1 \ldots \exp Y_N
$$

$$
= \exp \left[
\begin{pmatrix}
    \ln f & 0 \\
    \vdots & \ddots \\
    0 & & \ln f
\end{pmatrix} + Y_1
\right]
\exp Y_2 \ldots \exp Y_N,
$$
as required. □

4.2. Rings of holomorphic functions on Stein spaces.

Corollary 5. Let $\Omega$ be a Stein space of dimension $k$ and let $X \in \text{GL}_n(O(\Omega))$. Then there exists a number $E(k, n)$ such that the following properties are equivalent:

(i) $X$ is null-homotopic;

(ii) $X$ is a product of $E(k, n)$ exponentials.

Proof. By [10, Theorem 2.3], any null-homotopic $F \in \text{SL}_n(O(\Omega))$ is a product of $N(k, n)$ unipotent upper or lower triangular matrices. So, arguing as in the proof of Corollary 4, we conclude that (i) implies (ii) with $E(k, n) \leq N(k, n)$ The reverse implication is straightforward.

The numbers $N(k, n)$ are not known in general. If the dimension $k$ of the Stein space is fixed, then the dependence of $N(k, n)$ on the size $n$ of the matrix is easier to handle. Certain $K$-theory arguments guarantee that the number of unipotent matrices needed for factorizing an element in $\text{SL}_n(O(\Omega))$ is a non-increasing function of $n$ (see [12]). So, as done in [3], combining the above property and results from [11], we obtain the following estimates:

$$
E(1, n) \leq N(1, n) = 4 \text{ for all } n,
$$

$$
E(2, n) \leq N(2, n) \leq 5 \text{ for all } n, \text{ and}
$$

for each $k$, there exists $n(k)$ such that $E(k, n) \leq N(k, n) \leq 6$ for all $n \geq n(k)$.

In Section 4.4, we in fact improve on that: we show $E(1, 2) \leq 3$. In general, it seems that the number of exponentials $E(k, n)$ to factorize an element in $\text{GL}_n(O(\Omega))$ is less than the number $N(k, n)$ needed to write an element in $\text{SL}_n(O(\Omega))$ as a product of unipotent upper or lower triangular matrices.

Also, remark that (ii) implies (i) in Corollary 5 for any algebra $R$ in the place of the ring of holomorphic functions. Assume that the algebra $R$ has a topology. Then a topology on $\text{GL}_n(R)$ is naturally induced and the implication (i)$\Rightarrow$(ii) means that
any product of exponentials is contained in the connected component of the identity (also known as the principal component) of $\text{GL}_n(R)$. The reverse implication is a difficult question, even without a uniform bound on the number of exponentials.

4.3. Rings $R$ with $\text{bsr}(R) = 1$. Combining Theorem 2 and Corollary 4 we recover a more general version of Theorem 7.1(3) from [15], where $R$ is assumed to be a Banach algebra. Moreover, we obtain similar results for larger size matrices.

Corollary 6. Let $R$ be a commutative unital ring, $\text{bsr} R = 1$, and let every $x \in R^*$ admit a logarithm. Then every element in $\text{GL}_2(R)$ is a product of 4 exponentials.

Corollary 7. Let $R$ be a commutative unital ring, $\text{bsr} R = 1$, and let every $x \in R^*$ admit a logarithm. Then every element in $\text{GL}_n(R)$, $n \geq 3$, is a product of 6 exponentials.

Proof. For $n = 3$, it suffices to combine Theorem 2 and Corollary 4.

Now, assume that $n \geq 4$. Let $u_2$ denote the number of unipotent matrices needed to factorize any element in $\text{SL}_2(R)$ starting with an upper triangular matrix. Theorem 20(b) in [7] says that any element in $\text{SL}_3(R)$ is a product of 6 exponentials for

$$
n \geq \min \left( m \left[ \frac{u_2(R) + 1}{2} \right] \right),$$

where the minimum is taken over all $m \geq \text{bsr} R + 1$. In our case the minimum is taken over $m \geq 2$ and the number $u_2(R) = 4$ by the proof of Theorem 2. Since $n \geq 4$, the proof is finished. $\square$

Corollary 6 applies to the disk algebra and also to the rings $O(\mathbb{C})$ and $O(\mathbb{D})$ of holomorphic functions. Indeed, the identity $\text{bsr}(O(\Omega)) = 1$ for an open Riemann surface follows from the strengthening of the classical Wedderburn lemma (see [19, Chapter 6, Section 3]; see also [10] or [2]). However, for $R = O(\mathbb{C})$ and $R = O(\mathbb{D})$, the number 4 is not optimal; see Section 4.4 below. Also, it is known that the optimal number is at least 2 (see [15]). So, we arrive at the following natural question:

Problem 2. Is any element of $\text{GL}_2(O(\mathbb{D}))$ or $\text{GL}_2(O(\mathbb{C}))$ a product of two exponentials?

4.4. Products of 3 exponentials. In this section, we prove the following result.

Proposition 3. Let $\Omega$ be an open Riemann surface. Then every element in $\text{SL}_2(O(\Omega))$ is a product of 3 exponentials.

We will need several auxiliary results. The first theorem is a classical one [8].

Theorem 3 (Mittag-Leffler Interpolation Theorem). Let $\Omega$ be an open Riemann surface and let $\{z_i\}_{i=1}^{\infty}$ be a discrete closed subset of $\Omega$. Assume that a finite jet

$$J_i(z) = \sum_{j=0}^{N_i} b_j^{(i)}(z - z_i)^j$$

is defined in some local coordinates for every point $z_i$. Then there exists $f \in O(\Omega)$ such that

$$f(z) - J_i(z) = o(|z - z_i|^{N_i}) \quad \text{as} \quad z \to z_i, \quad i = 1, 2, \ldots.$$
Corollary 8. Under assumptions of Theorem \[\text{5}\], suppose that \(b^{(i)}_0 \neq 0\) in \(\text{4.1}\) for \(i = 1, 2, \ldots\). Then there exist \(f, g \in \mathcal{O}(\Omega)\) such that \(\text{4.2}\) holds and \(f = e^g\).

Proof. Let \(b_0 = b^{(i)}_0\) for some \(i\). Since \(b_0 \neq 0\), there exists a logarithm \(\ln\) in a neighborhood of \(b_0\). So, \(\ln\) is a local biholomorphism which induces a bijection between jets of \(f\) and \(g := \ln f\).

In “modern” language, the proof of Corollary 8 uses the fact that \(\mathbb{C}^*\) is an Oka manifold (we refer the interested reader to [9]). Thus for any Stein manifold \(X\) and an analytic subset \(Y \subset X\), a (jet of) holomorphic map \(f : Y \to \mathbb{C}^*\) (along \(Y\)) extends to a holomorphic map \(f : X \to \mathbb{C}^*\) if and only if it extends continuously. The obstruction for a continuous extension is an element of the relative homology group \(H_2(X, Y, Z)\). Observe that, for any discrete subset \(Y\) of a 1-dimensional Stein manifold \(X\), we have \(H_2(X, Y, Z) = 0\) because \(H_1(X, Z) = H_1(Y, Z) = 0\). This is the point where the proof of Proposition 3 below breaks down when we replace the Riemann surface \(\Omega\) by a Stein manifold of higher dimension. Even a nowhere vanishing continuous function \(\alpha\), as in the proof, does not exist in general.

Lemma 2. Let \(\Omega\) be an open Riemann surface and \(X \in \text{GL}_2(\mathcal{O}(\Omega))\). Assume that \(\lambda \in \mathcal{O}^*(\Omega)\) is the double eigenvalue of \(X\) and \(\det X\) has a logarithm in \(\mathcal{O}(\Omega)\). Then \(X\) is an exponential.

Proof. We consider two cases.
Case 1: \(X(z)\) is a diagonal matrix for all \(z \in \Omega\).
We have
\[
X(z) = \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda(z) \end{pmatrix} = \exp \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha(z) \end{pmatrix},
\]
Case 2: \(X(z)\) is not identically diagonal.
Either the first or the second line in \(X(z) - \lambda(z)I\), say \((h(z), g(z))\), is not identical zero. So,
\[
v_1(z) = \begin{pmatrix} -g(z) \\ h(z) \end{pmatrix}
\]
is a holomorphic eigenvector for \(X(z)\) except those points \(z \in \Omega\) for which \(v_1(z) = 0\). Construct a function \(f(z) \in \mathcal{O}(\Omega)\) such that its vanishing divisor is exactly \(\text{min}(\text{ord} g, \text{ord} h)\). Then
\[
v(z) = \frac{1}{f(z)} v_1(z)
\]
is a holomorphic eigenvector for \(X(z), z \in \Omega\).
Now, choose a matrix \(P(z) \in \text{GL}_2(\mathcal{O}(\Omega))\) with first column \(v(z)\). Then the matrix \(P^{-1}(z)X(z)P(z)\) has the following form:
\[
\begin{pmatrix} \lambda(z) & \beta(z) \\ 0 & \lambda(z) \end{pmatrix} = \exp \begin{pmatrix} \frac{1}{2} \gamma(z) & \frac{\beta(z)}{\lambda(z)} \\ 0 & \frac{1}{2} \gamma(z) \end{pmatrix}
\]
Thus,
\[
X(z) = \exp P(z) \begin{pmatrix} \frac{1}{2} \gamma(z) & \frac{\beta(z)}{\lambda(z)} \\ 0 & \frac{1}{2} \gamma(z) \end{pmatrix} P^{-1}(z),
\]
as required.
Proof of Proposition 3. Let

\[ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \]

that is, \( ad - bc = 1 \). We are looking for \( \alpha \in \mathbb{R}^* \) and \( \beta \in \mathbb{R} \) such that the matrix

\[ X \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^2a & \beta a + b \\ \alpha^2c & \beta c + d \end{pmatrix} := Y \]

has a double eigenvalue.

Case 1: \( c = 0 \). We have

\[ X = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}. \]

It suffice to observe that

\[ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & b \\ 0 & a^{-1} \end{pmatrix} \]

has the double eigenvalue \( a^{-1} \).

Case 2: \( c \neq 0 \). The matrix \( Y \) has a double eigenvalue if \( 4 \det Y = (\text{tr} Y)^2 \), that is,

\[ (\alpha^2a + \beta c + d)^2 = 4\alpha^2. \]

Put \( \beta = \frac{2\alpha - a\alpha^2 - d}{c} \).

Clearly, \( \beta \) is a formal solution of (4.3). Below we show how to construct \( \alpha(z) = \exp(\tilde{\alpha}(z)) \in \mathcal{O}^*(\Omega) \) such that \( \beta \) is holomorphic.

Let \( \{z_i\} \subset \Omega \) be the zero set of \( c(z) \). Fix \( i \) and \( z_i \in \Omega \). Let \( c(z_i) = \cdots = c^{(k)}(z_i) = 0 \), and \( c^{(k+1)}(z_i) \neq 0 \). Observe that \( a(z_i) \neq 0 \). So, define \( \alpha(z) \), in a neighborhood of \( z_i \), as \( 1/\alpha(z) \) up to a sufficiently high order, namely,

\[ a(z)\alpha(z) = 1 + (z - z_i)^k h(z), \]

where \( h(z) \) is holomorphic in a neighborhood of \( z_i \). Since \( ad - bc = 1 \), we have \( 1 - ad = (z - z_i)^k g(z) \). Therefore,

\[ 2\alpha a - a^2 \alpha^2 - ad = -(1 - a\alpha^2)^2 + 1 - ad \]

\[ = -(z - z_0)^{2k} h^2(z) + (z - z_0)^k g(z) \]

vanishes of order \( k \) at \( z_i \). Hence, \( 2\alpha - a\alpha^2 - d \) also vanishes of order \( k \) at \( z_i \).

So, we have constructed \( \alpha(z) \) locally as finite jets \( J_i(z) \) defined by (4.4) with \( b_0^{(i)} \neq 0 \) in some local coordinates for every point \( z_i, i = 1, 2, \ldots \). Now, Corollary 8 provides \( \tilde{\alpha} \in \mathcal{O}(\Omega) \) such that \( \alpha(z) = \exp(\tilde{\alpha}(z)) \in \mathcal{O}^*(\Omega) \) and (4.4) holds. Hence, \( \beta \) is holomorphic.

So, the matrix

\[ X \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} := Y \]

has a double eigenvalue and \( \det Y \) admits a logarithm. Thus, applying Lemma 2, we conclude that \( Y \) is an exponential. To finish the proof of the proposition, it remains observe that

\[ \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \alpha^{-1} \\ 0 & \alpha \end{pmatrix}, \]

where both multipliers on the right hand side are exponentials. \( \square \)
Corollary 9. Let $X \in \text{GL}_2(\mathcal{O}(\Omega))$. The following properties are equivalent:

(i) $X$ is a product of 3 exponentials;
(ii) $\det X$ is an exponential;
(iii) $X$ is null-homotopic.

Proof. Clearly, (i)⇒(iii). Now, assume that $X$ is null-homotopic. Then $\det X$ is homotopic to the function $f \equiv 1$. Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering, we conclude that $\det X(z) = \exp(h(z))$ with $h \in \mathcal{O}(\Omega)$. So, (iii) implies (ii). The implication (ii)⇒(i) is standard; see, for example, the proof of Corollary 4. □

References


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