For the MSSP Special Issue in Honour of Professor Lothar Gaul

Receptance-Based Robust Eigenstructure Assignment

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Abstract

The robustness of receptance-based active control techniques to uncertain parameters in fitted transfer function matrices is considered. Variability in assigned closed-loop poles, which arises from uncertain open-loop poles, zeros, and scaling parameters, is quantified by means of analytical sensitivity formulae, which are derived in this research. The sensitivity formulae are shown to be computationally efficient, even for a large number of random parameters, and require only measured receptances, thereby preserving the model-free superiority of receptance-based techniques. An evolution-based global optimisation procedure is used to perform eigenstructure assignment so that the robustness, as defined by a metric, is maximised. The robustness metric is designed to scale the relative importance of each closed-loop pole and their respective real and imaginary parts. The proposed technique is tested numerically on a multi-degree-of-freedom system. It is shown that, in both single- and multiple-input systems, it is possible to increase the robustness by optimally selecting a set of closed-loop poles. However, it is determined that the closed-loop eigenvectors of the system play a significant role in the propagation of uncertainty and hence, since multiple-input systems may independently assign both closed-loop poles and eigenvectors, multiple-input systems are able to reduce the uncertainty propagation to a greater extent.

Keywords: eigenstructure assignment, receptance method, robust control, uncertainty quantification

1. Introduction

The Receptance Method, first formulated by Ram and Mottershead [1], is a now well-established, experimental-based active vibration control technique that has been applied to numerous mechanical, aerospace and civil engineering systems [2, 3, 4, 5]. The technique involves using input-output transfer function matrices, obtained by rational transfer function fitting of measured frequency response function (FRF) data [6], to design a feedback controller that performs eigenstructure assignment [7]. That is, the placement of closed-loop poles and also, if the system is multiple-input, the assignment of closed-loop left or right eigenvectors [8, 9]. The method is advantageous in that, since only experimental data is used, there is no need to create a numerical model of the system and thus errors associated with numerical modelling are eliminated. However, despite this significant benefit, other sources of uncertainty are introduced, including: measurement errors, variability between supposedly nominal systems, issues due to controller performance and robustness, and transfer function misfitting.

In recent years, there have been several studies on the effect of such uncertainties on the Receptance Method. Mottershead et al. [10] and Tehrani et al. [11] considered the robustness of the closed-loop poles, assigned using the Receptance Method, to the feedback gains used in the receptance controller. A sensitivity-based approach was used and it was shown possible to assign either the closed-loop poles or their respective

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sensitivities. Similar work on closed-loop eigenvalue sensitivities has also been considered by Bernal [12] and Bernal and Ulriksen [13]. Tehrani et al. [14] also used a sensitivity-based approach to quantify the variability of closed-loop poles due to uncertainty in measured transfer functions. Small, complex perturbations were used to simulate measurement uncertainty and an optimum pole placement strategy was used to reduce the resulting spread of the closed-loop poles. Also in [14], multiple-input systems with uncertainty were considered for the first time using a sequential pole placement technique. Bai et al. [15] developed an approach to simultaneously maximise the robustness of the closed-loop poles and minimise the norm of the feedback gains using a nonlinear optimisation approach. However, the method required knowledge of the system matrices, which are generally not available without system identification techniques. Liang et al. [16, 17] considered, for the first time, the sensitivity of eigenvalues to physical sources, such as friction coefficients and contact stiffness parameters, and also used an optimisation approach to minimise such sensitivities. Adamson et al. [18, 19] investigated variability between supposedly nominal systems, which arises from manufacturing tolerances, damage and degradation. Variability in the closed-loop poles were minimised, according to their variance in the real and imaginary parts, using a global optimisation algorithm.

Despite the above-mentioned works, little research effort has been given to the issue of transfer function misfitting. This issue arises when parameters, such as poles and zeros, in the fitted transfer functions are uncertain, which leads to the random placement of poles in the closed-loop system. Physical sources that lead to such misfitting include difficulty in estimating large damping values, modes that have very close frequencies, and poor fitting algorithms. This is the subject of the present work.

In this research, variability in the assigned, closed-loop poles is quantified by means of local sensitivity expressions, which are derived for the first time. The expressions are shown to be dependent only on the nominal transfer function matrix, the control gains, and the left and right eigenvectors. Thus, there is no need to know the system matrices. A robustness metric is then defined, similar to [18], which serves to weight the relative importance of the robustness of each closed-loop pole and its respective real and imaginary parts. A global optimisation procedure is then developed to optimise the robustness metric. In single-input systems, this is done by optimally assigning the poles within rectangular regions in the complex plane. In multiple-input systems, this is done by either assigning the closed-loop eigenvectors, the closed-loop poles, or a combination of both.

The remainder of this paper is divided as follows. In Section 2, the theory of the Receptance Method is outlined and the problem associated with parameter misfitting in the transfer function matrices is formalised. Following this, in Section 3, analytical expressions for the closed-loop pole sensitivities are derived for each parameter in the transfer function fittings. Next, in Section 4, a robustness metric is defined, which uses the earlier derived sensitivity expressions. In Section 5, the global optimisation procedure is described for single-input and multiple-input systems. Finally, in Section 6, the optimisation procedure is tested on a multi-degree-of-freedom system in the form of four numerical examples.

2. Introductory theory

2.1. Receptance-based formulation

Consider a linear, multiple-input, multiple-output dynamic system governed by the frequency domain equation

$$\mathbf{Z}(s)\mathbf{y}(s) = \mathbf{B}(s)\mathbf{u}(s) \tag{1}$$

where $\mathbf{Z}(s) \in \mathbb{C}^{n\times n}$ is the dynamic stiffness matrix, $\mathbf{y}(s) \in \mathbb{C}^{n}$ is the output vector, $\mathbf{B}(s) \in \mathbb{C}^{n\times m}$ is the force distribution matrix, and $\mathbf{u}(s) \in \mathbb{C}^{m}$ is the input vector. When a static, linear, proportional plus derivative output feedback controller is used, $\mathbf{u}(s)$ is related to $\mathbf{y}(s)$ by

$$\mathbf{u}(s) = \left(s\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{y}(s) \tag{2}$$

where $\mathbf{F}, \mathbf{G} \in \mathbb{R}^{n\times m}$ are matrices of control gains, which are constant. The closed-loop dynamic is therefore described by

$$\left(\mathbf{Z}(s) - \mathbf{B}(s)\left(s\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{y}(s) = \mathbf{0} \tag{3}$$

Pre-multiplying by the receptance matrix, $\mathbf{H}(s) = \mathbf{Z}(s)^{-1}$, gives that

$$\left(\mathbf{I} - \mathbf{R}(s)\left(s\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{y}(s) = \mathbf{0} \tag{4}$$

where $\mathbf{R}(s) = \mathbf{H}(s)\mathbf{B}(s)$ is the input-output transfer function matrix. This is illustrated in block-diagram form in Fig. 1.

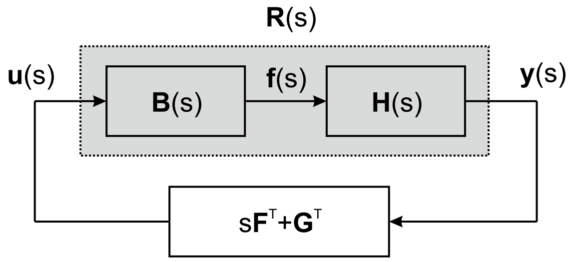

Figure 1: Closed-loop system.

The eigenvalue problem associated with the closed-loop system is

$$\left(\mathbf{I} - \mathbf{R}(\mu_i)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_{R,i} = \mathbf{0}, \quad i = 1, 2, \ldots, 2n \tag{5}$$

$$\mathbf{w}_{L,i}^T\left(\mathbf{I} - \mathbf{R}(\mu_i)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right) = \mathbf{0}^T, \quad i = 1, 2, \ldots, 2n \tag{6}$$

where μ_i is the i^{th} closed-loop pole, and $\mathbf{w}_{L,i}$ and $\mathbf{w}_{R,i}$ are the left and right closed-loop eigenvectors belonging to the i^{th} pole, respectively.

In the above formulation, the poles are a function of only the open-loop transfer function matrix $\mathbf{R}(s)$ and the control gains $\mathbf{F}$ and $\mathbf{G}$. Therefore, it is possible to determine a set of control gains to assign desired closed-loop poles using only the transfer function matrix. The process of determining such gains is known as the Receptance Method and is outlined below.

2.2. The Receptance Method

Suppose that the desired set of closed-loop poles is $\mu = \{\mu_1, \mu_2, \ldots, \mu_{2n}\}$, which is closed under conjugation[1]. From Eq. 5, the control gains $\mathbf{F}$ and $\mathbf{G}$ must be chosen so that

$$\mathbf{w}_{R,i} = \mathbf{R}(\mu_i)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_{R,i}, \quad i = 1, 2, \ldots, 2n \tag{7}$$

By setting

$$\boldsymbol{\alpha}_i = \left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_{R,i}, \quad i = 1, 2, \ldots, 2n \tag{8}$$

Eq. 7 reduces to

$$\mathbf{w}_{R,i} = \mathbf{R}(\mu_i)\boldsymbol{\alpha}_i, \quad i = 1, 2, \ldots, 2n \tag{9}$$

and hence from Eq. 8

$$\boldsymbol{\alpha}_i = \left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{R}(\mu_i)\boldsymbol{\alpha}_i, \quad i = 1, 2, \ldots, 2n \tag{10}$$

By denoting

$$\mathbf{F} = [\mathbf{f}_1 \quad \mathbf{f}_2 \quad \ldots \quad \mathbf{f}_m], \quad \mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \ldots \quad \mathbf{g}_m] \tag{11}$$

Eq. 10 may be written as

$$\mathbf{P}_i\mathbf{y} = \boldsymbol{\alpha}_i, \quad i = 1, 2, \ldots, 2n \tag{12}$$

[1]This condition ensures that the control gains are strictly real, which is required for physical implementation.

where

$$\mathbf{P}_i = \begin{pmatrix} \mu_i\boldsymbol{\alpha}_i^T\mathbf{R}(\mu_i)^T & \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{\alpha}_i^T\mathbf{R}(\mu_i)^T & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mu_i\boldsymbol{\alpha}_i^T\mathbf{R}(\mu_i)^T & \dots & \mathbf{0} & \mathbf{0} & \boldsymbol{\alpha}_i^T\mathbf{R}(\mu_i)^T & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mu_i\boldsymbol{\alpha}_i^T\mathbf{R}(\mu_i)^T & \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\alpha}_i^T\mathbf{R}(\mu_i)^T \end{pmatrix} \quad (13)$$

$$\mathbf{y} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_m \end{pmatrix} \quad (14)$$

By defining

$$\mathbf{T} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_{2n} \end{pmatrix} \quad (15)$$

and

$$\mathbf{z} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_{2n} \end{pmatrix} \quad (16)$$

the gains necessary to assign the closed-loop poles are given by

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{z}. \quad (17)$$

It is to be noted that Eq. 17 is valid if, and only if, all closed-loop poles do not coincide with any of the open-loop poles. This is because $\mathbf{H}(\mu_i)$ is ill-defined at the open-loop poles. Further details on the resolution of this problem are given in [8].

It has been shown in [8] that the choice of $\boldsymbol{\alpha}_i$ is arbitrary. That is, the poles will always be assigned to the desired positions in the complex plane for any $\boldsymbol{\alpha}_i$. However, one may choose $\boldsymbol{\alpha}_i$ such that the resulting control gains are minimum norm [20] or so that a desired eigenstructure assignment is achieved [8]. Furthermore, it is necessary that $\boldsymbol{\alpha}_i$ is closed under conjugation for any eigenvalue pair $\{\mu_i, \bar{\mu}_i\}$[20].

2.3. Problem formulation

As shown above, the transfer matrix $\mathbf{R}(s)$ must be evaluated at each closed-loop pole ($s = \mu_i, i = 1, 2, \dots, 2n$) in order to find $\mathbf{F}$ and $\mathbf{G}$. In the case that $\text{Re}(\mu_i) = 0$, $\mathbf{R}(\mu_i)$ is simply a matrix of frequency response functions (FRFs), which can be measured experimentally. However, in the more general case that $\text{Re}(\mu_i) \neq 0$, $\mathbf{R}(\mu_i)$ is not available directly from experimental measurements. Therefore, it is usually required to perform rational function fitting to measured FRF data in order to estimate $\mathbf{R}(\mu_i)$.

Consider the general form of $\mathbf{R}(s)$, given by

$$\mathbf{R}(s) = \frac{\mathbf{N}(s)}{d(s)} \quad (18)$$

where $\mathbf{N}(s)$ is the matrix numerator term; and $d(s)$ is the scalar denominator term, which is also known as the characteristic polynomial. Expanding the numerator and denominator allows $\mathbf{R}(s)$ to be written in the form

$$\mathbf{R}(s) = \frac{1}{\prod_{i=1}^{2n}(s-\lambda_i)} \begin{pmatrix} n_{11}(s) & n_{12}(s) & \dots & n_{1m}(s) \\ n_{21}(s) & n_{22}(s) & \dots & n_{2m}(s) \\ \dots & \dots & \dots & \dots \\ n_{n1}(s) & n_{n2}(s) & \dots & n_{nm}(s) \end{pmatrix} \tag{19}$$

where

$$n_{ij}(s) = \eta_{ij} \prod_{r=1}^{q<n} (s - z_{ij_r}) \tag{20}$$

Therefore, the transfer matrix is described entirely by the parameters λ_i, z_{ij_r} and η_{ij}, which are referred to as the poles, zeros, and scaling parameters, respectively. Due to uncertainty in the measured FRFs, it is difficult to estimate the fitting parameters precisely and, consequently, there is usually an error between the true and estimated value of $\mathbf{R}(s)$.

Suppose that one of the fitting parameters, denoted θ, is uncertain. The new eigenvalue problem, considering the influence of parameter uncertainty, is

$$\left(\mathbf{I} - \mathbf{R}(\mu_i,\theta)\left(\mu_i \mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_{R,i} = \mathbf{0}, \quad i = 1,2,\dots,2n \tag{21}$$

$$\mathbf{w}_{L,i}^T\left(\mathbf{I} - \mathbf{R}(\mu_i,\theta)\left(\mu_i \mathbf{F}^T + \mathbf{G}^T\right)\right) = \mathbf{0}^T, \quad i = 1,2,\dots,2n \tag{22}$$

Therefore, for a fixed set of control gains, the eigenvalue condition depends explicitly on the fitting parameter and thus its choice affects the eigenvalue solution. That is, the closed-loop poles now also depend on the fitting parameter. In the remainder of this paper, the influence of the fitting parameter on the closed-loop poles is investigated.

3. Closed-loop pole sensitivities

Consider a small perturbation of one of the fitting parameters about its nominal value, so that: $\theta \to \theta + \delta\theta$, $\mu_i \to \mu_i + \delta\mu_i$ and $\mathbf{w}_{R,i} \to \mathbf{w}_{R,i} + \delta\mathbf{w}_{R,i}$. The new eigenvalue problem associated with the perturbed system is given by

$$\left(\mathbf{I} - \mathbf{R}(\mu_i + \delta\mu_i, \theta + \delta\theta)\left((\mu_i + \delta\mu_i)\mathbf{F}^T + \mathbf{G}^T\right)\right)(\mathbf{w}_{R,i} + \delta\mathbf{w}_{R,i}) = \mathbf{0}, \quad i = 1,2,\dots,2n \tag{23}$$

Assuming that $\delta\theta$ and $\delta\mu_i$ are sufficiently small,

$$\mathbf{R}(\mu_i + \delta\mu_i, \theta + \delta\theta) \approx \mathbf{R}(\mu_i,\theta) + \frac{\partial \mathbf{R}}{\partial s}\delta\mu_i + \frac{\partial \mathbf{R}}{\partial \theta}\delta\theta, \quad i = 1,2,\dots,2n \tag{24}$$

and therefore

$$\left(\mathbf{I} - \left(\mathbf{R}(\mu_i,\theta) + \frac{\partial \mathbf{R}}{\partial s}\delta\mu_i + \frac{\partial \mathbf{R}}{\partial \theta}\delta\theta\right)\left((\mu_i + \delta\mu_i)\mathbf{F}^T + \mathbf{G}^T\right)\right)(\mathbf{w}_{R,i} + \delta\mathbf{w}_{R,i}) = \mathbf{0}, \quad i = 1,2,\dots,2n \tag{25}$$

Expanding Eq. 25 and neglecting second-order terms leads to

$$\begin{aligned} &\left(\mathbf{I} - \mathbf{R}(\mu_i,\theta)\left(\mu_i \mathbf{F}^T + \mathbf{G}^T\right)\right)(\mathbf{w}_{R,i} + \delta\mathbf{w}_{R,i}) \\ &+ \left(-\mathbf{R}(\mu_i,\theta)\mathbf{F}^T\delta\mu_i - \left(\frac{\partial \mathbf{R}}{\partial s}\delta\mu_i + \frac{\partial \mathbf{R}}{\partial \theta}\delta\theta\right)\left(\mu_i \mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_{R,i} = \mathbf{0}, \quad i = 1,2,\dots,2n \end{aligned} \tag{26}$$

Pre-multiplying by $\mathbf{w}_{L,i}^T$ gives

$$\begin{aligned} &\mathbf{w}_{L,i}^T\left(\mathbf{I} - \mathbf{R}(\mu_i,\theta)\left(\mu_i \mathbf{F}^T + \mathbf{G}^T\right)\right)(\mathbf{w}_{R,i} + \delta\mathbf{w}_{R,i}) \\ &+ \mathbf{w}_{L,i}^T\left(-\mathbf{R}(\mu_i,\theta)\mathbf{F}^T\delta\mu_i - \left(\frac{\partial \mathbf{R}}{\partial s}\delta\mu_i + \frac{\partial \mathbf{R}}{\partial \theta}\delta\theta\right)\left(\mu_i \mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_{R,i} = 0, \quad i = 1,2,\dots,2n \end{aligned} \tag{27}$$

and hence by definition of Eq. 22

$$\mathbf{w}_{L,i}^T \left(-\mathbf{R}\left(\mu_i,\theta\right)\mathbf{F}^T\delta\mu_i - \left(\frac{\partial \mathbf{R}}{\partial s}\delta\mu_i + \frac{\partial \mathbf{R}}{\partial \theta}\delta\theta\right)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_{R,i} = 0, \quad i = 1, 2, \ldots, 2n \tag{28}$$

Re-arranging and taking the limit as $\delta\theta$ tends to zero gives

$$\frac{\partial \mu_i}{\partial \theta} = -\frac{\mathbf{w}_{L,i}^T \frac{\partial \mathbf{R}}{\partial \theta}\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_{R,i}}{\mathbf{w}_{L,i}^T\left(\mathbf{R}\left(\mu_i,\theta\right)\mathbf{F}^T + \frac{\partial \mathbf{R}}{\partial s}\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_{R,i}}, \quad i = 1, 2, \ldots, 2n \tag{29}$$

which is the sensitivity of the closed-loop pole μ_i to a change in some fitting parameter θ.

In practice, the term $\frac{\partial \mathbf{R}}{\partial s}$ of Eq. 29 is evaluated by differentiating the nominal, fitted transfer function matrix. Also, the left and right eigenvectors are found from the null space of

$$\mathbf{I} - \mathbf{R}(\mu_i,\theta)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right) \tag{30}$$

and

$$\left(\mathbf{I} - \mathbf{R}(\mu_i,\theta)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right)^T \tag{31}$$

respectively. In the following subsections, expressions are derived for the term $\frac{\partial \mathbf{R}}{\partial \theta}$, which depends explicitly on the fitting parameter of interest.

3.1. Sensitivity to fitted poles

Let the transfer function matrix be expressed as

$$\mathbf{R}\left(s\right) = \frac{\mathbf{N}(s)}{\prod_{i=1}^{2n}\left(s-\lambda_i\right)} \tag{32}$$

and suppose that one of the fitted poles, λ_j, is uncertain. Separating out the bracketed term involving λ_j gives that

$$\mathbf{R}\left(s,\lambda_j\right) = \frac{\mathbf{N}(s)}{\left(s-\lambda_j\right)\prod_{i=1,i\neq j}^{2n}\left(s-\lambda_i\right)} \tag{33}$$

and hence differentiating with respect to λ_j gives

$$\frac{\partial \mathbf{R}}{\partial \lambda_j} = \frac{\mathbf{N}(s)}{\left(s-\lambda_j\right)^2\prod_{i=1,i\neq j}^{2n}\left(s-\lambda_i\right)} = \frac{\mathbf{R}(s)}{s-\lambda_j} \tag{34}$$

Substituting Eq. 34 into Eq. 29 yields

$$\frac{\partial \mu_i}{\partial \lambda_j} = -\frac{\mathbf{w}_L^T\mathbf{R}(\mu)\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{w}_R}{\left(\mu_i-\lambda_j\right)\mathbf{w}_L^T\left(\mathbf{R}\left(\mu_i,\theta\right)\mathbf{F}^T + \frac{\partial \mathbf{R}}{\partial s}\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_R}, \quad i = 1, 2, \ldots, 2n \tag{35}$$

which is the sensitivity of a closed-loop pole μ_i to one of the fitted open-loop poles λ_j. Using the eigenvalue condition of Eq. 5, the sensitivity can be reduced to

$$\frac{\partial \mu_i}{\partial \lambda_j} = -\frac{\mathbf{w}_L^T\mathbf{w}_R}{\left(\mu_i-\lambda_j\right)\mathbf{w}_L^T\left(\mathbf{R}\left(\mu_i,\theta\right)\mathbf{F}^T + \frac{\partial \mathbf{R}}{\partial s}\left(\mu_i\mathbf{F}^T + \mathbf{G}^T\right)\right)\mathbf{w}_R}, \quad i = 1, 2, \ldots, 2n \tag{36}$$

3.2. Sensitivity to fitted zeros

Let the transfer matrix be expressed as

$$\mathbf{R}(s) = \frac{\mathbf{N}(s)}{d(s)} \tag{37}$$

and suppose that the k^{th} zero $z_{ij,k}$ belonging to the ij^{th} element of $\mathbf{N}(s)$ is variable. The derivative of $\mathbf{R}(s)$ with respect to $z_{ij,k}$ is given by

$$\frac{\partial \mathbf{R}}{\partial z_{ij,k}} = \frac{1}{d(s)} \frac{\partial \mathbf{N}(s)}{\partial z_{ij,k}} = \frac{1}{d(s)} \frac{\partial n_{ij}(s)}{\partial z_{ij,k}} \mathbf{e}_i \mathbf{e}_j^T \tag{38}$$

where $\mathbf{e}_i \in \mathbb{R}^n$ and $\mathbf{e}_j \in \mathbb{R}^m$ are vectors with unit entries belonging to the i^{th} and j^{th} coordinates of the zero, respectively. Since

$$n_{ij}(s) = \eta_{ij} \prod_{p=1}^{q \leq 2n} (s - z_{ij,p}) \tag{39}$$

Eq. 38 may be written as

$$\frac{\partial \mathbf{R}}{\partial z_{ij,k}} = -\frac{\eta_{ij}}{d(s)} \prod_{p=1, p \neq k}^{q \leq 2n} (s - z_{ij,p}) \, \mathbf{e}_i \mathbf{e}_j^T = -\frac{r_{ij}(s)}{s - z_{ij,k}} \mathbf{e}_i \mathbf{e}_j^T. \tag{40}$$

Therefore, by substituting Eq. 40 into Eq. 29, the sensitivity of a closed-loop pole with respect to a fitted zero is given by

$$\frac{\partial \mu_i}{\partial z_{ij,k}} = \frac{\mathbf{w}_{L,i}^T r_{ij}(s) \mathbf{e}_i \mathbf{e}_j^T \left(\mu_i \mathbf{F}^T + \mathbf{G}^T \right) \mathbf{w}_{R,i}}{(s - z_{ij,k}) \, \mathbf{w}_{L,i}^T \left(\mathbf{R}(\mu_i, \theta) \mathbf{F}^T + \frac{\partial \mathbf{R}}{\partial s} \left(\mu_i \mathbf{F}^T + \mathbf{G}^T \right) \right) \mathbf{w}_{R,i}} \tag{41}$$

3.3. Sensitivity to fitted scaling parameters

Finally, suppose that the ij^{th} scaling parameter η_{ij} of $\mathbf{N}(s)$ is variable. Using Eq. 37, the derivative of $\mathbf{R}(s)$ with respect to η_{ij} is given by

$$\frac{\partial \mathbf{R}}{\partial \eta_{ij}} = \frac{1}{d(s)} \frac{\partial \mathbf{N}(s)}{\partial \eta_{ij}} = \frac{1}{d(s)} \frac{\partial n_{ij}(s)}{\partial \eta_{ij}} \mathbf{e}_i \mathbf{e}_j^T \tag{42}$$

Since

$$n_{ij}(s) = \eta_{ij} \prod_{p=1}^{q \leq 2n} (s - z_{ij,p}) \tag{43}$$

Eq. 42 may be written as

$$\frac{\partial \mathbf{R}}{\partial \eta_{ij}} = \frac{1}{d(s)} \prod_{p=1}^{q \leq 2n} (s - z_{ij,p}) \, \mathbf{e}_i \mathbf{e}_j^T = \frac{r_{ij}(s)}{\eta_{ij}} \mathbf{e}_i \mathbf{e}_j^T \tag{44}$$

Therefore, by substituting Eq. 44 into Eq. 29, the sensitivity of a closed-loop pole with respect to a fitted scaling parameter is given by

$$\frac{\partial \mu_i}{\partial \eta_{ij}} = \frac{\mathbf{w}_{L,i}^T r_{ij}(s) \mathbf{e}_i \mathbf{e}_j^T \left(\mu_i \mathbf{F}^T + \mathbf{G}^T \right) \mathbf{w}_{R,i}}{\eta_{ij} \mathbf{w}_{L,i}^T \left(\mathbf{R}(\mu_i, \theta) \mathbf{F}^T + \frac{\partial \mathbf{R}}{\partial s} \left(\mu_i \mathbf{F}^T + \mathbf{G}^T \right) \right) \mathbf{w}_{R,i}} \tag{45}$$

Special case: No zeros

If the matrix element $n_{ij}(s)$ does not contain any zeros and is simply a constant term (i.e. $n_{ij}(s) = \eta_{ij}$)

$$\frac{\partial \mathbf{R}}{\partial \eta_{ij}} = \frac{\eta_{ij}}{\eta_{ij}} \mathbf{e}_i \mathbf{e}_j^T = \mathbf{e}_i \mathbf{e}_j^T \tag{46}$$

and hence

$$\frac{\partial \mu_i}{\partial \eta_{ij}} = \frac{\mathbf{w}_{L,i}^T \mathbf{e}_i \mathbf{e}_j^T \left(\mu_i \mathbf{F}^T + \mathbf{G}^T \right) \mathbf{w}_{R,i}}{\mathbf{w}_{L,i}^T \left(\mathbf{R}(\mu_i, \theta) \mathbf{F}^T + \frac{\partial \mathbf{R}}{\partial s} \left(\mu_i \mathbf{F}^T + \mathbf{G}^T \right) \right) \mathbf{w}_{R,i}} \tag{47}$$

4. Robustness metric

With analytical sensitivity expressions derived, it is now possible to consider strategies to maximise the robustness of the assigned closed-loop poles to variations in the fitting parameters. In order to do so, however, one must first define a robustness metric. The robustness metric is used to quantify: (i) the combined effect of all fitting parameters on the movement of a chosen closed-loop pole, and (ii) the relative importance of the robustness of each closed-loop pole. In this work, the robustness metric is defined by using a total differential approach. In this way, the combined effect of all uncertain parameters is considered simultaneously using the previously derived sensitivity formulae.

Consider the total differential of a closed-loop pole μ_i, which is written as

$$d\mu_i = \sum_j \frac{\partial \mu_i}{\partial \theta_j} d\theta_j, \quad i = 1, 2, \ldots, 2n \tag{48}$$

By extending the differential of each parameter, it is shown that

$$\Delta\mu_i \approx \sum_j \frac{\partial \mu_i}{\partial \theta_j} \Delta\theta_j, \quad i = 1, 2, \ldots, 2n \tag{49}$$

and hence, provided the perturbations of the fitting parameters are sufficiently small, the movement of the closed-loop pole is determined uniquely by a weighted linear sum of the sensitivity associated with each fitting parameter. Taking the real and imaginary part of Eq. 49 leads to

$$\mathrm{Re}\left(\Delta\mu_i\right) \approx \sum_j \mathrm{Re}\left(\frac{\partial \mu_i}{\partial \theta_j}\Delta\theta_j\right), \quad i = 1, 2, \ldots, 2n \tag{50}$$

$$\mathrm{Im}\left(\Delta\mu_i\right) \approx \sum_j \mathrm{Im}\left(\frac{\partial \mu_i}{\partial \theta_j}\Delta\theta_j\right), \quad i = 1, 2, \ldots, 2n \tag{51}$$

and hence by expanding

$$\mathrm{Re}\left(\Delta\mu_i\right) \approx \sum_j \left(\mathrm{Re}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\mathrm{Re}\left(\Delta\theta_j\right) - \mathrm{Im}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\mathrm{Im}\left(\Delta\theta_j\right)\right), \quad i = 1, 2, \ldots, 2n \tag{52}$$

$$\mathrm{Im}\left(\Delta\mu_i\right) \approx \sum_j \left(\mathrm{Re}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\mathrm{Im}\left(\Delta\theta_j\right) + \mathrm{Im}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\mathrm{Re}\left(\Delta\theta_j\right)\right), \quad i = 1, 2, \ldots, 2n \tag{53}$$

If one assumes that each fitting parameter is within an interval specified by

$$\Delta\theta_j = \pm x_j \pm y_j \mathrm{i} \tag{54}$$

where $x_j \in \mathbb{R}^+$ and $y_j \in \mathbb{R}^+$, the maximum values of the real and imaginary parts of Eq. 52 and Eq. 53 are given by

$$\mathrm{Re}\left(\Delta\mu_i\right)_{\max} \approx \sum_j \left(\left|\mathrm{Re}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\right| x_j + \left|\mathrm{Im}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\right| y_j\right), \quad i = 1, 2, \ldots, 2n \tag{55}$$

and

$$\mathrm{Im}\left(\Delta\mu_i\right)_{\max} \approx \sum_j \left(\left|\mathrm{Re}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\right| y_j + \left|\mathrm{Im}\left(\frac{\partial \mu_i}{\partial \theta_j}\right)\right| x_j\right), \quad i = 1, 2, \ldots, 2n \tag{56}$$

For each pole μ_i a *pole metric* is defined as

$$J_i = \beta_i \frac{\mathrm{Re}\,(\Delta\mu_i)_{\max}}{|\mathrm{Re}\,(\mu_i)|} + \gamma_i \frac{\mathrm{Im}\,(\Delta\mu_i)_{\max}}{|\mathrm{Im}\,(\mu_i)|}, \quad i = 1, 2, \ldots, 2n \tag{57}$$

where $\beta_i, \gamma_i \in \mathbb{R}^+$ are constants. The constants serve to weight the relative importance of the percentage change of the real and imaginary part of each pole. In other words, $\beta_i > \gamma_i$ is chosen if it is more important for the real part of the pole to be robust, and $\beta_i < \gamma_i$ is chosen if it is more important for the imaginary part of the pole to be robust. The *total robustness metric* is then defined as

$$J_T = \sum_{i=1}^{2n} \rho_i J_i \tag{58}$$

where $\rho_i \in \mathbb{R}^+$ are constants. The constants here serve to weight the relative importance of each pole; that is, they weight the robustness of each mode.

It is important to point out that, in general, the variation of a closed-loop pole to a random parameter is not linear. Therefore, Eqs. 55-56 may not accurately predict the maximum real and imaginary part. However, even if this is the case, the purpose of the values x_j and y_j is still to weight the relative confidence in each parameter. That is, to adjust the total robustness metric so that parameters that are well known do not have such a large impact.

5. Sensitivity reduction

In this section, three optimisation procedures are developed to minimise the total robustness metric defined in Section 4.

5.1. Single-input systems

In single-input systems, the force distribution matrix $\mathbf{B}(s)$ reduces to a vector $\mathbf{b}(s) \in \mathbb{C}^n$. This means that the transfer matrix also reduces to a vector $\mathbf{r}(s) \in \mathbb{C}^n$ and thus so too do the control gains $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$. As a consequence, the $\boldsymbol{\alpha}_i$ term in Eq. 8 becomes scalar and hence, since the eigenvector in Eq. 9 may be scaled arbitrarily, the single-input system is unable to independently assign eigenvectors if the closed-loop poles are assigned. Therefore, the only way in which to minimise the robustness metric is by optimally choosing where to place the closed-loop poles in the complex plane.

Of course, one cannot simply allow an optimisation to place the closed-loop poles arbitrarily in the complex plane. Firstly, a condition of stability must be enforced ($\mathrm{Re}\,(\mu_i) < 0$). Secondly, the poles must be placed in a location that is physically achievable by the controller (i.e. the control gains must not be too large). Thirdly, the poles must usually be placed so as to achieve desired natural frequencies and damping values, which are chosen by the user.

In this work, the closed-loop poles μ_i are constrained to rectangular regions defined by

$$\mu_i = [\underline{a}_i, \overline{a}_i] + [\underline{b}_i, \overline{b}_i]\mathrm{i}, \quad i = 1, 2, \ldots, 2n \tag{59}$$

where $\underline{a}_i, \overline{a}_i, \underline{b}_i, \overline{b}_i \in \mathbb{R}$. With such a restriction, the optimisation problem is defined by:

Optimisation 1: *Minimise J_T in Eq. 58 by placing the closed-loop poles μ_i subject to the constraints given by Eq. 59.*

5.2. Multiple-input systems

In multiple-input systems, it is possible to independently assign both the eigenvalues and eigenvectors. Therefore, in theory, there is greater flexibility to reduce the robustness metric compared to the single-input equivalent system. Here, two optimisation strategies are considered.

5.2.1. Strategy 1: fixed closed-loop poles

In this first strategy, the closed-loop poles are assigned to fixed locations in the complex plane. That is, they are not optimisation variables, as was the case in the single-input system. Instead, only the vectors $\boldsymbol{\alpha}_i$ are chosen optimally. This is equivalent to optimally choosing the closed-loop eigenvectors, and hence the displacement pattern, of each mode.

Since $\boldsymbol{\alpha}_i$ may be chosen arbitrarily, the optimisation may produce values of $\boldsymbol{\alpha}_i$ that yield very large control gains and hence an unfeasible control effort. Therefore, constraints on the control gains are enforced. In this work, the gain constraints take the form

$$w||\mathbf{F}||_F + ||\mathbf{G}||_F \leq c_{max} \tag{60}$$

where $||.||_F$ represents the Frobenius norm, and $c_{max} \in \mathbb{R}^+$ and $w \in \mathbb{R}^+$ are constants. The term w is used to scale the gains associated with the feedback derivative term to comparable levels with the proportional term. This is because $\mathbf{F}$ is, in general, much smaller than $\mathbf{G}$. For general purposes, the scaling parameter w may be selected as the largest imaginary part from the set of all poles. This is equivalent to scaling the $\mathbf{F}$ matrix by the damped natural frequency.

In this case, the optimisation problem may be summarised by:

Optimisation 2: *Minimise J_T in Eq. 58 by placing the closed-loop poles at the desired locations in the complex plane and choosing optimum $\boldsymbol{\alpha}_i$, subject to the gain constraints in Eq. 60.*

5.2.2. Strategy 2: variable closed-loop poles

An alternative, and more flexible, strategy is a combination of the two above-mentioned optimisation problems. Now, both the poles and the eigenvectors are optimally selected.

Optimisation 3: *Minimise J_T in Eq. 58 by selecting optimum closed-loop poles and $\boldsymbol{\alpha}_i$ vectors, subject to the gain constraints in Eq. 60 and the pole constraints in Eq. 59.*

6. Numerical examples

The optimisation strategies discussed in Section 5 are applied numerically on a three-degree-of-freedom mass-spring-damper system. Throughout all examples, the mass, damping and stiffness matrices are given respectively by

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{C} = 0.5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{pmatrix},$$

and hence the open-loop poles are as given in Table 1.

Table 1: Open-loop poles.

Pole	Value
λ_1	-0.0166 + 0.5516i
λ_2	-0.0166 − 0.5516i
λ_3	-0.1890 + 1.6044i
λ_4	-0.1890 − 1.6044i
λ_5	-0.2528 + 2.2289i
λ_6	-0.2528 − 2.2289i

6.1. Single-input

In the case that

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

the transfer matrix is given by

$$\mathbf{R}(s) = \frac{1}{\prod_{i=1}^{6}(s-\lambda_i)} \begin{pmatrix} \frac{1}{2}\left(s^4 + \frac{2}{3}s^3 + \frac{22}{3}s^2 + \frac{3}{2}s + 8\right) \\ s^4 + \frac{7}{12}s^3 + \frac{23}{4}s^2 + \frac{5}{3}s + \frac{13}{2} \\ \frac{1}{3}\left(s^4 + \frac{5}{4}s^3 + \frac{39}{4}s^2 + \frac{21}{4}s + 21\right) \end{pmatrix}$$

and thus the open-loop zeros and scaling parameters are as given in Table 2 and Table 3, respectively.

Table 2: Open-loop zeros (single-input).

Element ij	$z_{ij,1}$	$z_{ij,2}$	$z_{ij,3}$	$z_{ij,4}$
11	-0.2678+2.4161i	-0.2678-2.4161i	-0.0656+1.1617i	-0.0656-1.1617i
21	-0.1429+2.0115i	-0.1429-2.0115i	-0.1487+1.2555i	-0.1487-1.2555i
31	-0.1690+1.8863	-0.1690-1.8863	-0.4560+2.3764i	-0.4560-2.3764i

Table 3: Scaling parameters (single-input).

Scaling parameter	Value
η_{11}	$\frac{1}{2}$
η_{21}	1
η_{31}	$\frac{1}{3}$

If the closed-loop poles were to be assigned to the nominal points

$$\mu_{1,2} = -0.2 \pm 0.8i$$
$$\mu_{3,4} = -0.5 \pm 2i$$
$$\mu_{5,6} = -1 \pm 2.5i$$

the required control gains, computed by the Receptance Method, would be

$$\mathbf{f} = \begin{pmatrix} -4.2265 \\ -0.5163 \\ 0.4387 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -0.3915 \\ -6.8904 \\ 4.2002 \end{pmatrix} \tag{61}$$

for any choice of α [8]. However, in the following two examples, the pole placement procedure is relaxed so that the poles are instead assigned to any points inside the regions defined by

$$\mu_{1,2} = [-0.5, -0.2] \pm [0.6, 1]\text{i}$$
$$\mu_{1,2} = [-0.7, -0.3] \pm [1.7, 2.3]\text{i}$$
$$\mu_{1,2} = [-1.2, -0.8] \pm [2.2, 2.8]\text{i}.$$

These regions have been chosen both to contain the nominal closed-loop poles given earlier and also to ensure that entire rectangular regions have lower (more negative) real parts than the open-loop poles, thus representing a stabilising control objective. Moreover, the upper limit of the real part of the rectangular region of the first pole pair is equal to the corresponding nominal closed-loop pole pair. This is to enforce that the real part of the first pole pair in the optimisation is always less than or equal to the nominal pole, as would likely be used in practice for poles with low damping.

This relaxation allows the poles to be placed to minimise a total robustness metric. Here, two metrics are investigated and are given in the proceeding examples.

Example 1: eigenvalue assignment 1 - equal weighting

Consider the case where the constants β_i, γ_i and ρ_i in Eq. 57 and Eq. 58 are set to one, for all i. That is, equal priority of the robustness of each closed-loop pole and equal priority of the robustness of the real and imaginary part. It is assumed that each fitting parameter is variable by 3% of its nominal value. In other words, 3% in the real (x_j) and imaginary part (y_j) of the values given in Tables 1, 2 and 3. The initial robustness metric, using the gains in Eq. 61, is calculated as 4.64. Using the Differential Evolution global optimisation algorithm by Storn and Price [21], the poles are placed optimally, satisfying the constraints, to minimise the robustness metric. After running, the optimum set of closed-loop poles are computed as

$$\begin{aligned}\mu_{1,2} &= -0.2 \pm 1i\\ \mu_{3,4} &= -0.3 \pm 2.245i\\ \mu_{5,6} &= -1.2 \pm 2.8i\end{aligned}$$

which corresponds to control gains of

$$\mathbf{f} = \begin{pmatrix} -3.1986 \\ -0.2951 \\ -1.7668 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -9.6234 \\ -4.0595 \\ 2.7671 \end{pmatrix}$$

and a robustness metric of 2.19 (a reduction of 53%). Figure 2 shows the variability of the poles before and after optimisation. Note that the rectangles represent the constraints of the nominal closed-loop poles. The pole variability is shown visually by means of a Monte-Carlo simulation with 1000 samples between ±3% variation of the fitting parameters. For simplicity, a uniform distribution is chosen; however, this is merely for visualisation purposes and an underlying probability distribution is not needed in this method.

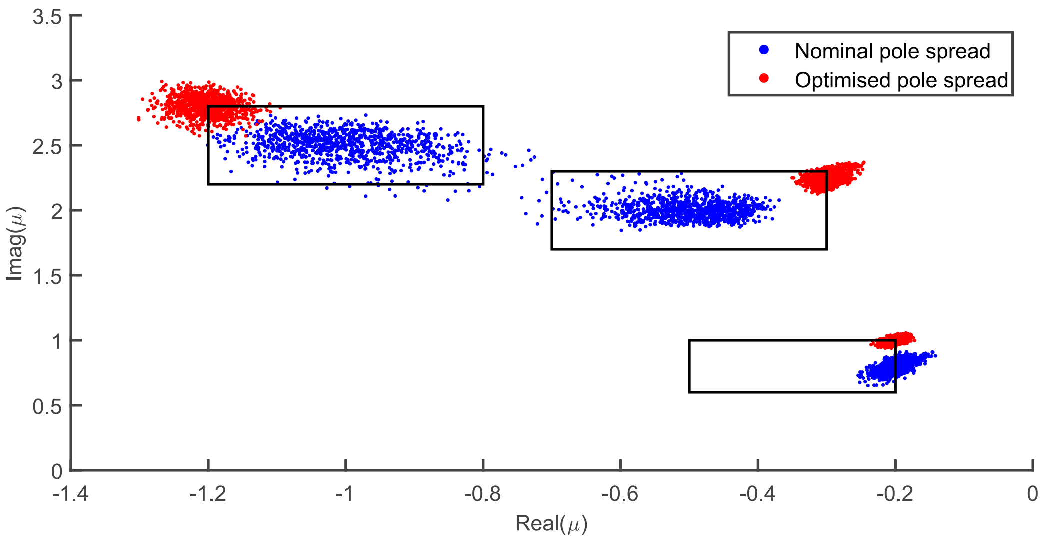

Figure 2: Pole spread (Single-input: optimisation 1).

An interesting point to consider is that the optimum closed-loop poles lie in the corners, or extreme points, of the rectangular constraints. This suggests that the global minimum of the optimum poles lies beyond the constraints enforced by the rectangles and thus one is likely to obtain better solutions if one were able to relax the constraints. However, we consider such a relaxation to yield control gains that are beyond the performance limits of the controller.

Overall, it was found that the Differential Evolution algorithm provided a robust solution to the defined optimisation problem. Regardless of the starting positions of the closed-loop poles, the solutions always converged to the one given above. Of course, this may be specific to this particular problem and one cannot always guarantee that the constrained minimum is always found [21].

Example 2: eigenvalue assignment 2 - first pole pair priority, real part priority

Now, consider the case where β_1,β_2, ρ_1, ρ_2 are set to one and all other weighting parameters are set to zero. That is, the robustness is defined only by the real part variation of the first pole pair (μ_1, μ_2). Again assuming that each fitting parameter is variable by 3% of its nominal value, the initial robustness metric is calculated as 1.29. After running, using the same algorithm as in optimisation 1, the optimum set of closed-loop poles are

$$\begin{aligned}\mu_{1,2} &= -0.2 \pm 1i \\ \mu_{3,4} &= -0.3 \pm 1.94i \\ \mu_{5,6} &= -0.8 \pm 2.2i\end{aligned}$$

which corresponds to control gains of

$$\mathbf{f} = \begin{pmatrix} -0.6273 \\ -1.3065 \\ -0.1896 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -0.3197 \\ -2.9580 \\ 0.3633 \end{pmatrix}$$

and a robustness metric of 0.38 (a reduction of 71%). Figure 3 shows the new variability of the poles before and after optimisation. By comparison with Fig. 2, the spread of the real part of the first pole pair is now slightly smaller. This is, however, at a small expense to the variability of the other pole pairs $(\mu_3, \mu_4, \mu_5, \mu_6)$. Interestingly, the placement of the first pole pair is identical to optimisation one. However, the placement of the other poles is different, especially in the case of the third pole pair (μ_5, μ_6). This implies that there is an interdependence between *all* of the poles in relation to the propagation of uncertainty from the fitted parameters to the pole spreads. In addition, since the first pole pair does not move, this suggests that the variability of a closed-loop pole is governed most by its own placement in the complex plane. That is, the placement of the other poles, whilst having an effect, do not have such a large impact. However, one cannot guarantee that this is always true and may only be valid for this specific example.

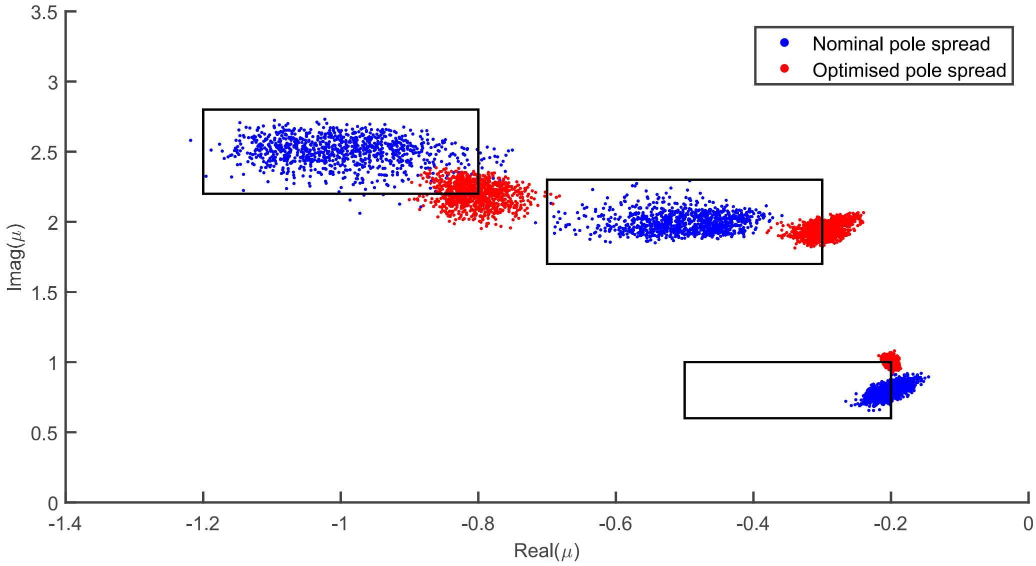

Figure 3: Pole spread (Single-input: optimisation 2).

6.2. Multiple-input

Now, consider the case where

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

so that the system has three inputs. The new transfer matrix is given by

$$\mathbf{R}(s) = \frac{1}{\prod_{i=1}^{6}(s-\lambda_i)} \begin{pmatrix} \frac{1}{2}\left(s^4+\frac{2}{3}s^3+5s^2+\frac{1}{2}s+\frac{8}{3}\right) & 2\left(s^2+\frac{1}{4}s+\frac{4}{3}\right) & \frac{1}{2}\left(s^2+\frac{3}{2}s+8\right) \\ s^2+\frac{1}{4}s+\frac{4}{3} & 2\left(s^4+\frac{5}{12}s^3+\frac{97}{24}s^2+\frac{3}{4}s+\frac{17}{6}\right) & \frac{1}{2}\left(s^3+\frac{17}{4}s^2+4s+14\right) \\ \frac{1}{6}\left(s^2+\frac{3}{2}s+8\right) & \frac{1}{3}\left(s^3+\frac{17}{4}s^2+4s+14\right) & s^4+\frac{3}{4}s^3+\frac{57}{8}s^2+\frac{5}{2}s+10 \end{pmatrix}$$

and hence the new zeros and scaling parameters are as given in Table 4 and Table 5, respectively. As

Table 4: Open-loop zeros (multiple-input).

Element ij	$z_{ij,1}$	$z_{ij,2}$	$z_{ij,3}$	$z_{ij,4}$
11	-0.3209+2.0668i	-0.3209+2.0668i	-0.0124+0.7807i	-0.0124-0.7807i
12	-0.125+1.1479i	-0.125-1.1479i		
13	-0.75+2.7272i	-0.75-2.7272i		
21	-0.125+1.1479i	-0.125-1.1479i		
22	-0.1234+1.7506i	-0.1234-1.7506i	-0.0849+0.9554i	-0.0849-0.9554i
23	-4.1062	-0.0719+1.8451i	-0.0719-1.8451i	
31	-0.75+2.7272i	-0.75-2.7272i		
32	-4.1062	-0.0719+1.8451i	-0.0719-1.8451i	
33	-0.2077+2.2211i	-0.2077-2.2211i	-0.1673+1.4076i	-0.1673-1.4076i

before, we consider all fitting parameters to be variable by 3% of their nominal value.

Table 5: Scaling parameters (multiple-input).

Scaling parameter	Value
η_{11}	$\frac{1}{2}$
η_{12}	2
η_{13}	$\frac{1}{2}$
η_{21}	1
η_{22}	2
η_{23}	$\frac{1}{2}$
η_{31}	$\frac{1}{6}$
η_{32}	$\frac{1}{3}$
η_{33}	1

Example 3: eigenvector assignment

In this example, we consider the case where the closed-loop poles are assigned to the fixed points

$$\begin{aligned}\mu_{1,2} &= -0.2 \pm 0.8i \\ \mu_{3,4} &= -0.5 \pm 2i \\ \mu_{5,6} &= -1 \pm 2.5i\end{aligned}$$

and the eigenvectors are optimally assigned by appropriate selection of the $\boldsymbol{\alpha}_i$ vectors in Eq. 12, subject to the the gain constraint

$$3||\mathbf{F}||_F + ||\mathbf{G}||_F \leq 7$$

The weighting parameter of the $||\mathbf{F}||_F$ term in the gain constraint is chosen to be based roughly on the maximum imaginary part of the closed-loop poles.

The same robustness metric and optimisation procedure as in example 1 is used throughout this example. In other words, all weighting constants are set to one.

The optimum set of $\boldsymbol{\alpha}_i$ vectors are computed as

$$\boldsymbol{\alpha}_{1,2} = \begin{pmatrix} 1 \\ -0.16 \\ -1.6613 \pm 4.2788i \end{pmatrix}, \quad \boldsymbol{\alpha}_{3,4} = \begin{pmatrix} 1 \\ 0.0541 \pm 0.2543 \\ -1.3277 \pm 1.1091i \end{pmatrix}, \quad \boldsymbol{\alpha}_{5,6} = \begin{pmatrix} 1 \\ -9.6961 \pm 2.4055i \\ 0 \end{pmatrix},$$

which corresponds to control gains of

$$\mathbf{F} = \begin{pmatrix} -0.5758 & 0.3000 & 0.1308 \\ 0.0510 & -0.7657 & -0.1349 \\ 0.3297 & 0.6033 & -0.6640 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} -0.5572 & 1.5169 & 1.4346 \\ -0.1247 & -1.2435 & -0.0245 \\ 0.5684 & 0.1341 & -1.4786 \end{pmatrix}$$

and right eigenvectors of

$$\mathbf{w}_{1,2} = \begin{pmatrix} 1 \\ 1.4959 \pm 0.0385i \\ 1.5297 \pm 0.2102i \end{pmatrix}, \quad \mathbf{w}_{3,4} = \begin{pmatrix} 1 \\ 0.0283 \pm 0.9508i \\ -0.97 \pm 0.3930i \end{pmatrix}, \quad \mathbf{w}_{5,6} = \begin{pmatrix} 1 \\ -3.0714 \pm 4.2318i \\ 0.3231 \pm 0.4319i \end{pmatrix},$$

Note that, since eigenvectors may be scaled arbitrarily, the first element in the vector of $\boldsymbol{\alpha}_i$ is always selected as one and hence this reduces number of optimisation variables.

Figure 4 shows the closed-loop pole spread of the optimised system. Note here that the rectangular boxes from the previous examples do not appear since the nominal closed-loop poles are fixed to set locations and may not move. To compare the results, the pole spreads are also shown for a reference multiple-input system where $\alpha_{k,i} = 1 \forall k, i$. As shown, the position of the nominal poles are the same. However, by assigning different values of $\boldsymbol{\alpha}_i$, the pole spreads are vastly different. Indeed, the robustness metric decreases from 7.73 (in the reference case) to 2.77 (a decrease of 64%).

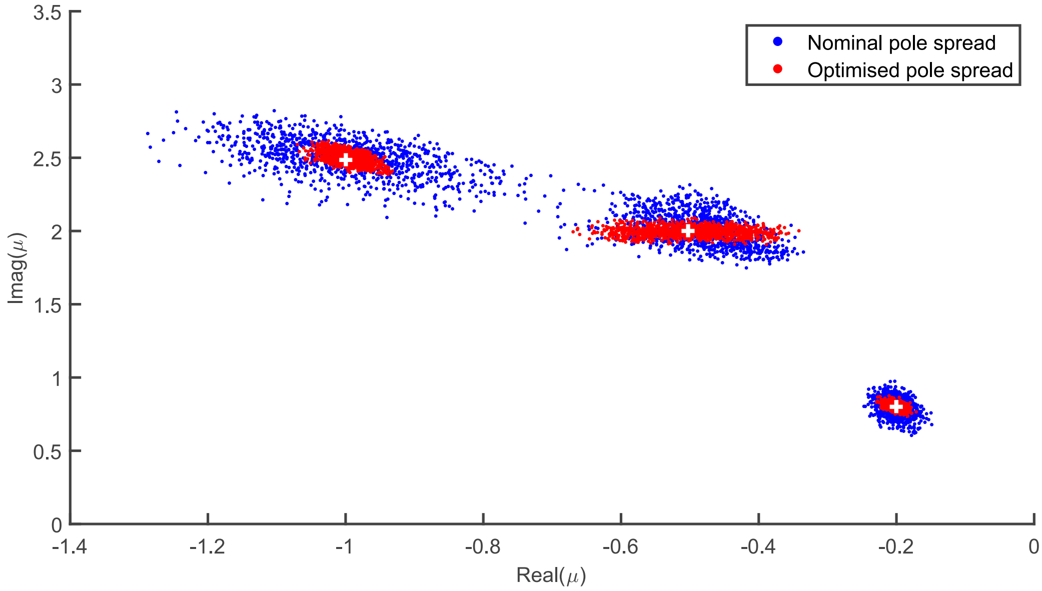

Figure 4: Pole spread (Multiple-input: optimisation 1).

At this point, the need to use the gain constraint becomes clear. Simply by substituting the control gains into the gain constraint equation, the gain constraint is evaluated as 7.1. This is a slight violation of the constraint given above and arises since the constraint is implemented as a penalty function in the objective function (see [22]). Therefore, the controller is at its maximum limit and therefore it is likely that relaxing the constraint will yield a better optimisation result. However, again, we consider such a relaxation to by physically unfeasible.

It is to be noted here that the initial robustness metric in example 1 and example 3 are not identical, despite the closed-loop poles initially being placed as the same locations in the complex plane. This is due to the additional uncertain parameters in the transfer function matrix, which are introduced by the two additional inputs. An interesting point to consider is whether the introduction of more inputs leads to a situation where the robustness metric is always increased, regardless of the optimisation. However, this is left as an area of future research.

Example 4: eigenstructure assignment

Finally, we consider the case of eigenstructure assignment; that is, where both the poles and eigenvectors are assigned. In addition to choosing optimal $\boldsymbol{\alpha}_k$ values, the poles are now placed subject to the constraints

$$\mu_{1,2} = [-0.5, -0.2] \pm [0.6, 1]i$$
$$\mu_{1,2} = [-0.7, -0.3] \pm [1.7, 2.3]i$$
$$\mu_{1,2} = [-1.2, -0.8] \pm [2.2, 2.8]i$$

To enable a comparison to the previous example, the same optimisation procedure and gain constraints are used.

After running, the optimum solution set was computed as

$$\mu_{1,2} = -0.2 \pm 0.6i$$
$$\mu_{3,4} = -0.3 \pm 1.7i$$
$$\mu_{5,6} = -0.8 \pm 2.3631i$$

and

$$\boldsymbol{\alpha}_{1,2} = \begin{pmatrix} 1 \\ -6.1699 \pm 0.0249i \\ -8.0772 \pm 0.7286i \end{pmatrix}, \quad \boldsymbol{\alpha}_{3,4} = \begin{pmatrix} 1 \\ -0.0099 \pm 0.0017 \\ 0 \end{pmatrix}, \quad \boldsymbol{\alpha}_{5,6} = \begin{pmatrix} 1 \\ -9.57 \\ 1.6021 \pm 0.8635 \end{pmatrix},$$

which corresponds to control gains of

$$\mathbf{F} = \begin{pmatrix} -0.5795 & 0.4785 & -0.1507 \\ 0.0662 & -0.5496 & -0.0176 \\ 0.2963 & 0.0985 & -0.2943 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} -0.9245 & 0.6526 & -0.2264 \\ -0.1820 & -0.5489 & 0.1528 \\ 0.6079 & 0.0347 & -0.0998 \end{pmatrix}$$

and right-eigenvectors of

$$\mathbf{w}_{1,2} = \begin{pmatrix} 1 \\ 1.6061 \pm -0.0803i \\ 2.0365 \pm -0.0730i \end{pmatrix}, \quad \mathbf{w}_{3,4} = \begin{pmatrix} 1 \\ 0.9976 \pm -0.0985i \\ -0.5046 \pm 0.0752i \end{pmatrix}, \quad \mathbf{w}_{5,6} = \begin{pmatrix} 1 \\ -2.5269 \pm 3.2459i \\ 0.4014 \pm 0.6070i \end{pmatrix},$$

With these new control gains, the robustness metric decreases from 7.73 to 1.67 (a reduction of 79%). This is illustrated visually in Fig. 5, which shows a significant reduction in the size of the pole spreads for all three pole pairs. Interestingly, the optimal poles found in example 1 do not match those found in this example. This suggests that the optimal eigenvectors and poles are not independent and are, in some way, related to each other. Furthermore, it is clear that the pole spread corresponding to the first pole pair is

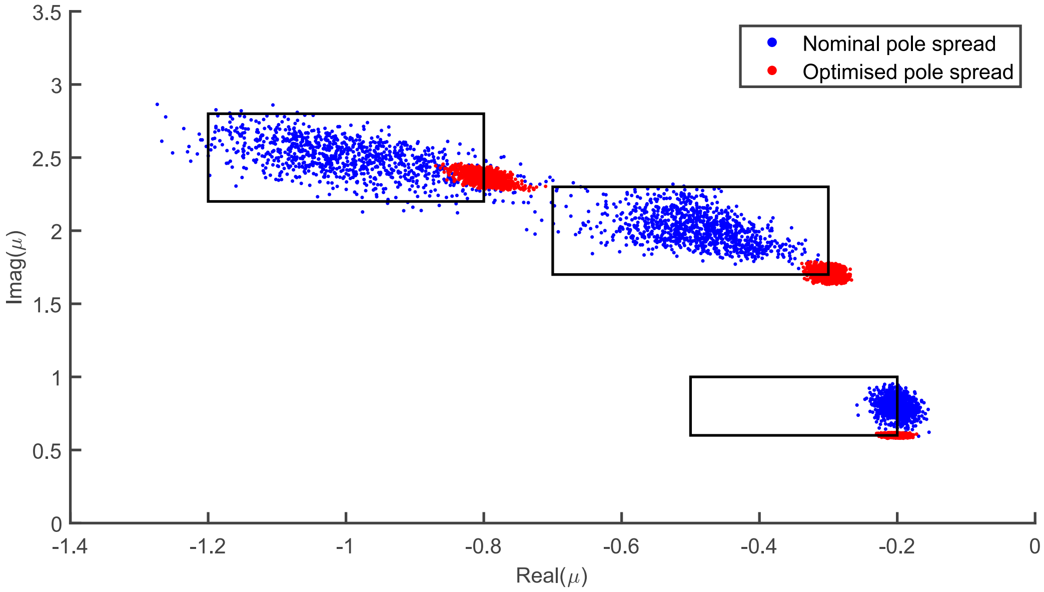

Figure 5: Pole spread (Multiple-input: optimisation 2).

flatter and seems to be aligned in one particular direction, rather than a circular or elliptical spread. Such an effect may occur when the pole's sensitivity to particular fitting parameters is vastly decreased. In this case, the pole's spread becomes dependent on only a smaller number of parameters and hence the pole spreads appear more akin to a root locus of the most dominant fitting parameters.

7. Conclusions

This work considers the effect of transfer function parameter misfitting on eigenstructure assignment using the Receptance Method. Closed-loop pole sensitivity formulae are derived for variable fitted open-loop poles, zeros and scaling parameters. A global optimisation approach is then used to minimise the sensitivities by either eigenvalue and/or eigenvector assignment. The robustness of the system is measured by a metric, which serves to weight the relative importance of each mode and the variation of each pole's real and imaginary parts. In single-input systems, the robustness metric is reduced by placing the poles optimally within rectangular regions in the complex plane. In multiple-input systems, the robustness metric is reduced either by modifying the closed-loop eigenvectors, closed-loop poles, or a combination of both.

The method is applied numerically to a three-degree-of-freedom mass, spring, damper system. It is shown that the definition of the robustness metric affects the optimal solution found by the global optimisation and hence the location of the placed closed-loop poles in the complex plane and the assigned closed-loop eigenvectors. In addition, it is demonstrated that multiple-input systems are capable of reducing the pole spreads more than single-input systems.

The work presented in this paper highlights several areas of research that the authors suggest may be of interest for future work. Firstly, the examples presented in this paper are based on a simple mechanical system with a small number of degrees of freedom. When the number of degrees of freedom increases, the number of fitting parameters largely increases and hence the computational effort in the optimisation will be greater. It is suggested that it may be possible to decrease the size of the problem by identifying poles, zeros and scaling parameters with little influence on the robustness metric. For example, poles that are far away from each other in terms of frequency are less likely to influence one another than poles that are nearby. Secondly, it was shown that using multiple inputs led to a reduction of the spreads associated with the closed-loop poles by optimally assigning the closed-loop eigenvectors. One may also investigate whether the placement of the inputs, and hence the form of the force distribution matrix, can also reduce the uncertainty and, if so, whether a similar procedure developed in this paper can be used in this case. Furthermore, it would be interesting to consider the influence of the number of sensors used and their respective locations. Thirdly, the numerical examples only considered the case where the inputs had equal uncertainty in their parameters. In other words, the parameters of each column of the transfer function matrix were weighted equally. In practice, this is not likely to be the case and some inputs may have parameters that are more uncertain than others. Therefore, it would be interesting to investigate whether the introduction of highly uncertain inputs would actually increase the robustness metric, regardless of the optimisation. Finally, further research is needed to investigate how to select the upper and lower bounds of the fitting parameters that are used to define the robustness metric.

Acknowledgements

The authors gratefully acknowledge the financial support provided by the Engineering and Physical Sciences Research Council (EPSRC) grant EP/N017897/1. LJA also acknowledges the support provided by the EPSRC Doctoral Training Scholarship.

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