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# Forbidden subgraphs for Hamiltonian problems on 

2-trees
by

Caitlin Owens

A Dissertation<br>Presented to the Graduate Committee of Lehigh University in Candidacy for the Degree of Doctor of Philosophy<br>in<br>Mathematics

Lehigh University
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Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Caitlin Owens
Forbidden subgraphs for Hamiltonian problems on 2-trees

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#### Abstract

The Hamiltonian path problem is a well-known NP-complete graph theory problem which is to determine whether or not it is possible to find a spanning path in a graph. Some variations on this problem include the 1HP and 2HP problems, which are to determine whether or not it is possible to find a Hamiltonian path in a graph if one or two endpoints of the path are fixed, respectively. Both problems are also NP-complete for graphs in general, though like the Hamiltonian path problem, they are polynomially solvable on certain types of graphs. 2-trees are a specific type of graph for which the $1 \mathrm{HP}, 2 \mathrm{HP}$, and traditional Hamiltonian path problems are polynomially solvable. It is known that 2-trees have a Hamiltonian cycle if and only if they are 1-tough. However, the analogous statement for Hamiltonian paths does not hold. We will structurally characterize 2 HP on 2-trees, and then use these results to structurally characterize 1 HP and HP on 2-trees. We will define a family of 2 -trees such that any 2-tree has a Hamiltonian path if and only if it does not contain any graph from that family as an induced graph.


## Chapter 1

## Introduction

In this chapter, we will review basic information regarding 2-trees and Hamiltonian problems. In Section 1.1, we will review basic terminology and definitions and in Section 1.2 we will provide basic results. In Chapter 2 we will introduce the new definitions and techniques that will be used in this dissertation. Chapter 3 and 4 will have our main results regarding the Hamiltonian path problems on 2-trees, and Chapter 5 will conclude the dissertation with future work.

### 1.1 Problem Description

The Hamiltonian path problem (HP) is to determine whether or not a given graph has a Hamiltonian path, i.e., a spanning path in the graph. Two variations of this problem, 1HP and 2HP, determine whether a given graph has a Hamiltonian path fixing one or two given vertices, respectively, as endpoints.

The 1HP problem is known to be polynomially solvable on interval graphs, cographs, and biconvex graphs. It is known to be NP-complete on chordal and comparability graphs. The complexity of the 1HP problem is unknown on both permutation graphs and convex graphs. The complexity of the 2 HP problem is unknown on interval graphs [2], though it is known to be polynomially solvable on cographs [3].

For $k$-trees, a subclass of chordal graphs, HP, 1HP, and 2HP problems are polynomially solvable. They fall into the class of partial $k$-trees, graphs with treewidth at most $k$. The Hamiltonian path problem is polynomially solvable on graphs with
bounded treewidth [11], using FPT algorithms, or algorithms which are fixed parameter tractable. Since adding a pendant edge to a $k$-tree keeps the graph in the class of partial $k$-trees, we can solve the 1HP and 2HP problems by adding a pendant edge to a given path endpoint and running the Hamiltonian path algorithm for partial $k$-trees on the resulting graph.

In [23], Renjith and Sadagopan give a linear-time algorithm for Hamiltonian paths in 2-trees. They also discuss some structural qualities of 2-trees having a Hamiltonian path. Most of their results involve multiple algorithms and structures of a graph, which is not a 2 -tree, formed from the algorithms. In Chapter 4, we will take a different approach, by using toughness properties and results from the 2HP problem to give a list of forbidden induced sub-2-trees for which a 2-tree will not have a Hamiltonian path.

In this dissertation, we will give structural conditions of 2-trees and forbidden induced subgraphs for Hamiltonian paths not to exist in 1HP, 2HP, and the traditional Hamiltonian path problem.

### 1.2 Basic Definitions and Results

Definition 1.2.1. A graph, $G$, is Hamiltonian if $G$ contains a Hamiltonian cycle.
We will use the following notations for the 1HP and 2HP problems.
Definition 1.2.2. Given a graph $G$ and $x_{1} \in V$, an $\boldsymbol{x}_{1}$-Hamiltonian Path in $G$ is a Hamiltonian path which either begins or ends with $x_{1}$.

Definition 1.2.3. Given a graph $G$ and $x_{1}, x_{2} \in V$, an ( $\left.\boldsymbol{x}_{1}, \boldsymbol{x}_{2}\right)$-Hamiltonian Path in $G$ is a Hamiltonian path between $x_{1}$ and $x_{2}$.

Our focus for these problems will be on 2-trees, which are $k$-trees for $k=2$.
Definition 1.2.4. [24] Define a $\boldsymbol{k}$-tree as follows:

- $K_{k}$, the complete graph on $k$ vertices, is a $k$-tree, and
- If $G$ is a $k$-tree, then the graph formed by adding a vertex adjacent to all vertices in a $k$-clique in $G$ is a $k$-tree.

Subgraphs of $k$-trees are called partial $\boldsymbol{k}$-trees. A simplicial vertex is a vertex whose neighbors form a clique. Simplicial vertices in a $k$-tree have degree $k$.

Notation 1.2.5. We will use the notation from [7] where $S_{1}(G)$ is the set of simplicial vertices in $G$.

Definition 1.2.6. For any graph $G=(V, E)$, and $X \subset V, G[X]$, the graph induced by $X$, has vertex set $X$, and edge set $E^{\prime} \subset E$ such that $u v \in E^{\prime}$ iff $u v \in E$ and $u, v \in X$.

Notation 1.2.7. For any graph $G=(V, E)$ and $v \in V, G-v$ denotes $G[V-\{v\}]$. Likewise, for $S \subset V, G-S$ denotes $G[V-S]$.

In [24], Rose also characterizes $k$-trees as connected graphs which contain a $k$ clique but no $k+2$-clique, and such that every minimal $x, y$ separator of $G$ is a $k$ clique. An $x, y$ separator is a set $S \subset V$ such that $x$ and $y$ lie in different components of $G-S$.

Stemming from Chvátal's conjecture that there exists a $t_{0}$ such that every $t_{0}$-tough graph is hamiltonian [10], many known results, including those from [7], regarding Hamiltonian problems in $k$-trees involve toughness conditions.

Definition 1.2.8. For a graph $G=(V, E)$ and $S \subset V$, let $c(G-S)$ denote the number of components in $G-S$. Then $G$ is t-tough if $|S| \geq t(c(G-S)$ ) for all cut-sets, $S$, i.e., $S \subset V$ such that $c(G-S)>1$. A set, $S$ such that $|S|=t(c(G-S))$ is called a tough set.

Definition 1.2.9. A graph $G$ is 1-path-tough if $|S| \geq(c(G-S)-1)$ for all $S \subseteq V$.
The following theorem, originally stated by Chvátal in [10], is well known and can be found in many graph theory textbooks.

Theorem 1.2.10. [10] If a graph $G$ has a Hamiltonian cycle, then $G$ is 1-tough.
Theorem 1.2.11. If $G$ has a Hamiltonian path, then $G$ is 1-path-tough.
Path tough has also been used in [12 to describe a graph, $G$, such that for any nonempty $S \subset V, G-S$ can be covered by at most $|S|$ vertex disjoint paths.

Closely related to toughness, we will often use the scattering number of a graph when proving that Hamiltonian paths do not exist in a graph.

Definition 1.2.12. [16] The scattering number of a graph $G$ is

$$
s(G)=\max _{S \subseteq V, c(G-S) \neq 1}\{c(G-S)-|S|\}
$$

Hence, if $G$ is 1-tough, then $s(G) \leq 0$ and if $G$ is 1-path-tough, then $s(G) \leq 1$. Furthermore, if $G$ is a graph for which $s(G) \geq 2$, then $G$ does not have a Hamiltonian path.

Additionally, for graphs with scattering number at least one, the scattering number of a graph gives a well known lower bound for the path partition number of a graph.

Notation 1.2.13. $P P(G)$ denotes the path partition number of a graph, $G$, the minimum number of vertex disjoint paths required to cover the vertices of $G$.

The path partition number has also been referred to as the path cover number.
Lemma 1.2.14. For any graph $G$,

$$
P P(G) \geq \max _{U \subseteq V}\{c(G-U)-|U|\}
$$

The related $k$-fixed endpoint path partition problem is to determine the minimum number of vertex disjoint paths required to cover the vertices of $G$ given that each vertex in a set $T$ of $k$ vertices are each endpoints of a path. In [4], Baker gives the following lower bound for the $k$-fixed endpoint path partition number of a graph $G$ with respect to $T \subset V(G)$. This will be helpful when we look at 2HP.

Notation 1.2.15. $P P(G ; T)$ denotes the $k$-fixed endpoint path partition number of $a$ graph $G$ with respect to $T \subset V(G)$.

Lemma 1.2.16 (Baker, 2013). [4] For any graph $G$ and a set $T \subset V(G)$,

$$
P P(G ; T) \geq \max _{U \subseteq V}\{c(G-U)-|S|\}
$$

for $S=U-T$.
We will begin looking at the Hamiltonian path problems on 2-trees by looking at the toughness conditions regarding Hamiltonian cycles in $k$-trees from [7].

Theorem 1.2.17 (Broersma, Xiong, Yoshimoto, 2005). [7] If $G \neq K_{2}$ is a $\frac{k+1}{3}$-tough $k$-tree $(k \geq 2)$, then $G$ is Hamiltonian.

For $k=2$, the above theorem proves that 1-toughness is also a sufficient condition for 2-trees to have a Hamiltonian cycle. In their proof, the Broersma, Xiong, and Yoshimoto also prove that there is a cycle which contains all of the edges, $e=u v$, for which $c(G-\{u, v\})=1$. For 1-tough 2-trees, this is the only Hamiltonian cycle in the graph. In Theorem 1.2.23, we will restate and prove Theorem 1.2 .17 for the special case when $k=2$.

Knowing that 1-toughness is a sufficient condition for a 2-tree to have a Hamiltonian cycle, it seemed natural to check if there was a similar 1-path-toughness condition for 2-trees having Hamiltonian paths. For a cocomparability graph, $G, G$ has a Hamiltonian cycle iff it is 1-tough, and likewise, $G$ has a Hamiltonian path iff it is 1-path-tough [13]. However, while 1-path-toughness is a necessary condition, it is not a sufficient condition for a 2 -tree to have a Hamiltonian path. We build an infinite class of 1-path-tough 2-trees which do not contain a Hamiltonian path, as demonstrated in Figure 1.3. These 1-path-tough 2 -trees will not be 1-tough, since clearly if a 1-path-tough 2 -tree is also a 1-tough 2-tree, then it will have a Hamiltonian path by Theorem 1.2.17. So, first we will discuss a few structural conditions to identify 2-trees which are and are not 1-tough. In [19], Markenzon, Justel, and Paciornik refer to a 1-tough 2-tree as a simple-clique 2-tree or SC 2-tree, but we will refer to these 2-trees by their toughness condition.

Definition 1.2.18. The open neighborhood, $N_{G}(v)$, of a vertex $v$, is the set of vertices adjacent to $v$ in $G$. We will drop the $G$, when the graph in question is clear. The closed neighborhood of a vertex $v$ is $N[v]=N(v) \cup\{v\}$.

Definition 1.2.19. We will say a vertex, $v$, is adjacent to an edge, uw, if $v$ is adjacent to both $u$ and $w$. Furthermore, for any edge, $e=u w$, the closed neighborhood of e, $N[e]$, will be defined as $N[e]=N[u] \cap N[w]$, and the open neighborhood of $\mathbf{e}, N(e)$, will be defined as $N(e)=N(u) \cap N(w)$.

Definition 1.2.20. A $\boldsymbol{t}$-edge is an edge, e such that $|N(e)|=t$.
Remark 1.2.21. A t-edge will be shared by $t$ distinct induced $K_{3}$ 's, or triangles.
The following lemma and its proof are similar to that found in [23] with new notation. We provide an additional proof here for clarity and completeness.

Lemma 1.2.22. Suppose $G \neq K_{2}$ is a 2-tree. If $x y \in E(G)$ is a t-edge then $c(G-$ $\{x, y\})=t$.

Proof. We proceed by induction on $|V(G)|=n$. If $n=3$, then $G=K_{3}$. Furthermore, all edges are 1-edges, and the claim is true. Suppose the claim is true for graphs with $n-1$ vertices. Now, consider $G$ a 2 -tree with $|V(G)|=n$. Then there exists a simplicial vertex $v$, such that $G^{\prime}=G-v$ is a 2-tree with $\left|V\left(G^{\prime}\right)\right|=n-1$. Suppose that $v$ is adjacent to $u w$ in $G$. If $x y \neq u w, u v, v w$, then $x y \in E\left(G^{\prime}\right)$, and by the induction hypothesis, if $x y$ is a $t$-edge then $c\left(G^{\prime}-\{x, y\}\right)=t$. Since $v$ is adjacent to $u$ and $w$, then $v$ is in the same component as $u$ if $x=w$ or $y=w$, and $v$ is in the same component as $w$ if $x=u$ or $y=u$. So, $c(G-\{x, y\})=c\left(G^{\prime}-\{x, y\}\right)=t$. If $u w$ is a $t$-edge in $G^{\prime}$, then $c\left(G^{\prime}-\{u, w\}\right)=t$, and so in $G$, since $u w$ is also adjacent to $v$, then $u w$ is a $(t+1)$-edge. Additionally, $c(G-\{u, w\})=t+1$ as $v$ is only adjacent to $u$ and $w$. In $G$, both $u v$ and $v w$ are 1-edges. Furthermore, $c(G-\{u, v\})=1$ since $c(G-\{u, v\})=c\left(G^{\prime}-\{u\}\right)=1$, as $G^{\prime}$ is a 2-tree and minimal separators of 2-trees are 2-cliques. Likewise, $c(G-\{v, w\})=1$.

Using our new terminology, we can restate Theorem 1.2 .17 with a structural condition, as Theorem 1.2.23 below.

Theorem 1.2.23. If $G \neq K_{2}$ is a 2-tree, then the following are equivalent:

1. $G$ is 1-tough,
2. $G$ contains no $t$-edges for $t \geq 3$, and
3. $G$ is Hamiltonian.

Proof. (1) $\Longrightarrow(2)$
We will prove the contrapositive. If $G$ contains a $t$-edge, $x y$, for $t \geq 3$, then $c(G-$ $\{x, y\})=t \geq 3>2=|\{x, y\}|$, then $G$ is not 1-tough.
(2) $\Longrightarrow(3)$

We will prove, by induction on $|V(G)|=n$, that if $G$ contains no $t$-edges for $t \geq 3$, then $G$ contains a Hamiltonian cycle containing all of the 1-edges of $G$, and hence is Hamiltonian. If $n=3$, then $G=K_{3}$. $G$ only contains $t$-edges where $t=1$ and $G$ is Hamiltonian with Hamiltonian cycle containing all 1-edges. Suppose that all 2-trees with $n-1$ vertices and only $t$-edges for $t \leq 2$ are Hamiltonian with Hamiltonian cycle containing all 1-edges. Now consider $G$ a 2-tree with $|V(G)|=n$ such that $G$ contains only $t$-edges for $t \leq 2$. Let $v$ be a simplicial vertex of $G$, adjacent to $u w \in E(G)$. Then by the induction hypothesis, $G^{\prime}=G-v$ is Hamiltonian and hence contains a Hamiltonian cycle, $C$, containing all 1-edges. Furthermore, $u w$ must be a 1-edge in
$G^{\prime}$, since $G$ contains no 3-edges. Therefore, replacing, $u w$ in $C$ with $(u, v, w)$ creates a Hamiltonian cycle $C^{\prime}$ in $G$, containing all 1-edges.
$(3) \Longrightarrow(1)$ Theorem 1.2 .10 .
From Theorem 1.2.23, if $G$ is 2 -tree which is not 1-tough, then $G$ contains at least one $t$-edge, $x y$, for $t \geq 3$. If $G$ is 2 -tree which is 1 -path-tough, then by the lemma below, $G$ cannot contain a $t$-edge, $x y$, for $t \geq 4$. However, there are 2 -trees which are not 1 -path-tough which do not contain a $t$-edge, $x y$, for $t \geq 4$. Furthermore, there are 1-path-tough 2-trees which do not contain a Hamiltonian path. So, for Hamiltonian paths in 2-trees, we will not have a necessary and sufficient condition using $t$-edges as in Theorem 1.2.23. In Theorem 4.1.15, we will prove necessary and sufficient conditions for a 2-tree to have a Hamiltonian path, using induced subgraphs. We could also restate (2) in Theorem 1.2 .23 using an induced subgraph condition instead. If $G$ is a 2-tree which contains a $t$-edge for $t \geq 3$, then $G$ contains an induced $K_{2} \vee 3 K_{1}$.


Figure 1.1: $K_{2} \vee 3 K_{1}$

Lemma 1.2.24. If $G$ is a 2-tree and contains a $t$-edge for $t \geq 4$, then $G$ does not contain a Hamiltonian path.

Proof. Let $x y \in E(G)$ be a $t$-edge for $t \geq 4$. Then, $c(G-\{x, y\})-|\{x, y\}|=t-2 \geq 2$, and $G$ is not 1-path-tough.

Lemma 1.2.25. Suppose $G$ is a 2-tree and contains a 3-edge, ab, such that ab is adjacent to two simplicial vertices, $v_{1}$ and $v_{2}$. Then $G$ has a Hamiltonian path iff $G-v_{1}$ has a Hamiltonian path with either $a$ or $b$ as an endpoint of the path.

Proof. $\Longleftarrow$ Without loss of generality, assume $G-v_{1}$ has a Hamiltonian path, $P$, which begins with $a$ as an endpoint. Then $\left(v_{1}, P\right)$ is a Hamiltonian path in $G$.
$\Longrightarrow$ Suppose $G$ has a Hamiltonian path, $P$. Since $c(G-\{a, b\})=3$, then the endpoints of the Hamiltonian path must lie in two of the three components. Hence
at least one of the simplicial vertices must be an endpoint of the path. Without loss of generality, let $v_{1}$ be an endpoint of the path. Since $v_{1}$ is only adjacent to $a$ and $b$, then either $a$ or $b$ follows $v_{1}$ on the path. Since we cannot use $v_{1}$ again on the path, then the rest of the path must be in $G-v_{1}$, and hence $P-v_{1}$ is a Hamiltonian path in $G-v_{1}$ which has either $a$ or $b$ as an endpoint.

From the above lemma, we can see that when a 2-tree contains a 3-edge, if there is a Hamiltonian path, there will be endpoint restrictions. Because of this, in Chapter 3 , we begin our investigation looking at the 2HP problem on 2-trees, to extend these results to the the Hamiltonian path problem on 2-trees.

Definition 1.2.26. A pair of vertices, $u, v$ are called false twins if $N(u)=N(v)$. Vertices, $u, v$ are called twins if $N[u]=N[v]$, i.e., the vertices are also adjacent.

Definition 1.2.27. Let $P_{n}$ be a path with $n$ vertices. Then $\mathbf{P}_{\mathbf{n}}^{\mathbf{k}}$, the $k^{\text {th }}$ power of $P_{n}$, is a graph which has the same vertex set as $P_{n}$, but has edges between any vertices whose distance in $P_{n}$ is at most $k$.

Note that $P_{n}^{2}$ is a 2 -tree. In particular, it is a special case of a 2-path graph. Originally introduced in [22] and further characterized in [19], a 2-tree with exactly two simplicial vertices is a 2-path graph.


Figure 1.2: A specific example of a 2 -path: $P_{15}^{2}$, with simplicial vertices $u$ and $v$

Now consider $G=P_{17}^{2}$. We will form $H$ from $G$ by first adding a false twin of each of the simplicial vertices of $G$. Then we will add a pair of simplicial vertices adjacent to a 1-edge, $a b$, such that $(N[a] \cup N[b]) \cap(N[e] \cup N[f])=\emptyset$, for any 3-edge, ef, as in Figure 1.3 .


Figure 1.3: Example of a 1-path tough graph with no Hamiltonian path: $H$ was constructed by adding simplicial vertices to $G=P_{17}^{2}$, shown in bold, such that $(N[a] \cup N[b]) \cap(N[e] \cup N[f])=\emptyset$, for any 3-edges, $a b$, ef.

This is an example of a 1-path-tough graph which does not have a Hamiltonian path. The main idea, which will be formally proved in Chapter 4, for why there is no Hamiltonian path comes from Lemma 1.2 .25 and that based on the construction of $H$, a Hamiltonian path in $H$ would have three distinct endpoints. Furthermore, if we construct $H$ from $G=P_{n}^{2}$ with larger $n$ and add more pairs of simplicial vertices with the same properties as before, then we can create an infinite class of 1-path-tough graphs which do not contain a Hamiltonian path.

Definition 1.2.28. A graph $G$ is Hamilton-connected if there is a Hamiltonian path between all pairs of vertices of $G$.

Theorem 1.2.29 (Kabela, preprint 2017). 17 Let $k \geq 3$. Every $k$-tree of toughness greater than $\frac{k}{3}$ is Hamilton-connected.

The above theorem does not hold for $k=2$. While 1-tough 2-trees are Hamiltonian, and contain a Hamiltonian path, they are not Hamilton-connected. Furthermore, even for $k=3$, equality does not hold in the above theorem. In [17], Kabela gives examples of 1 -tough planar 3 -trees which do not contain a Hamiltonian path.

Since 1-tough 2-trees are not Hamilton-connected, in Chapter 3, we will discuss the 2HP problem on 1-tough 2-trees and which pairs of vertices will not be ends of a Hamiltonian path. We will then use these results to characterize the rest of the 2-trees with fixed endpoints which do not contain a Hamiltonian path. In Chapter 4 and 5 we will extend the results from 2HP on 2-trees to the traditional Hamiltonian path problem, and 1 HP , respectively. In order to describe our results on 2HP, HP, and 1HP, we will begin the next chapter with a new toughness definition and special induced subgraphs of 2-trees which will help us define induced subgraphs which will not contain a Hamiltonian path. We will also discuss the types of induced subgraphs,
which do not contain a Hamiltonian path, and which will prevent a general chordal graph from having a Hamiltonian path.

## Chapter 2

## New Approach

Since toughness and $t$-edges alone will not be enough to characterize 2HP, HP, and 1HP on 2-trees, we will take a new approach for a characterization by looking at induced subgraphs of 2-trees and by defining a new property regarding toughness.

In the next chapters, we will introduce families of 2-trees for which there do not exist Hamiltonian paths, or Hamiltonian paths with specified fixed endpoints. We will prove in Theorems 3.1.24, 3.2.10, 4.1.15, and 4.2.12, that if a 2 -tree contains a graph from these families as an induced subgraph, the 2-tree will not have a Hamiltonian path. In this chapter, we will define special types of 2-trees, which will be useful in describing our families of graphs which do not contain Hamiltonian paths.
In general if a graph has an induced subgraph which is not Hamiltonian, we will not know whether or not our graph is Hamiltonian. In Section 2.1 we will define a special type of induced subgraph. If a graph has one of these induced subgraphs and is not Hamiltonian, then our graph will not be Hamiltonian as well. In Section 2.2, we will define specific 2-trees which will be the building blocks of our families of 2-trees in the later chapters. In Section 2.3, we will we define our new toughness property which will help us to prove graphs do not have Hamiltonian paths in later chapters.

### 2.1 Induced Subgraphs

In [15], Goodman and Hedetniemi prove that 2-connected graphs which do not contain an induced $K_{1,3}$ or $N(1,0,0)$ are Hamiltonian. This is only a sufficient condition for a 2-connected graph to be Hamiltonian, whereas we will prove both necessary and sufficient conditions for Hamiltonian paths in 2-trees.

(a) $K_{1,3}$

(b) $N(1,0,0)$

Figure 2.1: A 2-connected graph not containing 2.1(a) and 2.1(b) as an induced subgraph is Hamiltonian.

Since 2-trees are 2-connected, 2-trees which do not contain an induced $K_{1,3}$ or $N(1,0,0)$ are Hamiltonian. Note that a 2-tree which contains a $t$-edge for $t \geq 3$ will contain an induced $K_{1,3}$, so if we are looking at 2-trees which do not contain an induced $K_{1,3}$, they will be 1-tough. However, we can also have 1-tough, and hence Hamiltonian, 2-trees which contain an induced $K_{1,3}$ and an induced $N(1,0,0)$, as in Figure 2.2.

(a) induced $K_{1,3}$

(b) induced $N(1,0,0)$

Figure 2.2: Example of a Hamiltonian 2-tree containing an induced $K_{1,3}$ and $N(1,0,0)$ where the induced $K_{1,3}$ and $N(1,0,0)$ are bolded

In general, a graph can be Hamiltonian even if it contains an induced subgraph which is not Hamiltonian. For example, a cycle is Hamiltonian, but no induced subgraph of a cycle is Hamiltonian. Even for the class of 2-trees, a Hamiltonian 2-tree can contain an induced subgraph which is not Hamiltonian, and so in this chapter we will present sufficiency conditions for which an induced subgraph which is not Hamiltonian will mean the chordal graph which contains it will not be Hamiltonian.

Consider the following 2-path, $G$, where all vertices are adjacent to a degree seven vertex.


Figure 2.3: A 2-path, $G$, for which all vertices are adjacent to a degree seven vertex

Then $G[V(G)-v]$ below is an induced subgraph. Furthermore, $G[V(G)-v]$ is not Hamiltonian as it contains a cut-vertex, and hence is not 1-tough.


Figure 2.4: $G[V(G)-v]$ corresponding to $G$ in Figure 2.3

Definition 2.1.1. Let $G$ be a $k$-tree. If $H$ is an induced subgraph of $G$, which is also a $k$-tree, then $H$ will be called an induced sub-k-tree.

If we consider a 2 -tree, $G$, which has an induced sub-2-tree, $H$, such that $H$ does not contain a Hamiltonian path, then $G$ also does not contain a Hamiltonian path. We will prove this below, though it is worthwhile to note that a parallel statement will not hold for chordal graphs in general.

Consider the class of chordal partial 3 -trees, $\mathscr{C}$, which like 2 -trees can be constructed recursively as follows:

1. $K_{2}$ is in $\mathscr{C}$, and
2. If $G$ is in $\mathscr{C}$, then the graph formed by adding a vertex adjacent to all vertices in a 2 -clique or a 3-clique in $G$ is in $\mathscr{C}$.

Now consider $G$ in Figure 2.5 below. We can see that $G \in \mathscr{C}$ by considering the edge labelled 87, the 'start' and adding vertices to the graph in decreasing consecutive order follows (2) in the recursive definition. Then $G$ is Hamiltonian, with Hamiltonian
cycle $(6,5,1,8,2,7,4,3,6)$. However, we can find an induced subgraph of $G$ which remains in the class of $\mathscr{C}$, which is not Hamiltonian.


Figure 2.5: Hamiltonian $G \in \mathscr{C}$

The graph $G[V(G)-4]$ is a well known 1-tough graph which is not Hamiltonian and furthermore, $G[V(G)-4]$ is an induced subgraph of $G$ which is also in $\mathscr{C}$.


Figure 2.6: $G[V(G)-4] \in \mathscr{C}$ corresponding to $G$ in Figure 2.5 which is not Hamiltonian

A well known property of chordal graphs, which will help us distinguish between types of induced subgraphs, is that the vertices of a chordal graph can be labelled with a simplicial elimination ordering. A simplicial elimination ordering is also often called a perfect elimination ordering.

Definition 2.1.2. A labelling $\left(v_{1}, \ldots, v_{n}\right)$ is a simplicial elimination ordering of a graph $G$, if $v_{i}$ is a simplicial vertex in $G_{i-1}$ where $G=G_{0}$ and $G_{i}=G_{i-1}-v_{i}$.

Definition 2.1.3. Let $H$ be an induced subgraph of $G$. $H$ will be called an $\boldsymbol{S E O}$ induced subgraph if there exists a simplicial elimination ordering, $\left(v_{1}, \ldots, v_{n}\right)$, of $G$, such that $H=G\left[\left\{v_{i}, v_{i+1}, \ldots, v_{n}\right\}\right]$.

Lemma 2.1.4. Let $G$ be a chordal graph with simplicial vertex $v$.

1. If $G-v$ does not have a Hamiltonian path, then $G$ does not have a Hamiltonian path.
2. If $G-v$ does not have a Hamiltonian cycle, then $G$ does not have a Hamiltonian cycle.
3. If $G-v$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Proof. (a) Suppose that $G$ has a Hamiltonian path, $P$. If $v$ is an endpoint of the $P$, then $P-v$ is a Hamiltonian path in $G-v$. Now suppose that $v$ is preceded by $u$ and followed by $w$ on $P$. Since $v$ is simplicial, then $u$ and $w$ must be part of a clique, and hence are adjacent. Thus, replacing $u v, v w$ on $P$ with $u w$ yields a Hamiltonian path in $G-v$. The proofs of (b) and (c) are similar.

Corollary 2.1.5. Let $G$ be a chordal graph. If $G$ contains an SEO-induced subgraph which does not contain a Hamiltonian path, then $G$ does not contain a Hamiltonian path.

Remark 2.1.6. Note that $G[V(G)-4]$ in Figure 2.6 is not an SEO-induced subgraph of $G$ in Figure 2.5 for any simplicial elimination ordering since the vertex labelled 4 is not simplicial in $G$.

Proposition 2.1.7 (Proskurowski, 1980). [22] Given a $k$-tree $Q$ and any $k$-clique $B$ of $Q, Q$ can be constructed from $B$ by the iterative method of Definition 1.2.4.

Remark 2.1.8. Labelling the base subgraph ( $n, \ldots, n-k+1$ ), in Proposition 2.1.7 and successive simplicial vertices in the construction in decreasing consecutive order will yield a simplicial elimination ordering.

Lemma 2.1.9. Let $G$ be a $k$-tree. If $H$ is any induced sub- $k$-tree of $G$, then $H$ is an SEO-induced subgraph of $G$.

Proof. We will induct on $|V(H)|=m \leq|V(G)|=n$. If $m=k$, then $H$ is a $k$ clique. Then from Proposition 2.1.7, the claim is true. Now suppose that for any induced sub- $k$-tree with $m-1$ vertices, that the claim is true. Now, suppose $H$ is an induced sub- $k$-tree of $G$ such that $|V(H)|=m$. Let $w$ be a simplicial vertex in $H$. Then $H-w$ is an induced sub- $k$-tree of $G$ such that $|V(H-w)|=m-1$. By
the induction hypothesis, there exists a simplicial elimination ordering, $\left(v_{1}, \ldots, v_{n}\right)$, of $G$, such that $H-w=G\left[\left\{v_{n-m+2}, v_{n-m+3}, \ldots, v_{n}\right\}\right]$. If $w$ is labelled $v_{n-m+1}$, then $H=G\left[\left\{v_{n-m+1}, v_{n-m+2}, v_{n-m+3}, \ldots, v_{n}\right\}\right]$ is an SEO-induced subgraph with the same labelling. If $w$ is labelled $v_{j} \neq v_{n-m+1}$, then reduce by one all labels from $v_{j+1}$ to $v_{n-m+1}$ and relabel $w$ as $v_{n-m+1}$. Note that $w$ cannot be adjacent to any vertices with labels from $v_{j+1}$ to $v_{n-m+1}$ or $w=v_{j}$ would have degree more than $k$ in $G_{j-1}$, contradicting that $w$ is simplicial. So, the new labelling will still be a simplicial elimination ordering and $H=G\left[\left\{v_{n-m+1}, v_{n-m+2}, v_{n-m+3}, \ldots, v_{n}\right\}\right]$ is an SEO-induced subgraph under the new labelling.

Corollary 2.1.10. Let $H$ be a $k$-tree which does not contain a Hamiltonian path. If $H$ is an induced sub-k-tree of $G$, then $G$ also does not contain a Hamiltonian path.

Corollary 2.1.11. Let $H$ be a $k$-tree, with $x_{1}, x_{2} \in V(G)$, which does not contain an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. If $H$ is an induced sub- $k$-tree of $G$, then $G$ also does not contain an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Corollary 2.1.12. If $H$ is an induced sub-k-tree of $G$, then $G$ can be constructed from $H$ by the iterative method of Definition 1.2.4.

### 2.2 Special Induced Sub-2-trees

In order to describe the sub-2-trees which we will later prove prevent 2-trees from having a Hamiltonian path, we will use graph amalgamation on disjoint graphs to create a connected graph as defined in the following definition.

Definition 2.2.1. Given two disjoint graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$, an amalgamation, $G$, of $G_{1}$ and $G_{2}$, will be constructed by identifying $x \in V_{1}$ and $y \in V_{2}$ such that if $G=(V, E)$,
then $V=\left(V_{1}-x\right) \cup\left(V_{2}-y\right) \cup\{z\}$
and $E=\left\{a b: a, b \neq x, y\right.$ and $\left.a b \in E_{1} \cup E_{2}\right\} \cup\left\{a z: a \neq x, y\right.$ and $a x \in E_{1}$ or $\left.a y \in E_{2}\right\}$. This will be called the the amalgamation of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $z$ will be called the $(x, y)$-amalgamated vertex.

Definition 2.2.2. A diamond graph is a $K_{4}$ with one edge removed. The 2-edge of the diamond will be called the central edge.


Figure 2.7: $D_{0}$, a diamond graph

Definition 2.2.3. Let $D_{0}=D_{0}(\emptyset)$, the 0 -split diamond, be a diamond graph with the vertices on the central edge labelled $c_{0}$ and $c_{1}$, and the other two vertices with labels $t_{0}$ and $b_{0}$.
Given an $s \geq 1$ and $R \subseteq\{1,2, \ldots, s\}$, such that $|R|=r$, the $\boldsymbol{s}$-split diamond with respect to $R$ is denoted $D_{s}(R)$ and is formed from $D_{s-1}(R-s)$ by adding $c_{s+1}$ adjacent to
(a) $t_{s-r} c_{s}$ and adding $b_{r}$ adjacent to $c_{s} c_{s+1}$ if $s \in R$, and
(b) $b_{r} c_{s}$ and adding $t_{s-r}$ adjacent to $c_{s} c_{s+1}$ if $s \notin R$

The vertices $\left\{c_{0}, c_{1}, \ldots ., c_{s+1}\right\}$ will be called central vertices, $c_{0}$ and $c_{s+1}$ will be called exterior central vertices, and the path the central vertices form will be called the central path of the s-split diamond. The vertices $\left\{t_{0}, t_{1}, \ldots, t_{s-r}\right\}$ will be called top vertices and $\left\{b_{0}, b_{1}, \ldots, b_{r}\right\}$ will be called bottom vertices.

Remark 2.2.4. We could create isomorphic graphs using different sets for $R$. For example, if $R^{\prime}=\{s+1-i: i \in R\}$. Then $D_{s}(R)$ is isomorphic to $D_{s}\left(R^{\prime}\right)$.

Remark 2.2.5. The diamond graph is a 1-tough 2-tree, and since $D_{s}(R)$ and is formed from $D_{s-1}(R-s)$ by adding two simplicial vertices to $D_{s-1}(R-s)$, such that there are no $t$-edges for $t \geq 3$, then $D_{s}(R)$ is a 1-tough 2-tree, $\forall s, R$.

We can see an example of this recursion as follows. Consider the 4 -split diamond, $D_{4}(\{1,3,4\})$ below.


Figure 2.8: Example of an $s$-split diamond : $D_{4}(\{1,3,4\})$

From $D_{4}(\{1,3,4\})$ we can create two different 5 -split diamonds, $D_{5}(\{1,3,4\})$, and $D_{5}(\{1,3,4,5\})$, pictured below. In either case, we are adding two simplicial vertices to create an additional diamond which shares an edge with the 4 -split diamond, and whose central edge extends the central path of the 4 -split diamond.


Figure 2.9: Example of an $s$-split diamond : $D_{5}(\{1,3,4\})$ constructed from $D_{4}(\{1,3,4\})$


Figure 2.10: Example of an $s$-split diamond : $D_{5}(\{1,3,4,5\})$ constructed from $D_{4}(\{1,3,4\})$

Definition 2.2.6. Let $D_{s_{1}}^{1}\left(R_{1}\right), \ldots . D_{s_{m}}^{m}\left(R_{m}\right)$ be disjoint $s_{1}, \ldots, s_{m}$-split diamonds respectively with $\left|R_{i}\right|=r_{i}$. Denote the central vertices of $D^{i},\left\{c_{0}^{i}, c_{1}^{i}, . ., c_{s_{i}+1}^{i}\right\}$, the top vertices of $D^{i},\left\{t_{0}^{i}, t_{1}^{i}, . ., t_{s_{i}-r_{i}}^{i}\right\}$, and the bottom vertices of $D^{i},\left\{b_{0}^{i}, b_{1}^{i}, . ., b_{r_{i}}^{i}\right\}$. Then an $\ell$-string of diamonds, for $\ell=s_{1}+s_{2}+\ldots+s_{m}+m$ will be formed as follows:

1. Amalgamate $c_{s_{i}+1}^{i}$ with $c_{0}^{i+1}$, to form $z_{i}$ and call $z_{i}$
the ( $\left.D^{i}, D^{i+1}\right)$-amalgamated vertex.
2. Then add exactly one of the following:
(a) A path between $b_{r_{i}}^{i}$ and $b_{0}^{i+1}$ such that each vertex of the path is also adjacent to $z_{i}$, or
(b) A path between $t_{s_{i}-r_{i}}^{i}$ and $t_{0}^{i+1}$ such that each vertex of the path is also adjacent to $z_{i}$

An l-string of diamonds will be denoted

$$
D_{s_{1}}^{1} ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2} ;\left(x_{2}, \ell_{2}\right) ; \ldots ; D_{s_{m-1}}^{m-1} ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}
$$

where $x_{i}=t$ if there is a path between $t_{s_{i}-r_{i}}^{i}$ and $t_{0}^{i+1}, x_{i}=b$ if there is a path between $b_{r_{i}}^{i}$ and $b_{0}^{i+1}$, and $\ell_{i}$ is the length of that path.
The path formed from the central paths of the $s_{1}, \ldots, s_{m}$-split diamonds and the amalgamated vertices, $\left(c_{0}^{1}, \ldots, c_{s_{1}}^{1}, z_{1}, \ldots, z_{m-1}, c_{1}^{m}, \ldots, c_{s_{m}+1}^{m}\right)$ will be called the central path of the $\ell$-string of diamonds.

Remark 2.2.7. The paths in (2a) and (2b) above are added so that an $\ell$-string of diamonds is a 2-tree.


Figure 2.11: Example of an 8 -string of diamonds, $D_{0} ;(t, 1) ; D_{5}(\{1,3,4\}) ;(t, 3) ; D_{0}$, with amalgamated vertices shown as larger vertices

In this dissertation, we will be introducing families of forbidden induced sub-2trees, with and without fixed endpoints, such that if $G$ is a 2-tree, which contains
an induced sub-2-tree in the family, then $G$ will not have a Hamiltonian path. Note, however, that these will be families of forbidden induced sub-2-trees, as was the case in our example of a 1-path-tough graph which does not contain a Hamiltonian path (See Figure 1.3). In that example, our base graph was $P_{n}^{2}$, where we could create an infinite family of such graphs just by increasing $n$. Similarly in our lists, we will be able create infinite families of forbidden sub-2-trees, by increasing the number of vertices in a graph and the distance between an endpoint of a Hamiltonian path and a forbidden substructure. So, in order to create a primitive list of forbidden sub-2-trees, for which a 2 -tree not having a Hamiltonian path must contain, then we will perform the following graph amalgamation.

Definition 2.2.8. Suppose $G$ is a 2-tree with $a b \in E(G)$. Let $H$ be a 2-path with simplicial vertices $x, y$, such that $x$ is adjacent to uv. Amalgamate $G$ and $H-x$ by performing a vertex amalgamation of $a$ and $u$ and then $b$ and $v$ as in Definition 2.2.1. This process will be called an amalgamation of a $\boldsymbol{y}$-2-path with $\boldsymbol{a b}$.

If, for a graph $G$, we have amalgamated a $y$-2-path with $a b \in E(G)$, it will be represented with a single curve between $y$ and $a$ and $y$ and $b$, where $G[\{a, b, y\}]$ is some 2-path. An example of an amalgamation of a $y$-2-path with $t_{0}^{2} c_{1}^{2}$ in $D_{0} ;(t, 1) ; D_{0}$ is below.


Figure 2.12: An amalgamation of a $y$-2-path with $t_{0}^{2} c_{1}^{2}$ in $D_{0} ;(t, 1) ; D_{0}$

Then Figures 2.13 and 2.14 below are both included in Figure 2.12. Also, if $y$ is an endpoint fixed for a Hamiltonian path, then neither graphs in Figures 2.13 and 2.14 are induced subgraphs of one another.


Figure 2.13: Specific example of a graph represented by Figure 2.12


Figure 2.14: Specific example of a graph represented by Figure 2.12

### 2.3 A New Toughness Definition

We define a new toughness property which will be helpful in describing when there will not be a Hamiltonian path between two vertices. This definition will also relate to the $\ell$-strings of diamonds defined in the previous section.

Definition 2.3.1. A tough path from $\boldsymbol{v}_{\boldsymbol{1}}$ to $\boldsymbol{v}_{\boldsymbol{n}}$ is a path $P=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ such that for all $i, j \in 1, \ldots, n$, with $i<j$ and $S_{v_{i}, v_{j}}=\left\{v_{i}, v_{i+1}, \ldots v_{j-1}, v_{j}\right\},\left|S_{v_{i}, v_{j}}\right|=$ $c\left(G-S_{v_{i}, v_{j}}\right)$.

Remark 2.3.2. If $G$ is a 1-tough graph then $S_{v_{i}, v_{j}}$ is a tough set.


Figure 2.15: Example of a tough path: $\left(c_{0}, c_{1}, c_{2}, c_{3}, c_{4}, c_{5}, c_{6}\right)$

The following Lemma and proof are similar to a Lemma and proof for a toughness inequality in [7].

Lemma 2.3.3. If $v$ is a simplicial vertex in $H$ and $G=H-v$, then $c(H-S) \geq$ $c(G-S)$.

Proof. If $c(H-S)<c(G-S)$, then $v$ is adjacent to at least two components of $G-S$. But since $v$ is simplicial, then $N(v)$ is a clique, and hence all neighbors not in $S$ lie in the same component, a contradiction.

Corollary 2.3.4. Let $H$ be a $k$-tree and $S \subset V(H)$ such that $c(H-S)=t$. If $H$ is an induced sub-k-tree of a $k$-tree $G$, then $c(G-S) \geq t$.

Proof. From Lemma 2.1.9, an induced sub- $k$-tree is an SEO-induced subgraph, so $H$ can be formed by iteratively removing simplicial vertices.

Lemma 2.3.5. Let $G$ be a 1-tough $k$-tree and $H$ be an induced sub-k-tree of $G$. If $P$ is a tough path in $H$, then $P$ is a tough path in $G$.

Proof. Since $P=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ is a tough path, then for all $i, j \in 1, \ldots, n$, with $i<j,\left|S_{v_{i}, v_{j}}\right|=c\left(H-S_{v_{i}, v_{j}}\right)$. From Corollary 2.3.4, $c\left(G-S_{v_{i}, v_{j}}\right) \geq\left|S_{v_{i}, v_{j}}\right|$. But if $c\left(G-S_{v_{i}, v_{j}}\right)>\left|S_{v_{i}, v_{j}}\right|$, then $G$ is not 1-tough, so we must have $\left|S_{v_{i}, v_{j}}\right|=c\left(G-S_{v_{i}, v_{j}}\right)$, and hence $P$ is a tough path in $G$.

Lemma 2.3.6. Let $G$ be a 1-tough 2-tree. If there exists a tough set, $U$, such that $x_{1}, x_{2} \in U$, then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Proof. Since $U$ is a tough set, then $|U|=c(G-U)$. Then $c(G-U)-\left|U-\left\{x_{1}, x_{2}\right\}\right|=$ $c(G-U)-(|U|-2)=c(G-U)-|U|+2=2$. Hence, from Lemma 1.2.16. $P P\left(G ;\left\{x_{1}, x_{2}\right\}\right) \geq 2$, and there cannot be a Hamiltonian path between $x_{1}$ and $x_{2}$.

Lemma 2.3.7. The central path of an s-split diamond is a tough path.
Proof. We will proceed by induction on $s$. If $s=0$, then we have the diamond graph $D_{0}$ with central path, $\left(c_{0}, c_{1}\right)$, and $c\left(D_{0}-c_{0}\right)=c\left(D_{0}-c_{1}\right)=1$. Furthermore, since $t_{0}$ and $b_{0}$ are both simplicial vertices and adjacent to $c_{0} c_{1}$, then $c\left(D_{0}-\left\{c_{0}, c_{1}\right\}=2\right.$, and so the central path is a tough path. Suppose that the central path of an $(s-1)$-split diamond is a tough path. Now, consider $D_{s}(R)$. If $s \in R$ then $b_{r}$ is simplicial, and $D_{s}(R)-\left\{b_{r}, c_{s+1}\right\}=D_{s-1}(R-s)$ is an $s-1$-split diamond, and hence $\left(c_{0}, c_{1}, \ldots, c_{s}\right)$ is a tough path in $D_{s-1}(R-s)$. From Lemma 2.3.5. $\left(c_{0}, c_{1}, \ldots, c_{s}\right)$ is also a tough path in $D_{s}(R)$. Furthermore, removing $c_{s+1}$ from $D_{s}(R)-\left\{c_{i}, c_{i+1}, . ., c_{s}\right\}$ for $0 \leq i \leq s$ will increase the number of components of the graph by one, as $b_{r}$ and $t_{s-r}$ will be in the same component of $D_{s}(R)-\left\{c_{i}, c_{i+1}, . ., c_{s}\right\}$, as they are both adjacent to $c_{s+1}$, but different components of $D_{s}(R)-\left\{c_{i}, c_{i+1}, . ., c_{s}, c_{s+1}\right\}$, since $b_{r}$ is only adjacent $c_{s} c_{s+1}$. Hence the central path of $D_{s}(R)$ is a tough path. The proof is similar if $s \notin R$.

## Lemma 2.3.8. The central path of an $\ell$-string of diamonds is a tough path.

Proof. We will proceed by induction on the number of amalgamated vertices, $j$. If $j=0$, then the $\ell$-string of diamonds is an $(l+1)$-split diamond, and by Lemma 2.3.7, the claim is true. Now suppose that when there are $j-1$ amalgamated vertices that the central path of an $\ell$-string of diamonds is a tough path. Now, consider an $\ell$ string of diamonds, $G=D_{s_{1}}^{1} ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2} ;\left(x_{2}, \ell_{2}\right) ; \ldots . ;\left(x_{j-1}, \ell_{j-1}\right) ; D_{s_{j}}^{j} ;\left(x_{j}, \ell_{j}\right) ; D_{s_{j+1}}^{j+1}$. By the induction hypothesis, the central path, $P=\left(c_{0}^{1}, \ldots, c_{s_{j}+1}^{j}\right)$, of $D_{s_{1}}^{1} ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2} ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{j-1}, \ell_{j-1}\right) ; D_{s_{j}}^{j}$ is a tough path in $D_{s_{1}}^{1} ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2} ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{j-1}, \ell_{j-1}\right) ; D_{s_{j}}^{j}$. Also, by Lemma 2.3.7, the central path, $P^{\prime}=\left(c_{0}^{j+1}, \ldots, c_{s_{j}+1}^{j+1}\right)$, of $D_{s_{j+1}}^{j+1}$, is a tough path in $D_{s_{j+1}}^{j+1}$. From Lemma 2.3.5. $\left(c_{0}^{1}, \ldots, c_{s_{j}+1}^{j}=z_{j}\right)$ and $\left(z_{j}=c_{0}^{j+1}, \ldots, c_{s_{j}+1}^{j+1}\right)$ are also tough paths in $G$. Let $S_{w, c_{s_{j}}^{j}}$ be any consecutive subset of vertices from the tough path, $P$ which ends with $c_{s_{j}}^{j}$. In $G-S_{w, c_{s_{j}}^{j}}, D_{s_{j+1}}^{j+1}$ is in one component, with some additional vertices. Hence, the combined path, $\left(P, P^{\prime}-z_{j}\right)$, will be a tough path in $G$.

Lemma 2.3.9. Let $P=\left(v_{1}, v_{2}, \ldots, v_{n-1}, v_{n}\right)$ be a tough path in a 1-tough 2-tree, $G$. If $v_{i-1}$ is adjacent to $v_{i+1}$ in $G$, for some $2 \leq i \leq n-1$, then replacing $\left(v_{i-1}, v_{i}, v_{i+1}\right)$ with $\left(v_{i-1}, v_{i+1}\right)$ forms a tough path $P^{\prime}$.

Proof. Let $C_{v_{i+1}}$ be the component of $G-\left\{v_{i-1}, v_{i}\right\}$ which contains $v_{i+1}$. Since $v_{i-1}$ is adjacent to $v_{i+1}$, then $v_{i-1} \in N\left(v_{i} v_{i+1}\right)$. If $v_{i-1} v_{i+1}$ is a 1-edge, then $v_{i-1}$ is an isolated
vertex in $G\left[C_{v_{i+1}} \cup v_{i-1} v_{i}\right]-v_{i+1}$, and then $c\left(G-\left\{v_{i-1}, v_{i}, v_{i+1}\right\}\right)=2$, contradicting that $P$ is a tough path. So, $v_{i-1} v_{i+1}$ must be a 2-edge, and $c\left(G-\left\{v_{i-1}, v_{i+1}\right\}\right)=2$. Furthermore, since $c\left(G-S_{v_{i+1}, v_{j}}\right)=\left|S_{v_{i+1}, v_{j}}\right|$ and $c\left(G-S_{v_{k}, v_{i-1}}\right)=\left|S_{v_{k}, v_{i-1}}\right|$, then $c\left(G-\left(S_{v_{k}, v_{j}}-v_{i}\right)\right)=\left|S_{v_{k}, v_{j}}-v_{i}\right|$.

Definition 2.3.10. A short tough path is a tough path $P=\left(v_{1}, v_{2}, \ldots, v_{n-1}, v_{n}\right)$, in a 2-tree, $G$, for which $v_{i-1} v_{i+1} \notin E(G)$ for any $2 \leq i \leq n-1$.

Lemma 2.3.11. If $P$ is a short tough path in a 1-tough 2-tree, $G$, then $P$ is the central path of an induced $\ell$-string of diamonds in $G$.

Proof. We will proceed by induction on the length, $\mathscr{L}$, of the tough path. If $\mathscr{L}=1$, then $P=\left(v_{1}, v_{2}\right)$, and since $G$ is a 1 -tough 2 -tree, then $v_{1} v_{2}$ is a 2 -edge in $G$. Hence, $G\left[N\left[v_{1}, v_{2}\right]\right]$ is a diamond graph, and a 1-string of diamonds, with central path $\left(v_{1}, v_{2}\right)$. Suppose that the claim is true for tough paths of length $\mathscr{L}-1$. Now, suppose $G^{\prime}$ is a 1 -tough 2 -tree with short tough path $P^{\prime}=\left(v_{1}, v_{2}, \ldots, v_{\mathscr{L}}, v_{\mathscr{L}+1}\right)$ of length $\mathscr{L}$. $P^{\prime \prime}=\left(v_{1}, v_{2}, \ldots, v_{\mathscr{L}}\right)$ is a short tough path of length of $\mathscr{L}-1$, and by the induction hypothesis, $P^{\prime \prime}$ is the central path of an induced $\ell$-string of diamonds in $G^{\prime}$. Let $H=D_{s_{1}}^{1} ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2} ;\left(x_{2}, \ell_{2}\right) ; \ldots ; D_{s_{m-1}}^{m-1} ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}$ be the induced $\ell$-string of diamonds, and $t^{\prime}, b^{\prime}$ be the top and bottom vertices, respectively, adjacent to $v_{\mathscr{L}}$ in $D_{s_{m}}^{m}$. Since $H$ is an induced sub-2-tree, then $H$ is an SEO-induced subgraph of $G^{\prime}$, and hence there is a labelling of the vertices of $G^{\prime}$ such that $G^{\prime}$ can be constructed from $H$ as in Definition 1.2 .4 by iteratively adding vertices $\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$ in order of the labelling. Since $P^{\prime}$ is a short tough path, then $v_{\mathscr{L}-1} v_{\mathscr{L}+1} \notin E\left(G^{\prime}\right)$ and so $v_{\mathscr{L}+1} \in\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$. If $v_{\mathscr{L}+1}=y_{i}$ such that $i<j$ for any $y_{j}$ a neighbor of $v_{\mathscr{L}}$, then $v_{\mathscr{L}+1} t^{\prime} \in E\left(G^{\prime}\right)$ or $v_{\mathscr{L}+1} b^{\prime} \in E\left(G^{\prime}\right)$. Furthermore, $v_{\mathscr{L}}, v_{\mathscr{L}+1}$ must be a 2-edge in $G^{\prime}$ and so there is another vertex, $x^{\prime} \in\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$ adjacent to $v_{\mathscr{L}}, v_{\mathscr{L}+1}$, and this forms an $l+1$-string of diamonds. Now suppose $v_{\mathscr{L}+1}=y_{i}$ such that $i>j$ for at least one $y_{j}$ a neighbor of $v_{\mathscr{L}}$. Let $\left\{y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k^{\prime}}}\right\}$ be the vertices that are adjacent to $v_{\mathscr{L}}$ where $j_{i}<i$. Adding the vertices in $\left\{y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k^{\prime}}}\right\}$ followed by $v_{\mathscr{L}+1}$ as in Definition 1.2 .4 forms a path $\left(t^{\prime}, y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k^{\prime}}}, v_{\mathscr{L}+1}\right)$ or $\left(b^{\prime}, y_{j_{1}}, y_{j_{2}}, \ldots, y_{j_{k^{\prime}}}, v_{\mathscr{L}+1}\right)$ where all vertices on the path are adjacent to $v_{\mathscr{L}}$. Furthermore, $v_{\mathscr{L}}, v_{\mathscr{L}+1}$ must be a 2-edge in $G^{\prime}$ and so there is another vertex, $z^{\prime} \in\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$ adjacent to $v_{\mathscr{L}}, v_{\mathscr{L}+1}$, and this forms an $l+1$-string of diamonds which is an induced subgraph of $G^{\prime}$.

Lemma 2.3.12. Let
$G=D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$.

Then there is a tough path from $t_{0}^{1}$ to $t_{0}^{m}$, a tough path from $c_{0}^{1}$ to $t_{0}^{m}$, and a tough path from $t_{0}^{1}$ to $c_{1}^{m}$.

Proof. $G-\left\{c_{0}^{1}, t_{0}^{1}, b_{0}^{1}\right\}$ is an $(l-1)$-string of diamonds, and hence, by Lemma 2.3.8, the central path, $P$, is a $\left(z_{1}, c_{1}^{m}\right)$-tough path. Furthermore, by Lemma 2.3.5, it is a tough path in $G$. Let $S_{z_{1}, w}$ be any consecutive subset of vertices from the tough path which begins with $z_{1}$, the $\left(D^{1}, D^{2}\right)$-amalgamated vertex. Then, $c_{0}^{1}, t_{0}^{1}, b_{0}^{1}$, and $t_{0}^{2}$ are in the same component of $G-S_{z_{1}, w}$, but since $c_{0}^{1}$ is only adjacent to $t_{0}^{1}, b_{0}^{1}$, and $z_{1}$, then removing $t_{0}^{1}$ from $G-S_{z_{1}, w}$ will add a component. Hence $\left(t_{0}^{1}, P\right)$ is a $\left(t_{0}^{1}, c_{1}^{m}\right)$-tough path. Similarly, there is a $\left(c_{0}^{1}, t_{0}^{m}\right)$-tough path which uses the central path from $c_{0}^{1}$ to $z_{m-1}$, the $\left(D^{m-1}, D^{m}\right)$-amalgamated vertex. Hence, there is a $\left(z_{1}, t_{0}^{m}\right)$-tough path, $P^{\prime}$ which uses the central path from $z_{1}$ to $z_{m-1}$. Let $S_{z_{1}, w}^{\prime}$ be any consecutive subset of vertices from the tough path $P^{\prime}$ which begins with $z_{1}$. Then, $c_{0}^{1}, t_{0}^{1}, b_{0}^{1}$, and $t_{0}^{2}$ are in the same component of $G-S_{z_{1}, w}^{\prime}$, but since $c_{0}^{1}$ is only adjacent to $t_{0}^{1}, b_{0}^{1}$, and $z_{1}$, then removing $t_{0}^{1}$ from $G-S_{z_{1}, w}^{\prime}$ will add a component. Hence $\left(t_{0}^{1}, P^{\prime}\right)$ is a $\left(t_{0}^{1}, t_{0}^{m}\right)$-tough path.


Figure 2.16: A general example of
$G=D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$ in Lemma 2.3.12
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 2.17: Specific example of $G$ in Lemma 2.3.12. $D_{0} ;(t, 1) ; D_{5}(\{1,3,4\}) ;(t, 1) ; D_{0}$

Corollary 2.3.13. Let
$G=D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$. Then there is a tough path from $t_{0}^{1}$ to $b_{0}^{m}$, a tough path from $c_{0}^{1}$ to $b_{0}^{m}$, and a tough path from $t_{0}^{1}$ to $c_{1}^{m}$.


Figure 2.18: A general example of
$G=D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots . ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$ in Corollary 2.3.13
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

Corollary 2.3.14. Let
$G=D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$. Then $G$ does not have a $\left(t_{0}^{1}, t_{0}^{m}\right),\left(c_{0}^{1}, t_{0}^{m}\right)$, or $\left(t_{0}^{1}, c_{1}^{m}\right)$-Hamiltonian path.

Corollary 2.3.15. Let
$G=D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$. Then $G$ does not have a $\left(t_{0}^{1}, b_{0}^{m}\right),\left(c_{0}^{1}, b_{0}^{m}\right)$, or $\left(t_{0}^{1}, c_{1}^{m}\right)$-Hamiltonian path.

In addition to tough paths, some 2-trees will not have a Hamiltonian path because there exists a $t$-edge, $a b, t \geq 2$, and a component $C$ of $G-\{a, b\}$, such that $G[C \cup\{a, b\}]$ does not have a Hamiltonian path. The following lemmas describe these cases.

Lemma 2.3.16. Let $G$ be a 2-tree, and $x_{1}, x_{2} \in V(G)$. If there exists $a b \in E(G)$ such that:

1. $x_{1}$ and $x_{2}$ lie in different components, $C_{x_{1}}, C_{x_{2}}$, respectively, of $G-\{a, b\}$ and
2. In $G\left[V\left(C_{x_{1}}\right) \cup\{a, b\}\right]$ there is no $\left(x_{1}, a\right)$-Hamiltonian path and no $\left(x_{1}, b\right)$-Hamiltonian path,
then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.


Figure 2.19: Graph $G$ corresponding to Lemma 2.3.16 where the dotted section of the graph represents any 2-tree to preserve generality

Proof. Suppose that $G$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, $P$, but in $G\left[V\left(C_{x_{1}}\right) \cup\{a, b\}\right]$ there is no $\left(x_{1}, a\right)$-Hamiltonian path and no $\left(x_{1}, b\right)$-Hamiltonian path. Then, $P$ must alternate between vertices from $V\left(C_{x_{1}}\right)$ and $V\left(C_{x_{2}}\right)$ using $\{a, b\}$, beginning with $x_{1}$ and ending with $x_{2}$. However, $P$ cannot switch from vertices in $V\left(C_{x_{1}}\right)$ to $V\left(C_{x_{2}}\right)$ and then back to $V\left(C_{x_{1}}\right)$, as then $a$ and $b$ would be used in the path already, and there would be no path back to $C_{x_{2}}$. So either:
(a) There is an $x_{1}$-Hamiltonian path, $P_{1}$, in $C_{x_{1}}$ and an $x_{2}$-Hamiltonian path, $P_{2}$, in $C_{x_{2}}$, and $P=\left(P_{1}, a, b, P_{2}\right)$ or $P=\left(P_{1}, b, a, P_{2}\right)$, or
(b) There is an $x_{1}$-Hamiltonian path, $P_{1}$, in $C_{x_{1}}$ and two paths, $P_{21}$ and $P_{22}$ in $C_{x_{2}}$, and $P=\left(P_{1}, a, P_{21}, b, P_{22}\right)$ or $P=\left(P_{1}, b, P_{21}, a, P_{22}\right)$, or
(c) There are two paths, $P_{11}$ and $P_{12}$ in $C_{x_{1}}$, and an $x_{2}$ - Hamiltonian path, $P_{2}$, in $C_{x_{2}}$, and $P=\left(P_{11}, a, P_{12}, b, P_{2}\right)$ or $P=\left(P_{11}, b, P_{12}, a, P_{2}\right)$.

In (a) and (c), P begins with an $\left(x_{1}, a\right)$ or $\left(x_{1}, b\right)$-Hamiltonian path in $G\left[V\left(C_{x_{1}}\right) \cup\right.$ $\{a, b\}]$, a contradiction. In (b), there is an $x_{1}$-Hamiltonian path, $P_{1}$ in $C_{x_{1}}$ which connects to either $a$ or $b$. Since $a$ is adjacent to $b$, then $\left(P_{1}, a, b\right)$ and ( $P_{1}, b, a$ ) are ( $x_{1}, b$ ) and $\left(x_{1}, a\right)$-Hamiltonian paths, respectively, in $G\left[V\left(C_{x_{1}}\right) \cup\{a, b\}\right]$, a contradiction.

Corollary 2.3.17. Let $G$ be a 2-tree, and $x_{1}, x_{2} \in V(G)$. If there exists $a b \in E(G)$ such that:

1. $x_{1}$ and $x_{2}$ lie in different components, $C_{x_{1}}, C_{x_{2}}$, respectively, of $G-\{a, b\}$, and
2. In $G\left[V\left(C_{x_{1}}\right) \cup\{a, b\}\right]$ there is a tough path from $x_{1}$ to $a$ and a tough path from $x_{1}$ to $b$,
then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Lemma 2.3.18. Let $G$ be a 2-tree, and $x_{1}, x_{2} \in V(G)$. If there exist ab, $c d \in E(G)$ such that:
3. $x_{1}$ and $x_{2}$ lie in different components of $G-\{a, b\}$ and $G-\{c, d\}$, and
4. In $G-\{a, b, c, d\}, x_{1}, x_{2}$ lie in $C_{x_{1}}, C_{x_{2}}$, respectively, such that in $G[V(G)-$ $\left.V\left(C_{x_{1}}\right)-V\left(C_{x_{2}}\right)\right]$ there are no $(a, c),(a, d),(b, c)$, or $(b, d)$-Hamiltonian paths, then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.


Figure 2.20: Graph $G$ corresponding to Lemma 2.3.18, where the dotted section of the graph represents any 2-tree to preserve generality

Proof. From [24, $G\left[V(G)-V\left(C_{x_{1}}\right)\right]$ is a 2-tree. By Lemma 2.3.16, there is no Hamiltonian path between $a$ and $x_{2}$ in $G\left[V(G)-V\left(C_{x_{1}}\right)\right]$ and there is no Hamiltonian path between $b$ and $x_{2}$ in $G\left[V(G)-V\left(C_{x_{1}}\right)\right]$. Hence, Lemma 2.3.16, there is no Hamiltonian path between $x_{1}$ and $x_{2}$.

Corollary 2.3.19. Let $G$ be a 2-tree, and $x_{1}, x_{2} \in V(G)$. If there exist ab, cd $\in E(G)$ such that:

1. $x_{1}$ and $x_{2}$ lie in different components of $G-\{a, b\}$ and $G-\{c, d\}$, and
2. In $G-\{a, b, c, d\}, x_{1}, x_{2}$ lie in $C_{x_{1}}, C_{x_{2}}$, respectively, such that in $G[V(G)-$ $\left.V\left(C_{x_{1}}\right)-V\left(C_{x_{2}}\right)\right]$ there are tough paths from a to $c$, a to $d$, $b$ to $c$, and $b$ to $d$, then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

## Chapter 3

## 2HP

Since the Hamiltonian path problem on 2-trees can be reduced to the 2HP problem on 2-trees, we will first prove results for 2HP on 2-trees to later extend to the Hamiltonian path problem on 2-trees. Since we know that 1-tough 2-trees contain a Hamiltonian path, we begin our investigation of 2HP with 1-tough 2-trees in Section 3.1. We will begin this section by defining a family, $\mathscr{F}^{1}$, of 1-tough 2 -trees, with specified vertices, $x_{1}$ and $x_{2}$, which we will later prove contain no ( $x_{1}, x_{2}$ )-Hamiltonian path. In Theorem 3.1.24, we will also prove that a 1-tough 2 -tree which does not contain, as an induced sub-2-tree, one of the graphs, with specified vertices, $x_{1}$ and $x_{2}$, in $\mathscr{F}^{1}$, will have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

We will then extend the results from Section 3.1 to the 2HP problem on 2-trees with scattering number at most one, in Section 3.2. We will begin this section by defining a family, $\mathscr{F}^{2}$, of graphs, with specified vertices, $x_{1}$ and $x_{2}$, which contains $\mathscr{F}^{1}$, and for which we will later prove contain no $\left(x_{1}, x_{2}\right)$-Hamiltonian path. In Theorem 3.2.10, we will also prove that a 2 -tree with scattering number at most one, which does not contain, as an induced sub-2-tree, one of the graphs, with specified vertices, $x_{1}$ and $x_{2}$, in $\mathscr{F}^{2}$, will have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

### 3.12 HP on 1 -tough 2 -trees

Definition 3.1.1. Define $\mathscr{F}^{1}=\left\{F_{a}^{1}, F_{b}^{1}, F_{c}^{1}, F_{d}^{1}, F_{e}^{1}, F_{f}^{1}\right\}$ where:
(a) $F_{a}^{1}$ is constructed from $D_{0}$ by:
(i) Adding a simplicial vertex adjacent to $c_{0} t_{0}$, and
(ii) Amalgamating an $x_{2}$-2-path with $t_{0} c_{1}$.


Figure 3.1: An example of $F_{a}^{1}$
(b) $F_{b}^{1}$ is an $\ell$-string of diamonds,
$D_{s_{1}}^{1} ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2} ;\left(x_{2}, \ell_{2}\right) ; \ldots ; D_{s_{m-1}}^{m-1} ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}$, with $x_{1}=c_{0}^{1}$ and $x_{2}=c_{s_{m}+1}^{m}$.


Figure 3.2: An example of $F_{b}^{1}: D_{5}(\{1,3,4,5\})$ with $x_{1}=c_{0}^{1}$ and $x_{2}=c_{6}^{1}$
(c) $F_{c}^{1}$ is constructed from
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 2$, where $x_{1}=c_{0}^{1}$, by amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$.


Figure 3.3: A general example of $F_{c}^{1}$ :
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$
$m \geq 2$, with an amalgamated $x_{2}$-2-path and such that $x_{1}=c_{0}^{1}$
where $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$, with $x_{1}=c_{0}^{1}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 3.4: An example of $F_{c}^{1}: D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with an amalgamated $x_{2}$-2-path and such that $x_{1}=c_{0}^{1}$
(d) $F_{d}^{1}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 3$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$, and
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$.

OR
$F_{d}^{1}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$,
$m \geq 3$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$, and
(ii) Amalgamating an $x_{2}$-2-path with $b_{0}^{m} c_{1}^{m}$.


Figure 3.5: A general example of $F_{d}^{1}$ :
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$ $m \geq 3$, with amalgamated $x_{1}$ and $x_{2}$-2-paths where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 3.6: An example of $F_{d}^{1}: D_{0} ;(t, 1) ; D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with amalgamated $x_{1}$ and $x_{2}-2$-paths


Figure 3.7: A general example of $F_{d}^{1}$ :

$$
D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}
$$

$m \geq 3$, with amalgamated $x_{1}$ and $x_{2}$-2-paths
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 3.8: An example of $F_{d}^{1}: D_{0} ;(t, 1) ; D_{5}(\{1,3,4,5\}) ;(b, 1) ; D_{0}$ with amalgamated $x_{1}$ and $x_{2}$-2-paths
(e) $F_{e}^{1}$ is constructed from $G=D_{0}^{1} ;(t, 1) ; D_{0}^{2}$ by amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$, amalgamating an $x_{2}$-2-path with $t_{0}^{2} c_{1}^{2}$, and by adding a simplicial vertex adjacent to $t_{0}^{1}, t_{0}^{2}$.


Figure 3.9: Example of $F_{e}^{1}, D_{0}^{1} ;(t, 1) ; D_{0}^{2}$ with amalgamated $x_{1}$ and $x_{2}$-2-paths and a simplicial vertex adjacent to $t_{0}^{1}, t_{0}^{2}$
(f) $F_{f}^{1}$ is constructed from $G=D_{0}^{1} ;(t, \ell) ; D_{0}^{2}$, for $l \geq 2$ by amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$ and amalgamating an $x_{2}$-2-path with $t_{0}^{2} c_{1}^{2}$.


Figure 3.10: An example of $F_{f}^{1}, D_{0}^{1} ;(t, 2) ; D_{0}^{2}$ with amalgamated $x_{1}$ and $x_{2}$-2-paths

Lemma 3.1.2. The graph $F_{a}^{1}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. The paths $\left(x_{1}=c_{0}, t_{0}\right)$ and $\left(x_{1}=c_{0}, c_{1}\right)$ are tough paths. Furthermore, $x_{1}$ and $x_{2}$ are in different components of $F_{a}^{1}-\left\{t_{0}, c_{1}\right\}$, and hence, by Lemma 2.3.16, there is no ( $x_{1}, x_{2}$ )-Hamiltonian path.

Lemma 3.1.3. The graph $F_{b}^{1}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. From Lemma 2.3.8, there is a tough path from $x_{1}=c_{0}^{1}$ to $x_{2}=c_{s_{m}+1}^{m}$. Hence, from Lemma 2.3.6, there is no $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Lemma 3.1.4. The graph $F_{c}^{1}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Proof. From the proof of Lemma 2.3.12, there is a tough path from $c_{0}^{1}$ to $t_{0}^{m}$. From Lemma 2.3.8, there is a tough path from $c_{0}^{1}$ to $c_{1}^{m}$. Furthermore, $c_{0}^{1}$ and $x_{2}$ are in different components of $F_{c}^{1}-\left\{t_{0}^{m}, c_{1}^{m}\right\}$ and hence, by Lemma 2.3.16, there is no ( $x_{1}=c_{0}^{1}, x_{2}$ )-Hamiltonian path.

Lemma 3.1.5. The graph $F_{d}^{1}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. If $F_{d}^{1}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{1}}^{2}\left(R_{1}\right) ;\left(x_{1}, \ell_{2}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-2}\right) ; D_{s_{m-2}}^{m-1}\left(R_{m-2}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$,
then from Lemma 2.3.12, there are tough paths from $t_{0}^{1}$ to $t_{0}^{m}$, from $t_{0}^{1}$ to $c_{1}^{m}$, and from $t_{0}^{m}$ to $c_{0}^{1}$. Likewise, if $F_{d}^{1}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{1}}^{2}\left(R_{1}\right) ;\left(x_{1}, \ell_{2}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-2}\right) ; D_{s_{m-2}}^{m-1}\left(R_{m-2}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$,
then from Corollary 2.3 .13 there are tough paths from $t_{0}^{1}$ to $b_{0}^{m}$, from $t_{0}^{1}$ to $c_{1}^{m}$, and from $c_{0}^{1}$ to $b_{0}^{m}$. From Lemma 2.3.8, there is a tough path from $c_{0}^{1}$ to $c_{1}^{m}$. Furthermore, if $F_{d}^{1}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{1}}^{2}\left(R_{1}\right) ;\left(x_{1}, \ell_{2}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-2}\right) ; D_{s_{m-2}}^{m-1}\left(R_{m-2}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$, then $x_{1}$ and $x_{2}$ are in different components of $F_{d}^{1}-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and $F_{d}^{1}-\left\{t_{0}^{m}, c_{1}^{m}\right\}$, and hence, by Lemma 2.3.18, there is no $\left(x_{1}, x_{2}\right)$-Hamiltonian path. Likewise, if $F_{d}^{1}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{1}}^{2}\left(R_{1}\right) ;\left(x_{1}, \ell_{2}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-2}\right) ; D_{s_{m-2}}^{m-1}\left(R_{m-2}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$, then $x_{1}$ and $x_{2}$ are in different components of $F_{d}^{1}-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and $F_{d}^{1}-\left\{b_{0}^{m}, c_{1}^{m}\right\}$, and hence, by Lemma 2.3.18, there is no $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Lemma 3.1.6. The graph $F_{e}^{1}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. The paths $\left(t_{0}^{1}, z_{1}, t_{0}^{2}\right),\left(c_{0}^{1}, z_{1}, c_{1}^{2}\right),\left(t_{0}^{1}, z_{1}, c_{1}^{2}\right)$, and $\left(c_{0}^{1}, z_{1}, t_{0}^{2}\right)$ are tough paths. Furthermore, $x_{1}$ and $x_{2}$ are in different components of $F_{e}^{1}-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and $F_{e}^{1}-\left\{t_{0}^{2}, c_{1}^{2}\right\}$, and hence, by Lemma 2.3.18, there is no $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Lemma 3.1.7. The graph $F_{f}^{1}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. The paths $\left(t_{0}^{1}, z_{1}, t_{0}^{2}\right),\left(c_{0}^{1}, z_{1}, c_{1}^{2}\right),\left(t_{0}^{1}, z_{1}, c_{1}^{2}\right)$, and $\left(c_{0}^{1}, z_{1}, t_{0}^{2}\right)$ are tough paths. Furthermore, $x_{1}$ and $x_{2}$ are in different components of $F_{f}^{1}-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and $F_{f}^{1}-\left\{t_{0}^{2}, c_{1}^{2}\right\}$, and hence, by Lemma 2.3.18, there is no $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

### 3.1.1 Paths in $\ell$-strings of diamonds

Definition 3.1.8. A forced edge, $e=u v$, is an edge that must be used in the Hamiltonian Path (if one exists). Incident forced edges form a forced path.

Since $G$ is a 2-tree, simplicial vertices have degree 2, and hence lie on a forced path if they are not endpoints of the path. To simplify the graphs we are considering, we use a reduction process on our graphs which contracts sections of the graph where there is a forced path. This process is similar to that used when reducing a seriesparallel network of resistors. These series-parallel networks are partial 2-trees and this reduction method has been used to find the resistance in the network. It has also been used to find the probability that a communication network will work. In both cases the edges are labelled with weights: resistance, and probabilities, respectively [1].

Notation 3.1.9. Let $(G, u, v)$ denote a 2-tree, $G$, with $u, v \in V(G)$, and let $S_{1}^{*}(G, u, v)$ denote the set of simplicial vertices in $G-\{u, v\}$.

Definition 3.1.10. Given a 1-tough 2-tree, $G$, such that $G \neq K_{3}$, then the reduced graph of $(G, u, v)$, is formed using the following algorithm:

1. Let $w \in S_{1}^{*}(G, u, v)$, and $x, y$ the neighbors of $w$. If $G-w \neq K_{2}$, remove $w$ and turn the edge xy into a forced edge.
2. Repeat (1) for all $w \in S_{1}^{*}(G, u, v)$. Define the resulting graph to be $G_{1}^{*}$.
3. For $i \geq 2$, let $S_{i}^{*}=S_{1}^{*}\left(G_{i-1}^{*}, u, v\right)$ where $G_{i-1}^{*} \neq K_{3}$ is the graph formed by repeating (1) for $G=G_{i-1}^{*}$ and for all $w \in S_{1}^{*}\left(G_{i-1}^{*}, u, v\right)$.

Repeat (3) for all $i=2,3, \ldots, j$ for $j$ such that $S_{j}^{*}=\emptyset$ or $G_{j}^{*}=K_{3}$. This is the reduced graph of $(G, u, v)$.
For $F$ the set of forced edges, let $(H, u, v, F)$ denote the reduced graph of $(G, u, v)$, for G a 1-tough 2-tree.

Since simplicial vertices in 2-trees are not adjacent [7], when we remove the vertices in each $S_{i}^{*}$, regardless of order, we will end up with the same graph, unless removing all vertices in $S_{i}^{*}$ results in $K_{2}$. In this case, if we change the order of removal of vertices, we will end up with different, but isomorphic graphs.

Furthermore, the reduction process removes all simplicial vertices other than the two given endpoints, and hence the resulting graph is a 2-path.

In order to describe a Hamiltonian path in a 2-path graph, we will use a specific simpliical elimination ordering to create a labelling for our vertices.

## Definition 3.1.11. Algorithm for labelling a 2-path:

Let $H$ be a 2-path with simplicial vertices, $\{u, v\}$, with $|V(H)|=n$.

1. Label $\{u, v\}$ with 1 and $n$. Remove vertex labelled 1.
2. Label the new simplicial vertex (not the one labelled $n$ ), consecutively and remove.
3. Repeat (2) until all that remains is a $K_{3}$.
4. Label the last $K_{3}$ by starting with the original 2-path (labels intact) and removing the vertex labelled $n$. Label the new simplicial vertex (which is not labelled) with $n-1$. Label the remaining vertex $n-2$.

Remark 3.1.12. Since simplicial vertices in $G-S_{1}(G)$ are adjacent to vertices in $S_{1}(G)$ [7], vertices that are consecutively labelled will be adjacent. Hence, following the ordering in the labelling algorithm consecutively will yield a Hamiltonian path.

Definition 3.1.13. [22] $A \boldsymbol{k}$-caterpillar, $P$, is a $k$-tree in which deletions of all simplicial vertices results in a $k$-path.

Definition 3.1.14. Let $\left(H, x_{1}, x_{2}, F\right)$ be the reduced graph of $\left(G, x_{1}, x_{2}\right)$. The caterpillar representation, $\left(H^{\prime}, x_{1}, x_{2}\right)$, of a graph $\left(G, x_{1}, x_{2}\right)$ is created by adding $|F|$ simplicial vertices to $\left(H, x_{1}, x_{2}, F\right)$, making each vertex adjacent to exactly one forced edge, and changing all forced edges back to regular edges.

Remark 3.1.15. $\left(H^{\prime}, x_{1}, x_{2}\right)$ could also have been constructed by removing one less simplicial vertex from each of the forced edges in the reduced graph algorithm, though it would be more difficult to define. Furthermore, since $H$ is a 2-tree, then $H^{\prime}$ is also a 2-tree and since the forced edges were changed back to regular edges, $\left(H^{\prime}, x_{1}, x_{2}\right)$ is an induced sub-2-tree of ( $G, x_{1}, x_{2}$ ).

Remark 3.1.16. Since $x_{1}$ and $x_{2}$ are simplicial in $\left(H, x_{1}, x_{2}, F\right)$, then they are incident to at most two forced edges, and hence in $\left(H^{\prime}, x_{1}, x_{2}\right)$, they have degree at most four.

Lemma 3.1.17. Let $G$ be a 1-tough 2-tree with $x_{1}, x_{2} \in V(G)$. Let $\left(H, x_{1}, x_{2}, F\right)$ be the reduced graph of $\left(G, x_{1}, x_{2}\right)$ and $\left(H^{\prime}, x_{1}, x_{2}\right)$ the caterpillar representation of $\left(G, x_{1}, x_{2}\right)$. Then the following are equivalent:

1. $G$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path,
2. $\left(H^{\prime}, x_{1}, x_{2}\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, and
3. $\left(H, x_{1}, x_{2}, F\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path which uses all of the edges in $F$.

Proof. (1) $\Longrightarrow(2)$
Suppose $\left(H^{\prime}, x_{1}, x_{2}\right)$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. Since $\left(H^{\prime}, x_{1}, x_{2}\right)$ is an induced sub-2-tree of $\left(G, x_{1}, x_{2}\right)$, then by Corollary 2.1.10, $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

$$
(2) \Longrightarrow(3)
$$

Suppose $\left(H^{\prime}, x_{1}, x_{2}\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, $P$. Let $v \neq x_{1}, x_{2}$ be a simplicial vertex with neighbors $u$ and $w$. Then $P=\left(x_{1}, \ldots, u, v, w, \ldots, x_{2}\right)$ or $P=$ $\left(x_{1}, \ldots, w, v, u, \ldots, x_{2}\right)$. Furthermore, because $H^{\prime}$ is a 2-tree, then $u w \in E\left(H^{\prime}\right)$, and from the reduction algorithm $u w \in F$. Replacing $(u, v, w)$ or $(w, v, u)$ by $(u, w)$ in $P$, then $P$ is a Hamiltonian path using exactly one forced edge. Repeating this process for all $S_{1}^{*}\left(H^{\prime}, x_{1}, x_{2}\right)$, then $P$ will be a Hamiltonian path in $\left(H, x_{1}, x_{2}, F\right)$.
$(3) \Longrightarrow(1)$ Suppose $\left(H, x_{1}, x_{2}, F\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, $P$, which uses all of the edges in $F$. Consider $x y \in F$. In $\left(G, x_{1}, x_{2}\right), x y$ is incident to at least one vertex, $v$, which is not in $\left(H, x_{1}, x_{2}, F\right)$ so that $c(G-\{x, y\})=2$. Let $C_{v}$ be the component of $G-\{x, y\}$ which contains $v$. From [24], $G\left[C_{v} \cup x y\right]$ is a 2-tree, and from Lemma 1.2.23, it is also 1 -tough and so it contains a Hamiltonian cycle $C$. In $G\left[C_{v} \cup x y\right], x y$ is a 1-edge and hence lies on $C$. Thus, there is a Hamiltonian path, $P^{\prime}$, in $G\left[C_{v} \cup x y\right]$ from $x$ to $y$, and we can replace $x y$ in $P$ with $P^{\prime}$. Repeating this process for all $f \in F$ will yield a Hamiltonian path in $\left(G, x_{1}, x_{2}\right)$.

Lemma 3.1.18. Let $\left(H^{\prime}, x_{1}, x_{2}\right)$ the caterpillar representation of $\left(G, x_{1}, x_{2}\right)$, where $G$ is a 1-tough 2-tree. If $\left(H^{\prime}, x_{1}, x_{2}\right)$ is a 2-path, then $\left(H^{\prime}, x_{1}, x_{2}\right)$ has an $\left(x_{1}, x_{2}\right)$ Hamiltonian path.

Proof. If $\left(H^{\prime}, x_{1}, x_{2}\right)$ is a 2-path, then the caterpillar representation is the same as the reduced graph, and hence the reduced graph does not have any forced edges. Thus,
taking the specified simplicial ordering from Definition 3.1.11 in consecutive order will yield a Hamiltonian path.

Lemma 3.1.19. Let $\left(H^{\prime}, x_{1}, x_{2}\right)$ the caterpillar representation of $\left(G, x_{1}, x_{2}\right)$, where $G$ is a 1-tough 2-tree. If $\left(H^{\prime}, x_{1}, x_{2}\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, and $x_{1} x_{2}$ is not a 1-edge in $\left(H^{\prime}, x_{1}, x_{2}\right)$, then $x_{1}, x_{2}$ have degree at most three in $\left(H^{\prime}, x_{1}, x_{2}\right)$.

Proof. Suppose that in $\left(H^{\prime}, x_{1}, x_{2}\right), x_{1}$ has degree four. If $x_{1} x_{2}$ is a 2-edge in $\left(H^{\prime}, x_{1}, x_{2}\right)$, then $\left(x_{1}, x_{2}\right)$ is a trivial tough path, and hence $\left(H^{\prime}, x_{1}, x_{2}\right)$ does not have an $\left(x_{1}, x_{2}\right)$ Hamiltonian path by Lemma 2.3.6. So, suppose $x_{1}$ is not adjacent to $x_{2}$ in $\left(H^{\prime}, x_{1}, x_{2}\right)$. Then $\left(H^{\prime}, x_{1}, x_{2}\right)$ contains $F_{a}^{1}$ as an induced sub-2-tree, and hence does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path by Corollary 2.1.10. Similarly if $x_{2}$ has degree four.

In Theorem 3.1.24, we will use the caterpillar representation of a 2-tree and the paths through $s$-split diamonds and $\ell$-strings of diamonds in the lemmas below to construct a path through any 2 -tree which does not contain $F_{x}^{1} \in \mathscr{F}^{1}$.

Lemma 3.1.20. Let $D_{s}(R)$ be an s-split diamond. Then there is a unique $\left(c_{0}, b_{r}\right)$ Hamiltonian path and a unique $\left(c_{0}, t_{s-r}\right)$-Hamiltonian path.


Figure 3.11: An example of a Hamiltonian path in an $s$-split diamond, $D_{5}(\{1,3,4\})$

Proof. If $s=0$ then both $b_{0}$ and $t_{0}$ are adjacent to $c_{0}$ on the unique Hamiltonian cycle, $C$. So, using the edges in $C$, there are unique $\left(c_{0}, b_{0}\right)$ and $\left(c_{0}, t_{0}\right)$-Hamiltonian paths. Now, suppose that the claim is true for an $(s-1)$-split diamond. Consider $D_{s}(R)$ an $s$-split diamond. Then $c_{0}$ is adjacent to a simplicial vertex, either $t_{0}$ or $b_{0}$. Without loss of generality, assume $b_{0}$ is simplicial. Then $\left(c_{0}, b_{0}, c_{1}\right)$ is a forced path and $D_{s}(R)-\left\{c_{0}, b_{0}\right\}$ is an $(s-1)$-split diamond. By the induction hypothesis, there
is a $\left(c_{1}, b_{r}\right)$ and $\left(c_{1}, t_{s-r}\right)$-Hamiltonian path, $P$ and $P^{\prime}$ respectively. Hence $\left(c_{0}, b_{0}, P\right)$ and $\left(c_{0}, b_{0}, P^{\prime}\right)$ are unique $\left(c_{0}, b_{r}\right)$ and $\left(c_{0}, t_{s-r}\right)$-Hamiltonian paths, respectively.

Remark 3.1.21. The path created uses all of the edges, other than the central path, except for $c_{i-1} t_{j}$ if $i \in R, c_{i} b_{k}$ if $i \notin R$ and $i+1 \notin R$, in addition to avoiding $c_{s} b_{r}$ in $a\left(c_{0}, b_{r}\right)$-Hamiltonian path, and $c_{s} t_{s-r}$ in a $\left(c_{0}, t_{s-r}\right)$-Hamiltonian path.

Lemma 3.1.22. Let
$G=D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m}, \ell_{m}\right) ; D_{s_{m+1}}^{m+1}\left(R_{m+1}\right)$ be an $\ell$-string of diamonds with $z_{i}$ the $\left(D^{i}, D^{i+1}\right)$-amalgamated vertex for all $i$. If $y$ is the simplicial vertex in $\left\{b_{r_{m+1}}^{m+1}, t_{s_{m+1}-r_{m+1}}^{m+1}\right\}$ and neither $t_{0}^{i} \neq y$ nor $b_{0}^{i} \neq y$ is simplicial for $i>1$, then there is a unique $\left(c_{0}, y\right)$-Hamiltonian path.


Figure 3.12: An example of a Hamiltonian path in an $\ell$-string of diamonds:

$$
D_{0} ;\left(t, \ell_{1}\right) ; D_{0} ;(b, 1) ; D_{0} ;\left(t, \ell_{3}\right) ; D_{0}
$$

Proof. We proceed by induction on the number of amalgamated vertices, $m$. If $m=0$, then $G$ is an $s$-split diamond, and by Lemma 3.1.20, the claim is true. Now, suppose that for an $\ell$-string of diamonds with $m-1$ amalgamated vertices, that the claim is true. Now, let $G^{\prime}$ be an $\ell$-string of diamonds with $m$ amalgamated vertices. Without loss of generality, let $x_{1}=t$. From Lemma 3.1.20, there is a $\left(c_{0}^{1}, t_{s_{1}-r_{1}}^{1}\right)$-path, $P$, which covers all of the vertices in $D_{s_{1}}^{1}$. Furthermore, in $G^{\prime}-\left(D_{s_{1}}^{1}-t_{s_{1}-r_{1}}^{1}\right)$, the path, $P^{\prime}$, from $t_{s_{1}-r_{1}}^{1}$ to $t_{0}^{2}$ is forced since the only other vertex adjacent to vertices on this path is $z_{1}$, which is not in $G^{\prime}-\left(D_{s_{1}}^{1}-t_{s_{1}-r_{1}}^{1}\right)$. Also, $G^{\prime}-\left(D_{s_{1}}^{1}-z_{1}\right)-\left(P^{\prime}-t_{0}^{2}\right)$ is a string of diamonds with $m-1$ amalgamated vertices where $t_{0}^{2}$ is simplicial but no other $t_{0}^{i} \neq y$ nor $b_{0}^{i} \neq y$ is simplicial for $i \geq 1$. By the induction hypothesis, there is a unique $\left(z_{1}=c_{0}^{2}, y\right)$-Hamiltonian path, $P^{\prime \prime}$. Furthermore, since $t_{0}^{2}$ is simplicial in $G^{\prime}-\left(D_{s_{1}}^{1}-z_{1}\right)-\left(P^{\prime}-t_{0}^{2}\right)$, the path must begin with $\left(c_{0}^{2}, t_{0}^{2}\right)$. Replacing $\left(c_{0}^{2}, t_{0}^{2}\right)$ with $\left(c_{0}^{2}, P^{\prime}\right)$ in $P^{\prime \prime}$ and preceding this path with $P$, yields a unique $\left(c_{0}^{1}, y\right)$-Hamiltonian path.

Remark 3.1.23. In addition to the unused edges from Lemma 3.1.20, this path will also avoid the edges $c_{0}^{i} b_{0}^{i}$ and $c_{0}^{i} t_{0}^{i}$.

Theorem 3.1.24. If $G$ is a 1-tough 2-tree with $x_{1}, x_{2} \in V(G)$, then the following are equivalent:

1. $G$ contains $F^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree,
2. One of the following tough conditions hold:
(a) There exists a tough path from $x_{1}$ to $x_{2}$,
(b) There exists $a b \in E(G)$ such that $x_{1}$ and $x_{2}$ lie in different components, $C_{x_{1}}, C_{x_{2}}$, respectively of $G-\{a, b\}$ and such that in $G\left[V\left(C_{x_{1}}\right) \cup\{a, b\}\right]$ there is a tough path from $x_{1}$ to $a$ and a tough path from $x_{1}$ to $b$, or
(c) There exists ab, cd $\in E(G)$ such that $x_{1}$ and $x_{2}$ lie in different components of $G-\{a, b\}$ and $G-\{c, d\}$ and such that if $x_{1}$ and $x_{2}$ lie in components, $C_{x_{1}}, C_{x_{2}}$, respectively of $G-\{a, b, c, d\}$ where in $G\left[V-V\left(C_{x_{1}}\right)-V\left(C_{x_{2}}\right)\right]$ there are tough paths from a to $c, a$ to $d, b$ to $c$, and $b$ to $d$.
3. $G$ does not have an $\left(x_{1}, x_{2}\right)$ Hamiltonian path.

Proof. $(1) \Longrightarrow(2)$
(A) If $G$ contains $F_{a}^{1}, x_{1} t_{0}$ and $x_{1} c_{1}$ are tough paths and $x_{1}$ and $x_{2}$ are in different components of $G-\left\{t_{0} c_{1}\right\}$.
(B) If $G$ contains $F_{b}^{1}$, the central path is a tough path from $x_{1}=c_{0}^{1}$ to $x_{2}=c_{s_{m}+1}^{m}$.
(C) If $G$ contains $F_{c}^{1}$, there is a tough path from $x_{1}=c_{0}^{1}$ to $c_{1}^{m}$ and to $t_{0}^{m}$ and $x_{1}=c_{0}^{1}$ and $x_{2}$ are in different components of $G-\left\{t_{0}^{m}, c_{1}^{m}\right\}$.
(D) If $G$ contains $F_{d}^{1}$, there is a tough path from $c_{0}^{1}$ to $c_{1}^{m}$ and to $t_{0}^{m}$ and a tough path from $t_{0}^{1}$ to $c_{1}^{m}$ and to $t_{0}^{m}$ and $x_{1}$ and $x_{2}$ are in different components of $G-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and of $G-\left\{t_{0}^{m}, c_{1}^{m}\right\}$ OR there is a tough path from $c_{0}^{1}$ to $c_{1}^{m}$ and to $b_{0}^{m}$ and a tough path from $t_{0}^{1}$ to $c_{1}^{m}$ and to $b_{0}^{m}$ and $x_{1}$ and $x_{2}$ are in different components of $G-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and of $G-\left\{b_{0}^{m}, c_{1}^{m}\right\}$.
(E) If $G$ contains $F_{e}^{1}$, there is a tough path from $c_{0}^{1}$ to $c_{1}^{2}$ and to $t_{0}^{2}$ and a tough path from $t_{0}^{1}$ to $c_{1}^{2}$ and to $t_{0}^{2}$ and $x_{1}$ and $x_{2}$ are in different components of $G-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and of $G-\left\{t_{0}^{2}, c_{1}^{2}\right\}$.
(F) If $G$ contains $F_{f}^{1}$, there is a tough path from $c_{0}^{1}$ to $c_{1}^{2}$ and to $t_{0}^{2}$ and a tough path from $t_{0}^{1}$ to $c_{1}^{2}$ and to $t_{0}^{2}$ and $x_{1}$ and $x_{2}$ are in different components of $G-\left\{t_{0}^{1}, c_{0}^{1}\right\}$ and of $G-\left\{t_{0}^{2}, c_{1}^{2}\right\}$.
$(2) \Longrightarrow(3)$
(a) Lemma 2.3.6
(b) Corollary 2.3.17
(c) Corollary 2.3.19
$(3) \Longrightarrow(1)$
Suppose $G$ does not contain any $F_{x}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree. Let ( $H^{\prime}, x_{1}, x_{2}$ ) be the caterpillar representation of $\left(G, x_{1}, x_{2}\right)$. Then $\left(H^{\prime}, x_{1}, x_{2}\right)$ does not contain any of $F_{x}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree. If $\left(H^{\prime}, x_{1}, x_{2}\right)$ is a 2-path, then $\left(H^{\prime}, x_{1}, x_{2}\right)$ will have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path by Lemma 3.1.18, so we will assume that $\left(H^{\prime}, x_{1}, x_{2}\right)$ is not a 2-path. Also since $\left(H^{\prime}, x_{1}, x_{2}\right)$ does not contain any $F_{x}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, then $x_{1}$ and $x_{2}$ have degree two or three in $\left(H^{\prime}, x_{1}, x_{2}\right)$. In the following cases we will construct paths in $\left(H^{\prime}, x_{1}, x_{2}\right)$.

Case (A) Suppose $x_{1}$ is a simplicial vertex in $\left(H^{\prime}, x_{1}, x_{2}\right)$. Since $\left(H^{\prime}, x_{1}, x_{2}\right)$ is not a 2-path, there is at least one simplicial vertex, other than $x_{1}$ and $x_{2}$. Let $v_{1}$ be the vertex with the smallest label from Definition 3.1.11, which is adjacent to a simplicial vertex, $s_{1} \neq x_{1}, x_{2}$. Using that same labelling, in consecutive order, there is a path, $P_{A}^{\prime}$, from $x_{1}$ to $y_{1}$, a vertex which is labelled one less than $v_{1}$. Since $v_{1}$ is adjacent to a simplicial vertex, then there is a tough path which starts at $v_{1}$. Let $P_{A}$ be a maximal short tough path beginning at $x_{1}$. By Lemma 2.3.11, $P_{A}$ is the central path of an $\ell$ string of diamonds,
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(w_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(w_{2}, \ell_{2}\right) ; \ldots ;\left(w_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$. Without loss of generality, suppose $w_{1}=t$. Since $\left(H^{\prime}, x_{1}, x_{2}\right)$ does not contain any $F_{x}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, then $x_{2} \neq c_{s_{m}+1}^{m}$ and neither $t_{0}^{i} \neq x_{2}$ nor $b_{0}^{i} \neq x_{2}$ is simplicial for $i>2$.
(I) Suppose $t_{0}^{1}$ is adjacent to a simplicial vertex as well. Then, $t_{0}^{2} \neq x_{2}$ and $b_{0}^{2} \neq x_{2}$ are not simplicial and continuing from $P_{A}^{\prime}$, we can take the path to $v_{1}=c_{0}^{1}$ and continue the path as in Case B with $x_{1}=$ $v_{1}=c_{0}^{1}$.
(II) Suppose $t_{0}^{1}$ is not adjacent to a simplicial vertex. Then, continuing from $P_{A}^{\prime}$, we can take the path $\left(t_{0}^{1}, v_{1}=c_{0}^{1}, s_{1}=b_{0}^{1}, c_{1}^{1}\right)$.
(a) If $c_{1}^{1}=z_{1}$ is an amalgamated vertex, and $\ell_{1}=1$, then removing all of the visited vertices other than $c_{1}^{1}=z_{1}$, we will have an $(l-1)$-string of diamonds. If $\ell_{1}>1$, then $b_{0}^{2}$ is not simplicial and we will have an $(l-1)$-string of diamonds with additional vertices, left from the path of length $\ell_{1}$ between $t_{0}^{1}$ and $t_{0}^{2}$. In either case, we can then extend $P_{A}^{\prime}$ by using the path given in Lemma 3.1.22 beginning at the amalgamated vertex, $c_{1}^{1}=z_{1}$. We can continue the construction of the path as in Case (B)(II).
(b) If $c_{1}^{1}=z_{1}$ is not an amalgamated vertex, then $R_{1} \neq \emptyset$, so let $R_{1}=\left\{q_{1}, q_{2}, \ldots, q_{r_{1}}\right\}$. Let $q_{i}$ be the first value such that $q_{i-1} \neq$ $q_{i}-1$.
(i) If no such value exists, then $c_{s_{1}+1}^{1}=z_{1}$ is an amalgamated vertex and $b_{j}^{1}$ is simplicial for all $j \in\left\{1, \ldots, s_{1}\right\}$. So there is a forced path from $c_{1}^{1}$ to $c_{s_{1}+1}^{1}=z_{1}$ which uses all edges $c_{k} b_{k}$ and $c_{k+1} b_{k}$ for $1 \leq k \leq s_{1}+1$. By assumption, $t_{0}^{1}$ is not adjacent to a simplicial vertex, so this path uses all possible edges which could have a simplicial vertex adjacent in $\left(H^{\prime}, x_{1}, x_{2}\right)$. Hence, replacing any edges of the path which are adjacent to simplicial vertices in $\left(H^{\prime}, x_{1}, x_{2}\right)$, with the path through the simplicial vertex, we have a path in ( $H^{\prime}, x_{1}, x_{2}$ ). Furthermore, removing the visited vertices, other than $c_{s_{1}+1}^{1}=z_{1}$, we will have an $\left(l-\left(s_{1}+1\right)\right)$-string of diamonds if $\ell_{1}=1$ and if $\ell_{1}>1$, then we will have an $\left(l-\left(s_{1}+1\right)\right)$-string of diamonds with additional vertices, left from the path of length $\ell_{1}$ between $t_{0}^{1}$ and $t_{0}^{2}$. In either case, we can then extend $P_{A}^{\prime}$ by using the path given in Lemma 3.1.22 beginning at the amalgamated vertex, $c_{s_{1}+1}^{1}=z_{1}$. We can continue the construction of the path as in Case (B)(II).
(ii) If a $q_{i}$ exists, then $b_{j}^{1}$ is simplicial for $0 \leq q_{i-1}-1=q_{i-2}$, and hence there is a forced path from $c_{1}^{1}$ to $c_{q_{i-1}}^{1}$, which uses all edges $c_{k} b_{k}$ and $c_{k+1} b_{k}$ for $1 \leq k \leq q_{i-2}$. By assumption,
$t_{0}^{1}$ is not adjacent to a simplicial vertex, so this path uses all possible edges which could have a simplicial vertex adjacent in $\left(H^{\prime}, x_{1}, x_{2}\right)$. Hence, replacing any edges of the path which are adjacent to simplicial vertices in $\left(H^{\prime}, x_{1}, x_{2}\right)$, with the path through the simplicial vertex, we have a path in ( $H^{\prime}, x_{1}, x_{2}$ ). Furthermore, removing the visited vertices, other than $c_{q_{i-1}}^{1}$, and removing the visited vertices, other than $c_{q_{i-1}}^{1}$, we will have a 1 -tough 2 -tree with $c_{q_{i-1}}^{1}$ simplicial, and we can continue this path by repeating Case (A) with $x_{1}=c_{q_{i-1}}^{1}$.

Case (B) Suppose $x_{1}$ is not a simplicial vertex in $\left(H^{\prime}, x_{1}, x_{2}\right)$. Then $x_{1}$ has degree three and hence is adjacent to a simplicial vertex. Hence, $x_{1}$ lies on a tough path. Let $P_{B}$ be a maximal short tough path beginning at $x_{1}$. By Lemma 2.3.11, $P_{B}$ is the central path of an $\ell$-string of diamonds, $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(w_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(w_{2}, \ell_{2}\right) ; \ldots ;\left(w_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$. Without loss of generality, suppose $w_{1}=t$. Since $\left(H^{\prime}, x_{1}, x_{2}\right)$, ad $\left(H^{\prime}, x_{1}, x_{2}\right)$ does not contain any $F_{x}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, then $x_{2} \neq c_{s_{m}+1}^{m}$ and neither $t_{0}^{i} \neq x_{2}$ nor $b_{0}^{i} \neq x_{2}$ is simplicial for $i>1$.
(I) If the $\ell$-string of diamonds contains no amalgamated vertices, then we have an $(l-1)$-split diamond, $D_{s_{1}}\left(R_{1}\right)$. From Lemma 3.1.20, there are $\left(x_{1}, b_{r_{m}}\right)$ and $\left(x_{1}, t_{s_{m}-r_{m}}\right)$ paths, $P_{B 1}^{\prime}$ and $P_{B 2}^{\prime}$, respectively, which cover all of the vertices in $D_{s_{1}}\left(R_{1}\right)$. Note that since the $(l-1)$ split diamond is an induced sub-2-tree, it is possible for 1-edges of the $(l-1)$-split diamond to be adjacent to simplicial vertices in $\left(H^{\prime}, x_{1}, x_{2}\right)$. Such edges would correspond to edges that would need to be used in a path through the $(l-1)$-split diamond. However, attaching simplicial vertices to any unused edges in the path from Lemma 3.1.20 would form an induced sub-2-tree in $\mathscr{F}^{1}$. And so, for any edges of the $(l-1)$-split diamond which are adjacent to a simplicial vertex in $\left(H^{\prime}, x_{1}, x_{2}\right)$ we can replace the edge on $P_{B 1}^{\prime}$ or $P_{B 2}^{\prime}$ with the path through the simplicial vertex to form a path, $P_{B 1}^{\prime \prime}$ and $P_{B 2}^{\prime \prime}$, respectively, in $\left(H^{\prime}, x_{1}, x_{2}\right)$. If $b_{r_{m}}=x_{2}$, then $P_{B 1}^{\prime \prime}$ is an $\left(x_{1}, x_{2}\right)$-Hamiltonian path in $\left(H^{\prime}, x_{1}, x_{2}\right)$. If $t_{s_{m}-r_{m}}=$ $x_{2}$, then $P_{B 2}^{\prime \prime}$ is an $\left(x_{1}, x_{2}\right)$-Hamiltonian path in $\left(H^{\prime}, x_{1}, x_{2}\right)$. So
now suppose that $b_{r_{m}}, t_{s_{m}-r_{m}} \neq x_{2}$. Let $y=b_{r_{m}}$ if $\left(H^{\prime}, x_{1}, x_{2}\right)-$ $\left\{b_{r_{m}} c_{s_{1}+1}\right\}$ leaves $x_{1}$ and $x_{2}$ in different components, and $y=t_{s_{m}-r_{m}}$ if $\left(H^{\prime}, x_{1}, x_{2}\right)-\left\{t_{s_{m}-r_{m}} c_{s_{1}+1}\right\}$ leaves $x_{1}$ and $x_{2}$ in different components. Let $P_{y}^{\prime \prime}=P_{B 1}^{\prime \prime}$ if $y=b_{r_{m}}$ and $P_{y}^{\prime \prime}=P_{B 2}^{\prime \prime}$ if $y=t_{s_{m}-r_{m}} . P_{y}^{\prime \prime}-y$ is an $\left(x_{1}, c_{s_{1}+1}\right)$-path and furthermore $\left(H^{\prime}, x_{1}, x_{2}\right)-\left(P_{y}^{\prime \prime}-\left\{y, c_{s_{1}+1}\right\}\right)$ is a 1-tough 2-tree. Additionally, $c_{s_{1}+1}$ is simplicial since if it weren't, then $P_{B}$ would not be maximal. So we can finish constructing the Hamiltonian path by finding an $\left(c_{s_{1}+1}, x_{2}\right)$-Hamiltonian path in $\left(H^{\prime}, x_{1}, x_{2}\right)-\left(P_{y}^{\prime \prime}-\left\{y, c_{s_{1}+1}\right\}\right)$ using Case (A).
(II) Suppose the $\ell$-string of diamonds contains at least one amalgamated vertex. Using the path in Lemma 3.1.22, we have a path, $P_{B}^{\prime}$ from $x_{1}$ to $y$, where $y=b_{r_{m}}^{m}$ if $y_{m-1}=t$ in $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(w_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(w_{2}, \ell_{2}\right) ; \ldots ;\left(w_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$, and $y=t_{s_{m}-r_{m}}^{m}$ if $y_{m-1}=b$ in $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(w_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(w_{2}, \ell_{2}\right) ; \ldots ;\left(w_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$, such that all vertices in the $\ell$-string of diamonds are covered. Note that since the $\ell$-string of diamonds is an induced sub-2-tree, it is possible for 1 -edges of the $\ell$-string of diamonds to be adjacent to simplicial vertices in $\left(H^{\prime}, x_{1}, x_{2}\right)$, which would correspond to edges that would need to be used in a path through the $\ell$-string of diamonds. However, attaching simplicial vertices to any unused edges in the path from Lemmas 3.1.20 and 3.1.22 would form an induced sub-2-tree in $\mathscr{F}^{1}$. And so, for any edges of the $\ell$-string of diamonds which are adjacent to a simplicial vertex in $\left(H^{\prime}, x_{1}, x_{2}\right)$ we can replace the edge on $P_{B}^{\prime}$ with the path through the simplicial vertex to form a path, $P_{B}^{\prime \prime}$ in $\left(H^{\prime}, x_{1}, x_{2}\right)$. If $y=x_{2}$, then $P_{B}^{\prime \prime}$ is an $\left(x_{1}, x_{2}\right)$-Hamiltonian path in $\left(H^{\prime}, x_{1}, x_{2}\right)$. If $y \neq x_{2}$, then $P_{B}^{\prime \prime}-y$ is an $\left(x_{1}, c_{s_{m}+1}^{m}\right)$-path. Furthermore, $\left(H^{\prime}, x_{1}, x_{2}\right)-\left(P_{B}^{\prime \prime}-\left\{y, c_{s_{m}+1}^{m}\right\}\right)$ is a 1-tough 2-tree and $c_{s_{m}+1}^{m}$ is simplicial since if it weren't, then $P_{B}$ would not be maximal. So we can finish constructing the Hamiltonian path by finding an $\left(c_{s_{m}+1}^{m}, x_{2}\right)$-Hamiltonian path in $\left(H^{\prime}, x_{1}, x_{2}\right)-\left(P^{\prime \prime}-\left\{y, c_{s_{m}+1}^{m}\right\}\right)$ using Case (A).

Remark 3.1.25. In the cases when $G$ is a 1-tough 2-tree, which do not contain an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, we can partition $G$ into two vertex disjoint paths with $x_{1}$
the end of one path and $x_{2}$ the end of the other. We can do this by breaking the Hamiltonian cycle in $G$ into two paths.

### 3.2 2HP on 2-trees

Definition 3.2.1. Define $\mathscr{F}^{2}=\left\{\mathscr{F}^{1}, F_{a}^{2}, F_{b}^{2}, F_{c}^{2}, F_{d}^{2}, F_{e}^{2}\right\}$ where:
(a) $F_{a}^{2}$ is a 2-tree with vertices $x_{1}, x_{2}$, and a 3-edge, ef, such that either:
(i) $x_{1}$ and $x_{2}$ are in the same component of $G-\{e, f\}$, or
(ii) $e \in\left\{x_{1}, x_{2}\right\}$.


Figure 3.13: General example of $F_{a}^{2}$ such that $x_{1}$ and $x_{2}$ are in the same component of $G-\{e, f\}$, and to preserve generality, the dotted section of the graph represents any 2 -tree with scattering number at most one


Figure 3.14: General example of $F_{a}^{2}$ such that $e \in\left\{x_{1}, x_{2}\right\}$ and to preserve generality, the dotted section of the graph represents any 2 -tree with scattering number at most one
(b) $F_{b}^{2}$ is a 2-tree with vertices $x_{1}, x_{2}$, which contains a 3-edge, ab, such that:
(i) $x_{1}$ and $x_{2}$ are in different components of $F_{b}^{2}-\{a, b\}$,
(ii) $N(a)-\left\{x_{1}, x_{2}\right\}$ contains two simplicial vertices,
(iii) $N(b)-\left\{x_{1}, x_{2}\right\}$ contains two simplicial vertices, and
(iv) In $F_{b}^{2}-\{a, b\}$ two of the simplicial vertices lie in the same component.


Figure 3.15: General example of $F_{b}^{2}$ and to preserve generality, the dotted section of the graph represents any 2 -tree with scattering number at most one
(c) $F_{c}^{2}$ is a 2-tree with vertices $x_{1}, x_{2}$, which contains a 3-edge, ab, such that $x_{1}$ and $x_{2}$ are in different components of $F_{c}^{2}-\{a, b\}$ and $N(a)-\left\{x_{1}, x_{2}\right\}$ contains three simplicial vertices.


Figure 3.16: General example of $F_{c}^{2}$ and to preserve generality, the dotted section of the graph represents any 2 -tree with scattering number at most one
(d) $F_{d}^{2}$ is constructed from $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$, $m \geq 2$, by:
(i) Amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$, and
(ii) Amalgamating an $x_{1}$-2-path with $c_{0}^{1} c_{1}^{1}$.


Figure 3.17: A general example of $F_{d}^{2}$ :
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots . ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$
$m \geq 2$, with an $x_{2}$-2-path amalgamated with $t_{0}^{m} c_{1}^{m}$, and $x_{1}-2$-path amalgamated with $c_{0}^{1} c_{1}^{1}$
where $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ is shown in gray to preserve generality


Figure 3.18: Specific example of $F_{d}^{2}: \quad D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with $x_{1}$-2-path amalgamated with $c_{0}^{1} c_{1}^{1}$ and $x_{2}$-2-path amalgamated with $t_{0}^{2} c_{1}^{2}$
(e) $F_{e}^{2}$ is constructed from an $\ell$-string of diamonds, with $x_{1}=c_{0}^{1}$, by amalgamating an $x_{2}$-2-path with $c_{s_{m}}^{m} c_{s_{m}+1}^{m}$.


Figure 3.19: A general example of $F_{e}^{2}: D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$
with an amalgamated $x_{2}$-2-path and such that $x_{1}=c_{0}^{1}$
where $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$ is shown in gray to preserve generality


Figure 3.20: Specific example of $F_{e}^{2}: D_{5}(\{1,3,4,5\})$ with $x_{1}=c_{0}$ and $x_{2}$-2-path amalgamated with $c_{5} c_{6}$

Lemma 3.2.2. The graph $F_{a}^{2}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. Since $e f$ is a 3-edge, then $c(G-\{e, f\})=3$ and hence if $G$ has a Hamiltonian path, then the ends of the path must lie in two of the three components of $G-\{e, f\}$. So, if $x_{1}$ and $x_{2}$ are in the same component of $G-\{e, f\}$, then $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. Similarly, if $e \in\left\{x_{1}, x_{2}\right\}$, then $x_{1}$ or $x_{2}$ is not in one of the components of $G-\{e, f\}$ and $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Lemma 3.2.3. The graph $F_{b}^{2}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. Let $u$ be the simplicial vertex in $N(a b)-\left\{x_{1}, x_{2}\right\}, v$ the simplicial vertex in $N(a)-\left\{x_{1}, x_{2}, u\right\}$, and $w$ the simplicial vertex in $N(b)-\left\{x_{1}, x_{2}, u\right\}$. Suppose that $G$ contains an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, $P$. Since $u, v$ and $w$ are simplicial and not endpoints of $P$, then $P$ must contain $(v, a, u, b, w)$. But since $v$ and $w$ are in the same component of $G-\{a, b\}$, then $x_{1}$ and $x_{2}$ need to be in the same component of $G-\{a, b\}$, a contradiction.

Lemma 3.2.4. The graph $F_{c}^{2}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. Let $u, v$ and $w$ be the simplicial vertices in $N(a)-\left\{x_{1}, x_{2}\right\}$, and suppose $G$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. Since $u, v$ and $w$ are not endpoints to the path, then $u, v$ and $w$ must all be either preceded or followed by $a$. But that means that $a$ must be used at least twice on the Hamiltonian path, a contradiction.

Lemma 3.2.5. The graph $F_{d}^{2}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. Since $c\left(H-c_{0}^{1} c_{1}^{1}\right)=3$, then if $H$ has a Hamiltonian path, there must be a Hamiltonian path in each of the components, and $c_{0}^{1}$ and $c_{1}^{1}$ must connect the paths. Furthermore, if $H$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, then the path must start in the component of $H-c_{0}^{1} c_{1}^{1}$ which contains $x_{1}$ and end in the component which contains $x_{2}$. But that would mean that $H$ has a $\left(c_{0}^{1}, x_{2}\right)$ or $\left(c_{1}^{1}, x_{2}\right)$-Hamiltonian path, a contradiction to Lemma 3.1.4.

Lemma 3.2.6. The graph $F_{e}^{2}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.
Proof. Since $c\left(H-c_{s_{m}}^{m} c_{s_{m}+1}^{m}\right)=3$, then if $H$ has a Hamiltonian path, there must be a Hamiltonian path in each of the components, and $c_{s_{m}}^{m}$ and $c_{s_{m}+1}^{m}$ must connect the paths. Furthermore, if $H$ has an $\left(c_{0}^{1}, x_{2}\right)$-Hamiltonian path, then the path must start in the component of $H-c_{s_{m}}^{m} c_{s_{m}+1}^{m}$ which contains $c_{0}^{1}$ and end in the component which contains $x_{2}$. But that would mean that $H$ has a $\left(c_{0}^{1}, c_{s_{m}}^{m}\right)$ or $\left(c_{0}^{1}, c_{s_{m}+1}^{m}\right)$-Hamiltonian path, a contradiction to Lemma 3.1.3.

Similar to the reduced graph of a 1-tough 2-tree with fixed endpoints, we will create a reduced graph of a 2-tree with scattering number one and fixed points, in order to more easily describe the paths in the 2-trees, as follows.

Definition 3.2.7. Given a 2-tree, $G$ with $s(G)=1$, then the reduced graph of $(G, u, v)$, is formed using the following algorithm:

1. For every 3-edge ab with components of $G-\{a, b\}, C_{a b}^{1}, C_{a b}^{2}, C_{a b}^{3}$, if $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ is 1-tough and does not contain $u$ or $v$, then replace $C_{a b}^{i}$ with a simplicial vertex adjacent to ab.
2. Let $w \in S_{1}^{*}(G, u, v)$, and $x$, $y$ the neighbors of $w$. If $x y$ is not a 3-edge, remove $w$ and turn the edge $x y$ into a forced edge.
3. Repeat (2) for all $w \in S_{1}^{*}(G, u, v)$. Define the resulting graph to be $G_{1}^{*}$.
4. For $i \geq 2$, let $S_{i}^{*}=S_{1}^{*}\left(G_{i-1}^{*}, u, v\right)$ where $G_{i-1}^{*}$ is the graph formed by repeating (2) for $G=G_{i-1}^{*}$ and for all $w \in S_{1}^{*}\left(G_{i-1}^{*}, u, v\right)$.

Repeat (4) for all $i=2,3, \ldots, j$ for $j$ such that $S_{j}^{*}=\emptyset$ or for all $s \in S_{j}^{*}, N(s)$ is a 3-edge. This is the reduced graph of $(G, u, v)$.
For $F$ the set of forced edges, let $(H, u, v, F)$ denote the reduced graph of $(G, u, v)$, for G a 2-tree containing at least one 3-edge.

Since simplicial vertices in 2-trees are not adjacent [7], when we remove the vertices in each $S_{i}$, regardless of order, we will end up with the same graph.

Remark 3.2.8. When creating the reduced graph of a 2-tree with scattering number one with no fixed endpoints, $S_{1}^{*}(G, u, v)$ will be replaced by $S_{1}(G)$.

We will form the corresponding caterpillar representation of $(G, u, v)$ as in Chapter 2.

Lemma 3.2.9. Let $G$ be a 2-tree with $x_{1}, x_{2} \in V(G)$ and $s(G)=1$. Let $\left(H, x_{1}, x_{2}, F\right)$ be the reduced graph of $\left(G, x_{1}, x_{2}\right)$, and $\left(H^{\prime}, x_{1}, x_{2}\right)$ the caterpillar representation of $\left(G, x_{1}, x_{2}\right)$. Then the following are equivalent:

1. $G$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path,
2. $\left(H^{\prime}, x_{1}, x_{2}\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, and
3. $\left(H, x_{1}, x_{2}, F\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path which uses all of the edges in $F$.

Proof. (1) $\Longrightarrow(2)$
Suppose ( $H^{\prime}, x_{1}, x_{2}$ ) does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. Since $\left(H^{\prime}, x_{1}, x_{2}\right)$ is an induced sub-2-tree of $\left(G, x_{1}, x_{2}\right)$, then by Corollary 2.1.10, $G$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path.

Suppose $\left(H^{\prime}, x_{1}, x_{2}\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, $P$. Let $v \neq x_{1}, x_{2}$ be a simplicial vertex with neighbors $u$ and $w$. Then $P=\left(x_{1}, \ldots, u, v, w, \ldots, x_{2}\right)$ or $P=$ $\left(x_{1}, \ldots, w, v, u, \ldots, x_{2}\right)$. Furthermore, because $H^{\prime}$ is a 2 -tree, then $u w \in E\left(H^{\prime}\right)$, and from the reduction algorithm $u w \in F$. Replacing $(u, v, w)$ or $(w, v, u)$ by $(u, w)$ in $P$, then $P$ is a Hamiltonian path using exactly one forced edge. Repeating this process for all $S_{1}^{*}\left(H^{\prime}, x_{1}, x_{2}\right)$, then $P$ will be a Hamiltonian path in $\left(H, x_{1}, x_{2}, F\right)$.
$(3) \Longrightarrow(1)$ Suppose $\left(H, x_{1}, x_{2}, F\right)$ has an $\left(x_{1}, x_{2}\right)$-Hamiltonian path, $P$, which uses all of the edges in $F$. Let $a b$ be a 3 -edge in $G$ with components of $G-\{a, b\}$, $C_{a b}^{1}, C_{a b}^{2}, C_{a b}^{3}$, where $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ and is 1-tough and does not contain $x_{1}$ or $x_{2}$. In $\left(H, x_{1}, x_{2}, F\right), C_{a b}^{i}$ has been replaced by the simplicial vertex, $v_{a b}^{i}$. Since $G\left[C_{a b}^{i} \cup\right.$ $\{a, b\}]$ is 1-tough, then $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ has a Hamiltonian cycle, $C$ using all 1-edges in $G\left[C_{a b}^{i} \cup\{a, b\}\right]$. Hence, since $a b$ is a 1-edge in $G\left[C_{a b}^{i} \cup\{a, b\}\right]$, then there is an $(a, b)$-Hamiltonian path $P^{\prime}$ in $G\left[C_{a b}^{i} \cup\{a, b\}\right]$. Since $v_{a b}^{i}$ is on the interior of $P$ in $\left(H, x_{1}, x_{2}, F\right)$, then we can replace $\left(a, v_{a b}^{i}, b\right)$ on $P$ with $P^{\prime}$. Now, consider $x y \in F$. In $G, x y$ is incident to at least one vertex, $v$, which is not in $\left(H, x_{1}, x_{2}, F\right)$ so that $c(G-\{x, y\})=2$. Let $C_{v}$ be the component of $G-\{x, y\}$ which contains $v$. From [24], $G\left[C_{v} \cup x y\right]$ is a 2-tree, and from the reduction algorithm, $G\left[C_{v} \cup x y\right]$ must be 1-tough and so it contains a Hamiltonian cycle $C$. In $G\left[C_{v} \cup x y\right], x y$ is a 1-edge and hence lies on $C$. Thus, there is a Hamiltonian path, $P^{\prime \prime}$, in $G\left[C_{v} \cup x y\right]$ from $x$ to $y$, and we can replace $x y$ in $P$ with $P^{\prime}$. Repeating these processes for all $f \in F$ and all 3-edges, $c d$ and all $C_{c d}^{i}$ such that $G\left[C_{c d}^{i} \cup\{c, d\}\right]$ is 1-tough, will yield an $\left(x_{1}, x_{2}\right)$-Hamiltonian path in $G$.

Theorem 3.2.10. If $G$ is a 2-tree with $x, y \in V(G)$, then $G$ has an $(x, y)$-Hamiltonian path iff $s(G) \leq 1$ and $(G, x, y)$ does not contain any $F^{2} \in \mathscr{F}^{2}$.

Proof. $\Longrightarrow$ If $s(G) \geq 2$, then $G$ is not 1-path-tough, and $G$ does not contain a Hamiltonian path.

1. If $(G, x, y)=F_{a}^{2}$, then $(G, x, y)$ does not have an $(x, y)$-Hamiltonian path by Lemma 3.2.2. If $(G, x, y)$ contains $F_{a}^{2}$ as an induced sub-2-tree, then $(G, x, y)$ does not have an $(x, y)$-Hamiltonian path by Corollary 2.1.11.
2. If $(G, x, y)=F_{b}^{2}$, then $(G, x, y)$ does not have an $(x, y)$-Hamiltonian path by Lemma 3.2.3. If $(G, x, y)$ contains $F_{b}^{2}$ as an induced sub-2-tree, then $(G, x, y)$ does not have an $(x, y)$-Hamiltonian path by Corollary 2.1.11.
3. If $(G, x, y)=F_{c}^{2}$, then $(G, x, y)$ does not have an $(x, y)$-Hamiltonian path by Lemma 3.2.4. If $(G, x, y)$ contains $F_{c}^{2}$ as an induced sub-2-tree, then $(G, x, y)$ does not have an $(x, y)$-Hamiltonian path by Corollary 2.1.11.
4. If $(G, x, y)$ contains an $F_{x}^{1} \in \mathscr{F}^{1} \subset \mathscr{F}^{2}$, then $G$ does not have an $(x, y)$ Hamiltonian path by Theorem 3.1.24 and Corollary 2.1.11.
$\Longleftarrow$
Suppose $G$ does not have an $(x, y)$-Hamiltonian path, but that $s(G) \leq 1$. Since $s(G) \leq 1$, then $G$ contains no $t$-edges for $t \geq 4$. We will proceed by induction on the number of 3 -edges, $m$. If $m=0$, then by Theorem 3.1.24, $(G, x, y)$ contains an $F_{x}^{1} \in \mathscr{F}^{1} \subset \mathscr{F}^{2}$. Suppose the claim is true for all graphs with $(m-1) 3$-edges. Now consider $G$ a 2-tree with $s(G) \leq 1$ such that $G$ does not have an $(x, y)$-Hamiltonian path with $m$ 3-edges. Let $\left(H^{\prime}, x, y\right)$ be the caterpillar representation of $G$. Then $s\left(H^{\prime}\right) \leq 1$ and $H^{\prime}$ does not have an $(x, y)$-Hamiltonian path. Suppose $H^{\prime}$ does not contain $F_{a}^{2}$. Denote the 3 -edges in $H^{\prime}, S_{i}=s_{i} s_{i}^{\prime}$ for all $1 \leq i \leq m$. Then the 3edges in $H^{\prime}$ can be ordered $S_{1}, S_{2}, \ldots . S_{m}$ so that for all $i, x$ and $y$ are in different components of $H^{\prime}-S_{i}$, in $H^{\prime}-S_{1}, x$ is in a different component than $s_{i}$ and $s_{i}^{\prime}$ for all $i$, in $H^{\prime}-S_{m}, y$ is in a different component than $s_{i}$ and $s_{i}^{\prime}$ for all $i$, and such that for all $i \in\{1,2, \ldots, m-2\}, s_{i}$ and $s_{i+2}$ are in different components of $H^{\prime}-S_{i+1}$. Let $C_{1}$ be the component of $H^{\prime}-S_{1}$ which contains $x$. Let $C_{2}$ be the component of $H^{\prime}-S_{1}$ which contains $y$. Let $H_{1}^{\prime}$ be the graph constructed from $G\left[C_{1} \cup S_{1}\right]$ by adding a simplicial vertex, $v_{1}$, adjacent to $S_{1}$. Let $H_{2}^{\prime}$ be the graph constructed from $G\left[C_{2} \cup S_{1}\right]$ by adding a simplicial vertex, $v_{2}$, adjacent to $S_{1}$. Let $S_{1}=a b$. If $H_{1}^{\prime}$ has an $(x, a)$-Hamiltonian path, $P$, then because $v_{1}$ is simplicial, then $P$ ends with $\left(b, v_{1}, a\right)$. Likewise, if $H_{2}^{\prime}$ has a $(b, y)$-Hamiltonian path, $P^{\prime}$, then $P^{\prime}$ begins with $\left(b, v_{2}, a\right)$. So, in $H^{\prime},\left(P-\left\{a, v_{1}, b\right\}, P^{\prime}\right)$ is an $(x, y)$-Hamiltonian path. Similarly if $H_{1}^{\prime}$ has an $(x, b)$ Hamiltonian path and $H_{2}^{\prime}$ has an $(a, y)$-Hamiltonian path. So, since $H^{\prime}$ does not have an $(x, y)$-Hamiltonian path, either (1) $H_{1}^{\prime}$ has neither an $(x, a)$-Hamiltonian path nor an $(x, b)$-Hamiltonian path, or (2) $H_{2}^{\prime}$ has neither an $(a, y)$-Hamiltonian path nor an $(b, y)$-Hamiltonian path, or (3) $H_{1}^{\prime}$ only has an $(x, a)$-Hamiltonian path while $H_{2}^{\prime}$ only has an $(a, y)$-Hamiltonian path, or (4) $H_{1}^{\prime}$ only has an $(x, b)$-Hamiltonian path while $H_{2}^{\prime}$ only has an $(b, y)$-Hamiltonian path.
5. If $H_{1}^{\prime}$ does not have an $(x, a)$-Hamiltonian path, then by Theorem 3.1.24, then $\left(H_{1}^{\prime}, x, a\right)$ contains an $F_{x}^{1} \in \mathscr{F}^{1} \subset \mathscr{F}^{2}$. Likewise, if $H_{1}^{\prime}$ does not have an $(x, b)$-Hamiltonian path, then by Theorem 3.1.24, then $\left(H_{1}^{\prime}, x, b\right)$ contains an $F_{x}^{1} \in \mathscr{F}^{1} \subset \mathscr{F}^{2}$. Note first that if there is an ef such that $x$ and $a$ lie in different components of $H_{1}^{\prime}-\{e, f\}$, then since $a$ and $b$ are adjacent, then either $b$ is in the same component as $a$ in $H_{1}^{\prime}-\{e, f\}$, or $b \in\{e, f\}$. Also, if in $\left(H_{1}^{\prime}, x, a\right)$, and similarly for $\left(H_{1}^{\prime}, x, b\right)$, there is an $e f$ such that $x$ and $a$ lies in different components, $C_{x}, C_{a}$, respectively of $H_{1}^{\prime}-\{e, f\}$ and such that in
$G\left[V\left(C_{x}\right) \cup\{e, f\}\right]$ there is a tough path from $x$ to $e$ and a tough path from $x$ to $f$, then we have $F_{a}^{1}$ or $F_{c}^{1}$, with $x=c_{0}^{1}$. In either case, $y$ will be in the same component as $a$, and hence $\left(H^{\prime}, x, y\right)$ also contains $F_{a}^{1}$ or $F_{c}^{1}$. Similarly, if $\left(H_{1}^{\prime}, x, a\right)$ and/or $\left(H_{1}^{\prime}, x, b\right)$ contains $F_{d}^{1}, F_{e}^{1}$, or $F_{f}^{1}$, then $\left(H^{\prime}, x, y\right)$ also contains $F_{d}^{1}, F_{e}^{1}$, or $F_{f}^{1}$. Now, suppose that $\left(H_{1}^{\prime}, x, a\right)$ and $\left(H_{1}^{\prime}, x, b\right)$ contain $F_{b}^{1}$, so in $H_{1}^{\prime}$ there is a tough path from $x$ to $a$ and a tough path from $x$ to $b$. If the short tough path from $x$ to $a$ contains the short tough path from $x$ to $b$, then in $\left(H^{\prime}, x, y\right)$ we have $F_{e}^{2}$. Otherwise, we have $x$ and $y$ in different components of $H^{\prime}-\{a, b\}$ and so we have $F_{a}^{1}$ or $F_{c}^{1}$ in $\left(H^{\prime}, x, y\right)$, with $x=c_{0}^{1}$. If $\left(H_{1}^{\prime}, x, a\right)$ contains $F_{a}^{1}$ with $x_{1}=a$ and $\left(H_{1}^{\prime}, x, b\right)$ contains $F_{a}^{1}$ with $x_{1}=b$, then $\left(H^{\prime}, x, y\right)$ contains $F_{b}^{2}$. Now, suppose ( $H_{1}^{\prime}, x, a$ ) contains $F_{c}^{1}$ or $F_{a}^{1}$ with $x_{1}=a$ and $\left(H_{1}^{\prime}, x, b\right)$ contains $F_{c}^{1}$ with $x_{1}=b$. The case when $\left(H_{1}^{\prime}, x, b\right)$ contains $F_{c}^{1}$ or $F_{a}^{1}$ with $x_{1}=b$ and ( $H_{1}^{\prime}, x, a$ ) contains $F_{c}^{1}$ with $x_{1}=a$ is similar. If the tough paths starting at $a$ and $b$ do not intersect, then $\left(H^{\prime}, x, y\right)$ contains $F_{b}^{2}$. If $a b$ is an edge of one of the tough paths, then $\left(H^{\prime}, x, y\right)$ contains $F_{d}^{2}$. If the tough paths starting at $a$ and $b$ intersect, but $a b$ is not an edge of one of the tough paths, then $\left(H^{\prime}, x, y\right)$ contains $F_{f}^{1}$ or $F_{d}^{1}$.
6. If $H_{2}^{\prime}$ does not have an ( $a, y$ )-Hamiltonian path, then by the induction hypothesis, $\left(H_{2}^{\prime}, y, a\right)$ contains an $F_{x}^{2} \in \mathscr{F}^{2}$. Likewise, if $H_{2}^{\prime}$ does not have an $(b, y)$-Hamiltonian path, then by the induction hypothesis, $\left(H_{2}^{\prime}, y, b\right)$ contains an $F_{x}^{2} \in \mathscr{F}^{2}$. As above, if $\left(H_{2}^{\prime}, y, a\right)$ or $\left(H_{2}^{\prime}, y, b\right)$ contains an $F_{x}^{1} \in \mathscr{F}^{1} \subset \mathscr{F}^{2}$, then $\left(H^{\prime}, x, y\right)$ contains an $F_{x}^{2} \in \mathscr{F}^{2}$. If $\left(H_{2}^{\prime}, y, a\right)$ and/or $\left(H_{2}^{\prime}, y, b\right)$ contains $F_{b}^{2}$ or $F_{c}^{2}$, then since $x$ will be in the same component as $a$ and/or $b$, respectively, then $\left(H^{\prime}, x, y\right)$ will also contain $F_{b}^{2}$ or $F_{c}^{2}$. Similarly, if $\left(H_{2}^{\prime}, y, a\right)$ and/or $\left(H_{2}^{\prime}, y, b\right)$ contains $F_{d}^{2}$ or $F_{e}^{2},\left(H^{\prime}, x, y\right)$ will also contain $F_{d}^{2}$ or $F_{e}^{2}$, respectively.
7. Without loss of generality, assume $H_{1}^{\prime}$ only has an $(x, b)$-Hamiltonian path while $H_{2}^{\prime}$ only has a $(b, y)$-Hamiltonian path. Then $\left(H_{1}^{\prime}, x, a\right)$ and $\left(H_{2}^{\prime}, y, a\right)$ contain an $F_{x}^{2} \in \mathscr{F}^{2}$. But since $H_{1}^{\prime}$ has an $(x, b)$-Hamiltonian path while $H_{2}^{\prime}$ has a $(b, y)$ Hamiltonian path, then $\left(H_{1}^{\prime}, x, b\right)$ and $\left(H_{2}^{\prime}, y, b\right)$ cannot contain an $F_{x}^{2} \in \mathscr{F}^{2}$. Then, $\left(H_{1}^{\prime}, x, a\right)$ must contain $F_{a}^{1}, F_{b}^{1}, F_{c}^{1}$ and $\left(H_{2}^{\prime}, y, a\right)$ must contain $F_{a}^{1}, F_{b}^{1}, F_{c}^{1}$, or $F_{e}^{2}$. If $\left(H_{1}^{\prime}, x, a\right)$ and $\left(H_{2}^{\prime}, y, a\right)$ both contain $F_{a}^{1}$, then $\left(H^{\prime}, x, y\right)$ contains $F_{c}^{2}$. If $\left(H_{1}^{\prime}, x, a\right)$ and $\left(H_{2}^{\prime}, y, a\right)$ both contain $F_{c}^{1}$, then $\left(H^{\prime}, x, y\right)$ contains $F_{d}^{1}$. If $\left(H_{1}^{\prime}, x, a\right)$ and $\left(H_{2}^{\prime}, y, a\right)$ both contain $F_{b}^{1}$, then there is a tough path from $x$ to $y$ and hence $\left(H^{\prime}, x, y\right)$ also contains $F_{b}^{1}$. If one contains $F_{b}^{1}$ and the other
contains $F_{a}^{1}$ or $F_{c}^{1}$, then we have $F_{c}^{1}$ in $\left(H^{\prime}, x, y\right)$. If one contains $F_{a}^{1}$ and the other contains $F_{c}^{1}$, then we have $F_{d}^{1}$ in $\left(H^{\prime}, x, y\right)$. If $\left(H_{2}^{\prime}, y, a\right)$ contains $F_{e}^{2}$ and the other contains $F_{a}^{1}$ or $F_{c}^{1}$, then we have $F_{d}^{2}$ in $\left(H^{\prime}, x, y\right)$. Lastly if $\left(H_{2}^{\prime}, y, a\right)$ contains $F_{e}^{2}$ and the other contains $F_{b}^{1}$, then we have $F_{e}^{2}$ in $\left(H^{\prime}, x, y\right)$.

## Chapter 4

## Using 2HP to Characterize HP and 1HP

As mentioned earlier in this dissertations, the Hamiltonian path problem on 2-trees is closely related to 2 HP on 2-trees, and will use the results from the previous chapter on 2 HP to prove necessary and sufficient conditions for which a 2 -tree will not have a Hamiltonian path in Theorem 4.1.15 in section 4.1. We will begin as in the previous chapter by defining a family, $\mathscr{H}$, of 2 -trees which will not have a Hamiltonian path. In Theorem4.1.15, we will prove that any 2 -tree with scattering number at most one, which does not contain one of the graphs in $\mathscr{H}$ as an induced sub-2-tree, will have a Hamiltonian path. In section 4.2, we will use the results from 2HP on 2-trees to prove necessary and sufficient conditions for which a 2 -tree with a specified vertex, $x_{2}$, will not have an $x_{2}$-Hamiltonian path in Theorem4.2.12. We will begin as in the previous chapters by defining a family, $\mathscr{I}$, of 2-trees, with a specified vertex, $x_{2}$, which will not have an $x_{2}$-Hamiltonian path. In Theorem 4.2.12, we will prove that any 2-tree with scattering number at most one, which does not contain one of the graphs in $\mathscr{I}$ as an induced sub-2-tree, will have an $x_{2}$-Hamiltonian path.

### 4.1 Hamiltonian Path Problem

Definition 4.1.1. Define $\mathscr{H}=\left\{H_{a}, H_{b}, H_{c}, H_{d}, H_{e}, H_{f}, H_{g}\right\}$ where:
(a) $H_{a}$ is a 2-tree which contains three 3-edges, ab, cd, and ef, none of which are incident, such that:
(i) cd and ef are in the same component of $G-\{a, b\}$
(ii) $a b$ and $c d$ are in the same component of $G-\{e, f\}$
(iii) $a b$ and $e f$ are in the same component of $G-\{c, d\}$


Figure 4.1: A general example of $H_{a}$ where the dotted section of the graph represents any 2 -tree with scattering number at most one to preserve generality


Figure 4.2: A specific example of $H_{a}: P_{15}^{2}$ with three pairs of simplicial vertices added


Figure 4.3: A specific example of $H_{a}$
(b) $H_{b}$ is a 2-tree which contains exactly two 3-edges, ab and cd, such that:
(i) $a b$ is not incident to $c d$,
(ii) $a b$ and cd are each adjacent to two simplicial vertices, and
(iii) $N(a b)$ contains two simplicial vertices.


Figure 4.4: A general example of $H_{b}$. To preserve generality, the dotted section of the graph represents any 2 -tree with scattering number at most one.
(c) $H_{c}$ is a 2-tree which contains three 3-edges such that for one of the 3-edges, ef,:
(i) Two of the three components of $G-\{e, f\}$ contain a 3-edge, and
(ii) e is adjacent to three simplicial vertices which are all in different components of $G-\{e, f\}$.


Figure 4.5: A general example of $H_{c}$ where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality
(d) $H_{d}$ is constructed from
$G=D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 2$, by :
(i) Amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$,
(ii) Adding a false twin, $x_{2}^{\prime}$, of $x_{2}$, and
(iii) Adding a simplicial vertex adjacent to $c_{0}^{1} c_{1}^{1}$.


Figure 4.6: A general example of $H_{d}$ :
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots . ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$
$m \geq 2$, with an $x_{2}$-2-path amalgamated with $t_{0}^{2} c_{1}^{2}$, and a simplicial vertex added to $c_{0}^{1} c_{1}^{1}$
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ is shown in gray to preserve generality


Figure 4.7: Specific example of $H_{d}: D_{5}(\{1,3,4\}) ;(t, 1) ; D_{0}$ with an $x_{2}$-2-path amalgamated with $t_{0}^{2} c_{1}^{2}$, and a simplicial vertex added to $c_{0}^{1} c_{1}^{1}$
(e) $H_{e}$ is constructed from $G=D_{0}^{1} ;(t, \ell) ; D_{0}^{2}$, for $l \geq 2$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{2} c_{1}^{2}$,
(iii) Adding a false twin, $x_{1}^{\prime}$, of $x_{1}$, and
(iv) Adding a false twin, $x_{2}^{\prime}$, of $x_{2}$.


Figure 4.8: Specific example of $H_{e}$ with $\ell=2$
(f) $H_{f}$ is constructed from $G=D_{0}^{1} ;(t, 1) ; D_{0}^{2}$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{2} c_{1}^{2}$,
(iii) Adding a false twin, $x_{1}^{\prime}$, of $x_{1}$,
(iv) Adding a false twin, $x_{2}^{\prime}$, of $x_{2}$, and
(v) Adding a simplicial vertex adjacent to $t_{0}^{1} t_{0}^{2}$.


Figure 4.9: Example of $H_{f}$
(g) $H_{g}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 3$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$,
(iii) Adding a false twin, $x_{1}^{\prime}$, of $x_{1}$, and
(iv) Adding a false twin, $x_{2}^{\prime}$, of $x_{2}$.

OR
$H_{g}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 3$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $b_{0}^{m} c_{1}^{m}$,
(iii) Adding a false twin, $x_{1}^{\prime}$, of $x_{1}$, and
(iv) Adding a false twin, $x_{2}^{\prime}$, of $x_{2}$.


Figure 4.10: A general example of $H_{g}$ :
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$ $m \geq 3$, with amalgamated $x_{1}$ and $x_{2}$-2-paths where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 4.11: Specific example of $H_{g}: D_{0} ;(t, 1) ; D_{5}(\{1,3,4\}) ;(t, 1) ; D_{0}$ by amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$, amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$, and adding false twins, $x_{1}^{\prime}, x_{2}^{\prime}$ of $x_{1}, x_{2}$


Figure 4.12: A general example of $H_{g}$ :
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$
$m \geq 3$, with amalgamated $x_{1}$ and $x_{2}$-2-paths.
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 4.13: An example of $H_{g}: D_{0} ;(t, 1) ; D_{5}(\{1,3,4,5\}) ;(b, 1) ; D_{0}$ with amalgamated $x_{1}$ and $x_{2}$-2-paths

Note that in $\mathscr{H}$, all graphs have at least two 3 -edges. In this section we will be discussing 2 -trees which contain at least two 3 -edges, but no $t$-edge for $t \geq 4$. From Lemma 1.2 .24 , if $G$ is a 2 -tree which contains a $t$-edge for $t \geq 4$, then $G$ does not contain a Hamiltonian path. Furthermore, from Lemma 1.2 .23 if $G$ is a 2 -tree which only contains $t$-edges for $t \leq 2$, then $G$ is 1 -tough and hence contains a Hamiltonian path. 2-trees with exactly one 3 -edge and no $t$-edges for $t \geq 4$ have a Hamiltonian path, by Lemma 4.1.2 below.

Lemma 4.1.2. If $G$ is a 2-tree which contains exactly one 3-edge and no t-edges for $t \geq 4$, then $G$ has a Hamiltonian path.

Proof. Let $a b$ be the 3-edge in $G$. Let $C_{1}, C_{2}, C_{3}$ be the components of $G-\{a, b\}$. From [24], $G\left[C_{1} \cup\{a, b\}\right]$ is a 2-tree, and since $G$ contains no other 3-edges and no $t$-edges for $t \geq 4$, then it is also 1-tough. Hence, $G\left[C_{1} \cup\{a, b\}\right]$ contains a Hamiltonian cycle $C$ which contains all 1-edges in $G\left[C_{1} \cup\{a, b\}\right]$. Since $a b$ is a 1-edge in $G\left[C_{1} \cup\{a, b\}\right]$, then $a b$ lies on $C$, so $G\left[C_{1} \cup\{a, b\}\right]$ has an $(a, b)$-Hamiltonian path, $P . G\left[C_{1} \cup C_{2} \cup\{a, b\}\right]$ is also a 1-tough 2-tree, so there is a $b$-Hamiltonian path, $P^{\prime}$, in $G\left[C_{1} \cup C_{2} \cup\{a, b\}\right]$. Taking $P-a$ followed by $P^{\prime}$ yields a Hamiltonian path in $G$.

Lemma 4.1.3. Let $G$ be a 1-tough 2-tree with tough path $P=\left(v_{1}, v_{2}, \ldots, v_{n-1}, v_{n}\right)$. If $H$ is constructed by adding a simplicial vertex adjacent to $v_{i} v_{i+1}$ and a simplicial vertex adjacent to $v_{j} v_{j+1}, i<j$, then $H$ does not contain a Hamiltonian path.

Proof. Let $S_{v_{i}, v_{j+1}}=\left\{v_{i}, v_{i+1}, \ldots, v_{j}, v_{j+1}\right.$. Since $P$ is a tough path, $G-S_{v_{i}, v_{j+1}}=$ $\left|S_{v_{i}, v_{j+1}}\right|$. Then $c\left(H-S_{v_{i}, v_{j+1}}\right)=\left|S_{v_{i}, v_{j+1}}\right|+2$ and hence $s(H) \geq 2$ and so $H$ does not have a Hamiltonian path.

Since in Theorem 4.1.15, we assume scattering number at most one, we do not include in $\mathscr{H}, \mathscr{F}^{2}$, or $\mathscr{I}$, graphs which have the properties of Lemma 4.1.3. However, in the cases of $H_{d}, H_{e}, H_{f}$, and $H_{g}$, if the $x_{2}$-2-path, and likewise $x_{1}$-2-path, that is amalgamated to our graphs is a diamond with simplicial vertex $x_{2}$, then the graph produced will have scattering number at least two. In the future, we would like to characterize the 2 -trees which have scattering number two or more, such that we could prove characterization theorems for HP, 1HP, and 2HP on 2-trees which rely only on a forbidden family and do not include scattering number conditions.

Corollary 4.1.4. If $H$ is constructed from an $\ell$-string of diamonds, by adding two simplicial vertices, each adjacent to a different edge on the central path, then $H$ does not contain a Hamiltonian path.


Figure 4.14: An example of Lemma 4.1.3. $D_{0} ;(t, 1) ; D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with a simplicial vertex added to $c_{0}^{1} z_{1}$ and a simplicial vertex added to $z_{2} c_{1}^{m}$

Lemma 4.1.5. The graph $H_{a}$ does not have a Hamiltonian path.
Proof. Suppose $G$ has a Hamiltonian path, $P$. By assumption, $c(G-\{a, b\})=3$, and $c d$ and $e f$ are in the same component of $G-\{a, b\}$. Hence, at least one endpoint, $x_{1}$, of $P$ must lie in a different component of $G-\{a, b\}$ than $c d$ and $e f$. Likewise at least one endpoint, $x_{2}$, of $P$ must lie in a different component of $G-\{c, d\}$ than $a b$ and $e f$, and at least one endpoint, $x_{3}$ of $P$ must lie in a different component of $G-\{e, f\}$ than $a b$ and $c d$. But since $x_{1}$ is in a different component of $G-\{a, b\}$ than $c d$ and
$e f$, then $x_{1}$ is in the same component as $a b$ in $G-\{c, d\}$ and in $G-\{e, f\}$, and so $x_{1} \neq x_{2}, x_{3}$. Similarly, $x_{2} \neq x_{3}$, and $P$ must have three distinct endpoints. Hence $G$ does not have a Hamiltonian path.

Lemma 4.1.6. The graph $H_{b}$ does not have a Hamiltonian path.
Proof. Let $a b$ and $c d$ be the only two 3 -edges in $G$. Let $s_{a b}^{1}$ be a simplicial vertex adjacent to $a b$ and $s_{c d}^{1}$ be a simplicial vertex adjacent to $c d$. Then $G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}$ is a 1-tough 2-tree. Furthermore, since $a b$ was adjacent to four simplicial vertices and $a b$ is not incident to $c d$, then in $G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}, a$ and $b$ are each adjacent to two simplicial vertices, none of which can be $c$ or $d$. Hence, $\left(G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}, a, c\right),\left(G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}, a, d\right)$, $\left(G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}, b, c\right)$, and $\left(G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}, b, d\right)$ all contain an induced forbidden sub-2tree $F_{a}^{1} \in \mathscr{F}^{1}$ from Chapter 2. Thus, $G-\left\{s_{a b}^{1}, s_{c d}^{1}\right\}$ does not have an $(a, c),(a, d)$, $(b, c)$, or $(b, d)$-Hamiltonian path. Hence, from Lemma 1.2.25, $G$ does not contain a Hamiltonian path.

Lemma 4.1.7. The graph $H_{c}$ does not have a Hamiltonian path.
Proof. Let $a b$ and $c d$ be 3-edges which lie in different components of $G-\{e, f\}$, and suppose that $G$ contains a Hamiltonian path, $P$. Then $c(G-\{a, b\})=3$, and $c d$ and $e f$ are in the same component of $G-\{a, b\}$. Hence, at least one endpoint of $P$ must lie in a different component of $G-\{a, b\}$ than $c d$ and $e f$. Likewise at least one endpoint of $P$ must lie in a different component of $G-\{c, d\}$ than $a b$ and $e f$. Let $u, v, w$ be the simplicial vertices adjacent to $e$. None of $u, v, w$ can be an endpoint of $P$ as they will either be in the same component of $G-\{a, b\}$ and $G-\{c, d\}$ as ef or they will be one of $\{a, b, c, d\}$. Thus, on $P, u, v, w$ must all be preceded or followed by $e$. But then $e$ must appear on $P$ at least twice, and hence $G$ does not have a Hamiltonian path.

Lemma 4.1.8. The graph $H_{d}$ does not have a Hamiltonian path.
Proof. Suppose $x_{2}, x_{2}^{\prime}$ are adjacent to $c d$ and $x_{1}^{\prime}$ the simplicial vertex which was added to $G$ which was made adjacent $c_{0}^{1} c_{1}^{1}$. Suppose $c d=t_{0}^{m} c_{1}^{m}$, and $S_{G}$ is the set of all vertices on the central path of $G$. Then, $c\left(H-S_{G}\right)=\left|S_{G}\right|+1$ and since $x_{2}$ and $x_{2}^{\prime}$ are adjacent to $t_{0}^{m} c_{0}^{m}$, then, $c\left(H-\left(S_{G} \cup\left\{t_{0}^{m}\right\}\right)\right)=\left|S_{G} \cup\left\{t_{0}^{m}\right\}\right|+2$, and $H$ has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25. $H$ has a Hamiltonian path iff $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ has a $\left(c_{0}^{1}, c\right),\left(c_{0}^{1}, d\right),\left(c_{1}^{1}, c\right)$, or $\left(c_{1}^{1}, d\right)$ Hamiltonian path. But no such path exists in $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ by Theorem 3.1.24, since
$\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, c_{0}^{1}, c\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, c_{0}^{1}, d\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, c_{1}^{1}, c\right)$, and $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, c_{0}^{1}, d\right)$ have induced subtrees from $F_{a}^{1} \in \mathscr{F}^{1}$ or $F_{c}^{1} \in \mathscr{F}^{1}$. So $H$ does not have a Hamiltonian path.

Lemma 4.1.9. The graph $H_{e}$ does not have a Hamiltonian path.
Proof. Suppose $x_{1}, x_{1}^{\prime}$ are adjacent to $a b$ and $x_{2}, x_{2}^{\prime}$ are adjacent to $c d$. If $a b=t_{0}^{1} c_{0}^{1}$, $c d=t_{0}^{2} c_{1}^{2}$, and $S=\left\{t_{0}^{1}, c_{0}^{1}, t_{0}^{2}, c_{1}^{2}, z_{1}\right\}$, then $c(H-S)=7=|S|+2$ and hence $H$ has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25. $H$ has a Hamiltonian path iff $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ has a $(a, c),(a, d),(b, c)$, or $(b, d)$-Hamiltonian path. But no such path exists in $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ by Theorem 3.1.24, since $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, a, c\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, a, d\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, b, c\right)$, and $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, b, d\right)$ have induced subtrees from $F_{f}^{1} \in \mathscr{F}^{1}$. So $H$ does not have a Hamiltonian path.

## Lemma 4.1.10. The graph $H_{f}$ does not have a Hamiltonian path.

Proof. Suppose $x_{1}, x_{1}^{\prime}$ are adjacent to $a b$ and $x_{2}, x_{2}^{\prime}$ are adjacent to $c d$. If $a b=t_{0}^{1} c_{0}^{1}$, $c d=t_{0}^{2} c_{1}^{2}$, and $S=\left\{t_{0}^{1}, c_{0}^{1}, t_{0}^{2}, c_{1}^{2}, z_{1}\right\}$, then $c(H-S)=7=|S|+2$ and hence $H$ has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25, $H$ has a Hamiltonian path iff $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ has a $(a, c),(a, d),(b, c)$, or $(b, d)$-Hamiltonian path. But no such path exists in $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ by Theorem 3.1.24, since $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, a, c\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, a, d\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, b, c\right)$, and $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, b, d\right)$ have induced subtrees from $F_{e}^{1} \in \mathscr{F}^{1}$. So $H$ does not have a Hamiltonian path.

Lemma 4.1.11. The graph $H_{g}$ does not have a Hamiltonian path.
Proof. Suppose $x_{1}, x_{1}^{\prime}$ are adjacent to $a b$ and $x_{2}, x_{2}^{\prime}$ are adjacent to $c d$. Suppose $a b=t_{0}^{1} c_{0}^{1}$ and $c d=t_{0}^{m} c_{1}^{m}$, and $S_{G}$ is the set of all vertices on the central path of $G$. Then, $c\left(H-S_{G}\right)=\left|S_{G}\right|$ and since $x_{1}$ and $x_{1}^{\prime}$ are adjacent to $t_{0}^{1} c_{0}^{1}, x_{2}$ and $x_{2}^{\prime}$ are adjacent to $t_{0}^{m} c_{0}^{m}$, then, $c\left(H-\left(S_{G} \cup\left\{t_{0}^{1}, t_{0}^{m}\right\}\right)\right)=\left|S_{G} \cup\left\{t_{0}^{1}, t_{0}^{m}\right\}\right|+2$, and $H$ has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25, $H$ has a Hamiltonian path iff $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ has a $(a, c),(a, d),(b, c)$, or $(b, d)$-Hamiltonian path. But no such path exists in $H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}$ by Theorem 3.1.24, since $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, a, c\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, a, d\right),\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, b, c\right)$, and $\left(H-\left\{x_{1}^{\prime}, x_{2}^{\prime}\right\}, b, d\right)$ have induced subtrees from $F_{c}^{1} \in \mathscr{F}^{1}$. So $H$ does not have a Hamiltonian path.

Similar to the reduced graph of a 2-tree with scattering number one with fixed endpoints, we will create a reduced graph of a 2 -tree with scattering number one, without fixed endpoints, as follows.

Definition 4.1.12. Given a 2-tree, $G$ with $s(G)=1$, then the reduced graph of $G$, is formed using the following algorithm:

1. For every 3-edge ab with components of $G-\{a, b\}, C_{a b}^{1}, C_{a b}^{2}, C_{a b}^{3}$, if $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ is 1-tough then replace $C_{a b}^{i}$ with a simplicial vertex adjacent to $a b$.
2. Let $w \in S_{1}(G)$, and $x, y$ the neighbors of $w$. If $x y$ is not a 3-edge, remove $w$ and turn the edge $x y$ into a forced edge.
3. Repeat (2) for all $w \in S_{1}(G)$. Define the resulting graph to be $G_{1}$.
4. For $i \geq 2$, let $S_{i}=S_{1}\left(G_{i-1}\right)$ where $G_{i-1}$ is the graph formed by repeating (2) for $G=G_{i-1}$ and for all $w \in S_{1}\left(G_{i-1}\right)$.

Repeat (4) for all $i=2,3, \ldots, j$ for $j$ such that $S_{j}=\emptyset$ or for all $s \in S_{j}, N(s)$ is a 3-edge. This is the reduced graph of $G$.
For $F$ the set of forced edges, let $(H, F)$ denote the reduced graph of $G$, for $G$ a 2-tree containing at least one 3-edge.

Since simplicial vertices in 2-trees are not adjacent [7], when we remove the vertices in each $S_{i}$, regardless of order, we will end up with the same graph.

We will form the corresponding caterpillar representation of $G$ as in Chapter 2.
Definition 4.1.13. Let $G$ be a 2-tree with $s(G)=1$ and $(H, F)$ be the reduced graph of $G$. The caterpillar representation, $H^{\prime}$, of $G$ is created by adding $|F|$ simplicial vertices to $(H, F)$, making each vertex adjacent to exactly one forced edge, and changing all forced edges back to regular edges.

Lemma 4.1.14. Let $G$ be a 2-tree with $s(G)=1,(H, F)$ be the reduced graph of $G$, and $H^{\prime}$ the caterpillar representation of $G$. Then the following are equivalent:

1. G has a Hamiltonian path,
2. $H^{\prime}$ has a Hamiltonian path, and
3. $(H, F)$ has a Hamiltonian path which uses all of the edges in $F$.

Proof. (1) $\Longrightarrow(2)$
Suppose $H^{\prime}$ does not have a Hamiltonian path. Since $H^{\prime}$ is an induced sub-2-tree of $G$, then by Corollary 2.1.10, $G$ does not have a Hamiltonian path.
$(2) \Longrightarrow(3)$
Suppose $H^{\prime}$ has a Hamiltonian path, $P$. Let $v$ be a simplicial vertex with neighbors $u$ and $w$, such that $u w$ is not a 3-edge. Then $P=\left(x_{1}, \ldots, u, v, w, \ldots, x_{2}\right)$ or $P=\left(x_{1}, \ldots, w, v, u, \ldots, x_{2}\right)$. Furthermore, because $H^{\prime}$ is a 2-tree, then $u w \in E\left(H^{\prime}\right)$, and from the reduction algorithm $u w \in F$. Replacing $(u, v, w)$ or $(w, v, u)$ by $(u, w)$ in $P$, then $P$ is a Hamiltonian path using exactly one forced edge. Repeating this process for all $\left.s \in S_{1} \ell_{( } H^{\prime}\right)$, such that $s$ is not adjacent to a 3-edge, then $P$ will be a Hamiltonian path in $(H, F)$.
$(3) \Longrightarrow(1)$ Suppose $(H, F)$ has a Hamiltonian path, $P$, which uses all of the edges in $F$. Let $a b$ be a 3-edge in $G$ with components of $G-\{a, b\}, C_{a b}^{1}, C_{a b}^{2}, C_{a b}^{3}$, where $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ is 1-tough. In $(H, F), C_{a b}^{i}$ has been replaced by the simplicial vertex, $v_{a b}^{i}$. Since $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ is 1-tough, then $G\left[C_{a b}^{i} \cup\{a, b\}\right]$ has a Hamiltonian cycle, $C$ using all 1-edges in $G\left[C_{a b}^{i} \cup\{a, b\}\right]$. Hence, since $a b$ is a 1-edge in $G\left[C_{a b}^{i} \cup\{a, b\}\right]$, then there is an $(a, b)$-Hamiltonian path $P^{\prime}$ in $G\left[C_{a b}^{i} \cup\{a, b\}\right]$. So, if $v_{a b}^{i}$ is on the interior of $P$ in $(H, F)$, then we can replace $\left(a, v_{a b}^{i}, b\right)$ on $P$ with $P^{\prime}$. If $v_{a b}^{i}$ is an endpoint of $P$ in $(H, F)$, then we can replace $\left(v_{a b}^{i}, b\right)$ or $\left(v_{a b}^{i}, a\right)$ on $P$ with $P^{\prime}-a$ or $P^{\prime}-b$, respectively. Now, consider $x y \in F$. In $G, x y$ is incident to at least one vertex, $v$, which is not in $(H, F)$ so that $c(G-\{x, y\})=2$. Let $C_{v}$ be the component of $G-\{x, y\}$ which contains $v$. From [24], $G\left[C_{v} \cup x y\right]$ is a 2-tree, and from the reduction algorithm, $G\left[C_{v} \cup x y\right]$ must be 1-tough and so it contains a Hamiltonian cycle $C$. In $G\left[C_{v} \cup x y\right], x y$ is a 1-edge and hence lies on $C$. Thus, there is a Hamiltonian path, $P^{\prime \prime}$, in $G\left[C_{v} \cup x y\right]$ from $x$ to $y$, and we can replace $x y$ in $P$ with $P^{\prime}$. Repeating these processes for all $f \in F$ and all 3-edges, $c d$ and all $C_{c d}^{i}$ such that $G\left[C_{c d}^{i} \cup\{c, d\}\right]$ is 1-tough, will yield a Hamiltonian path in $G$.

Theorem 4.1.15. If $G$ is a 2-tree, then $G$ has a Hamiltonian path iff $s(G) \leq 1$ and $G$ does not contain any $H \in \mathscr{H}$ as an induced sub-2-tree.

Proof. $\Longrightarrow$
If $s(G) \geq 2$, then $G$ is not 1-path-tough, and $G$ does not contain a Hamiltonian path.

1. If $G=H_{a}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.5. If $G$ contains $H_{a}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
2. If $G=H_{b}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.6. If $G$ contains $H_{b}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
3. If $G=H_{c}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.7. If $G$ contains $H_{c}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
4. If $G=H_{d}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.8. If $G$ contains $H_{d}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
5. If $G=H_{e}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.9. If $G$ contains $H_{e}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
6. If $G=H_{f}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.10. If $G$ contains $H_{f}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
7. If $G=H_{g}$, then $G$ does not have a Hamiltonian path by Lemma 4.1.11. If $G$ contains $H_{g}$ as an induced sub-2-tree, then $G$ does not have a Hamiltonian path by Corollary 2.1.10.
$\Longleftarrow$
Suppose $G$ does not have a Hamiltonian path, but that $s(G) \leq 1$. Since $s(G) \leq 1$, then $G$ contains no $t$-edges for $t \geq 4$. If $G$ contains $m 3$-edges for $m \leq 1$, then $G$ has a Hamiltonian path. So $G$ has $m$-edges for $m \geq 2$. Let $H^{\prime}$ be the caterpillar representation of $G$. Then $s\left(H^{\prime}\right) \leq 1$ and $H^{\prime}$ does not have a Hamiltonian path by Lemma 4.1.14. Suppose that $H^{\prime}$ does not contain $H_{a}$. Denote the 3-edges in $H^{\prime}$, $S_{i}=s_{i} s_{i}^{\prime}$ for all $1 \leq i \leq m$. Then the 3-edges in $H^{\prime}$ can be ordered $S_{1}, S_{2}, \ldots . S_{m}$ so that in $H^{\prime}-S_{1}$, all $s_{i}, s_{i}^{\prime} \neq s_{1}, s_{1}^{\prime}$ are in the same component, in $H^{\prime}-S_{m}$, all $s_{i}, s_{i}^{\prime} \neq s_{m}, s_{m}^{\prime}$ are in the same component, and such that for all $i \in\{1,2, \ldots, m-2\}, s_{i}$ and $s_{i+2}$ are in different components of $H^{\prime}-S_{i+1}$. From the reduction algorithm, $S_{1}$ and $S_{m}$ are each
adjacent to two simplicial vertices. Furthermore, since $c\left(G-S_{i}\right)=3$, then, if $H^{\prime}$ has a Hamiltonian path, one of the simplicial vertices adjacent to $S_{1}$ must be an endpoint of the path, and likewise, one of the simplicial vertices adjacent to $S_{m}$ must be an endpoint of the path. Without loss of generality, label one of the simplicial vertices adjacent to $S_{1}, x_{1}$, and one of the simplicial vertices adjacent to $S_{m}, x_{2}$. So since $H^{\prime}$ does not have a Hamiltonian path, then $H^{\prime}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. So, by Theorem 3.2.10, $\left(H^{\prime}, x_{1}, x_{2}\right)$ must contain an $F^{2} \in \mathscr{F}^{2}$ as an induced sub-2-tree. Also, since $x_{1}$ and $x_{2}$ are simplicial, then ( $H^{\prime}, x_{1}, x_{2}$ ) must contain $F_{a}^{2}$, $F_{b}^{2}, F_{c}^{2}, F_{d}^{2}, F_{f}^{1}, F_{e}^{1}$, or $F_{d}^{1}$. Adding a false twin of $x_{1}$ and $x_{2}$ and removing the labels, we will get the forbidden induced sub-2-trees for $H^{\prime}$ without fixed endpoints. Using this process on $F_{f}^{1}$ forms $H_{e}$, on $F_{e}^{1}$ forms $H_{f}$, on $F_{d}^{1}$ forms $H_{g}$, on $F_{a}^{2}$ forms $H_{a}$, and on $F_{c}^{2}$ forms $H_{c}^{2}$. For $F_{d}^{2}$ and $F_{b}^{2}$, we can leave $x_{1}$ and just remove the label, as the $x_{1}$ is amalgamated with a 3 -edge and hence already forcing $x_{1}$ or the other simplicial vertex as an end. Using this process on $F_{d}^{2}$ forms $H_{d}^{2}$, and on $F_{b}^{2}$ forms $H_{b}$.

### 4.2 1HP

Definition 4.2.1. Define $\mathscr{I}=\left\{I_{a}, I_{b}, I_{c}, I_{d}, I_{e}, I_{f}, I_{g}, I_{h}, I_{i}, I_{j}\right\}$ where:
(a) $I_{a}$ is a 2-tree with vertex $x_{2}$, which contains two 3-edges, ab and cd, which are not incident, such that:
(i) $c d$ and $x_{2}$ are in the same component of $G-\{a, b\}$, and
(ii) ab and $x_{2}$ are in the same component of $G-\{c, d\}$, or
(iii) $x_{2} \in\{a, b, c, d\}$.


Figure 4.15: A general example of $I_{a}$ where the dotted section of the graph represents any 2 -tree with scattering number at most one to preserve generality
(b) $I_{b}$ is a 2-tree which contains exactly one 3-edge, ab, such that:
(i) $N(a b)-x_{2}$ contains two simplicial vertices,
(ii) $N(a)-x_{2}$ contains two simplicial vertices, and
(iii) $N(b)-x_{2}$ contains two simplicial vertices.


Figure 4.16: General example of $I_{b}$ where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality
(c) $I_{c}$ is a 2-tree which contains at least two 3-edges such that for one of the 3-edges, $e f$ :
(i) One component of $G-\{e, f\}$ contains a 3-edge, which is in a different component of $G-\{e, f\}$ than $x_{2}$, and
(ii) $e$ is adjacent to three simplicial vertices in $G-x_{2}$.


Figure 4.17: General example of $I_{c}$ where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality
(d) $I_{d}$ is constructed from an $\ell$ string of diamonds by adding a simplicial vertex adjacent to $c_{0}^{1} c_{1}^{1}$ and where $x_{2}=c_{s_{m}+1}^{m}$.


Figure 4.18: A general example of $I_{d}$ :
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$
with added simplicial vertex adjacent to $c_{0}^{1} c_{1}^{1}$, where $x_{2}=c_{s_{m}+1}^{m}$, and where $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(x_{m-1}, \ell_{m-1}\right) ; D_{s_{m}}^{m}\left(R_{m}\right)$, is shown in gray to preserve generality


Figure 4.19: Specific example of $I_{d}: D_{5}(\{1,3,4,5\})$ with $x_{2}=c_{6}^{1}$
(e) $I_{e}$ is constructed from $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$, $m \geq 2$, by amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$ and adding a simplicial vertex adjacent to $c_{0}^{1} c_{1}^{1}$.


Figure 4.20: A general example of $I_{e}$ :
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$
$m \geq 2$, with an amalgamated $x_{2}$-2-path and an added simplicial vertex adjacent to $c_{0}^{1} c_{1}^{1}$
where $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$, with $x_{1}=c_{0}^{1}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 4.21: Specific example of $I_{e}$ : $D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with an added simplicial vertex adjacent to $c_{0}^{1} c_{1}^{1}$ and an $x_{2}$-2-path amalgamated with $t_{0}^{2} c_{1}^{2}$
(f) $I_{f}$ is constructed from
$D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 2$, with $x_{1}=$ $c_{0}^{1}$, by amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$ and adding a false twin $x_{2}^{\prime}$ of $x_{2}$.


Figure 4.22: A general example of
$I_{f}: D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$ $m \geq 2$, with an amalgamated $x_{2}$-2-path, such that $x_{1}=c_{0}^{1}$, and where $D_{s_{1}}^{1}\left(R_{1}\right) ;\left(x_{1}, \ell_{1}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$, with $x_{1}=c_{0}^{1}$ and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 4.23: Specific example of $I_{f}: D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with $x_{1}=c_{0}^{1}$ and an $x_{2}-2-$ path amalgamated with $t_{0}^{2} c_{1}^{2}$
(g) $I_{g}$ is constructed from $D_{0}^{1} ;(t, \ell) ; D_{0}^{2}$, for $l \geq 2$ by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{2} c_{1}^{2}$, and
(iii) Adding a false twin $x_{1}^{\prime}$ of $x_{1}$.


Figure 4.24: Example of $I_{g}$
(h) $I_{h}$ is constructed from $D_{0}^{1} ;(t, 1) ; D_{0}^{2}$ by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{2} c_{1}^{2}$,
(iii) Adding a false twin $x_{1}^{\prime}$ of $x_{1}$, and
(iv) Adding a simplicial vertex adjacent to $t_{0}^{1} t_{0}^{2}$.


Figure 4.25: Example of $I_{h}$.
(i) $I_{i}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{1}}^{2}\left(R_{1}\right) ;\left(x_{1}, \ell_{2}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-2}\right) ; D_{s_{m-2}}^{m-1}\left(R_{m-2}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 3$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $t_{0}^{m} c_{1}^{m}$, and
(iii) Adding a false twin $x_{1}^{\prime}$ of $x_{1}$.

OR
$I_{i}$ is constructed from
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{1}}^{2}\left(R_{1}\right) ;\left(x_{1}, \ell_{2}\right) ; \ldots ;\left(x_{m-1}, \ell_{m-2}\right) ; D_{s_{m-2}}^{m-1}\left(R_{m-2}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}, m \geq 3$, by:
(i) Amalgamating an $x_{1}$-2-path with $t_{0}^{1} c_{0}^{1}$,
(ii) Amalgamating an $x_{2}$-2-path with $b_{0}^{m} c_{1}^{m}$, and
(iii) Adding a false twin $x_{1}^{\prime}$ of $x_{1}$.


Figure 4.26: A general example of $I_{i}$ :
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(t, \ell_{m-1}\right) ; D_{0}^{m}$ $m \geq 3$, with amalgamated $x_{1}$ and $x_{2}$-2-paths, and
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$
and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 4.27: Specific example of $I_{i}: D_{0} ;(t, 1) ; D_{5}(\{1,3,4,5\}) ;(t, 1) ; D_{0}$ with $x_{1}=c_{0}^{1}$ and an $x_{2}$-2-path amalgamated with $t_{0}^{2} c_{1}^{2}$


Figure 4.28: A general example of $I_{i}$ :
$D_{0}^{1} ;\left(t, \ell_{1}\right) ; D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots . ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right) ;\left(b, \ell_{m-1}\right) ; D_{0}^{m}$
$m \geq 3$, with amalgamated $x_{1}$ and $x_{2}$-2-paths, and
where $D_{s_{2}}^{2}\left(R_{2}\right) ;\left(x_{2}, \ell_{2}\right) ; \ldots ;\left(x_{m-2}, \ell_{m-2}\right) ; D_{s_{m-1}}^{m-1}\left(R_{m-1}\right)$ with $z_{1}=c_{0}^{2}$
and $z_{m-1}=c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality


Figure 4.29: An example of $I_{i}: D_{0} ;(t, 1) ; D_{5}(\{1,3,4,5\}) ;(b, 1) ; D_{0}$ with amalgamated $x_{1}$ and $x_{2}-2$-paths
(j) $I_{j}$ is constructed from $D_{0}$ by:
(i) Adding a simplicial vertex adjacent to $c_{0} t_{0}$,
(ii) Amalgamating an $x_{1}$-2-path with $t_{0} c_{1}$, and
(iii) Adding a false twin, $x_{1}^{\prime}$, or $x_{1}$.


Figure 4.30: Example of $I_{j}$

Lemma 4.2.2. The graph $I_{a}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Suppose $G$ has an $x_{2}$-Hamiltonian path, $P$. Since $c(G-\{a, b\})=3$, then the one endpoint of $P, x_{1}$, must be in one of the components of $G-\{a, b\}$ that does not contain $c d$ and $x_{2}$. Likewise, one endpoint of $P, x_{3}$, must be in one of the components of $G-\{c, d\}$ that does not contain $a b$ and $x_{2}$. Clearly, $x_{1}, x_{3} \neq x_{2}$. Additionally, since $x_{1}$ is in a different component of $G-\{a, b\}$ than $c d$, then it will be in the same component as $a b$ in $G-\{c, d\}$, and hence $x_{1} \neq x_{3}$. Thus, $P$ has three distinct endpoints, a contradiction.

Lemma 4.2.3. The graph $I_{b}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Let $v_{1}, v_{2}$ be the simplicial vertices adjacent to $a b$. By Lemma 1.2.25, $G$ has an $x_{2}$-Hamiltonian path iff $G-v_{1}$ has an $\left(a, x_{2}\right)$-Hamiltonian path or $G-v_{1}$ has a $\left(b, x_{2}\right)$-Hamiltonian path. However, $\left(G-v_{1}, a, x_{2}\right)$ and $\left(G-v_{1}, b, x_{2}\right)$ both contain $F_{a}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $G-v_{1}$ does not have an $\left(a, x_{2}\right)$-Hamiltonian path or a $\left(b, x_{2}\right)$-Hamiltonian path. Hence $G$ does not have an $x_{2}$-Hamiltonian path.

Lemma 4.2.4. The graph $I_{c}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Let $a b$ be the 3-edge which is in a different component of $G-\{e, f\}$ than $x_{2}$, and suppose that $G$ has an $x_{2}$-Hamiltonian path, $P$. Since $c(G-\{a, b\})=3$, then the one endpoint of $P, x_{1}$, must be in one of the components of $G-\{a, b\}$ that does not contain $e f$ and $x_{2}$. Let $u, v, w \neq x_{2}$ be the simplicial vertices adjacent to $e$. None of $u, v, w$ can be an endpoint of $P$ as they will either be in the same component of $G-\{a, b\}$ as $e f$ or they will be one of $\{a, b\}$. Thus, on $P, u, v, w$ must all be preceded or followed by $e$. But then $e$ must appear on $P$ at least twice, and hence $G$ does not have an $x_{2}$-Hamiltonian path.

Lemma 4.2.5. The graph $I_{d}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Let $v$ be the simplicial vertex made adjacent to $c_{0}^{1} c_{1}^{1}$. Since $c_{0}^{1}, c_{1}^{1}$, and $c_{s_{m}+1}^{m}$ are all vertices on the central path, then in $H-v$, there are $\left(c_{0}^{1}, c_{s_{m}+1}^{m}\right)$ and $\left(c_{1}^{1}, c_{s_{m}+1}^{m}\right)$ tough paths. Hence in $H-v$, there does not exist a $\left(c_{0}^{1}, c_{s_{m}+1}^{m}\right)$ or $\left(c_{1}^{1}, c_{s_{m}+1}^{m}\right)$ Hamiltonian path. Thus, by Lemma 1.2.25, there is no $x_{2}=c_{s_{m}+1}^{m}$-Hamiltonian path in $H$.

Lemma 4.2.6. The graph $I_{e}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Let $v_{1}, v_{1}^{\prime}$ the simplicial vertices adjacent to $c_{0}^{1} c_{1}^{1} .\left(H-v_{1}^{\prime}, c_{0}^{1}, x_{2}\right)$ and ( $H-$ $v_{1}^{\prime}, c_{1}^{1}, x_{2}$ ) have $F_{c}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H-v_{1}^{\prime}$ does not have a $\left(c_{0}^{1}, x_{2}\right)$-Hamiltonian path or a $\left(c_{1}^{1}, x_{2}\right)$-Hamiltonian path. Thus, by Lemma 1.2.25, $H$ does not have a $x_{2}$-Hamiltonian path.

Lemma 4.2.7. The graph $I_{f}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Suppose $x_{2}, x_{2}^{\prime}$ is adjacent to $a b .\left(H-x_{2}^{\prime}, c_{0}^{1}, a\right)$ and $\left(H-x_{2}^{\prime}, c_{0}^{1}, b\right)$ have $F_{b}^{1} \in \mathscr{F}^{1}$ or $F_{c}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H-x_{2}^{\prime}$ does not have a $\left(c_{0}^{1}, a\right)$-Hamiltonian path or a $\left(c_{0}^{1}, b\right)$-Hamiltonian path. Thus, by Lemma 1.2.25, $H$ does not have an $x_{2}=c_{0}^{1}$-Hamiltonian path.

Lemma 4.2.8. The graph $I_{g}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Suppose $x_{1}, x_{1}^{\prime}$ is adjacent to $a b$. $\left(H-x_{1}^{\prime}, x_{2}, a\right)$ and $\left(H-x_{1}^{\prime}, x_{2}, b\right)$ have $F_{b}^{1} \in \mathscr{F}^{1}$ or $F_{f}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H-x_{1}^{\prime}$ does not have an $\left(x_{2}, a\right)$-Hamiltonian path or an $\left(x_{2}, b\right)$-Hamiltonian path. Thus, by Lemma 1.2.25, $H$ does not have an $x_{2}$-Hamiltonian path.

Lemma 4.2.9. The graph $I_{h}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Suppose $x_{1}, x_{1}^{\prime}$ is adjacent to $a b$. $\left(H-x_{1}^{\prime}, x_{2}, a\right)$ and $\left(H-x_{1}^{\prime}, x_{2}, b\right)$ have $F_{b}^{1} \in \mathscr{F}^{1}$ or $F_{e}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H-x_{1}^{\prime}$ does not have an $\left(x_{2}, a\right)$-Hamiltonian path or an $\left(x_{2}, b\right)$-Hamiltonian path. Thus, by Lemma 1.2.25, $H$ does not have an $x_{2}$-Hamiltonian path.

Lemma 4.2.10. The graph $I_{i}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Suppose $x_{1}, x_{1}^{\prime}$ is adjacent to $a b$. $\left(H-x_{1}^{\prime}, x_{2}, a\right)$ and $\left(H-x_{1}^{\prime}, x_{2}, b\right)$ have $F_{c}^{1} \in \mathscr{F}^{1}$ or $F_{d}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H-x_{1}^{\prime}$ does not have an $\left(x_{2}, a\right)$-Hamiltonian path or an $\left(x_{2}, b\right)$-Hamiltonian path. Thus, by Lemma 1.2.25, $H$ does not have an $x_{2}$-Hamiltonian path.

Lemma 4.2.11. The graph $I_{j}$ does not have an $x_{2}$-Hamiltonian path.
Proof. Suppose $x_{1}, x_{1}^{\prime}$ is adjacent to $a b$. $\left(H-x_{1}^{\prime}, x_{2}, a\right)$ and $\left(H-x_{1}^{\prime}, x_{2}, b\right)$ have $F_{a}^{1} \in \mathscr{F}^{1}$ or $F_{b}^{1} \in \mathscr{F}^{1}$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H-x_{1}^{\prime}$ does not have an $\left(x_{2}, a\right)$-Hamiltonian path or an $\left(x_{2}, b\right)$-Hamiltonian path. Thus, by Lemma 1.2.25, $H$ does not have an $x_{2}$-Hamiltonian path.

Theorem 4.2.12. If $G$ is a 2-tree with $x_{2} \in V(G)$, then $\left(G, x_{2}\right)$ has an $x_{2}$-Hamiltonian path iff $s(G) \leq 1$ and $G$ does not contain any $I \in \mathscr{I}$ as an induced sub-2-tree.

Proof. $\Longrightarrow$ If $s(G) \geq 2$, then $G$ is not 1-path-tough, and $G$ does not contain a Hamiltonian path.

1. If $\left(G, x_{2}\right)=I_{a}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.2. If $\left(G, x_{2}\right)$ contains $I_{a}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
2. If $\left(G, x_{2}\right)=I_{b}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.3. If ( $G, x_{2}$ ) contains $I_{b}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
3. If $\left(G, x_{2}\right)=I_{c}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.4. If $\left(G, x_{2}\right)$ contains $I_{c}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
4. If $\left(G, x_{2}\right)=I_{d}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.5. If $\left(G, x_{2}\right)$ contains $I_{d}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
5. If $\left(G, x_{2}\right)=I_{e}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.6. If ( $G, x_{2}$ ) contains $I_{e}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
6. If $\left(G, x_{2}\right)=I_{f}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.7. If ( $G, x_{2}$ ) contains $I_{f}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
7. If $\left(G, x_{2}\right)=I_{g}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.8. If $\left(G, x_{2}\right)$ contains $I_{g}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
8. If $\left(G, x_{2}\right)=I_{h}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.9. If ( $G, x_{2}$ ) contains $I_{h}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
9. If $\left(G, x_{2}\right)=I_{i}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.10. If ( $G, x_{2}$ ) contains $I_{i}$ as an induced sub-2-tree, then ( $G, x_{2}$ ) does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.
10. If $\left(G, x_{2}\right)=I_{j}$, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Lemma 4.2.11. If $\left(G, x_{2}\right)$ contains $I_{j}$ as an induced sub-2-tree, then $\left(G, x_{2}\right)$ does not have an $x_{2}$-Hamiltonian path by Corollary 2.1.11.

## $\Longleftarrow$

Suppose $G$ does not have an $x_{2}$-Hamiltonian path, but that $s(G) \leq 1$. Since $s(G) \leq 1$, then $G$ contains no $t$-edges for $t \geq 4$. If $G$ contains $m 3$-edges for $m=0$, then $G$ has an $x_{2}$-Hamiltonian path. So $G$ has $m 3$-edges for $m \geq 1$. Let $H^{\prime}$ be the caterpillar representation of $G$. Then $s\left(H^{\prime}\right) \leq 1$ and $H^{\prime}$ does not have a $x_{2^{-}}$ Hamiltonian path by Lemma 4.1.14. Suppose that $\left(H^{\prime}, x_{2}\right)$ does not contain $I_{a}$.

Denote the 3-edges in $H^{\prime}, S_{i}=s_{i} s_{i}^{\prime}$ for all $1 \leq i \leq m$. Then the 3-edges in $H^{\prime}$ can be ordered $S_{1}, S_{2}, \ldots . S_{m}$ so that in $H^{\prime}-S_{m}, x_{2}$ is in a different component than $s_{i}$ and $s_{i}^{\prime}$ for all $i$, in $H^{\prime}-S_{1}$, all $s_{i}, s_{i}^{\prime} \neq s_{1}, s_{1}^{\prime}$ are in the same component, in $H^{\prime}-S_{m}$, all $s_{i}, s_{i}^{\prime} \neq s_{m}, s_{m}^{\prime}$ are in the same component, and such that for all $i \in\{1,2, \ldots, m-2\}$, $s_{i}$ and $s_{i+2}$ are in different components of $H^{\prime}-S_{i+1}$. From the reduction algorithm, $S_{1}$ is adjacent to two simplicial vertices. Furthermore, since $c\left(G-S_{1}\right)=3$, then, if $H^{\prime}$ has an $\left(x_{2}\right)$-Hamiltonian path, one of the simplicial vertices adjacent to $S_{1}$ must be an endpoint of the path. Without loss of generality, label one of the simplicial vertices adjacent to $S_{1}, x_{1}$. So since $H^{\prime}$ does not have an $\left(x_{2}\right)$-Hamiltonian path, then $H^{\prime}$ does not have an $\left(x_{1}, x_{2}\right)$-Hamiltonian path. So, by Theorem 3.2.10, $\left(H^{\prime}, x_{1}, x_{2}\right)$ must contain an $F_{x}^{2} \in \mathscr{F}^{2}$ as an induced sub-2-tree. Also, since $x_{2}$ is simplicial, then $\left(H^{\prime}, x_{1}, x_{2}\right)$ must contain $F_{a}^{2}, F_{b}^{2}, F_{c}^{2}, F_{d}^{2}, F_{e}^{2}, F_{f}^{1}, F_{e}^{1}, F_{d}^{1}, F_{c}^{1}$, or $F_{a}^{1}$. Adding a false twin of $x_{1}$ and removing the label, we will get the forbidden induced sub-2-trees for $H^{\prime}$ with one fixed endpoint. Using this process on $F_{f}^{1}$ forms $I_{f}$, on $F_{e}^{1}$ forms $I_{g}$, on $F_{d}^{1}$ forms $I_{h}$, on $F_{c}^{1}$ forms $I_{e}$, on $F_{a}^{1}$ forms $I_{i}$, on $F_{a}^{2}$ forms $I_{a}$, on $F_{c}^{2}$ forms $I_{c}^{2}$, on $F_{e}^{2}$ forms $I_{e}^{2}$. For $F_{d}^{2}$ and $F_{b}^{2}$, we can leave $x_{1}$ and just remove the label, as the $x_{1}$ is amalgamated with a 3 -edge and hence already forcing $x_{1}$ or the other simplicial vertex as an end. Using this process on $F_{d}^{2}$ forms $I_{d}$, and on $F_{b}^{2}$ forms $I_{b}$.

## Chapter 5

## Conclusion

In Chapter 2, we introduced a new toughness condition and introduced a new approach for characterizing Hamiltonian problems on 2-trees by describing a forbidden list of induced sub-2-trees for which 2-trees will not have Hamiltonian paths. While the approach of defining a forbidden list of induced subgraphs will not work for graphs in general, this approach will work for induced $k$-trees in a $k$-tree, as proved in Chapter 2. In Chapter 3, we characterized 2HP on 1-tough 2-trees by giving necessary and sufficient conditions for a 1-tough 2 -tree with fixed vertices, $x_{1}, x_{2}$, to have an $\left(x_{1}, x_{2}\right)$-Hamiltonian using both toughness conditions and defining a family, $\mathscr{F}^{1}$ of 2 -trees for which a 1 -tough 2 -tree containing a graph in $\mathscr{F}^{1}$ as an induced subgraph will not have a Hamiltonian path. Additionally, in Chapter 3, we used the results for 2 HP on 1 -tough 2 -trees to similarly characterize the 2 -trees which are not 1 -tough as containing a 2 -tree in a family, $\mathscr{F}^{2}$, as an induced subgraph. Furthermore, we used the results in Chapters 2 and 3 to characterize the Hamilonian path problem on 2-trees in Chapter 4 and 1HP on 2-trees in Chapter 5, by defining forbidden families of 2-trees, $\mathscr{H}$ and $\mathscr{I}$, respectively.

In the future, it is possible that we could extend these methods on 2-trees to other generalizations of the Hamiltonian path problem, like the Path Partition problem or the $k$-Fixed Endpoint Path Partition problem. It is also possible that we could try to extend these results to 3 -trees or $k$-trees. Since adding a vertex adjacent to all vertices in a 2 -tree would form a 3 -tree, our forbidden lists would be a starting point for investigating these problems on 3 -trees.

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## Curriculum vitae

Caitlin Owens<br>Christmas-Saucon Hall, Lehigh University<br>14 E Packer Ave<br>Bethlehem, PA, 18015<br>cmo312@lehigh.edu<br>\section*{EDUCATION}

Lehigh University Bethlehem, PA 18015
Ph.D. candidate in Mathematics
Advisor: Garth Isaak
August 2012-Present

GPA: 3.76
M.S. in Mathematics

August 2014

Kutztown University Kutztown, PA 19530
Post-Baccalaureate Certification - Secondary Ed Math May 2011-May 2012

Graduate curriculum GPA: 4.00
Undergraduate curriculum GPA: 3.84

PA Instructional I Certification: Mathematics 7-12
July 2012

Ithaca College
Ithaca, NY 14850
B.A. in Mathematics August 2005-May 2008

Mathematics GPA: 4.00
Minors: Economics, Honors Humanities and Sciences
Overall GPA: 3.89

## TEACHING EXPERIENCE

Adjunct Faculty Instructor, Calculus II

Muhlenberg College
Summer 2018
Allentown, PA 18104

Instructor, Calculus with Business Applications
Lehigh University
Spring 2018
Bethlehem, PA 18015

Adjunct Faculty Instructor, Calculus II
Muhlenberg College
Summer 2017
Allentown, PA 18104

Instructor, PreCalculus, Calculus I
Summer 2016, 2017, 2018
Lehigh University Summer Scholars Institute (LUSSI)
Lehigh University
Bethlehem, PA 18015

Instructor, PreCalculus
Lehigh University

Adjunct Faculty Instructor, Calculus II
Moravian College
Summer 2016
Bethlehem, PA 18018

Teaching Assistant
Lehigh University
August 2014-Present

Recitations Taught: Calculus I, II, III, and
Calculus with Business Applications

Graduate Assistant (Tutor)
August 2012-May 2014
Lehigh University
Bethlehem, PA 18015
Group tutor for: Calculus II and III

Student Teacher March 2012-May 2012
Lehighton High School
Lehighton, PA 18235
Courses Taught: Algebra I, Algebra II, inclusion class

Student Teacher
January 2012-March 2012
Palmerton Jr. High School
Palmerton, PA 18071
Courses Taught: Pre-Algebra, Algebra I, inclusion class

## SERVICE

Proctor/Grader for Lehigh University High School Math Contest (annual)
Lehigh University

Tutor in Lehigh University Math Help Center
August 2014-May 2018
Lehigh University Mathematics Department

Graduate Student Mentor May 2015-May 2018
Lehigh University Mathematics Department

Mathematics Graduate Liaison Committee Member
Lehigh University Mathematics Department
September 2016-May 2018

Building Committee Member
March 2017-May 2018
Lehigh University

Graduate Student Senate Senator August 2015-May 2018
Lehigh University

Graduate Student Intercollegiate Mathematics Seminar Co-President
Lehigh University Mathematics Department
April 2015-April 2017

Graduate Student Senate Treasurer
April 2015-May 2016
Lehigh University

Dean's Graduate Student Advisory Council Member
Lehigh University College of Arts and Sciences September 2015-May 2016

## OTHER WORK EXPERIENCE

Operations Analyst/ Cost Accountant, Hatfield Quality Meats, Hatfield, PA 19440

## HONORS

- American Mathematical Society (AMS) Travel Grant Recipient
- High Scorer on the Praxis II Content Area Mathematics
- Dean's List all semesters, Ithaca College
- Oracle Honor Society, Ithaca College
- Phi Kappa Phi National Honor Society, Ithaca College
- Pi Mu Epsilon National Mathematics Honor Society, Ithaca College
- Omicron Delta Epsilon International Economics Honor Society, Ithaca College
- Senior Mathematics Award, Ithaca College
- Ramanujan Mathematics Award, Ithaca College
- First Year Mathematics Award, Ithaca College
- Graduated Magna cum Laude, Ithaca College


## INVITED TALKS

- Pi Mu Epsilon Induction Ceremony, Ithaca College, March 2018
- Epsilon Series, Moravian College, November 2017
- Cedar Crest College, November 2017
- Penn State University-Hazleton, October 2017
- Mathematics Seminar, University of Scranton, October 2017
- Mathematics/Computer Science Colloquium Series, Muhlenberg College, April 2017
- Graduate Student Intercollegiate Mathematics Seminar, Lehigh University, February 2017
- Mathematics and Computer Science Lecture Series, DeSales University, November 2016
- Graduate Student Intercollegiate Mathematics Seminar, Lehigh University, September 2015


## OTHER TALKS

- Joint Mathematics Meetings, San Diego, January 2018
- Hudson River Undergraduate Mathematics Conference, Sienna College, 2007
- Ithaca College James J. Whalen Academic Symposium, Ithaca College, 2007


## SKILLS

- Proficient in: Microsoft Office; Activstudio Professional/ Promethean Board; SMARTBoard; Geometer's Sketchpad; TI Connect; LaTeX; Coursesite; MyLab; Canvas
- Working Knowledge of: Mathematica, MATLAB, EViews, SPSS

