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PROBABILITY FUNCTIONS FOR RANDOM RESPONSES: PREDICTION OF PEAKS, FATIGUE DAMAGE, AND CATASTROPHIC FAILURES

by Julius S. Bendat

Prepared under Contract No. NAS 5-1590 by
MEASUREMENT ANALYSIS CORPORATION
Los Angeles, California
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1. INTRODUCTION

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This report reviews a number of theoretical matters in random process theory which can be applied to physical problems such as predicting peaks, structural fatigue damage, and catastrophic structural failures. The presentation emphasizes the basic assumptions which are involved, and discusses how to properly interpret the theoretical results. Various engineering examples are given as illustrations. AUTHOR

The material is divided into nine sections as follows: Section 2, Zero Crossings and Threshold Crossings, summarizes certain known important results which enable one to estimate the expected number of threshold crossings at any level per unit time. Simple quantitative formulas are shown which apply only to Gaussian random processes. Section 3, Peak Probability Functions for Narrow Band Noise, derives the familiar result that for narrow band Gaussian noise, the peak probability density function follows a Rayleigh distribution. A more general result is derived for arbitrary non-Gaussian narrow band noise if the random process and its derivative random process are statistically independent.

Section 4, Expected Number and Spacing of Positive Peaks, discusses pertinent formulas for estimating the expected number of positive peaks per unit time which lie above any level, and the average time between peaks at any level. The latter quantity is equal to the average time required to exceed a given peak level. A simple result is shown which applies only to Gaussian random processes. The next Section 5, Measurement of Peak Probability Functions, contains a new result not appearing elsewhere which enables one to estimate the normalized standard error (defined here as the ratio of standard deviation of the measurement to the expected value of the measurement) in measuring a peak probability distribution function associated with a Gaussian narrow band random process. The result is expressed in terms of the BT product for a sample record, where T is the record length and B is its equivalent noise bandwidth. Sections 2 through 5 are all short sections where results are stated concisely.

The next Section 6, Expected Fatigue Damage and its Variance, discusses in some detail statistical criteria for estimating the expected value and the variance for the damage associated with typical narrow band stress records. These results are then applied in Section 7, Structural Fatigue Problems, to single degree-of-freedom engineering systems. It is assumed here that stress records are directly proportional to the response of the system. For convenience in obtaining quantitative results, and as a reasonable approximation to many physical problems, it is assumed that the damage autocorrelation function is of a damped exponential form. These assumptions lead to new useful practical formulas for estimating the standard error in structural fatigue measurements.

The remaining three sections of the report take up special topics which are related to the previous material but which have important distinctions. Section 8, Peak Probability Functions for Wideband Gaussian Noise, reviews some important not widely known formulas, which extend the familiar narrow band Rayleigh result. It is shown that the peak probability density function for determining the probability that a positive peak will be found among the population of all positive peaks, is in general neither Rayleigh nor Gaussian but a mixture of them both. A criteria for establishing the precise nature of the peak probability density function is the ratio of the expected number of zero crossings per unit time to the expected number of maxima per unit time. Section 9, Envelope Probability Density Functions, discusses briefly the topic of envelope probability density functions where the probability in question represents the probability per unit time that the envelope will fall inside different envelope levels. It is shown that envelope probability density functions are equivalent to peak probability density functions for narrow band Gaussian processes.

The final Section 10, Probability of Catastrophic Failures, explains how to formulate these questions mathematically, and derives basic probability relations. Results are shown to depend upon knowledge of the expected number of threshold crossings per unit time, the topic discussed in Section 2. The probability of nonfailure is calculated also and interpreted as the reliability of the structure to perform properly for a specified length of time. Its reciprocal yields the mean time failure for catastropic events.

2. ZERO CROSSINGS AND THRESHOLD CROSSINGS

Let $x(t)$ be a random record from a stationary random process $\{x(t)\}$ whose instantaneous amplitude probability density function is defined by $p(x)$. No assumption is made that $p(x)$ is necessarily Gaussian. However, for simplicity, it will be assumed that the mean value is zero.

At an arbitrary threshold level $x = \alpha$, the expected number of crossings per unit time through the interval $(\alpha, \alpha + d\alpha)$, where $d\alpha$ is arbitrary small, will be denoted by N_α. The expected number of crossings per unit time through the interval $(\alpha, a + d\alpha)$ with positive slope will be denoted by N_α^+. Since, on the average, there should be an equal number of crossings with positive and negative slope, $N_\alpha^+ = (1/2) N_\alpha$. See Figures 1 and 2.

Denote the time derivative of $x(t)$ by $v(t) = (dx/dt)$. Let $p(\alpha, \beta)$ represent the joint probability density function of $x(t)$ and $v(t)$. By definition

$$p(\alpha, \beta)\, d\alpha\, d\beta \approx \underset{\text{all } t}{\text{Probability}} \Big[\alpha < x(t) \leq \alpha + d\alpha \text{ and } \beta \leq v(t) < \beta + d\beta \Big] \tag{1}$$

For unit total time, Eq. (1) gives the amount of time per unit time that $x(t)$ lies in the interval $(\alpha, \alpha + d\alpha)$ when its velocity $v(t) \approx \beta$ since $d\beta$ is a negligibly small quantity.

The expected number of crossings per unit time through the interval $(\alpha, \alpha + d\alpha)$ for velocity β is estimated by dividing the amount of time per unit time spent inside this interval by the time required to cross this interval. If τ is the crossing time for a particular velocity β, then $\tau = d\alpha / |\beta|$ where the absolute value of β is used since crossing time must be a positive quantity. Hence, the

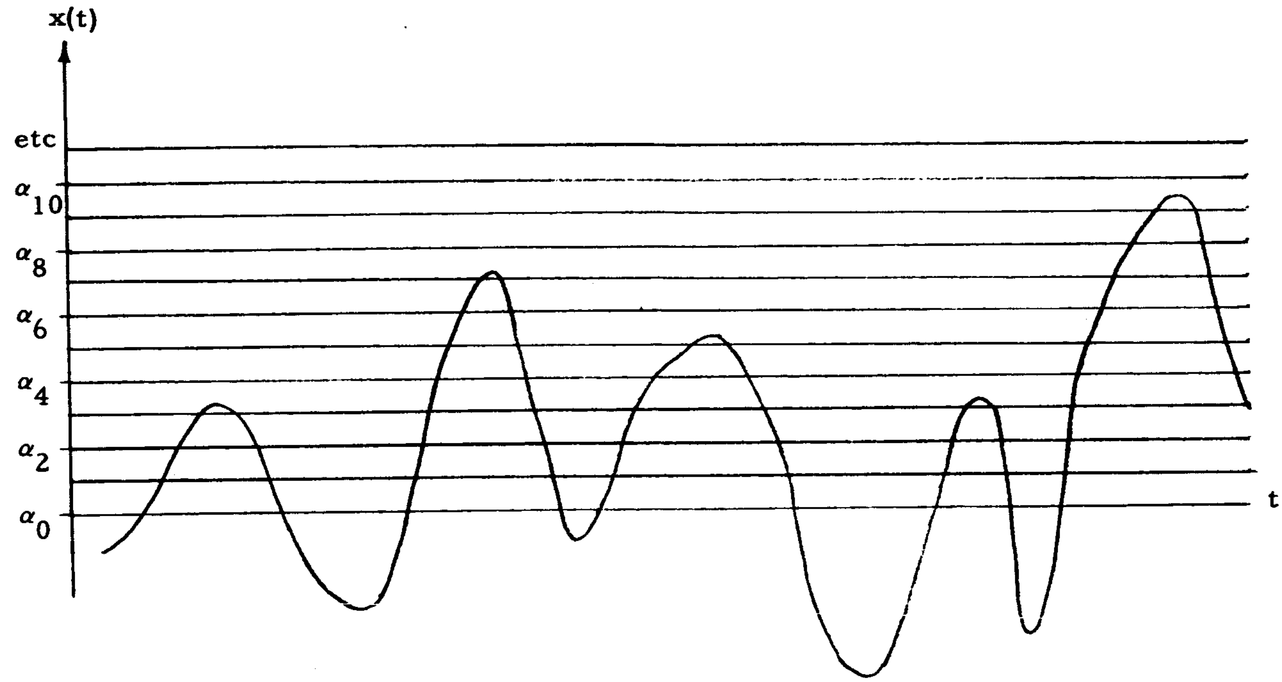


Figure 1. Threshold Crossing Analysis. Measure the number of times per unit time that x(t) crosses the levels α_0, α_1, α_2, ...

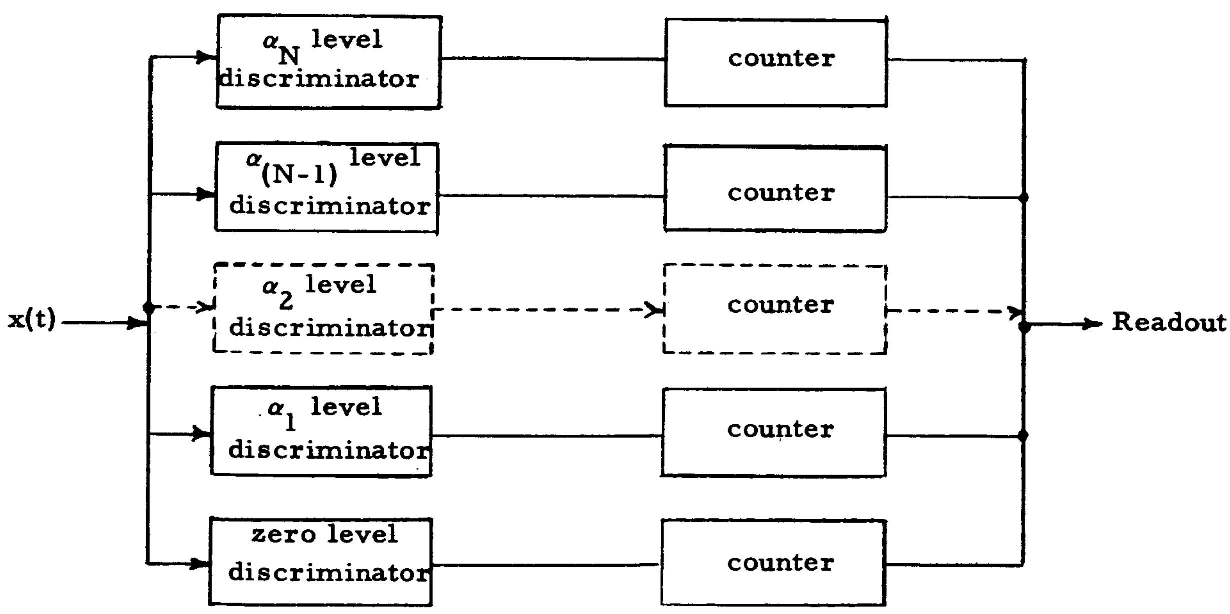


Figure 2. Block Diagram of Threshold Crossing Analyzer. (The counters register one count each time the level of a particular discriminator is exceeded. Division of each count by the zero level count yields the peak probability distribution function $P_p(\alpha)$.)

expected number of crossings per unit time through the level $x(t) = \alpha$ for velocity β is

$$\frac{p(\alpha, \beta)\, d\alpha\, d\beta}{\tau} = |\beta|\, p(\alpha, \beta)\, d\beta \tag{2}$$

Then, the total expected number of crossings per unit time through the level $x(t) = \alpha$ for all possible velocities β is

$$N_\alpha = \int_{-\infty}^{\infty} |\beta|\, p(\alpha, \beta)\, d\beta \tag{3}$$

The expected number of zero crossings per unit time is given by the expected number of crossings of the level $x(t) = 0$, namely,

$$N_0 = \int_{-\infty}^{\infty} |\beta|\, p(0, \beta)\, d\beta \tag{4}$$

The quantity

$$N_\alpha^+ = \frac{1}{2} \int_{-\infty}^{\infty} |\beta|\, p(\alpha, \beta)\, d\beta \tag{5}$$

Equation (5), in general, may be difficult to evaluate. However, if x and v are statistically independent, then $p(x, v) = p(x)\, q(v)$ where $q(v)$ is used instead of $p(v)$ to avoid confusion with $p(x)$. Now,

$$N_\alpha^+ = \frac{p(\alpha)}{2} \int_{-\infty}^{\infty} |\beta|\, q(\beta)\, d\beta \tag{6}$$

Suppose also that $q(\beta)$ is an even function of β, that is, $q(\beta) = q(-\beta)$ for all β. Then

$$N_\alpha^+ = p(\alpha) \int_{0}^{\infty} \beta\, q(\beta)\, d\beta \tag{7}$$

To illustrate Eq. (7), suppose that the velocity v is normally distributed with mean zero and variance σ_v^2 so that

$$q(v) = \frac{1}{\sigma_v \sqrt{2\pi}} \, e^{-v^2/2\sigma_v^2} \tag{8}$$

Then

$$N_\alpha^+ = p(\alpha) \int_0^\infty \frac{\beta}{\sigma_v \sqrt{2\pi}} \, e^{-\beta^2/2\sigma_v^2} \, d\beta = \frac{\sigma_v}{\sqrt{2\pi}} \, p(\alpha) \tag{9}$$

If, now, x is normally distributed with mean zero and variance σ_x^2 so that

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \, e^{-x^2/2\sigma_x^2} \tag{10}$$

Then

$$N_\alpha^+ = \frac{1}{2\pi} \left(\frac{\sigma_v}{\sigma_x} \right) e^{-\alpha^2/2\sigma_x^2} = N_0^+ \, e^{-\alpha^2/2\sigma_x^2} \tag{11}$$

From Eq. (5), one notes that for arbitrary $p(\alpha, \beta)$,

$$\frac{N_\alpha^+}{N_0^+} = \frac{\int_{-\infty}^{\infty} |\beta| \, p(\alpha, \beta) \, d\beta}{\int_{-\infty}^{\infty} |\beta| \, p(0, \beta) \, d\beta} \tag{12}$$

If x and v are statistically independent with $p(x,v) = p(x)\,q(v)$, Eq. (12) reduces to

$$\frac{N_\alpha^+}{N_0^+} = \frac{p(x)\rfloor_{x=\alpha}}{p(x)\rfloor_{x=0}} = \frac{p(\alpha)}{p(0)} \tag{13}$$

regardless of the distributions of $p(x)$ and $q(v)$.

From Eq. (11), which applies only to a Gaussian process,

$$N_0^+ = \frac{1}{2\pi}\left(\frac{\sigma_v}{\sigma_x}\right) \tag{14}$$

For a stationary random process $\{x(t)\}$ with a realizable power spectral density function $G_x(f)$ defined for $f \geqq 0$, the quantities

$$\sigma_x^2 = \int_0^\infty G_x(f)\,df \qquad = R_x(0) \tag{15}$$

$$\sigma_v^2 = \int_0^\infty (2\pi f)^2 G_x(f)\,df = -R_x''(0) \tag{16}$$

where $R_x(\tau)$ is the stationary autocorrelation function defined by

$$R_x(\tau) = \int_0^\infty G_x(f) \cos 2\pi f\tau \, df \tag{17}$$

Thus,

$$N_0^+ = \frac{1}{2\pi}\left[\frac{-R_x''(0)}{R_x(0)}\right]^{1/2} \tag{18}$$

Applications of these results which were derived first by Rice are given in Reference [1].

Example

Consider a bandwidth limited white noise where the power spectrum $G_x(f)$ = constant for $f_a \le f \le f_b$, and zero elsewhere. What is the expected number of zero crossings with positive slope per unit time for the signal?

From Eqs. (15) and (16),

$$R_x(0) = \int_0^\infty G_x(f)\, df = (f_b - f_a) G_x$$

$$-R_x''(0) = \int_0^\infty (2\pi f)^2 G_x(f)\, df = (f_b^3 - f_a^3) \frac{(2\pi)^2 G_x}{3}$$

Then, from Eq. (18)

$$N_0^+ = \frac{1}{2\pi} \left[\frac{(2\pi)^2 (f_b^3 - f_a^3) G_x}{3(f_b - f_a) G_x} \right]^{1/2} = \left[\frac{f_b^3 - f_a^3}{3(f_b - f_a)} \right]^{1/2}$$

As special cases, it follows that

$$N_0^+ = 0.577 f_b \qquad \text{if } f_a \to 0 \quad \text{(low pass case)}$$

$$N_0^+ = f_b \qquad \text{if } f_a \to f_b \quad \text{(narrow band case)}$$

3. PEAK PROBABILITY FUNCTIONS FOR NARROW BAND NOISE

The peak probability density function $p_p(\alpha)$ describes the probability of positive peaks occurring within the population of all positive peaks. To be specific,

$$p_p(\alpha)\, d\alpha = \text{Prob}\Big[\alpha < \text{positive peak} \leq \alpha + d\alpha\Big] \tag{19}$$

Then, the probability of a positive peak being greater than α is given by

$$P_p(\alpha) = \text{Prob}\Big[\text{positive peak} > \alpha\Big] = \int_\alpha^\infty p_p(\alpha)\, d\alpha \tag{20}$$

and

$$\frac{dP_p(\alpha)}{d\alpha} = -p_p(\alpha) \tag{21}$$

It should be noted that quantity $\left[1 - P_p(\alpha)\right]$, which defines the probability that a peak amplitude is less than α, is often called the distribution function for peaks.

Of course, since $p_p(\alpha)$ is a probability density over $(0, \infty)$,

$$\int_0^\infty p_p(\alpha)\, d\alpha = 1 \tag{22}$$

The quantity N_0 gives an indication of the "apparent frequency" of the noise record. For example, if $x(t)$ were a sine wave of frequency f_0 cps, then N_0 would be $2f_0$ zeros per second (e.g., a 60 cps sine wave has 120 zeros per second). The quantity $N_0^+ = (1/2)N_0$ estimates the expected number of cycles per unit time. For example, if $x(t)$ were a sine wave of frequency f_0 cps, then $N_0^+ = f_0$ cps. If each cycle leads to a single positive peak, as occurs for extremely narrow band noise processes, then N_α^+ estimates the expected number of cycles per unit time with peaks above the level $x(t) = \alpha$. Thus, for narrow band noise processes, an estimate of the fraction of cycles having peaks greater than $x(t) = \alpha$ is given by

$$P_p(\alpha) = \text{Prob}\left[\text{positive peak} > \alpha\right] = \frac{N_\alpha^+}{N_0^+} \tag{23}$$

A generalization of this result for arbitrary Gaussian noise processes which are not necessarily narrow band is given in Section 8.

Comparing the results in Eqs. (13) and (23), the following simple result is obtained for the peak probability for narrow band noise,

$$P_p(\alpha) = \text{Prob}\left[\text{positive peak} > \alpha\right] = \frac{p(\alpha)}{p(0)} \tag{24}$$

By taking the derivative of Eq. (24) with respect to the amplitude level $x(t) = \alpha$, the following result is obtained for the peak probability <u>density</u> function for narrow band noise, using Eq. (21),

$$p_p(\alpha) = \frac{\text{Prob}\left[\alpha < \text{positive peak} \leq \alpha + d\alpha\right]}{d\alpha} = \frac{-d\left[p(\alpha)\right]/d\alpha}{p(0)} \tag{25}$$

Note that Eqs. (24) and (25) are general for all $p(x)$, at least to the extent that no direct assumption is made concerning the form of $p(x)$. Of course, these relationships do involve the assumption from Eq. (13) that $x(t)$ and $v(t) = \dot{x}(t)$ are independent, which might be questioned if $p(x)$ is not Gaussian. However, past experience indicates that Eqs. (24) and (25) produce acceptable results for most practical applications even when $p(x)$ is not Gaussian.

If $p(x)$ is Gaussian with a mean zero and a variance of σ_x^2, then

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-x^2/2\sigma_x^2} \tag{26}$$

Now, from Eqs. (24) and (26), the peak probability for narrow band Gaussian noise is

$$P_p(\alpha) = \text{Prob}\left[\text{positive peak} > \alpha\right] = e^{-\alpha^2/2\sigma_x^2} \tag{27}$$

From Eqs. (25) and (26), the corresponding peak probability density function for narrow band Gaussian noise is

$$p_p(\alpha) = \frac{\alpha}{\sigma_x^2} e^{-\alpha^2/2\sigma_x^2} \tag{28}$$

Thus, for the special case of narrow band noise where the probability density function for the instantaneous amplitudes, $p(x)$, is the Gaussian function given in Eq. (26), the resulting probability density function for the peak amplitudes, $p_p(\alpha)$, will be the Rayleigh function shown in Eq. (28).

Example

Consider a narrow band random signal with an rms amplitude of $\sigma_x = 1$ volt. Assuming the signal is Gaussian, what is the probability of a peak occurring with an amplitude greater than $\alpha = 4$ volts?

From Eq. (27)

$$P_p(4) = \text{Prob}\left[\text{positive peak} > 4\right] = e^{-8} = 0.00033$$

Hence, there is about one chance in 3000 that any given peak will have an amplitude greater than $\alpha = 4$ volts.

4. EXPECTED NUMBER AND SPACING OF POSITIVE PEAKS

Let M denote the total expected number of positive peaks of x(t) per unit time, and M_α denote the expected number of positive peaks per unit time which lie above $x(t) = \alpha$. Then

$$M_\alpha = M P_p(\alpha) \tag{29}$$

where $P_p(\alpha)$ is the probability that a positive peak exceeds $x(t) = \alpha$, as defined in Eq. (20). Hence, if T is the total time during which x(t) is observed, the expected number of positive peaks which exceed the level α in time T is given by

$$M_\alpha T = M P_p(\alpha) T \tag{30}$$

Clearly, the average time between positive peaks above the level α will be equal to the reciprocal of the expected number of peaks above that level per unit time. That is,

$$T_\alpha = \frac{1}{M_\alpha} = \frac{1}{M P_p(\alpha)} \tag{31}$$

where T_α is the average time between positive peaks above the level α.

Consider now the special case where x(t) is a narrow band random signal. For this case, each peak above the level $x(t) = \alpha$ will be associated with a crossing of the level α. Then, the average time between crossings (with positive slope) of the level α is T_α as given in Eq. (31), where $P_p(\alpha)$ is as given by Eq. (23).

Again for narrow band noise, the expected number of positive peaks of x(t) per unit time, denoted by M, is equal to one-half of the expected number of zeros of $v(t) = \dot{x}(t)$ per unit time; that is, the number of crossings by v(t) of the level $v(t) = 0$. The factor one-half stems from the observation that half of the zeros of $v(t)$, on the average, represent negative peaks. By analogy with Eq. (11), if $a(t) = \dot{v}(t) = \ddot{x}(t)$, and if $\left[x(t),\ v(t)\right]$ and $\left[v(t),\ a(t)\right]$ are pairwise independent, have zero means, and follow normal distributions, then

$$M = \frac{1}{2\pi}\left(\frac{\sigma_a}{\sigma_v}\right) \tag{32}$$

where σ_a^2 is the variance associated with $a(t)$.

A general expression to determine M which is valid for arbitrary probability density functions is given by

$$M = \int_{-\infty}^{\infty} g(\alpha)\, da = \int_{-\infty}^{\infty}\left[\int_{0}^{-\infty} \gamma\, p(\alpha,\ 0,\ \gamma)\, d\gamma\right] d\alpha \tag{33}$$

where $p(\alpha, 0, \gamma)$ is the third-order probability density function associated with $x(t) = \alpha$, $v(t) = 0$, and $a(t) = \gamma$. This result is discussed in Ref. [1].

Example

Consider a narrow band random signal with an rms amplitude of $\sigma_x = 1$ volt and a center frequency of $f_b = 100$ cps. Assuming the signal is Gaussian, what is the expected number of positive peaks per second with an amplitude greater than $\alpha = 4$ volts, and what is the average time between such peaks?

From the example in Section 2, the expected number of positive peaks per second is $M = N_0^+ = f_b = 100$ cps. From the example in Section 3, the peak probability $P_p(\alpha)$ for $\alpha = 4$ is $P_p(4) = 0.00033$. Then, the expected number of positive peaks per second above $\alpha = 4$ is

$$M_{(4)} = M P_p(4) = 0.033$$

Hence, the average time between positive peaks above $\alpha = 4$ is

$$T_{(4)} = 1/M_{(4)} = 30 \text{ seconds}$$

5. MEASUREMENT OF PEAK PROBABILITY FUNCTIONS

Referring to Eq. (24), the peak probability distribution function for narrow band noise is given by

$$P_p(\alpha) = \text{Prob}\left[\text{positive peak} > \alpha\right] = \frac{p(\alpha)}{p(0)} \tag{34}$$

Hence, the probability of peaks above any given amplitude level $x(t) = \alpha$ may be determined from measurements of the amplitude probability density function $p(x)$ at the levels $x(t) = \alpha$ and $x(t) = 0$.

The amplitude probability density $p(x)$ at any amplitude level $x(t) = \alpha$ is measured using the following relationship.

$$\hat{p}(\alpha) = \frac{1}{\Delta x} \frac{t_{\Delta x}(\alpha)}{T} \tag{35}$$

Here, $t_{\Delta x}(\alpha)$ is the total time spent by the signal $x(t)$ within a narrow amplitude interval between α and $\alpha + \Delta x$, and T is the total observation time. The hat ($\wedge$) over $\hat{p}(\alpha)$ means that this is only an estimate of $p(\alpha)$. An exact measurement would be obtained in the limit as $\Delta x \to 0$ and $T \to \infty$.

The expected deviation of $\hat{p}(\alpha)$ from $p(\alpha)$ may be defined in terms of a normalized variance, $\epsilon^2(\alpha)$, for the measurement as follows.

$$\epsilon^2(\alpha) = \frac{\sigma^2\left[\hat{p}(\alpha)\right]}{p^2(\alpha)} \tag{36}$$

The quantity $\sigma^2\left[\cdot\right]$ represents the variance of the term in the brackets. The positive square root of the normalized variance is the normalized standard deviation $\epsilon(\alpha)$, which is often called the normalized standard error of the measurement.

For the case where p(x) is approximately Gaussian, it has been shown by previous theoretical and experimental work [2, 3] that

$$\epsilon^2(\alpha) = \frac{0.04}{BT(\Delta x)\,\hat{p}(\alpha)} \tag{37}$$

Here, B is the noise bandwidth for the random signal being measured, T is the total observation time, and Δx is the amplitude interval for the measurement. The question that now arises is as follows. What is the variance associated with a peak probability measurement $\hat{P}_p(\alpha)$ based upon measurements of $\hat{p}(\alpha)$ and $\hat{p}(0)$, as shown in Eq. (34)?

Let the normalized variance associated with a measurement $P_p(\alpha)$ be defined by

$$\epsilon_p^2(\alpha) = \frac{\sigma^2\left[\hat{P}_p(\alpha)\right]}{P_p^2(\alpha)} \tag{38}$$

From Eq. (34), the variance in a measurement $\hat{P}_p(\alpha)$ is

$$\sigma^2\left[\hat{P}_p(\alpha)\right] = \sigma^2\left[\hat{p}(\alpha)/\hat{p}(0)\right] \tag{39}$$

Referring to Eq. (37), for large values of α where $\hat{p}(\alpha) << \hat{p}(0)$, the variance associated with the measurement of $\hat{p}(0)$ will be insignificant compared to the variance of $\hat{p}(\alpha)$. Then, $\hat{p}(0)$ may be considered to be an exact measurement of p(0) and Eq. (39) becomes

$$\sigma^2\left[\hat{P}_p(\alpha)\right] = \frac{1}{p^2(0)}\,\sigma^2\left[\hat{p}(\alpha)\right] \tag{40}$$

The normalized variance becomes

$$\epsilon_P^2(\alpha) = \frac{\sigma^2\left[\hat{p}(\alpha)\right]}{p^2(0)\,P_p^2(\alpha)} \tag{41}$$

Using the relationships in Eqs. (34) and (36), the following result is obtained.

$$\epsilon_P^2(\alpha) = \frac{\sigma^2\left[\hat{p}(\alpha)\right]}{p^2(\alpha)} = \epsilon^2(\alpha) \tag{42}$$

Hence, the normalized variance associated with the peak probability measurement $\hat{P}_p(\alpha)$ is effectively the same as the normalized variance for the probability density measurement $\hat{p}(\alpha)$, as given by Eq. (37).

One should keep in mind that the above result applies only to the special case where $x(t)$ is narrow band noise with an approximately Gaussian probability density function. For this important special case, however, it is a convenient practical result which can be used to (1) evaluate the accuracy of peak probability measurements previously made, and (2) guide the design of future experiments to obtain peak probability measurements.

Example

Consider a narrow band random signal with an rms amplitude of $\sigma_x = 1$ volt and a noise bandwidth of $B = 8$ cps. It is desired to measure the probability of peaks in the signal occurring with an amplitude greater than $\alpha = 4$ volts. The problem is to determine how long a sample record should be obtained and analyzed to obtain a measurement of $\hat{P}_p(4)$ with a normalized standard error of $\epsilon_P(\alpha) = 0.10$ or 10%.

If it is assumed the signal is approximately Gaussian, then the measured probability density of $\alpha = 4$ volts will be about $\hat{p}(x) \approx 0.0001$. If the probability density analyzer to be used has an amplitude window of $\Delta x = 0.1$ volts, then the required sample record length for a 10% accuracy in the measurement of $\hat{P}_p(4)$ is

$$T = \frac{0.04}{\epsilon_p^2(\alpha)\, B(\Delta x)\, \hat{p}(x)} = \frac{0.04}{(0.01)(8)(0.1)(0.0001)}$$

$$\approx 50{,}000 \text{ seconds} \approx 14 \text{ hours}$$

After the measurement $\hat{p}(4)$ is obtained, this value may be used to determine a more accurate value for the actual standard error $\epsilon_p(4)$ for the measurement. This example illustrates the important fact that very long sample records are needed to obtain accurate probability measurements for extreme values.

6. EXPECTED FATIGUE DAMAGE AND ITS VARIANCE

Consider a stationary stress process $\{s(t)\}$ with zero mean which is not necessarily normal. A damage $D(T)$ can be associated with a time interval T of $s(t)$ by using the Palmgren-Miner criterion [4, 5]. This damage is a random variable taking on different values for each sample stress history. Consider a typical stress sample function $s(t)$ to be composed of "half-cycles" with varying peak stress amplitudes $|S_i|$, $i = 1, 2, 3, \ldots,$ as represented by typical narrow band noise. See Figure 3.

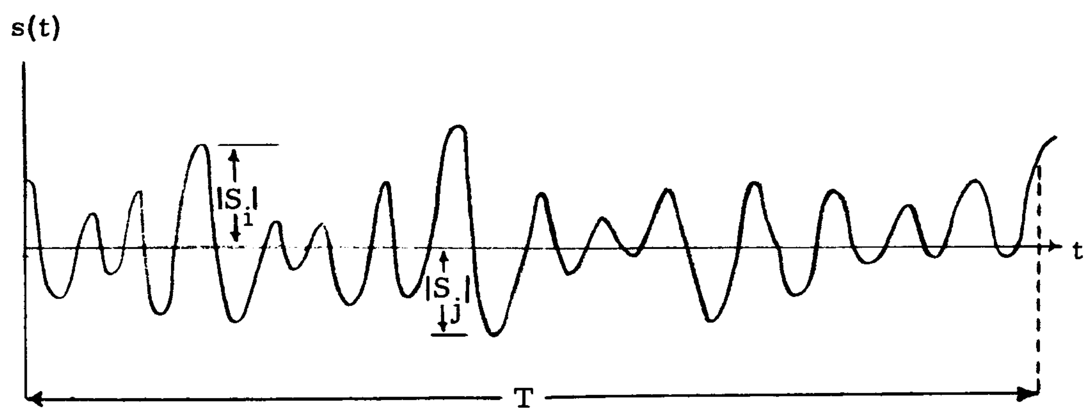


Figure 3. Narrow Band Stress Time History

For a single half-cycle which has a peak stress amplitude $|S_i|$, assume the damage is given by

$$d_i = \frac{1}{2} K |S_i|^b \tag{43}$$

where K and b are positive dimensional constants of the material. The factor (1/2) comes from considering the damage d_i to be associated with a half-cycle. The absolute value $|S_i|$ is used since damage is a positive quantity. If the damage accumulates in a linear fashion, then after m half-cycles, the total damage is

$$D = \sum_{i=0}^{m-1} d_i = \sum_{i=0}^{m-1} \frac{K}{2} |S_i|^b \tag{44}$$

Parameters should be chosen so that fatigue failure occurs when $D = 1$.

By the Palmgren-Miner hypothesis, the constant K is required to satisfy the condition

$$K |S_i|^b = \frac{1}{N_i} \tag{45}$$

where N_i is the number of complete cycles until failure with a peak stress amplitude $|S_i|$. Corresponding values of S_i and N_i are found from an S-N diagram.

6.1 EXPECTED FATIGUE DAMAGE

Letting the absolute value for the instantaneous stress $|s| = \alpha$, the expected values of d_i and D are determined by

$$\mu_d = E(d_i) = \frac{K}{2} \int_0^{\infty} \alpha^b p_p(\alpha)\, d\alpha \tag{46}$$

$$\mu_D = E(D) = m\, E(d_i) = \frac{mK}{2} \int_0^{\infty} \alpha^b p_p(\alpha)\, d\alpha \tag{47}$$

where $p_p(\alpha)$ is the peak probability density function associated with $s(t)$.

In the time interval T, the expected number of half-cycles m is estimated by

$$m = 2N_0^+ T \tag{48}$$

where N_0^+ is the expected number of zero crossings with positive slope of $s(t)$ per unit time. Thus, Eq. (47) gives for the expected fatigue damage

$$\mu_D = K N_0^+ T \int_0^\infty \alpha^b p_p(\alpha)\, d\alpha \tag{49}$$

6.2 VARIANCE IN DAMAGE ESTIMATE

Now consider the variance associated with an estimate for the actual damage D based upon the expected damage μ_D. By definition, the variance in a damage estimate is given by

$$\begin{aligned} \sigma_D^2 &= \mathrm{var}\,(D) = E(D - \mu_D)^2 = E(D^2) - \mu_D^2 \\ &= \sum_{i,j=0}^{m-1} E\left[d_i\, ,\, d_j\right] - \mu_D^2 \end{aligned} \tag{50}$$

Assuming the damage process $\left\{d_i\right\}$ is stationary, the double sum appearing in Eq. (50) becomes

$$\begin{aligned} \sum_{i,j=0}^{m-1} E(d_i d_j) &= \sum_{i=0}^{m-1} E(d_i^2) + \sum_{\substack{i,j=0 \\ i \neq j}}^{m-1} E(d_i d_j) \\ &= m\, E(d_0^2) + \sum_{\substack{i,j=0 \\ i \neq j}}^{m-1} E(d_0 d_{j-i}) \\ &= m\, E(d_0^2) + 2 \sum_{k=1}^{m-1} (m-k)\, E(d_0 d_k) \end{aligned} \tag{51}$$

where the substitution $k = j-i$ has been made.

From Eq. (47),

$$\mu_D = E(D) = m\,E(d) \tag{52}$$

where the subscript for d may be dropped since the process is stationary and, therefore, $E\left[d_0\right] = E\left[d_1\right] = \ldots = E\left[d_{m-1}\right]$.

One may verify directly that

$$2\sum_{k=1}^{m-1} (m-k) = m^2 - m \tag{53}$$

Hence the variance in fatigue damage is

$$\sigma_D^2 = m\left[E(d^2) - E^2(d)\right] + 2\sum_{k=1}^{m-1}(m-k)\left[E(d_0 d_k) - E^2(d_0)\right]$$

$$= m\,\sigma_d^2 + 2\sum_{k=1}^{m-1}(m-k)\,R_d(k) \tag{54}$$

where

$$\sigma_d^2 = \mathrm{var}\,(d) = E\left[d - E(d)\right]^2 = E(d^2) - E^2(d) \tag{55}$$

and

$$R_d(k) = E\left[d_0 d_k\right] - E^2(d) \tag{56}$$

The quantity $R_d(k)$ is the autocovariance function for the damage $\left\{d_i\right\}$. Note that $R_d(0) = \sigma_d^2$.

Referring to Eq. (54), the positive square root of the variance σ_D, is the standard deviation for D. The interpretation of the standard deviation is as follows. Assume that the fatigue damage to be expected for a particular type of structure under a given set of vibration conditions is μ_D, as defined in Eq. (49). Now assume that many different samples of that type of structure are subjected to those given set of vibration conditions. If the damage under those conditions has a standard deviation of σ_D as given by Eq. (54), and if it is assumed that the distribution of D is normal, then about 68% of the sample structures would be expected to have accumulated a damage D such that

$$|\mu_D - D| < \sigma_D \tag{57}$$

Furthermore, about 95% of the sample structures would be expected to have accumulated a damage D such that

$$|\mu_D - D| < 2\sigma_D \tag{58}$$

It is convenient to express the standard deviation in terms of a dimensionless parameter ϵ, called the normalized standard error.

$$\epsilon = \frac{\sigma_D}{\mu_D} \tag{59}$$

In terms of ϵ, the relationship for the 68% case in Eq. (57) becomes

$$\left| 1 - \frac{D}{\mu_D} \right| < \epsilon \tag{60}$$

Similarly, the relationship for the 95% case in Eq. (58) becomes

$$\left| 1 - \frac{D}{\mu_D} \right| < 2\epsilon \tag{61}$$

There are two possible ways in which the above relationships may be applied. The first involves the prediction of actual damage D based upon a calculated expected damage μ_D. The second involves the prediction of an expected damage μ_D based upon an actual damage measurement, D = 1. Note that damage cannot be measured for any value of

D other than unity since a failure is the only level of damage which is observable.

Consider first the prediction of D based upon a calculated value for μ_D, where it is assumed that the pertinent material properties for the structure are known. For this case, Eqs. (60) and (61) may be used to form a probability interval for the value of D based upon a calculation of μ_D. For example, assume the expected damage for a particular type of structure under a given set of conditions is μ_D. Then, from Eq. (61), the 95% probability interval for the actual damage D which would occur under those conditions is

$$(1 - 2\epsilon)\mu_D < D < (1 + 2\epsilon)\mu_D \tag{62}$$

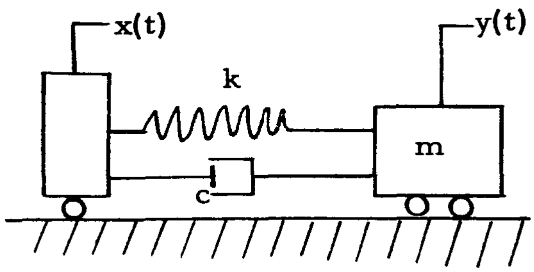
Consider now the prediction of an estimated value μ_D based upon an actual damage measurement, D = 1. This situation would arise when one is attempting to determine the material properties for a structure from empirical data. For this case, Eqs. (60) and (61) may be used to form confidence intervals for μ_D based upon the time and conditions required to produce a failure (D = 1). For example, assume the actual conditions which produced failure in a given structure are used to compute a variance σ_D^2 from Eq. (54). Then, from Eq. (61), the 95% confidence interval for the true value for μ_D under those conditions is

$$1/(1 + 2\epsilon) < \mu_D < 1/(1 - 2\epsilon) \tag{63}$$

Technically speaking, the above confidence interval for μ_D is fictitious since μ_D can never be greater than unity. However, the interval does permit meaningful bounds to be placed upon the value for the material constant needed to produce that value of μ_D.

7. STRUCTURAL FATIGUE PROBLEMS

Assume a structure can be represented by a simple linear lumped parameter system consisting of a mass, spring, and dashpot, as shown in Figure 4 below.



$$z(t) = y(t) - x(t)$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

Figure 4. Simple Linear Structure

Here, k is the spring constant in lbs/inch, c is the viscous damping coefficient in lb-sec/inch, m is the mass in lb-sec^2/inch, x(t) is a time varying foundation motion in inches measured from the mean position of the foundation, and y(t) is the response in inches measured from the position of equilibrium. The cyclical frequency term f_n is the undamped natural frequency for the structure, and the dimensionless damping term ζ is the damping ratio for the structure.

Assuming a linear stress-strain relationship, the stress level s(t) will be directly proportional to the strain level $z(t) = y(t) - x(t)$. That is,

$$s(t) = C\, z(t) \tag{64}$$

where C is a positive constant for the particular structure under consideration. Assume the strain z(t) has a mean square value of $\overline{z^2}$. Since x(t) and y(t) are measured from mean positions, the mean value for the strain will be zero. Hence, the mean square value for the strain will equal the variance σ_z^2 of the strain, that is $\overline{z^2} = \sigma_z^2$.

It follows that the stress $s(t)$ will have a mean value of zero (assuming no pre-stress) and a mean square value (variance) σ_s^2 where

$$\sigma_s^2 = C^2 \sigma_z^2 \tag{65}$$

Referring back to the model in Figure 4, the differential equation of motion for this structure is

$$\ddot{z}(t) + 4\pi\zeta f_n \dot{z}(t) + (2\pi f_n)^2 z(t) = -\ddot{x}(t) \tag{66}$$

Consider the strain (relative displacement) response $z(t)$ produced by an acceleration excitation $\ddot{x}(t)$. A gain factor $|H(f)|$ for the structure will be given by the ratio of the strain magnitude to the excitation magnitude when a sinusoidal acceleration $\ddot{x}(t) = \ddot{X} e^{j2\pi ft}$ is applied to the foundation.

Assume the response strain will be sinusoidal with the general form $z(t) = Z e^{j(2\pi ft + \phi)}$. Substituting this assumed solution into Eq. (66) and solving for $z(t)$ yields the following result.

$$z(t) = \frac{-\ddot{x}(t)}{(2\pi f_n)^2 - (2\pi f)^2 + j(2\pi f)(2\pi f_n)2\zeta} \tag{67}$$

Hence, the gain factor for the structure which relates an acceleration excitation to a strain (relative displacement) response is

$$|H(f)| = \frac{1}{(2\pi f_n)^2 \sqrt{\left[1 - (f/f_n)^2\right]^2 + \left[2\zeta f/f_n\right]^2}} \tag{68}$$

Given a random acceleration excitation $\ddot{x}(t)$ with a power spectral density function of $G_{\ddot{x}}(f)$ (inches/sec^2)^2/cps, the power spectral density function for the response strain in inches^2/cps becomes

$$G_z(f) = |H(f)|^2 G_{\ddot{x}}(f) = \frac{G_{\ddot{x}}(f)}{(2\pi f_n)^4 \left\{ \left[1 - (f/f_n)^2\right]^2 + \left[2\zeta f/f_n\right]^2 \right\}} \tag{69}$$

The mean square value σ_z^2 for the response strain is given by the area under the power spectral density function $G_z(f)$ for all positive frequencies f.

Consider the special case where the excitation acceleration has a uniform power spectrum; i.e., $G_{\ddot{x}}(f)$ = constant $G_{\ddot{x}}$ which hypothetically exists over all frequencies. Contour integration of Eq. (69) yields

$$\sigma_z^2 = \int_0^\infty G_z(f)\, df = \frac{G_{\ddot{x}}}{64\pi^3 \zeta f_n^3} \tag{70}$$

Hence, referring to Eq. (64), the mean square stress level will be

$$\sigma_s^2 = \frac{C^2 G_{\ddot{x}}}{64\pi^3 \zeta f_n^3} \tag{71}$$

Corresponding to the power spectral density function of Eq. (69) when $G_{\ddot{x}}(f)$ = constant, the associated autocorrelation function is

$$R_z(\tau) = \sigma_z^2 e^{-2\pi f_n \zeta \tau} \left(\cos p\tau + \frac{2\pi f_n \zeta}{p} \sin p\tau \right) \tag{72}$$

where σ_z^2 is given by Eq. (70), and $p = 2\pi f_n \sqrt{1 - \zeta^2}$.

For small ζ, $(\zeta << 1)$, $R_z(\tau)$ is approximated closely by

$$R_z(\tau) = \sigma_z^2 e^{-2\pi f_n \zeta \tau} \cos 2\pi f_n \tau \tag{73}$$

Note that the cosine term is of period $(1/f_n)$.

Consider Eq. (73) in succeeding half-cycles, $k = 0, 1, 2, \ldots$, for times τ of the form $(k/2f_n)$. At these times, $R_z(\tau)$ takes on its extremal values

$$R_z(k) = R_z(k/2f_n) = (-1)^k \sigma_z^2 e^{-k\pi\zeta} \quad ; \quad k = 0, 1, 2, \ldots \tag{74}$$

Also, $R_z(\tau)$ is essentially zero when $\tau = (2n - 1)/4f_n$, $n = 1, 2, 3, \ldots$ From Eq. (64) it follows that the stress function $s(k) = s(k/2f_n)$ has an autocorrelation function given by

$$R_s(k) = C^2 R_z(k) = C^2(-1)^k \sigma_z^2 e^{-k\pi\zeta} \tag{75}$$

From Eqs. (43) and (64), the incremental damage d_k becomes

$$d_k = \frac{K}{2} |S_k|^b = \frac{KC^b}{2} |z_k|^b \tag{76}$$

where $|S_k|$ is the peak stress in the half-cycle k and $|z_k|$ is the corresponding peak strain. The absolute value of stress is used since damage is a positive quantity. Equation (76) replaces the previous Eq. (43) in succeeding analysis.

As a reasonable approximation to many physical problems, and in order to obtain convenient closed-form results, it will now be assumed from analogy with Eqs. (74) and (75) that the incremental damage autocovariance function in Eq. (56) is given by the damped exponential function

$$R_d(k) = E(d_0 d_k) - E^2(d) = \sigma_d^2 \, e^{-2k\pi\zeta} \tag{77}$$

where

$$\sigma_d^2 = R_d(0) = E(d^2) - E^2(d) \tag{78}$$

Note that the exponent in Eq. (77) is $(-2k\pi\zeta)$ whereas the exponent in Eq. (75) is $(-k\pi\zeta)$. The factor of 2 results from the fact that the damage is always positive while the stress may be positive or negative. Substitution of Eq. (77) into Eq. (54) then yields a useful variance formula for the accumulated damage after m half-cycles,

$$\sigma_D^2 = m\sigma_d^2 + 2\sigma_d^2 \sum_{k=1}^{m-1} (m - k)\, e^{-2k\pi\zeta} \tag{79}$$

The remainder of the analysis in this section is devoted to evaluating Eq. (79) for arbitrary σ_d^2 and ζ, and to evaluating σ_d^2 for the special important case of a narrow band stress process where the damage D satisfies the Palmgren-Miner criterion of Eq. (43). Results which are obtained agree closely with similar results obtained in Refs. [4, 5].

7.1 EXPECTED VALUE IN DAMAGE ESTIMATES

Referring to Eq. (69), the response strain (and stress) for a single degree-of-freedom structure to random excitation will be narrow band as long as $\zeta \ll 1$, which is usually true in actual practice. Assuming the excitation has a Gaussian probability density function, the peak probability density function will be approximately as given by the Rayleigh probability density function of Eq. (28).

From Eqs. (28) and (46), and letting $|z| = \alpha$, it follows that

$$\mu_d = E(d) = \frac{KC^b}{2} \int_0^\infty \alpha^b p_p(\alpha)\, d\alpha$$

$$= \frac{KC^b}{2\sigma_z^2} \int_0^\infty \alpha^{b+1} e^{-\alpha^2/2\sigma_z^2}\, d\alpha$$

$$= \frac{KC^b}{2} (\sigma_z\sqrt{2})^b \left[2 \int_0^\infty \alpha^{b+1} e^{-\alpha^2}\, d\alpha \right]$$

$$= \frac{KC^b}{2} (\sigma_z\sqrt{2})^b\, \Gamma(1 + \frac{b}{2}) \tag{80}$$

where $\Gamma\,(1 + \frac{b}{2})$ is a gamma function which is defined usually as

$$\Gamma(n) = \int_0^\infty \alpha^{n-1} e^{-\alpha}\, d\alpha \tag{81}$$

Then from Eqs. (47) and (80), the expected value

$$\mu_D = E(D) = m\,E(d) = \frac{mKC^b}{2} (\sigma_z\sqrt{2})^b\, \Gamma(1 + \frac{b}{2}) \tag{82}$$

where $m = 2N_0^+T \approx 2f_nT$ for the response of a lightly damped single degree-of-freedom structure. Fatigue failure will occur when $\mu_D = 1.0$. Hence, for a strain response $z(t)$ with a mean value of zero and an rms value (standard deviation) of σ_z, the time required for a fatigue failure to occur is given by

$$T = \frac{1}{f_n KC^b (\sigma_z\sqrt{2})^b\, \Gamma(1 + \frac{b}{2})} \tag{83}$$

This formula is considered to be an important result of this investigation.

Example

Consider an aluminum structure which may be approximated by a single degree-of-freedom system, as shown in Figure 4, with a natural frequency of $f_n = 50$ cps and a damping ratio of $\zeta = 0.03$. Assume the stress on the structure at some point of interest is related to strain by the constant $C = 10^5$ psi/inch. Further assume the material is such that $b = 6$ and $S_0 = (2)10^5$ psi. Then, from Eq. (45), $K = 1/|S_0|^b = (1.56)10^{-32}(\text{psi})^{-6}$. Now suppose the structure is subjected to a random excitation at its foundation with a relatively uniform acceleration power spectral density function of $G_{\ddot{x}}(f) = 0.5\, g^2/\text{cps}$ over the frequency range from $f = 0$ to $f \gg f_n = 100$ cps ($g = 386$ inch/sec^2). First, how much fatigue damage would be expected in one hour of such vibration? Second, what is the expected fatigue life for the structure?

The mean square value for the strain caused by the foundation motion is given by Eq. (70) as follows

$$\sigma_z^2 = \frac{G_{\ddot{x}}}{64\pi^3 \zeta f_n^3} = \frac{0.5(386)^2}{64\pi^3(0.03)(50)^3} = 0.01 \text{ inches}^2$$

Then, the rms response strain is

$$\sigma_z = 0.1 \text{ inches}$$

Now, the first question will be answered by Eq. (82) as follows.

$$\mu_D = \frac{m}{2} K C^b (\sigma_z \sqrt{2})^b \Gamma(1 + \frac{b}{2})$$

where

$$m = 2 f_n t$$

$$f_n = 50 \text{ cps}$$

$$t = 3600 \text{ seconds}$$

$$K = (1.56) 10^{-32} (\text{psi})^{-6}$$

$$\sigma_z = 0.1 \text{ inch}$$

$$\Gamma(1 + \frac{b}{2}) = 6$$

$$\mu_D = (50)(3600)(1.56) 10^{-32} 10^{30} 10^{-6} (8)(6) = 0.135$$

In words, about 13.5% of the total fatigue life of the structure is consumed by one hour of vibration with the noted excitation level.

The second question will be answered by Eq. (82). Noting that the accumulation of damage is assumed to be proportional to the time t, the total time T required for a fatigue failure to occur $[\mu_D = 1.0]$ is given by t/μ_D. For this problem,

$$T = t/\mu_D = 1/0.135 = 7.42 \text{ hours}$$

7.2 VARIANCE IN DAMAGE ESTIMATES

The mean square value for damage is given by

$$E(d_0^2) = \frac{K^2C^{2b}}{4}\int_0^\infty \alpha^{2b}\, p_p(\alpha)\, d\alpha$$

$$= \frac{K^2C^{2b}}{4\sigma_y^2}\int_0^\infty \alpha^{2b+1}\, e^{-\alpha^2/2\sigma_y^2}\, d\alpha$$

$$= \frac{K^2C^{2b}}{4}(\sigma_y\sqrt{2})^{2b}\left[2\int_0^\infty y^{2b+1}\, e^{-y^2}\, dy\right]$$

$$= \frac{K^2C^{2b}}{4}(\sigma_y\sqrt{2})^{2b}\,\Gamma(1+b) \qquad (84)$$

Now, from Eqs. (80) and (84)

$$\sigma_d^2 = E(d_0^2) - E^2(d_0)$$

$$= \frac{K^2C^{2b}}{4}(\sigma_y\sqrt{2})^{2b}\left[\Gamma(1+b) - \Gamma^2\left(1+\frac{b}{2}\right)\right] \qquad (85)$$

In order to simplify Eq. (79), let

$$f(\pi\zeta) = \sum_{k=1}^{m-1} e^{-2k\pi\zeta} = \frac{1 - e^{-2(m-1)\pi\zeta}}{e^{2\pi\zeta} - 1} \qquad (86)$$

Then one can verify directly that

$$F(\pi\zeta) = \sum_{k=1}^{m-1}(m-k)\, e^{-2k\pi\zeta} = \frac{(m-1)e^{2\pi\zeta} - m + e^{-2(m-1)\pi\zeta}}{(e^{2\pi\zeta} - 1)^2} \qquad (87)$$

In terms of $F(\pi\zeta)$, Eq. (79) becomes

$$\sigma_D^2 = m\sigma_d^2 + 2\sigma_d^2 F(\pi\zeta) = \sigma_d^2\left[m + 2F(\pi\zeta)\right] \tag{88}$$

This is a general result for arbitrary σ_d^2 and ζ.

The normalized variance in the estimation of the total damage is given by the ratio of σ_D^2 and $E^2(D) = \mu_D^2$. That is,

$$\epsilon_D^2 = \frac{\sigma_D^2}{\mu_D^2} \tag{89}$$

From Eqs. (82) and (88), it follows that

$$\epsilon_D^2 = \left[\frac{m + 2F(\pi\zeta)}{m^2}\right]\epsilon_d^2 \tag{90}$$

where $m = 2N_0^+T$ and the quantity $F(\pi\zeta)$ is given by Eq. (87).
For the narrow band case, from Eqs. (80) and (85),

$$\epsilon_d^2 = \frac{\sigma_d^2}{\mu_d^2} = \frac{\Gamma(1 + b) - \Gamma^2(1 + \frac{b}{2})}{\Gamma^2(1 + \frac{b}{2})} \tag{91}$$

From Eq. (91), it is seen that the normalized variance ϵ_d^2 is a function only of the material constant b, and is greater than unity for all values of b greater than two.

Now, for those situations where the term $m\zeta$ is large, the quantity

$$2F(\pi\zeta) = \frac{m}{\pi\zeta} \tag{92}$$

as can be determined by expanding Eq. (87), and Eq. (90) becomes

$$\epsilon_D^2 = \left(\frac{1}{\pi\zeta m}\right)\epsilon_d^2 \tag{93}$$

This expression may be replaced by the equivalent value

$$\epsilon_D^2 = \left(\frac{1}{2\pi\zeta N_0^+ T}\right)\epsilon_d^2 = \left(\frac{1}{2BT}\right)\epsilon_d^2 \tag{94}$$

since

$$m = 2N_0^+ T \approx 2 f_n T \tag{95}$$

and the bandwidth

$$B = \pi\zeta f_n \tag{96}$$

is the equivalent noise bandwidth of the system as computed by the formula

$$B = \frac{\int_0^\infty G_y(f)\, df}{G_y(f_n)} \tag{97}$$

Equation (96) results from applying Eq. (68) to Eq. (97).

In conclusion, the normalized standard error ϵ_D for a damage estimate D is

$$\epsilon_D = \frac{\epsilon_d}{\sqrt{2BT}} = \frac{\epsilon_d}{\sqrt{\pi\zeta m}} \tag{98}$$

where ϵ_d is given by Eq. (91). Hence, the error ϵ_D for a damage estimate is a function of the BT product ($\pi\zeta m/2$) associated with the estimate as well as the material constant b. A tabulation of values for ϵ_D as a function of the BT product and material constant b are presented in Table 1.

Normalized Standard Error for Damage Estimates in Percent as a Function of BT Product for Measurement and Constant b for Material					
BT Product ($\pi \zeta m/2$)	Material Constant, b				
	b = 2	b = 4	b = 6	b = 8	b = 10
10	22.4%	50.0%			
20	15.8	34.4			
30	12.9	28.9	56.3%		
50	10.0	22.4	43.6		
70	8.4	18.9	36.8		
100	7.1	15.8	30.8	58.7%	
200	5.0	11.2	21.8	41.5	
300	4.1	9.1	17.8	33.9	
500	3.2	7.1	13.8	26.3	50.1%
700	2.7	6.0	11.6	22.2	43.3
1000	2.2	5.0	9.7	18.6	35.4
2000	1.6	3.4	6.8	13.1	25.0
3000	1.3	2.9	5.6	10.7	20.4
5000	1.0	2.2	4.4	8.3	15.8
7000		1.9	3.7	7.0	13.4
10000		1.6	3.1	5.9	11.2
20000		1.1	2.2	4.2	7.9
30000			1.8	3.4	6.5
50000			1.4	2.6	5.0
70000			1.2	2.2	4.3
100000				1.9	3.5

Table 1. Normalized Standard Errors for Damage Estimates

Example

Consider the aluminum structure in the example for Section 7.1, where the natural frequency is $f_n = 50$ cps, the damping ratio is $\zeta = 0.03$, and the material constant is $b = 6$. For the given vibration environment, the expected value for the fatigue damage is $\mu_D = 0.135$ for T = one hour of vibration. What is the normalized standard error for this damage estimate?

From Eq. (96), the noise bandwidth for the structural vibration is

$$B = \pi \zeta f_n = 4.71$$

Since $T = 3600$ seconds, the BT product associated with the prediction is

$$BT = \pi \zeta m/2 = 17000$$

Referring to Table 1 and noting that $b = 6$, the normalized standard error for the damage estimate is

$$\epsilon_D = 0.025 \text{ or } 2.5\%$$

Thus, from Eq. (62) the 95% confidence interval for the actual damage D based upon the expected damage $\mu_D = 0.135$ is

$$0.128 < D < 0.142$$

8. PEAK PROBABILITY FUNCTIONS FOR WIDEBAND GAUSSIAN NOISE

Assume that $x(t)$ is a sample member from a stationary Gaussian noise process with zero mean value and variance σ_x^2. Let N_0 denote the expected number of zero crossings per unit time, both with positive and with negative slopes, and let M denote the expected number of positive peaks (maxima) per unit time. Then $2M$ denotes the expected number of both positive and negative peaks per unit time.

As derived in Section 2 of this report, and in Refs. [1, 2],

$$N_0 = \frac{1}{\pi}\left(\frac{\sigma_v}{\sigma_x}\right) \tag{99}$$

$$M = \frac{1}{2\pi}\left(\frac{\sigma_a}{\sigma_v}\right) \tag{100}$$

where

$$\sigma_x^2 = \int_0^\infty G_x(f)\, df \tag{101}$$

$$\sigma_v^2 = \int_0^\infty (2\pi f)^2\, G_x(f)\, df \tag{102}$$

$$\sigma_a^2 = \int_0^\infty (2\pi f)^4\, G_x(f)\, df \tag{103}$$

The peak probability density function represents the probability that a positive peak will be found among the population of all positive peaks. This is the familiar usage and is the one considered in Section 3 for narrow band noise. The next Section 9 takes up a different concept of envelope probability density functions.

In terms of a standardized variable z with zero mean and unit variance; namely,

$$z = \frac{x}{\sigma_x} \quad ; \quad \sigma_z^2 = 1 \tag{104}$$

the probability density function w(z) which defines the probability that a positive peak will fall between z and z + dz is expressed by the formula, Ref. [2]

$$w(z) = \frac{k_1}{\sqrt{2\pi}}\, e^{-z^2/2k_1^2} + \left(\frac{N_0}{2M}\right) z\, e^{-z^2/2}\left[1 - P_n(z/k_2)\right] \tag{105}$$

where

$$k_1 = \sqrt{1 - (N_0/2M)^2}$$

$$k_2 = \frac{k_1}{(N_0/2M)} \qquad \frac{N_0}{2M} = \frac{\sigma_v^2}{\sigma_x \sigma_a} \tag{106}$$

and

$$P_n(z/k_2) = \frac{1}{\sqrt{2\pi}} \int_{z/k_2}^{\infty} e^{-y^2/2}\, dy \tag{107}$$

Note that $P_n(z/k_2)$ is the probability for a standard normal distribution with zero mean and unit variance that the value (z/k_2) will be exceeded. This integral is readily available in statistical tables.

The shape of w(z) is determined by the parameter $(N_0/2M)$. It can be shown from basic considerations that $(N_0/2M)$ always falls between zero and unity; namely,

$$0 \leqq (N_0/2M) = (\sigma_v^2/\sigma_x \sigma_a) \leqq 1 \tag{108}$$

If $(N_0/2M) = 0$, then $w(z)$ reduces to a standardized normal (Gaussian) probability density function,

$$w(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{when} \quad (N_0/2M) = 0 \tag{109}$$

This case occurs in practice for wideband noise where the expected number of maxima and minima per second, $2M$, is much larger than the expected number of zero crossings per second, N_0, so that $(N_0/2M)$ approaches zero.

If $(N_0/2M) = 1$, then $w(z)$ becomes a standardized Rayleigh probability density function,

$$w(z) = z e^{-z^2/2} \quad \text{when} \quad (N_0/2M) = 1 \tag{110}$$

The case occurs in practice for narrow band noise where the expected number of maxima and minima per second, $2M$, is approximately equal to the expected number of zero crossings per second, N_0, so that $(N_0/2M)$ approaches unity. The general form of $w(z)$ from Eq. (105) is thus something between a Gaussian and a Rayleigh probability density function, and is plotted in Figure 5 as a function of z for three values of the dimensionless parameter $(N_0/2M)$ equal to 0, 0.5, and 1.0.

In terms of $w(z)$ the probability $P_p(z)$ that a positive peak chosen at random from among all the possible positive peaks will exceed the value z is given by the formula

$$\begin{aligned} P_p(z) &= \int_z^{\infty} w(z)\, dz \\ &= P_n(z/k_1) + \frac{N_0}{2M} e^{-z^2/2}\left[1 - P_n(z/k_2)\right] \end{aligned} \tag{111}$$

using the P_n of Eq. (107).

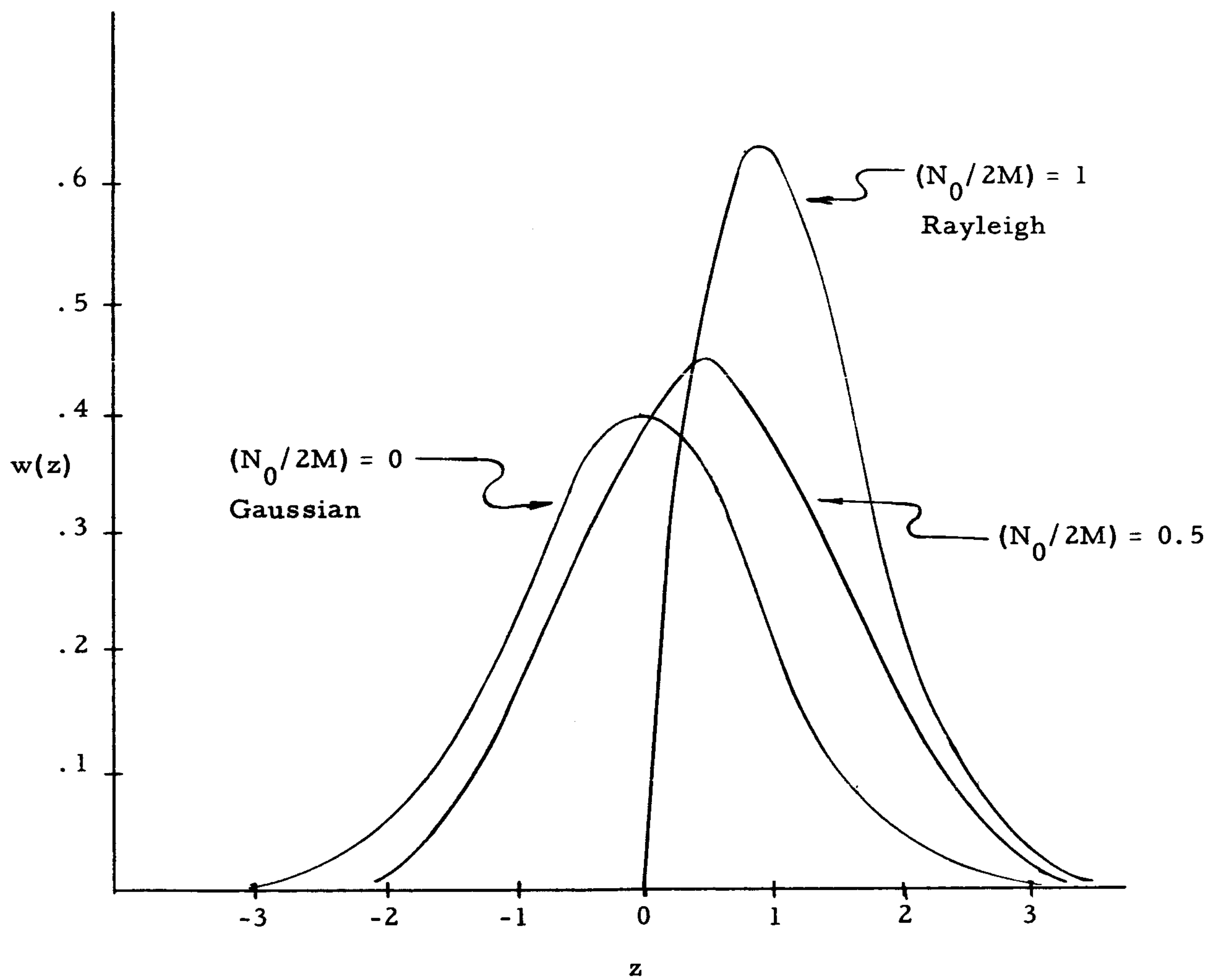


Figure 5. Peak Probability Density Function w(z) versus z

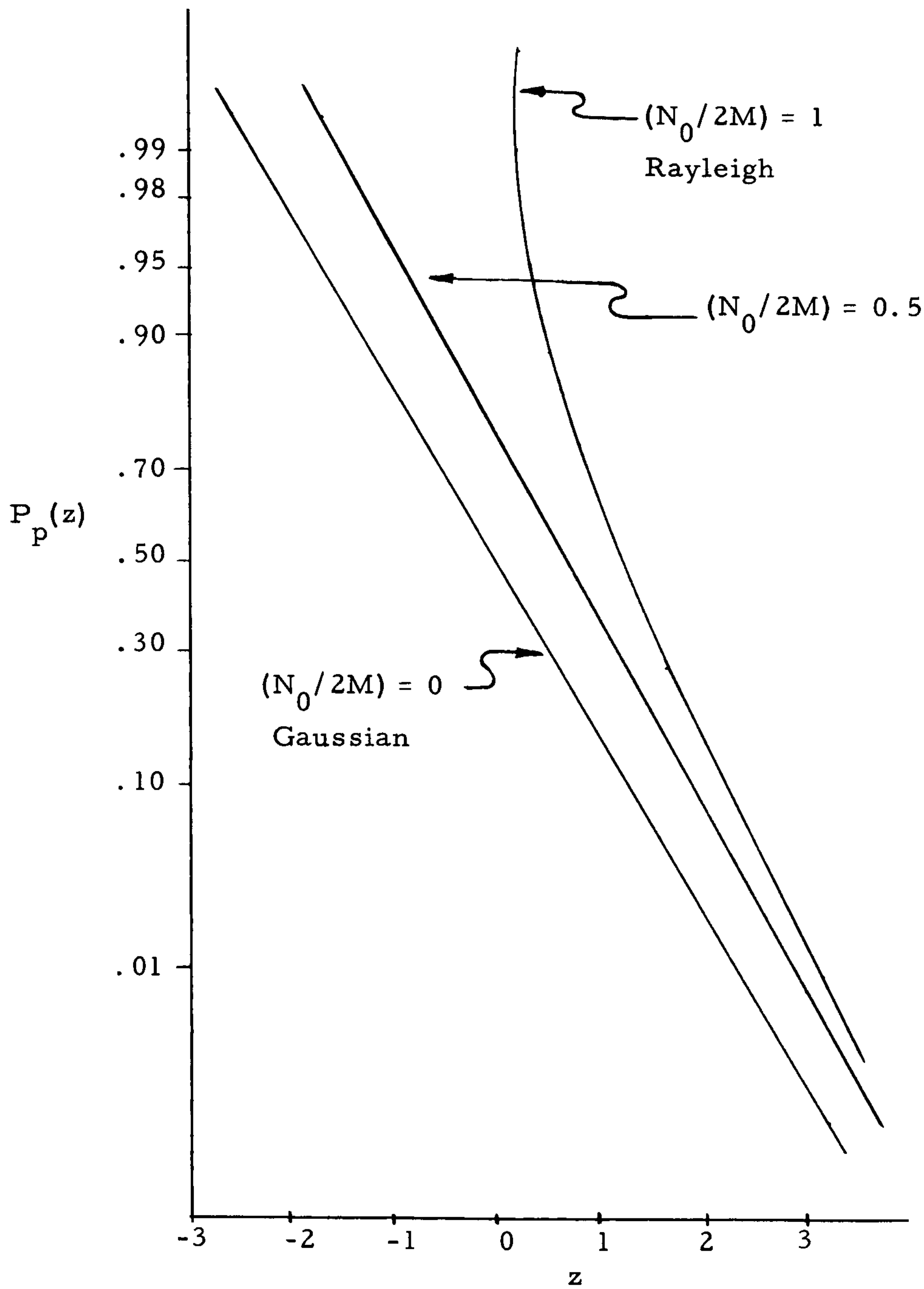


Figure 6. Graph of $P_p(z) = \int_z^\infty w(z)\,dz$ versus z

A graph of $P_p(z)$ as a function of z is plotted in Figure 6 for three fixed values of $(N_0/2M)$ equal to 0, 0.5, and 1.0.

From the above, it should be noted that the actual number of positive peaks per second which would exceed the value $\alpha = z\sigma$, denoted by M_α, may be estimated by the formula

$$M_\alpha = M P_p(\alpha/\sigma) = MP_p(z) \tag{112}$$

For large values of α relative to σ, one may verify

$$M_\alpha \approx (N_0/2)\, e^{-\alpha^2/2\sigma^2} \tag{113}$$

showing that for large α, the expected number of maxima per second lying above the line $x = \alpha$ is equal to the expected number of times per second that $x(t)$ crosses the line $x = \alpha$ with positive slope.

The expected number of peaks which exceed the value α in time T_1 is given by

$$M_\alpha T_1 = M T_1 P_p(\alpha/\sigma_1) \tag{114}$$

This can be set equal to the expected number of peaks which exceed the value α in time T_2 by introducing a different mean square value σ_2^2 such that

$$M_\alpha T_2 = M T_2 P_p(\alpha/\sigma_2) = M T_1 P_p(\alpha/\sigma_1)$$

Now,

$$\frac{T_2}{T_1} = \frac{P_p(\alpha/\sigma_1)}{P_p(\alpha/\sigma_2)} \tag{115}$$

Suppose the mean square value is such that σ_1^2 occurs for time T_1 followed by σ_2^2 for time T_2. What should be the equivalent σ^2 for time $T = T_1 + T_2$ if equivalence is based on having the same number of peaks exceeding α? The expected number of peaks which exceed the value α in time T_1 and the value α in time T_2 is given by

$$M_\alpha T_1 + M_\alpha T_2 = M T_1 P_p(\alpha/\sigma_1) + M T_2 P_p(\alpha/\sigma_2) \qquad (116)$$

The above should now be set equal to

$$M_\alpha T = M T P_p(\alpha/\sigma) \qquad (117)$$

yielding the relation

$$T = \left[\frac{P_p(\alpha/\sigma_1)}{P_p(\alpha/\sigma)}\right] T_1 + \left[\frac{P_p(\alpha/\sigma_2)}{P_p(\alpha/\sigma)}\right] T_2 \qquad (118)$$

In general, for N distinct mean square values in N time periods, one should set

$$T = \sum_{i=1}^{N} c_i T_i \quad \text{where} \quad c_i = \frac{P_p(\alpha/\sigma_i)}{P_p(\alpha/\sigma)} \qquad (119)$$

The general solution for $P_p(\alpha/\sigma)$ is

$$P_p(\alpha/\sigma) = \sum_{i=1}^{N} (T_i/T)\, P_p(\alpha/\sigma_i) \qquad (120)$$

Thus, knowledge of all quantities on the righthand side of Eq. (120) enables one to solve for $P_p(\alpha/\sigma)$, and in turn for the parameters (α/σ) and σ. The above relationships are useful for establishing equivalent stationary random signals for nonstationary random signals based upon a criteria of equivalent extreme values.

Example

Consider a narrow band random signal $x(t)$ with an rms amplitude σ_x which varies over a time interval T as follows.

$$t = 0 \text{ to } 0.05T \quad , \quad \sigma_x = 3 \text{ volts}$$

$$t = 0.05T \text{ to } 0.2T \quad , \quad \sigma_x = 2 \text{ volts}$$

$$t = 0.2T \text{ to } T \quad , \quad \sigma_x = 1 \text{ volt}$$

Assuming the random signal is Gaussian, what is the rms amplitude σ_y for the equivalent stationary random signal $y(t)$ that would produce the same number of peaks in the time T with an amplitude greater than $\alpha = 6$ volts.

Referring to Eq. (120), the probability of a signal with an rms amplitude of σ_y volts having a peak amplitude greater than $\alpha = 6$ volts in a time interval of T seconds is

$$P_p(6/\sigma_y) = \sum_{i=1}^{3} (T_i/T) P_p(\alpha/\sigma_i)$$

where

$$(T_1/T) = 0.05 \quad ; \quad P_p(\alpha/\sigma_1) = P_p(2)$$

$$(T_2/T) = 0.15 \quad ; \quad P_p(\alpha/\sigma_2) = P_p(3)$$

$$(T_3/T) = 0.80 \quad ; \quad P_p(\alpha/\sigma_3) = P_p(6)$$

From Eq. (27), for standardized variables, $P_p(z) = e^{-z^2/2}$,

$$P_p(2) = e^{-2} = 0.135$$

$$P_p(3) = e^{-4.5} = 0.0111$$

$$P_p(6) = e^{-18} \approx 1.67 \times 10^{-8}$$

Now, from Eq. (120),

$$P_p(6/\sigma_y) = (0.05)(0.135) + (0.15)(0.0111) + (0.80)(1.67)10^{-8} = 0.00842$$

Again, from Eq. (27), the following must be true

$$P_p(6/\sigma_y) = e^{-36/2\sigma_y^2} = 0.00842 = e^{-4.76}$$

Hence, the quantity $18/\sigma_y^2 = 4.76$, and the rms amplitude for the equivalent stationary signal is

$$\sigma_y = 1.94 \text{ volts}$$

9. ENVELOPE PROBABILITY DENSITY FUNCTIONS

The peak probability density function $p_p(\alpha)$ describes the distribution of positive peaks within the population of positive peaks. A different probability density function called the envelope probability density function can be defined by assuming that the envelope is a smooth curve joining the positive peaks, and by finding the average time spent by this envelope curve between the levels α and $a + d\alpha$. See Figure 7.

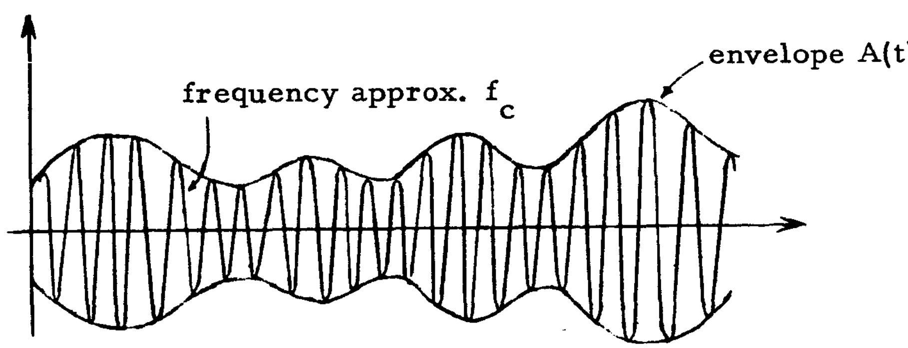


Figure 7. Envelope of Positive Peaks

For narrow band noise, the envelope $A(t)$ can be thought of as the amplitude of a slowly varying sine wave $u(t)$ of center frequency f_c as represented by

$$u(t) = A(t) \cos\left[2\pi f_c t + \theta(t)\right] \tag{121}$$

The envelope $A(t)$ and the phase angle $\theta(t)$ are assumed to vary slowly relative to the frequency f_c.

In order to determine the probability density function associated with $A(t)$ write $u(t)$ as

$$u(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \tag{122a}$$

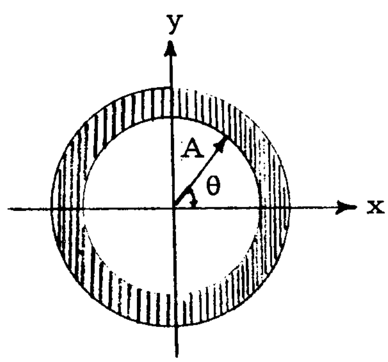
where

$$x(t) = A(t) \cos \theta(t)$$
$$y(t) = A(t) \sin \theta(t) \tag{122b}$$

Now, $A(t)$ is seen to be the non-negative radial quantity

$$A(t) = \sqrt{x^2(t) + y^2(t)} \tag{123}$$

when $x(t)$ and $y(t)$ are regarded as usual x and y coordinates, and $\theta(t)$ is seen to be corresponding angular quantity. The probability that the envelope $A(t)$ will lie between α and $\alpha + d\alpha$ is now equivalent to the joint probability that combinations of $x(t)$ and $y(t)$ will lie in the shaded area of the sketch below.



Suppose that the joint probability density function $p(x, y)$ is known. Then the joint probability density $q(A, \theta)$ must satisfy the relation

$$p(x, y)\, dx\, dy = p(A \cos \theta,\ A \sin \theta)\ A\, dA\, d\theta = q(A, \theta)\, dA\, d\theta \tag{124a}$$

where

$$q(A, \theta) = A p(A \cos \theta,\ A \sin \theta) \tag{124b}$$

since the element of area $dx\, dy$ in the x, y plane corresponds to the element of area $A\, dA\, d\theta$ in the A, θ plane. Now the probability density function $q(A)$ of the envelope $A(t)$ alone is obtained by summing over all possible phase angles and is

$$q(A) = \int_0^{2\pi} q(A, \theta)\, d\theta \tag{125}$$

Assume now that $x(t)$ and $y(t)$ are statistically independent random variables with zero means, equal variances σ^2, and follow a joint Gaussian distribution. Then their joint probability density function has the form

$$p(x, y) = p(x)\,p(y) = \frac{1}{2\pi\sigma^2} \exp\left[-(x^2 + y^2)/2\sigma^2\right] \tag{126}$$

From Eq. (124b), it follows that

$$q(A, \theta) = \frac{A}{2\pi\sigma^2} \exp\left[-A^2/2\sigma^2\right] \tag{127}$$

Then, from Eq. (125), one obtains

$$q(A) = \frac{A}{\sigma^2} \exp\left[-A^2/2\sigma^2\right] \qquad A \geq 0 \tag{128}$$

The probability density function governing the envelope $A(t)$ is the Rayleigh probability density function. Note that it has the same form as the peak probability density function $p_p(\alpha)$ discussed earlier, when A is replaced by α. This equality of $p_p(\alpha)$ and $q(\alpha)$ depends upon the Gaussian assumption for the distribution of the quantities $x(t)$ and $y(t)$. In general, for narrow band non-Gaussian quantities, or nonlinear systems, $p_p(\alpha)$ and $q(\alpha)$ would be quite different and should not be confused with one another. Reference [6] contains pertinent work on these matters.

10. PROBABILITY OF CATASTROPHIC FAILURES

In contrast with fatigue damage and fatigue failure criteria discussed in Section 6, a catastrophic failure occurs when the stress level exceeds a specified failure stress of the structural material. Catastrophic failures originate from severe and generally unpredictable stresses due to extreme environmental conditions. Previous work on these matters from Ref. [7] will now be reviewed. These results are deemed to yield conservative estimates of actual situations.

Let $s(t)$ be a sample function representing the stress time history in a structure. Let α be the stress level at which failure occurs. Then the probability of failure is given by

$$\text{Prob}\left[|s(t)| \geqq \alpha\right] \tag{129}$$

when failure occurs due to large excursions in either a positive or negative direction. The notation $|s(t)| \geqq \alpha$ represents the two cases $s(t) \geqq \alpha$ and $s(t) \leqq -\alpha$.

From earlier work in Section 2 on threshold crossings, when the input is assumed to be a member of a normally distributed stationary random process with zero mean value, the expected number of times per unit time (second) that the stress $s(t)$ will cross α with positive slope is

$$N_{\alpha}^{+} = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{s}}}{\sigma_s}\right) \exp\left(\frac{-\alpha^2}{2\sigma_s^2}\right) \tag{130}$$

where σ_s^2 is the mean square value of $s(t)$, and $\sigma_{\dot{s}}^2$ is the mean square value of ds/dt. In terms of the spectral density function $G_s(f)$ associated with $s(t)$,

$$\sigma_s^2 = \int_0^{\infty} G_s(f)\, df \tag{131}$$

$$\sigma_{\dot{s}}^2 = \int_0^\infty (2\pi f)^2 G_s(f)\, df \tag{132}$$

Hence, by the above operations, N_α^+ is a function of $G_s(f)$.

Assume now that s(t) has a probability density function which is symmetric about a zero mean value. Hence

$$\text{Prob}\Big[\,|s(t)| \geqq \alpha\Big] = 2\,\text{Prob}\Big[s(t) \geqq \alpha\Big] \tag{133}$$

and the expected number of times per unit time that $|s(t)|$ will exceed α becomes

$$N_\alpha = \frac{1}{\pi}\left(\frac{\sigma_{\dot{s}}}{\sigma_s}\right) \exp\left(\frac{-\alpha^2}{2\sigma_s^2}\right) \tag{134}$$

Equation (134) is the expected number of failures per unit time.

The probability that s(t) will cross α with positive slope in the time interval $(t,\ t+\Delta t)$ is given by

$$\text{Prob}\Big[s(t) \geqq \alpha\Big] = N_\alpha^+ \Delta t \tag{135}$$

This follows because the expected number of crossings of $s(t) = \alpha$ with positive slope during the arbitrarily small interval Δt is precisely the same as the fraction of favorable events $\{s(t) \geqq \alpha\}$ out of all possible events $\{s(t) > -\infty\}$. It is assumed here that a favorable event yields one crossing only since Δt is an arbitrarily small quantity. It is further assumed that the probability of a crossing during the small interval Δt is independent of the time when the interval started. Similarly, the probability that $|s(t)|$ will exceed α during the time interval $(t,\ t+\Delta t)$ is

$$\text{Prob}\Big[|s(t)| \geqq \alpha\Big] = N_\alpha \Delta t \tag{136}$$

Equation (136) gives the probability of failure in the time interval $(t, t+\Delta t)$.

The probability of nonfailure in a time interval $(0, t)$ will now be found. Let $P_0(t)$ represent the probability of nonfailure in the interval $(0, t)$, and let $P_0(t + \Delta t \mid t)$ represent the probability of nonfailure in the interval $(t, t+\Delta t)$ if no failure occured up to time t. Then

$$P_0(t + \Delta t) = P_0(t + \Delta t \mid t)\, P_0(t) \tag{137}$$

By definition,

$$P_0(t+\Delta t \mid t) = \mathrm{Prob}\left[\,|s(t)| < \alpha\right] = 1 - \mathrm{Prob}\left[\,|s(t)| \geqq \alpha\right] \tag{138}$$

Substituting Eq. (136) into Eq. (138) yields

$$P_0(t + \Delta t \mid t) = 1 - N_\alpha \Delta t \tag{139}$$

Then substituting Eq. (139) into Eq. (137),

$$P_0(t + \Delta t) = P_0(t)\left[1 - N_\alpha \Delta t\right] \tag{140}$$

or

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -N_\alpha P_0(t) \tag{141}$$

In the limit as $\Delta t \rightarrow 0$, Eq. (141) becomes a differential equation

$$\frac{dP_0(t)}{dt} = -N_\alpha P_0(t) \tag{142}$$

The solution to Eq. (142) is simply

$$P_0(T) = A_0 \exp\left[-N_\alpha T\right] \tag{143}$$

where A_0 is a constant of integration and T is the total time interval of interest. The initial condition $P_0(0) = 1$ requires that $A_0 = 1$. Hence

$$P_0(t) = \exp\left[-N_\alpha T\right] \tag{144}$$

Thus, $P_0(t)$, the probability of nonfailure in the time interval $(0, T)$, may be determined quite easily from N_α, the expected number of failures per unit time, given by Eq. (134).

Now, the probability of failure in the interval $(0, T)$ is given by

$$P_f(T) = 1 - P_0(T) = 1 - \exp\left[-N_\alpha T\right] \tag{145}$$

Let $p_f(t)$ represent the probability density function for the time to failure. Then

$$P_f(T) = \int_0^T p_f(t)\, dt \tag{146}$$

From Eqs. (145) and (146),

$$p_f(T) = \frac{dP_f(T)}{dT} = N_\alpha \exp\left[-N_\alpha T\right] \tag{147}$$

Now, the expected time to failure, and the variance in the time to failure, from Eq. (147) are

$$E(T) = \int_0^\infty T p_f(T)\, dT = \frac{1}{N_\alpha} \tag{148}$$

$$\sigma^2(T) = \int_0^\infty \left[T - E(T)\right]^2 p_f(T)\, dT = \frac{1}{N_\alpha} \tag{149}$$

These results of Eqs. (147), (148), and (149) may be derived also by assuming that the crossings $|s(t)| \geqq \alpha$, occur at a sequence of times following a Poisson distribution.

Returning to Eq. (144), this relationship may be interpreted as the probability that a structure will perform properly in the time interval (0, T) for any given excitation. This relationship can be used to define a criteria for structural reliability in terms of catastrophic failure (as opposed to long term fatigue failures). In more common reliability terminology, the reciprocal of N_α may be considered a mean-time-between-failure T_f. That is,

$$T_f = 1/N_\alpha \tag{150}$$

which agrees with Eq. (148). In terms of T_f, Eq. (144) becomes

$$P_0(T) = \exp(-T/T_f) \tag{151}$$

The proper value for N_α in the above relationships may be estimated using Eq. (134), where the values for σ_s and $\sigma_{\dot{s}}$ are obtained from the power spectrum for a structural response as indicated in Eqs. (131) and (132). The response power spectrum for a structure can be predicted from the power spectrum for the excitation as illustrated in Section 7.1. Thus, given a structure with a gain factor of $|H(f)|$ and an excitation of $G_x(f)$, the variance for the stress level and its first derivative are

$$\sigma_s^2 = C^2 \int_0^\infty |H(f)|^2 \, G_x(f) \, df \tag{152}$$

$$\sigma_{\dot{s}}^2 = C^2 \int_0^\infty (2\pi f)^2 \, |H(f)|^2 \, G_x(f) \, df \tag{153}$$

The values for σ_s and $\sigma_{\dot{s}}$ may be used in Eq. (134) to obtain the value for N_α associated with any extreme level of stress $s(t) = \alpha$ where a failure would occur.

It should be emphasized that Eqs. (150) and (151) assume that the probability of a failure in the interval $(0, T)$ is independent of the starting time for the interval. This assumption implies that the probability of any given peak value is independent of the value for the preceding peaks. This assumption is acceptable for those cases where the power spectrum for random signal $s(t)$ is relatively uniform over a very wide frequency range. However, the assumption becomes invalid when the power spectrum for $s(t)$ has a single sharp peak; i.e., when $s(t)$ is narrow band noise. For the case of narrow band noise, the autocorrelation function is an exponential cosine function as given by Eq. (72). It is clear from the relationship in Eq. (72) that the correlation between adjacent peaks increases as the noise bandwidth $B = \pi\zeta f_n$ decreases. Hence, for the case of narrow band random signals, Eqs. (150) and (151) are only approximations for the mean-time-between-failure T_f, and probability of no failures $P_0(T)$. Additional studies are needed to properly define these factors for the narrow band case.

Example

Consider a structure which will fail if the instantaneous stress level $s(t)$ exceeds the extreme value $\alpha = (2)10^5$ psi. Assume the structure is subjected to an excitation which produces a stress response with a reasonably uniform power spectrum having a density of $G_s(f) = 10^7$ psi^2/cps over the frequency range from 0 cps to 100 cps. First, what is the expected mean-time-between-failure, and, second, what is the probability of reliable operation (no failures) in a period of one hundred hours.

The mean-time-between-failure is given by T_f in Eq. (150) where N_α is given by Eq. (134). Referring to Eqs. (131) and (132), the variances for the stress response and its first derivative are

$$\sigma_s^2 = \int_0^\infty G_s(f)\, df = \int_0^{100} 10^7\, df = 10^9 \text{ psi}^2$$

$$\sigma_{\dot{s}}^2 = \int_0^\infty (2\pi f)^2 G_s(f)\, df = \int_0^{100} (2\pi)^2 10^7 f^2\, df = (13.1)10^{13} (\text{psi/sec})^2$$

Then, the expected number of crossings per second of the level $\alpha = (2)10^5$ psi is

$$N_\alpha = \frac{1}{2\pi}\left[\frac{(1.14)10^7}{(3.16)10^4}\right] e^{-20} = 14.3 \times 10^{-8} \text{ crossings/seconds}$$

The mean-time-between-failure is then

$$T_f = \frac{1}{N_\alpha} = (7)10^6 \text{ seconds} \approx 2000 \text{ hours}$$

and the probability of no failures in one hundred hours is given by Eq. (151) as follows.

$$P_0(100) = e^{-(100/2000)} = 0.95 \text{ or } 95\%$$

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