Intuitionistic non-normal modal logics: A general framework

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Abstract

We define a family of intuitionistic non-normal modal logics; they can bee seen as intuitionistic counterparts of classical ones. We first consider monomodal logics, which contain only one between Necessity and Possibility. We then consider the more important case of bimodal logics, which contain both modal operators. In this case we define several interactions between Necessity and Possibility of increasing strength, although weaker than duality. For all logics we provide both a Hilbert axiomatisation and a cut-free sequent calculus, on its basis we also prove their decidability. We then give a semantic characterisation of our logics in terms of neighbourhood models. Our semantic framework captures modularly not only our systems but also already known intuitionistic non-normal modal logics such as Constructive K (CK) and the propositional fragment of Wijesekera's Constructive Concurrent Dynamic Logic.

1 Introduction

Both intuitionistic modal logic and non-normal modal logic have been studied for a long time. The study of modalities with an intuitionistic basis goes back to Fitch in the late 40s (Fitch [7]) and has led to an important stream of research. We can very schematically identify two traditions: so-called Intuitionistic modal logics *versus* Constructive modal logics. Intuitionistic modal logics have been systematised by Simpson [23], whose main goal is to define an analogous of classical modalities justified from an intuitionistic point of view. On the other hand, constructive modal logics are mainly motivated by their applications to computer science, such as the type-theoretic interpretations (Curry–Howard correspondence, typed lambda calculi), verification and knowledge representation,[1] but also by their mathematical semantics (Goldblatt [11]).

On the other hand, non-normal modal logics have been strongly motivated on a philosophical and epistemic ground. They are called "non-normal" as they do not satisfy all the axioms and rules of the minimal normal modal logic K. They have been studied since the seminal works of Scott, Lemmon, and Chellas ([22], [2], see Pacuit [21] for a survey), and can be seen as generalisations of standard modal logics. They have found an interest in several areas such as

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[1] For a recent survey see Stewart *et al.* [25] and references therein.

epistemic and deontic reasoning, reasoning about games, and reasoning about probabilistic notions such as “truth in most of the cases”.

Although the two areas have grown up seemingly without any interaction, it can be noticed that some intuitionistic or constructive modal logics investigated in the literature contain non-normal modalities. The prominent example is the logic CCDL proposed by Wijesekera [27], whose propositional fragment (that we call $\mathsf{CCDL^p}$) has been recently investigated by Kojima [13]. This logic has a normal $\Box$ modality and a non-normal $\Diamond$ modality, where $\Diamond$ does not distribute over the $\vee$, that is

$$(\mathsf{C}_\Diamond) \qquad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$$

is not valid. The original motivation by Wijesekera comes from Constructive Concurrent Dynamic Logic, but the logic has also an interesting epistemic interpretation in terms of internal/external observers proposed by Kojima. A related system is Constructive K (CK), that has been proposed by Bellin *et al.* [1] and further investigated by Mendler and de Paiva [19], Mendler and Scheele [20]. This system not only rejects $\mathsf{C}_\Diamond$, but also its nullary version $\Diamond\bot \supset \bot$ ($\mathsf{N}_\Diamond$). In contrast all these systems assume a normal interpretation of $\Box$ so that

$$\Box(A \wedge B) \supset\subset (\Box A \wedge \Box B)$$

is always assumed. A further example is Propositional Lax Logic (PLL) by Fairtlough and Mendler [4], an intuitionistic monomodal logic for hardware verification where the modality does not validate the rule of necessitation.

Finally, all intuitionistic modal logics reject the interdefinability of the two operators:

$$\Box A \supset\subset \neg\Diamond\neg A$$

and its boolean equivalents.

To the best of our knowledge, no systematic investigation of non-normal modalities with an intuitionistic base has been carried out so far. Our aim is to lay down a general framework which can accommodate in a uniform way intuitionistic counterparts of the classical cube of non-normal modal logics, as well as $\mathsf{CCDL^p}$ and CK mentioned above. As we shall see, the adoption of an intuitionistic base leads to a finer analysis of non-normal modalities than in the classical case. In addition to the motivations for classical non-normal modal logics briefly recalled above, an intutionistic interpretation of non-normal modalities may be justified by more specific interpretations, of which we mention two:

- The **deontic interpretation**: The standard interpretation of deontic operators $\Box$ (Obligatory), $\Diamond$ (Permitted) is normal: but it has been known for a long time that the normal interpretation is problematic when dealing for instance with “Contrary to duty obligations".[2] One solution is to adopt a non-normal interpretation, rejecting in particular the monotonicity principle (from $A \supset B$ is valid infer $\Box A \supset \Box B$). Moreover, a constructive reading of the deontic modalities would further reject their interdefinability: one may require that the permission of A must be justified explicitly or positively (say by a proof from a corpus of norms) and

[2]For a survey on puzzles related to a normal interpretation of the deontic modalities see McNamara [18].

not just established by the fact that $\neg A$ is not obligatory (see for instance the distinction between weak and strong permissions in von Wright [31]).

- The **contextual interpretation**: A contextual reading of the modal operators is proposed in Mendler and de Paiva [19]. In this interpretation $\Box A$ is read as "A holds in all contexts" and $\Diamond A$ as "A holds in some context". This interpretation invalidates $\mathsf{C}_\Diamond$, while retaining the distribution of $\Box$ over conjunction ($\mathsf{C}_\Box$). But this contextual interpretation is not the only possible one. We can interpret $\Box A$ as A is "justified" (proved) in some context c, no matter what is meant by a context (for instance a knowledge base), and $\Diamond A$ as A is "compatible" (consistent) with every context. With this interpretation both operators would be non-normal as they would satisfy neither $\mathsf{C}_\Box$, nor $\mathsf{C}_\Diamond$.

As we said, our aim is to provide a general framework for non-normal modal logics with an intuitionistic base. However, in order to identify and restrain the family of logics of interest, we adopt some criteria, which partially coincide with Simpson's requirements (Simpson [23]):

- The modal logics should be conservative extensions of IPL.
- The disjunction property must hold.
- The two modalities should not be interdefinable.
- We do not consider systems containing the controversial $\mathsf{C}_\Diamond$.

Our starting point is the study of *monomodal* systems, which extend IPL with either $\Box$ or $\Diamond$, but not both. We consider the monomodal logics corresponding to the classical cube generated by the weakest logic E extended with conditions M, N, C (with the exception of $\mathsf{C}_\Diamond$). We give an axiomatic characterisation of these logics and equivalent cut-free sequent systems similar to the one by Lavendhomme and Lucas [15] for the classical case.

Our main interest is however in logics which contain *both* $\Box$ and $\Diamond$, and allow some form of interaction between the two. Their interaction is always weaker than interdefinability. In order to define logical systems we take a proof-theoretical perspective: the existence of a simple cut-free system, as in the monomodal case, is our criteria to identify meaningful systems. *A system is retained if the combination of sequent rules amounts to a cut-free system.*

It turns out that one can distinguish *three* degrees of interaction between $\Box$ and $\Diamond$, that are determined by answering to the question, for any two formulas A and B:

under what conditions $\Box A$ and $\Diamond B$ are jointly inconsistent?

Since there are *three* degrees of interaction, even the weakest classical logic E has *three* intuitionistic counterparts of increasing strength. When combined with M, N, C properties of the classical cube, we end up with a family of 24 distinct systems, all enjoying a cut-free calculus and, as we prove, an equivalent Hilbert axiomatisation. This shows that intuitionistic non-normal modal logic allows for finer distinctions whence a richer theory than in the classical case.

The existence of a cut-free calculus for each of the logics has some important consequences: We can prove that all systems are indeed distinct, that all of them are "good" extensions of intuitionistic logic, and more importantly that all of them are decidable.

We then tackle the problem of giving a semantic characterisation of this family of logics. The natural setting is to consider an intuitionistic version of neighbourhood models for classical logics. Since we want to deal with the language containing both $\Box$ and $\Diamond$, we consider neighbourhood models containing *two* distinct neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$. As in standard intuitionistic models, they also contain a partial order on worlds. Different forms of interaction between the two modal operators correspond to different (but natural) conditions relating the two neighbourhood functions. By considering further closure conditions of neighbourhoods, analogous to the classical case, we can show that this semantic characterises *modularly* the full family of logics. Moreover we prove, through a filtration argument, that most of the logics have the *finite model property*, thereby obtaining a semantic proof of their decidability.

It is worth noticing that in the (easier) case of intuitionistic monomodal logic with only $\Box$ a similar semantics and a matching completeness theorem have been given by Goldblatt [11]. More recently, Goldblatt's semantics for the intuitionistic version of system E has been reformulated and extended to axiom T by Witczak [29].

But our neighbourhood models have a wider application than the characterisation of the family of logics mentioned above. We show that adding suitable *interaction conditions* between $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ we can capture $\mathsf{CCDL^p}$ as well as CK. We show this fact first directly by proving that both $\mathsf{CCDL^p}$ and CK are sound and complete with respect to our models satisfying an additional condition. We then prove the same result by relying on some pre-existing semantics of these two logics and by transforming models. In case of $\mathsf{CCDL^p}$, there exists already a characterisation of it in terms of neighbourhood models, given by Kojima [13], although the type of models is different, in particular Kojima's models contain only one neighbourhood function.

The case of CK is more complicated, whence more interesting: this logic is characterised by a relational semantics defined in terms of Kripke models of a *peculiar* nature: they contain "fallible" worlds, *i.e.* worlds which force $\bot$. We are able to show directly that relational models can be transformed into our neighbourhood models satisfying a specific interaction condition and *vice versa*.

All in all, we get that the well-known CK can be characterised by neighbourhood models, after all rather standard structures, alternative to non-standard Kripke models with fallible worlds. This fact provides further evidence in favour of our neighbourhood semantics as a versatile tool to analyse intuitionistic non-normal modal logics.

2 Classical non-normal modal logics

2.1 Hilbert systems

Classical non-normal modal logics are defined on a propositional modal language $\mathcal{L}$ based on a set Atm of countably many propositional variables. Formulas are given by the following grammar, where p ranges over Atm:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A.$$

***a.* Modal axioms and rules defining non-normal modal logics**

$\mathsf{E}_\Box$	$\dfrac{A \supset B \quad B \supset A}{\Box A \supset \Box B}$	$\mathsf{E}_\Diamond$	$\dfrac{A \supset B \quad B \supset A}{\Diamond A \supset \Diamond B}$
$\mathsf{M}_\Box$	$\Box(A \wedge B) \supset \Box A$	$\mathsf{M}_\Diamond$	$\Diamond A \supset \Diamond(A \vee B)$
$\mathsf{C}_\Box$	$\Box A \wedge \Box B \supset \Box(A \wedge B)$	$\mathsf{C}_\Diamond$	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
$\mathsf{N}_\Box$	$\Box\top$	$\mathsf{N}_\Diamond$	$\neg\Diamond\bot$

***b.* Duality axioms**

$\mathsf{Dual}_\Box$	$\Diamond A \supset\subset \neg\Box\neg A$	$\mathsf{Dual}_\Diamond$	$\Box A \supset\subset \neg\Diamond\neg A$

***c.* Further relevant modal axioms and rules**

$\mathsf{K}_\Box$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$\mathsf{K}_\Diamond$	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
Nec $\dfrac{A}{\Box A}$	$\mathsf{Mon}_\Box$ $\dfrac{A \supset B}{\Box A \supset \Box B}$	$\mathsf{Mon}_\Diamond$	$\dfrac{A \supset B}{\Diamond A \supset \Diamond B}$

Figure 1: Modal axioms.

We use A, B, C as metavariables for formulas of $\mathcal{L}$. $\top$, $\neg A$ and $A \supset\subset B$ are abbreviations for, respectively, $\bot \supset \bot$, $A \supset \bot$ and $(A \supset B) \wedge (B \supset A)$. We take both modal operators $\Box$ and $\Diamond$ as primitive (as well as all boolean connectives), as it will be convenient for the intuitionistic case. Their duality in classical modal logics is recovered by adding to any system one of the duality axioms $\mathsf{Dual}_\Box$ or $\mathsf{Dual}_\Diamond$ (Figure 1), which are equivalent in the classical setting.

The weakest classical non-normal modal logic E is defined in language $\mathcal{L}$ by extending classical propositional logic (CPL) with a duality axiom and rule $\mathsf{E}_\Box$, and it can be extended further by adding any combination of axioms $\mathsf{M}_\Box$, $\mathsf{C}_\Box$ and $\mathsf{N}_\Box$. We obtain in this way eight distinct systems (Figure 2), which compose the family of classical non-normal modal logics.

Equivalent axiomatisations for these systems are given by considering the modal axioms in the right-hand column of Figure 1(a). Thus, logic E could be defined by extending CPL with axiom $\mathsf{Dual}_\Box$ and rule $\mathsf{E}_\Diamond$, and its extensions are given by adding combinations of axioms $\mathsf{M}_\Diamond$, $\mathsf{C}_\Diamond$ and $\mathsf{N}_\Diamond$.

It is worth recalling that axioms $\mathsf{M}_\Box$, $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Box$ are syntactically equivalent with the rules $\mathsf{Mon}_\Box$, $\mathsf{Mon}_\Diamond$ and Nec, respectively, and that axiom $\mathsf{K}_\Box$ is derivable from $\mathsf{M}_\Box$ and $\mathsf{C}_\Box$. As a consequence, we have that the top system MCN is equivalent to the weakest classical normal modal logic K.

2.2 Neighbourhood semantics

The standard semantics for classical non-normal modal logics is based on the so-called neighbourhood (or minimal, or Scott-Montague) models.

Definition 2.1. A *neighbourhood model* is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{N}$ is a neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$, and $\mathcal{V}$ is a valuation function $\mathcal{W} \longrightarrow Atm$. A neighbourhood model is supplemented,

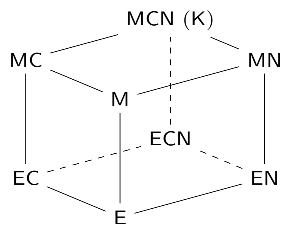

Figure 2: The classical cube.

closed under intersection, or contains the unit, if $\mathcal{N}$ satisfies the following properties:

If $\alpha \in \mathcal{N}(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}(w)$	(Supplementation);
If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$	(Closure under intersection);
$\mathcal{W} \in \mathcal{N}(w)$ for all $w \in \mathcal{W}$	(Containing the unit).

The forcing relation $w \Vdash A$ is defined inductively as follows:

$w \Vdash p$	iff	$p \in \mathcal{V}(w)$;
$w \nVdash \bot$;		
$w \Vdash B \wedge C$	iff	$w \Vdash A$ and $w \Vdash B$;
$w \Vdash B \vee C$	iff	$w \Vdash A$ or $w \Vdash B$;
$w \Vdash B \supset C$	iff	$w \Vdash B$ implies $w \Vdash C$;
$w \Vdash \Box B$	iff	$[B] \in \mathcal{N}(w)$;
$w \Vdash \Diamond B$	iff	$\mathcal{W} \setminus [B] \notin \mathcal{N}(w)$;

where $[B]$ denotes the set $\{v \in \mathcal{W} \mid v \Vdash B\}$, called the *truth set* of B.

We can also recall that in the supplemented case, the forcing conditions for modal formulas are equivalent to the following ones:

$w \Vdash \Box B$	iff	there is $\alpha \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [B]$;
$w \Vdash \Diamond B$	iff	for all $\alpha \in \mathcal{N}(w)$, $\alpha \cap [B] \neq \emptyset$.

Theorem 2.1 (Chellas [2])**.** Logic E(M,C,N) is sound and complete with respect to neighbourhood models (which in addition are supplemented, closed under intersection and contain the unit).

3 Intuitionistic non-normal monomodal logics

Our definition of intuitionistic non-normal modal logics begins with monomodal logics, that is logics containing only one modality, either $\Box$ or $\Diamond$. We first define the axiomatic systems, and then present their sequent calculi.

Under "intuitionistic modal logics" we understand any modal logic L that extends intuitionistic propositional logic (IPL) and satisfies the following requirements:

(R1) L is conservative over IPL: its non-modal fragment coincides with IPL.

(R2) L satisfies the disjunction property: if $A \vee B$ is derivable, then at least one formula between A and B is also derivable.

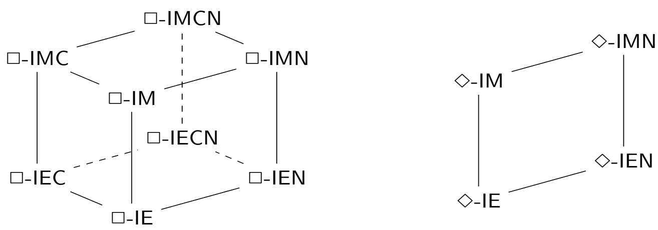

Figure 3: The lattices of intuitionistic non-normal monomodal logics.

3.1 Hilbert systems

From the point of view of axiomatic systems, two different classes of intuitionistic non-normal monomodal logics can be defined by analogy with the definition of classical non-normal modal logics (cf. Section 2). Intuitionistic modal logics are modal extensions of IPL, for which we consider the following axiomatisation:

$\supset$-1	$A \supset (B \supset A)$	$\wedge$-1	$A \wedge B \supset A$
$\supset$-2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$\wedge$-2	$A \wedge B \supset B$
$\vee$-1	$A \supset A \vee B$	$\wedge$-3	$A \supset (B \supset A \wedge B)$
$\vee$-2	$B \supset A \vee B$	efq	$\bot \supset A$
$\vee$-3	$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$	mp	$\dfrac{A \qquad A \supset B}{B}$

We define over IPL two families of intuitionistic non-normal monomodal logics, that depend on the considered modal operator, and are called therefore the $\Box$- and the $\Diamond$-family. The $\Box$-family is defined in language $\mathcal{L}_\Box := \mathcal{L} \setminus \{\Diamond\}$ by adding to IPL the rule $\mathsf{E}_\Box$ and any combination of axioms $\mathsf{M}_\Box$, $\mathsf{C}_\Box$ and $\mathsf{N}_\Box$. The $\Diamond$-family is instead defined in language $\mathcal{L}_\Diamond := \mathcal{L} \setminus \{\Box\}$ by adding to IPL the rule $\mathsf{E}_\Diamond$ and any combination of axioms $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Diamond$. It is work remarking that we don't consider intuitionistic non-normal modal logics containing axiom $\mathsf{C}_\Diamond$. We denote the resulting logics by, respectively, $\Box$-IE^* and $\Diamond$-IE^*, where E^* replaces any system of the classical cube (for $\Diamond$-logics, any system non containing $\mathsf{C}_\Diamond$).

Notice that, having rejected the definability of the lacking modality, $\Box$- and $\Diamond$-logics are distinct, as $\Box$ and $\Diamond$ behave differently. Moreover, as a consequence of the fact that the systems in the classical cube are pairwise non-equivalent, we have that the $\Box$-family contains eight distinct logics, while the $\Diamond$-family contains four distinct logics (something not derivable in a classical system is clearly not derivable in the corresponding intuitionistic system). It is also worth noticing that, as it happens in the classical case, axioms $\mathsf{M}_\Box$, $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Box$ are interderivable, respectively, with rules $\mathsf{Mon}_\Box$, $\mathsf{Mon}_\Diamond$ and Nec, and that $\mathsf{K}_\Box$ is derivable from $\mathsf{M}_\Box$ and $\mathsf{C}_\Box$ (as the standard derivations are intuitionistically valid).

3.2 Sequent calculi

We now present sequent calculi for intuitionistic non-normal monomodal logics. The calculi are defined as modal extensions of a given sequent calculus for IPL. We take G3ip as base calculus (Figure 4), and extend it with suitable combinations of the modal rules in Figure 5. The $\Box$-rules can be compared with the rules given in Lavendhomme and Lucas [15], where sequent calculi for

$\mathsf{Ax}\quad p, \Gamma \Rightarrow p$ $\qquad$ $\mathsf{L}\bot\quad \bot, \Gamma \Rightarrow A$

$$\mathsf{L}\wedge\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\mathsf{L}\vee\ \frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad \mathsf{R}\vee\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ (i = 0, 1)$$

$$\mathsf{L}\supset\ \frac{A \supset B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C} \qquad \mathsf{R}\supset\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Figure 4: Rules of G3ip (Troelstra and Schwichtenberg [26]).

$$\mathsf{E}^{\mathsf{seq}}_{\Box}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{E}^{\mathsf{seq}}_{\Diamond}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B}$$

$$\mathsf{M}^{\mathsf{seq}}_{\Box}\ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{M}^{\mathsf{seq}}_{\Diamond}\ \frac{A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B}$$

$$\mathsf{N}^{\mathsf{seq}}_{\Box}\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A} \qquad \mathsf{N}^{\mathsf{seq}}_{\Diamond}\ \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow B}$$

$$\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}\ \frac{A_1, ..., A_n \Rightarrow B \qquad B \Rightarrow A_1\ \ ...\ \ B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B}\ (n \geq 1)$$

$$\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}\ \frac{A_1, ..., A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B}\ (n \geq 1)$$

Figure 5: Modal rules for Gentzen calculi.

classical non-normal modal logics are presented. However, our rules are slightly different as (i) they have a single formula in the right-hand side of sequents; and (ii) contexts are added to the left-hand side of sequents appearing in the conclusion. Restriction (i) is adopted in order to have single-succedent calculi (as G3ip is), while with (ii) we implicitly embed weakening in the application of the modal rules. We consider the sequent calculi to be defined by the modal rules that are added to G3ip. The calculi are the following.

G.□-IE	:=	$\mathsf{E}^{\mathsf{seq}}_{\Box}$	G.□-IEC	:=	$\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$
G.□-IM	:=	$\mathsf{M}^{\mathsf{seq}}_{\Box}$	G.□-IMC	:=	$\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$
G.□-IEN	:=	$\mathsf{E}^{\mathsf{seq}}_{\Box} + \mathsf{N}^{\mathsf{seq}}_{\Box}$	G.□-IECN	:=	$\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}} + \mathsf{N}^{\mathsf{seq}}_{\Box}$
G.□-IMN	:=	$\mathsf{M}^{\mathsf{seq}}_{\Box} + \mathsf{N}^{\mathsf{seq}}_{\Box}$	G.□-IMCN	:=	$\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}} + \mathsf{N}^{\mathsf{seq}}_{\Box}$
G.◇-IE	:=	$\mathsf{E}^{\mathsf{seq}}_{\Diamond}$			
G.◇-IM	:=	$\mathsf{M}^{\mathsf{seq}}_{\Diamond}$			
G.◇-IEN	:=	$\mathsf{E}^{\mathsf{seq}}_{\Diamond} + \mathsf{N}^{\mathsf{seq}}_{\Diamond}$			
G.◇-IMN	:=	$\mathsf{M}^{\mathsf{seq}}_{\Diamond} + \mathsf{N}^{\mathsf{seq}}_{\Diamond}$			

Notice that - as in Lavendhomme and Lucas [15] - axiom $\mathsf{C}_{\Box}$ doesn't have a corresponding sequent rule, but it is captured by modifying the rules $\mathsf{E}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Box}$. In particular, these rules are replaced by $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$, respectively, that are the generalisations of $\mathsf{E}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Box}$ with n principal formulas (instead of just one) in the left-hand side of sequents. Observe that $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$ are non-standard, as they introduce an arbitrary number of modal formulas with a single application, and that $\mathsf{E}_\Box^{\mathsf{seq}}$ has in addition an arbitrary number of premisses. An other way to look at $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ and $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$ is to see them as infinite sets of rules, each set containing a standard rule for any $n \geq 1$. Under the latter interpretation the calculi are anyway non-standard as they are defined by infinite sets of rules.

We now prove the admissibility of some structural rules, and then show the equivalence between the sequent calculi and the Hilbert systems.

Proposition 3.1. The following weakening and contraction rules are height-preserving admissible in any monomodal calculus:

$$\mathsf{Lwk}\,\frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad \mathsf{Rwk}\,\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{ctr}\,\frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B}\,.$$

Proof. By induction on n, we show that whenever the premiss of an application of Lwk, Rwk or ctr has a derivation of height n, then its conclusion has a derivation of the same height. As usual, the proof considers the last rule applied in the derivation of the premiss (when the premiss is not an initial sequent). For rules of $\mathsf{G3ip}$ the proof is standard. For modal rules, left and right weakening are easily handled. For istance, the premiss $\Gamma \Rightarrow$ of Rwk is necessarily derived by $\mathsf{N}_\Diamond^{\mathsf{seq}}$. Then Γ contains a formula $\Diamond B$ that is principal in the application of $\mathsf{N}_\Diamond^{\mathsf{seq}}$, which in turn has $B \Rightarrow$ as premiss. By a different application of $\mathsf{N}_\Diamond^{\mathsf{seq}}$ to $B \Rightarrow$ we can derive $\Gamma \Rightarrow A$ for any A.

The proof is also immediate for contraction, where the most interesting case is possibly when both occurrences of A in the premiss $\Gamma, A, A \Rightarrow B$ of ctr are principal in the last rule applied in its derivation. In this case, the last rule is either $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ or $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$. If it is $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$, then $A \equiv \Box C$ for some C, and the sequent is derived from $D_1, ..., D_n, C, C \Rightarrow$ for some $\Box D_1, ..., \Box D_n$ in Γ. By i.h. we can apply ctr to the last sequent and obtain $D_1, ..., D_n, C \Rightarrow$, and then by $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$ derive sequent $\Gamma, A \Rightarrow B$, which is the conclusion of ctr (the proof is analogous for $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). ☐

We now show that the cut rule

$$\mathsf{Cut}\,\frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow B}{\Gamma \Rightarrow B}$$

is admissible in any monomodal calculus. The proof is based on the following notion of weight of formulas:

Definition 3.1 (Weight of formulas)**.** Function w assigning to each formula A its weight $w(A)$ is defined as follows: $w(\bot) = 0$; $w(p) = 1$; $w(A \circ B) = w(A) + w(B) + 1$ for $\circ \equiv \wedge, \vee, \supset$; and $w(\Box A) = w(\Diamond A) = w(A) + 2$.

Observe that, given the present definition, $\neg A$ has a smaller weight than $\Box A$ and $\Diamond A$. Although irrelevant to the next theorem, this will be used in Section 4 for the proof of cut elimination in bimodal calculi.

Theorem 3.2. Rule Cut is admissible in any monomodal calculus.

Proof. Given a derivation of a sequent with some applications of Cut, we show how to remove any such application and obtain a derivation of the same sequent without Cut. The proof is by double induction, with primary induction on the weight of the cut formula and subinduction on the cut height. We recall that, for any application of Cut, the cut formula is the formula which is deleted by that application, while the cut height is the sum of the heights of the derivations of the premisses of Cut.

We just consider the cases in which the cut formula is principal in the last rule applied in the derivation of both premisses of Cut. Moreover, we treat explicitly only the cases in which both premisses are derived by modal rules, as the non-modal cases are already considered in the proof of cut admissibility for G3ip, and because modal and non-modal rules don't interact in any relevant way.

• ($\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$; $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$. We have the following situation:

$$\dfrac{\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}\dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2 \Rightarrow D \quad D \Rightarrow B \quad D \Rightarrow C_1 \ \ ... \ \ D \Rightarrow C_m}{\Gamma, \Box B, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box D}\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}}{\Gamma, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box D}\mathsf{Cut}$$

The proof is converted as follows, with several applications of Cut with B as cut formula, hence with a cut formula of smaller weight. First we derive

$$\dfrac{\mathsf{wk}\dfrac{\Gamma_1 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B} \qquad \dfrac{B, \Gamma_2 \Rightarrow D}{B, \Gamma_1, \Gamma_2 \Rightarrow D}\mathsf{wk}}{\Gamma, \Gamma_1, \Gamma_2 \Rightarrow D}\mathsf{Cut}$$

Then for any $1 \leq i \leq n$, we derive

$$\dfrac{D \Rightarrow B \qquad \dfrac{B \Rightarrow A_i}{B, D \Rightarrow A_i}\mathsf{wk}}{D \Rightarrow A_i}\mathsf{Cut}$$

Finally we can apply $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ as follows

$$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow D \quad D \Rightarrow A_1 \ \ ... \ \ D \Rightarrow A_n \quad D \Rightarrow C_1 \ \ ... \ \ D \Rightarrow C_m}{\Gamma, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box D}\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$$

• ($\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$; $\mathsf{M}_\Box\mathsf{C}^{\mathsf{seq}}$) is analogous to ($\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$; $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). ($\mathsf{E}_\Box^{\mathsf{seq}}$; $\mathsf{E}_\Box^{\mathsf{seq}}$) and ($\mathsf{M}_\Box^{\mathsf{seq}}$; $\mathsf{M}_\Box^{\mathsf{seq}}$) are the particular cases where $n, m = 1$.

• ($\mathsf{N}_\Box^{\mathsf{seq}}$; $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$). Let $\Gamma_1 = B_1, ..., B_n$. The situation is as follows:

$$\dfrac{\mathsf{N}_\Box^{\mathsf{seq}}\dfrac{\Rightarrow A}{\Gamma, \Box\Gamma_1 \Rightarrow \Box A} \qquad \dfrac{A, \Gamma_1 \Rightarrow C \quad C \Rightarrow A \quad C \Rightarrow B_1 \ \ ... \ \ C \Rightarrow B_n}{\Gamma, \Box A, \Box\Gamma_1 \Rightarrow \Box C}\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}}{\Gamma, \Box\Gamma_1 \Rightarrow \Box C}\mathsf{Cut}$$

The proof is converted as follows, with an application of Cut on a cut formula of smaller weight.

$$\dfrac{\mathsf{Cut}\dfrac{\mathsf{wk}\dfrac{\Rightarrow A}{\Gamma_1 \Rightarrow A} \qquad A, \Gamma_1 \Rightarrow C}{\Gamma_1 \Rightarrow C} \qquad C \Rightarrow B_1 \ \ ... \ \ C \Rightarrow B_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box C}\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$$

• ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{M}_{\Box}\mathsf{C}^{\mathsf{seq}}$) is analogous to ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}$). ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{E}^{\mathsf{seq}}_{\Box}$) and ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{M}^{\mathsf{seq}}_{\Box}$) are the particular cases where $n = 1$.

• ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{E}^{\mathsf{seq}}_{\Diamond}$) and ($\mathsf{M}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{M}^{\mathsf{seq}}_{\Diamond}$) are analogous to ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{E}^{\mathsf{seq}}_{\Box}$) and ($\mathsf{M}^{\mathsf{seq}}_{\Box}$; $\mathsf{M}^{\mathsf{seq}}_{\Box}$), respectively.

• ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$). We have

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Diamond}\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \dfrac{B \Rightarrow}{\Gamma, \Diamond A, \Diamond B \Rightarrow C}\,\mathsf{N}^{\mathsf{seq}}_{\Diamond}}{\Gamma, \Diamond A \Rightarrow C}\,\mathsf{Cut}$$

which become

$$\dfrac{\dfrac{A \Rightarrow B \qquad \dfrac{B \Rightarrow}{A, B \Rightarrow}\,\mathsf{wk}}{A \Rightarrow}\,\mathsf{Cut}}{\Gamma, \Diamond A \Rightarrow C}\,\mathsf{N}^{\mathsf{seq}}_{\Diamond}$$

• ($\mathsf{M}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$) is analogous to ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$). □

As a consequence of the admissibility of Cut we obtain the equivalence between the sequent calculi and the axiomatic systems.

Proposition 3.3. Let L be any intuitionistic non-normal monomodal logic. Then calculus $\mathsf{G.L}$ is equivalent to system L.

Proof. The axioms and rules of L are derivable in $\mathsf{G.L}$. For the axioms of IPL and mp we can consider their derivations in $\mathsf{G3ip}$, as $\mathsf{G.L}$ enjoys admissibility of Cut. Here we show that any modal rule allows us to derive the corresponding axiom:

$$\dfrac{\dfrac{\dfrac{\mathsf{Cut}\;\dfrac{\mathsf{wk}\;\dfrac{\Rightarrow A \supset B}{A \Rightarrow A \supset B} \qquad A, A \supset B \Rightarrow B}{A \Rightarrow B} \qquad \dfrac{\dfrac{\Rightarrow B \supset A}{B \Rightarrow B \supset A}\,\mathsf{wk} \qquad B, B \supset A \Rightarrow A}{B \Rightarrow A}\,\mathsf{Cut}}{\Box A \Rightarrow \Box B}\,\mathsf{E}^{\mathsf{seq}}_{\Box}}{\Rightarrow \Box A \supset \Box B}\,\mathsf{R}{\supset}}{}$$

$$\dfrac{\dfrac{\dfrac{\mathsf{R}\wedge\;\dfrac{A, B \Rightarrow A \qquad A, B \Rightarrow B}{A, B \Rightarrow A \wedge B} \qquad \dfrac{A, B \Rightarrow A}{A \wedge B \Rightarrow A}\,\mathsf{L}\wedge \qquad \dfrac{A, B \Rightarrow B}{A \wedge B \Rightarrow B}\,\mathsf{L}\wedge}{\Box A, \Box B \Rightarrow \Box(A \wedge B)}\,\mathsf{E}_{\Box}\mathsf{C}^{\mathsf{seq}}}{\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)}\,\mathsf{L}\wedge}{\Rightarrow \Box A \wedge \Box B \supset \Box(A \wedge B)}\,\mathsf{R}{\supset}$$

$$\dfrac{\Rightarrow \top}{\Rightarrow \Box\top}\,\mathsf{N}^{\mathsf{seq}}_{\Box}$$

$$\dfrac{\dfrac{\bot \Rightarrow}{\Diamond\bot \Rightarrow}\,\mathsf{N}^{\mathsf{seq}}_{\Diamond}}{\Rightarrow \neg\Diamond\bot}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{\dfrac{A, B \Rightarrow A}{A \wedge B \Rightarrow A}\,\mathsf{L}\wedge}{\Box(A \wedge B) \Rightarrow \Box A}\,\mathsf{M}^{\mathsf{seq}}_{\Box}}{\Rightarrow \Box(A \wedge B) \supset \Box A}\,\mathsf{R}{\supset} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}\,\mathsf{R}\vee}{\Diamond A \Rightarrow \Diamond(A \vee B)}\,\mathsf{M}^{\mathsf{seq}}_{\Diamond}}{\Rightarrow \Diamond A \supset \Diamond(A \vee B)}\,\mathsf{R}{\supset}$$

Moreover, the rules of $\mathsf{G.L}$ are derivable in L. As before, it suffices to consider the modal rules. The derivations are in most cases straightforward, we just consider the following.

• If L contains $\mathsf{N}_{\Box}$, then $\mathsf{N}^{\mathsf{seq}}_{\Box}$ is derivable. Assume $\vdash_{\mathsf{L}} A$. Then by Nec (which is equivalent to $\mathsf{N}_{\Box}$), $\vdash_{\mathsf{L}} \Box A$.

• If L contains $\mathsf{N}_\Diamond$, then $\mathsf{N}_\Diamond^{\mathsf{seq}}$ is derivable. Assume $\vdash_\mathsf{L} A \supset \bot$. Since $\vdash_\mathsf{L} \bot \supset A$, by $\mathsf{E}_\Diamond^{\mathsf{seq}}$, $\vdash_\mathsf{L} \Diamond A \supset \Diamond\bot$. Then $\vdash_\mathsf{L} \neg\Diamond\bot \supset \neg\Diamond A$, and, since $\vdash_\mathsf{L} \neg\Diamond\bot$, we have $\vdash_\mathsf{L} \neg\Diamond A$.

• If L contains $\mathsf{C}_\Box$, then $\mathsf{E}_\Box\mathsf{C}^{\mathsf{seq}}$ is derivable. Assume $\vdash_\mathsf{L} A_1 \wedge ... \wedge A_n \supset B$ and $\vdash_\mathsf{L} B \supset A_i$ for all $1 \leq i \leq n$. Then $\vdash_\mathsf{L} B \supset A_1 \wedge ... \wedge A_n$. By $\mathsf{E}_\Box$, $\vdash_\mathsf{L} \Box(A_1 \wedge ... \wedge A_n) \supset \Box B$. In addition, by several applications of $\mathsf{C}_\Box$, $\vdash_\mathsf{L} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box(A_1 \wedge ... \wedge A_n)$. Therefore $\vdash_\mathsf{L} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box B$. ☐

4 Intuitionistic non-normal bimodal logics

In this section we present intuitionistic non-normal modal logics with both $\Box$ and $\Diamond$. In this case we first present their sequent calculi, and then give equivalent axiomatisations.

A simple way to define intuitionistic non-normal bimodal logics would be by considering the fusion of two monomodal logics that belong respectively to the $\Box$- and to the $\Diamond$-family. Given two logics $\Box$-IE^* and $\Diamond$-IE^*, their fusion in language $\mathcal{L}_\Box \cup \mathcal{L}_\Diamond$ is the smallest bimodal logics containing $\Box$-IE^* and $\Diamond$-IE^* (for the sake of simplicity we can assume that $\mathcal{L}_\Box$ and $\mathcal{L}_\Diamond$ share the same set of propositional variables, and differ only with respect to $\Box$ and $\Diamond$). The resulting logic is axiomatised simply by adding to IPL the modal axioms and rules of $\Box$-IE^*, plus the modal axioms and rules of $\Diamond$-IE^*.

It is clear, however, that in the resulting systems the modalities don't interact at all, as there is no axiom involving both $\Box$ and $\Diamond$. On the contrary, finding suitable interactions between the modalities is often the main issue when intuitionistic bimodal logics are concerned. In that case, by reflecting the fact that in IPL connectives are not interderivable, it is usually required that $\Box$ and $\Diamond$ are not dual. We take the lacking of duality as an additional requirement for the definition of intuitionistic non-normal bimodal logics:

(R3) $\Box$ and $\Diamond$ are not interdefinable.

In order to define intuitionistic non-normal bimodal logics by the axiomatic systems, we would need to select the axioms between a plethora of possible formulas satisfying (R3). If we look for instance at the literature on intuitionistic normal modal logics, we see that many different axioms have been considered, and the reasons for the specific choices are varied. We take therefore a different way, and define the logics starting with their sequent calculi. In particular we proceed as follows.

(i) Intuitionistic non-normal bimodal logics are defined by their sequent calculi. The calculi are conservative extensions of a given calculus for IPL, and have as modal rules some characteristic rules of intuitionistic non-normal monomodal logics, plus some rules connecting $\Box$ and $\Diamond$. In addition, we require that the Cut rule is admissible. As usual, this means that adding rule Cut to the calculus does not extend the set of derivable sequents.

(ii) To the purpose of defining the basic systems, we consider only interactions between $\Box$ and $\Diamond$ that can be seen as forms of "weak duality principles". In order to satisfy (R3), we require that these interactions are strictly weaker than $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$, in the sense that $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$ must not be derivable in any corresponding system.

$$\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}} \frac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad \mathsf{weak}_{\mathsf{b}}^{\mathsf{seq}} \frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

$$\mathsf{neg}_{\mathsf{a}}^{\mathsf{seq}} \frac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad \mathsf{neg}_{\mathsf{b}}^{\mathsf{seq}} \frac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

$$\mathsf{str}^{\mathsf{seq}} \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

Figure 6: Interaction rules for sequent calculi.

(iii) We will distinguish logics that are monotonic and logics that are non-monotonic. Moreover, the logics will be distinguished by the different strength of interactions between the modalities.

The above points are realised in practice as follows. As before, we take G3ip (Figure 4) as base calculus for intuitionistic logics. This is extended with combinations of the characteristic rules of intuitionistic non-normal monomodal logics in Figure 5. The difference is that now the calculi contain both some rules for $\Box$ and some rules for $\Diamond$. In order to distinguish monotonic and non-monotonic logics, we require that the calculi contain either both $\mathsf{E}_{\Box}^{\mathsf{seq}}$ and $\mathsf{E}_{\Diamond}^{\mathsf{seq}}$ (in this case the corresponding logic will be non-monotonic), or both $\mathsf{M}_{\Box}^{\mathsf{seq}}$ and $\mathsf{M}_{\Diamond}^{\mathsf{seq}}$ (corresponding to monotonic logics). In addition, the calculi will contain some of the interaction rules in Figure 6. Since the logics are also distinguished according to the different strenghts of the interactions between the modalities, we require that the calculi contain either both $\mathsf{weak}_{\mathsf{a}}^{\mathsf{seq}}$ and $\mathsf{weak}_{\mathsf{b}}^{\mathsf{seq}}$, or both $\mathsf{neg}_{\mathsf{a}}^{\mathsf{seq}}$ and $\mathsf{neg}_{\mathsf{b}}^{\mathsf{seq}}$, or $\mathsf{str}^{\mathsf{seq}}$.

In the following we present the sequent calculi for intuitionistic non-normal bimodal logics obtained by following our methodology. After that, for each sequent calculus we present an equivalent axiomatisation.

4.1 Sequent calculi

In the first part, we focus on sequent calculi for logics containing only axioms between $\mathsf{M}_{\Box}$, $\mathsf{M}_{\Diamond}$, $\mathsf{N}_{\Box}$ and $\mathsf{N}_{\Diamond}$ (that is, we don't consider axiom $\mathsf{C}_{\Box}$). The calculi are obtained by adding to G3ip (Figure 4) suitable combinations of the modal rules in Figures 5 and 6. Although in principle any combination of rules could define a calculus, we accept only those calculi that satisfy the restrictions explained above. This entails in particular the need of studying cut elimination. As usual, the first step to do towards the study of cut elimination is to prove the admissibility of the other structural rules.

Proposition 4.1. Weakening and contraction are height-preserving admissible in any sequent calculus defined by a combination of modal rules in Figures 5 and 6 that satisfies the restrictions explained above.

Proof. By extending the proof of Proposition 3.1 with the examination of the interaction rules in Figure 6. Due to their form, however, it is immediate to verify that if the premiss of wk or ctr is derivable by any interaction rule, then the conclusion is derivable by the same rule. $\Box$

We can now examine the admissibility of Cut. As it is stated by the following theorem, following our methodology we obtain 12 sequent calculi for intuitionistic non-normal bimodal logics.

Theorem 4.2. We let the sequent calculi be defined by the set of modal rules which are added to G3ip. The Cut rule is admissible in the following calculi:

$$\begin{array}{lll} \mathsf{G.IE}_1 & := & \mathsf{E}^{\mathsf{seq}}_{\Box} + \mathsf{E}^{\mathsf{seq}}_{\Diamond} + \mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}} + \mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}} \\ \mathsf{G.IE}_2 & := & \mathsf{E}^{\mathsf{seq}}_{\Box} + \mathsf{E}^{\mathsf{seq}}_{\Diamond} + \mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}} + \mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}} \\ \mathsf{G.IE}_3 & := & \mathsf{E}^{\mathsf{seq}}_{\Box} + \mathsf{E}^{\mathsf{seq}}_{\Diamond} + \mathsf{str}^{\mathsf{seq}} \\ \mathsf{G.IM} & := & \mathsf{M}^{\mathsf{seq}}_{\Box} + \mathsf{M}^{\mathsf{seq}}_{\Diamond} + \mathsf{str}^{\mathsf{seq}} \end{array}$$

Moreover, letting G^* be any of the previous calculi, Cut is admissible in

$$\begin{array}{lll} \mathsf{G}^*\mathsf{N}_{\Diamond} & := & \mathsf{G}^* + \mathsf{N}^{\mathsf{seq}}_{\Diamond} \\ \mathsf{G}^*\mathsf{N}_{\Box} & := & \mathsf{G}^* + \mathsf{N}^{\mathsf{seq}}_{\Diamond} + \mathsf{N}^{\mathsf{seq}}_{\Box} \end{array}$$

Proof. The structure of the proof is the same as the proof of Theorem 3.2. Again, we consider only the cases where the cut formula is principal in the last rule applied in the derivation of both premisses, with the further restriction that the last rules are modal ones.

The combinations between $\Box$-rules, or between $\Diamond$-rules, have been shown in the proof of Theorem 3.2. Here we consider the possible combinations of $\Box$- or $\Diamond$-rules with rules for interaction.

- ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$). We have

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Box}\,\dfrac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A, \Diamond C \Rightarrow \Box B} \qquad \dfrac{\Rightarrow B \quad C \Rightarrow}{\Gamma, \Box A, \Box B, \Diamond C \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{Cut}$$

which become

$$\dfrac{\mathsf{Cut}\,\dfrac{\Rightarrow B \quad B \Rightarrow A}{\Rightarrow A} \qquad C \Rightarrow}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$). We have

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Diamond}\,\dfrac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Diamond A, \Box C \Rightarrow \Diamond B} \qquad \dfrac{\Rightarrow C \quad B \Rightarrow}{\Gamma, \Diamond A, \Box C, \Diamond B \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}}{\Gamma, \Diamond A, \Box C \Rightarrow D}\,\mathsf{Cut}$$

which become

$$\dfrac{\Rightarrow C \qquad \dfrac{A \Rightarrow B \quad B \Rightarrow}{A \Rightarrow}\,\mathsf{Cut}}{\Gamma, \Diamond A, \Box C \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$). We have

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Box}\,\dfrac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A, \Diamond C \Rightarrow \Box B} \qquad \dfrac{B \Rightarrow \quad \Rightarrow C}{\Gamma, \Box A, \Box B, \Diamond C \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{Cut}$$

which become

$$\dfrac{\mathsf{Cut}\,\dfrac{A \Rightarrow B \quad B \Rightarrow}{A \Rightarrow} \qquad \Rightarrow C}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Diamond}$; $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$). We have

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Diamond}\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A, \Box C \Rightarrow \Diamond B} \qquad \dfrac{C \Rightarrow \qquad \Rightarrow B}{\Gamma, \Diamond A, \Box C, \Diamond B \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}}{\Gamma, \Diamond A, \Box C \Rightarrow D}\,\mathsf{Cut}$$

which become

$$\dfrac{\Rightarrow C \qquad \dfrac{\Rightarrow B \qquad B \Rightarrow A}{\Rightarrow A}\,\mathsf{Cut}}{\Gamma, \Diamond A, \Box C \Rightarrow D}\,\mathsf{weak}^{\mathsf{seq}}_{\mathsf{b}}$$

- ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}$). We have:

$$\dfrac{\mathsf{E}^{\mathsf{seq}}_{\Box}\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A, \Diamond C \Rightarrow \Box B} \qquad \dfrac{B, C \Rightarrow \qquad \neg B \Rightarrow C}{\Gamma, \Box A, \Box B, \Diamond C \Rightarrow D}\,\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{Cut}$$

which is converted into the following derivation:

$$\dfrac{\mathsf{Cut}\,\dfrac{\mathsf{wk}\,\dfrac{A \Rightarrow B}{A, C \Rightarrow B} \qquad \dfrac{B, C \Rightarrow A}{A, B, C \Rightarrow}\,\mathsf{wk}}{A, C \Rightarrow} \qquad \dfrac{\dfrac{B \Rightarrow A}{\neg A \Rightarrow \neg B} \qquad \dfrac{\neg B \Rightarrow C}{\neg A, \neg B \Rightarrow C}\,\mathsf{wk}}{\neg A \Rightarrow C}\,\mathsf{Cut}}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}$$

Observe that the former derivation has two application of Cut, both of them with a cut formula of smaller weight as, in particular, $w(\neg B) < w(\Box B)$ (cf. Definition 3.1).

- ($\mathsf{E}^{\mathsf{seq}}_{\Box}$; $\mathsf{str}^{\mathsf{seq}}$) is analogous to the next case ($\mathsf{M}^{\mathsf{seq}}_{\Box}$; $\mathsf{str}^{\mathsf{seq}}$).
- ($\mathsf{M}^{\mathsf{seq}}_{\Box}$; $\mathsf{str}^{\mathsf{seq}}$). We have:

$$\dfrac{\mathsf{M}^{\mathsf{seq}}_{\Box}\,\dfrac{A \Rightarrow B}{\Gamma, \Box A, \Diamond C \Rightarrow \Box B} \qquad \dfrac{B, C \Rightarrow}{\Gamma, \Box A, \Box B, \Diamond C \Rightarrow D}\,\mathsf{str}^{\mathsf{seq}}}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{Cut}$$

which is converted into the following derivation:

$$\dfrac{\dfrac{\mathsf{wk}\,\dfrac{A \Rightarrow B}{A, C \Rightarrow B} \qquad \dfrac{B, C \Rightarrow}{A, B, C \Rightarrow}\,\mathsf{wk}}{A, C \Rightarrow}\,\mathsf{Cut}}{\Gamma, \Box A, \Diamond C \Rightarrow D}\,\mathsf{str}$$

- ($\mathsf{N}^{\mathsf{seq}}_{\Box}$; $\mathsf{str}^{\mathsf{seq}}$). We have

$$\dfrac{\mathsf{N}^{\mathsf{seq}}_{\Box}\,\dfrac{\Rightarrow A}{\Gamma, \Diamond B \Rightarrow \Box A} \qquad \dfrac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C}\,\mathsf{str}^{\mathsf{seq}}}{\Gamma, \Diamond B \Rightarrow C}\,\mathsf{Cut}$$

which become

$$\dfrac{\dfrac{\mathsf{wk}\,\dfrac{\Rightarrow A}{B \Rightarrow A} \qquad A, B \Rightarrow}{B \Rightarrow}\,\mathsf{Cut}}{\Gamma, \Diamond B \Rightarrow C}\,\mathsf{N}^{\mathsf{seq}}_{\Diamond}$$

□

It is worth noticing that all cut-free calculi containing rule $\mathsf{N}^{\mathsf{seq}}_{\Box}$ also contain rule $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$. In fact, combinations of rules containing $\mathsf{N}^{\mathsf{seq}}_{\Box}$ and not $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$ would give calculi where the Cut rule is not admissible. This is due to the form of the interaction rules, that for instance allow us to derive the sequent $\Diamond\bot \Rightarrow$ using Cut and $\mathsf{N}^{\mathsf{seq}}_{\Box}$. A possible derivation is the following:

$$\dfrac{\mathsf{N}^{\mathsf{seq}}_{\Box}\ \dfrac{\Rightarrow \top}{\Diamond\bot \Rightarrow \Box\top} \qquad \dfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Box\top, \Diamond\bot \Rightarrow}\ \mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}}{\Diamond\bot \Rightarrow}\ \mathsf{Cut}$$

Instead, sequent $\Diamond\bot \Rightarrow$ doesn't have any cut-free derivation where $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$ is not applied, as no rule different from $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$ has $\Diamond\bot \Rightarrow$ in the conclusion. We will consider in Section 7 a calculus containing $\mathsf{N}^{\mathsf{seq}}_{\Box}$ and not $\mathsf{N}^{\mathsf{seq}}_{\Diamond}$. As we shall see, that calculus has interaction rules of a different form.

An additional remark concerns the possible choices of interaction rules in presence of $\mathsf{M}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Diamond}$. In particular, we notice that whenever we take $\mathsf{M}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Diamond}$, rule $\mathsf{str}^{\mathsf{seq}}$ is the only interaction that gives cut-free calculi. It can be interesting to consider a case of failure of cut elimination when other interaction rules are considered.

Example 4.1. Sequent $\Box\neg p, \Diamond(p \wedge q) \Rightarrow$ is derivable from $\mathsf{M}^{\mathsf{seq}}_{\Box}$ + $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}$ + $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$ + Cut, but it is not derivable from $\mathsf{M}^{\mathsf{seq}}_{\Box}$ + $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}$ + $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$ without Cut. A possible derivation is as follows:

$$\dfrac{\mathsf{wk}\ \dfrac{\mathsf{M}^{\mathsf{seq}}_{\Box}\ \dfrac{\mathsf{R}\neg\ \dfrac{\neg p, p \wedge q \Rightarrow}{\neg p \Rightarrow \neg(p \wedge q)}}{\Box\neg p \Rightarrow \Box\neg(p \wedge q)}}{\Box\neg p, \Diamond(p \wedge q) \Rightarrow \Box\neg(p \wedge q)} \qquad \dfrac{\dfrac{\neg(p \wedge q), p \wedge q \Rightarrow \qquad \neg(p \wedge q) \Rightarrow \neg(p \wedge q)}{\Box\neg(p \wedge q), \Diamond(p \wedge q) \Rightarrow}\ \mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}}{\Box\neg(p \wedge q), \Box\neg p, \Diamond(p \wedge q) \Rightarrow}\ \mathsf{wk}}{\Box\neg p, \Diamond(p \wedge q) \Rightarrow}\ \mathsf{Cut}$$

Let us now try to derive bottom-up the sequent without using Cut. As last rule we can only apply $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{a}}$ or $\mathsf{neg}^{\mathsf{seq}}_{\mathsf{b}}$, as they are the only rules with a conclusion of the right form. In the first case the premisses would be $\neg p, p \wedge q \Rightarrow$, and $\neg\neg p \Rightarrow p \wedge q$; while in the second case the premisses would be $\neg p, p \wedge q \Rightarrow$, and $\neg p \Rightarrow \neg(p \wedge q)$. It is clear, however, that in both cases the second premiss is not derivable.

We now consider sequent calculi for logics containing axiom $\mathsf{C}_{\Box}$. As it happens in the case of calculi for monomodal logics, $\mathsf{C}_{\Box}$ is not captured by adding a specific rule, but we need instead to modify most of the modal rules already given. In addition to the previous modifications of $\mathsf{E}^{\mathsf{seq}}_{\Box}$ and $\mathsf{M}^{\mathsf{seq}}_{\Box}$, we now need to modify also the interaction rules. In particular, we must take their generalisations that allow to introduce n boxed formulas by a single application (rules in Figure 7). Rule $\mathsf{weak}^{\mathsf{seq}}_{\mathsf{a}}$ is an exception as the boxed formula which is principal in the rule application occurs as unboxed in the right-hand side of the premiss, and therefore doesn't need to be changed.

As before, it can be easily shown that weakening and contraction are height-preserving admissible.

Proposition 4.3. Weakening and contraction are height-preserving admissible in any sequent calculus defined by a combination of modal rules in Figures 5 and 7 that satisfies our restrictions.

$$\mathsf{weak_bC^{seq}}\ \frac{A_1, ..., A_n \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C} \qquad \mathsf{strC^{seq}}\ \frac{A_1, ..., A_n, B \Rightarrow}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}$$

$$\mathsf{neg_aC^{seq}}\ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg B \Rightarrow A_1 \ \ ... \ \ \neg B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}$$

$$\mathsf{neg_bC^{seq}}\ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg A_i \Rightarrow B \ \ ... \ \ \neg A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow C}$$

Figure 7: Modified interaction rules for $\mathsf{C}_\Box$. In any rule $n \geq 1$.

Following our methodology we obtain again 12 sequent calculi, as it is stated by the following theorem:

Theorem 4.4. The Cut rule is admissible in the following calculi:

$$\begin{array}{lcl} \mathsf{G.IE_1C} & := & \mathsf{E_\Box C^{seq}} + \mathsf{E_\Diamond^{seq}} + \mathsf{weak_a^{seq}} + \mathsf{weak_bC^{seq}} \\ \mathsf{G.IE_2C} & := & \mathsf{E_\Box C^{seq}} + \mathsf{E_\Diamond^{seq}} + \mathsf{neg_aC^{seq}} + \mathsf{neg_bC^{seq}} \\ \mathsf{G.IE_3C} & := & \mathsf{E_\Box C^{seq}} + \mathsf{E_\Diamond^{seq}} + \mathsf{strC^{seq}} \\ \mathsf{G.IMC} & := & \mathsf{M_\Box C^{seq}} + \mathsf{M_\Diamond^{seq}} + \mathsf{strC^{seq}} \end{array}$$

Moreover, letting GC^* be any of the previuos calculi, Cut is admissible in

$$\begin{array}{lcl} \mathsf{GC^*N_\Diamond} & := & \mathsf{GC^*} + \mathsf{N_\Diamond^{seq}} \\ \mathsf{GC^*N_\Box} & := & \mathsf{GC^*} + \mathsf{N_\Diamond^{seq}} + \mathsf{N_\Box^{seq}} \end{array}$$

Proof. As before, we only show some relevant cases.

• ($\mathsf{E_\Box C^{seq}}$; $\mathsf{weak_a^{seq}}$). Let Γ_1 be the multiset $A_1, ..., A_n$, and $\Box\Gamma_1$ be $\Box A_1, ..., \Box A_n$. We have:

$$\mathsf{Cut}\ \frac{\mathsf{E_\Box C^{seq}}\ \dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1, \Diamond C \Rightarrow \Box B} \qquad \dfrac{\Rightarrow B \qquad C \Rightarrow}{\Gamma, \Box\Gamma_1, \Box B, \Diamond C \Rightarrow D}\ \mathsf{weak_a^{seq}}}{\Gamma, \Box\Gamma_1, \Diamond C \Rightarrow D}$$

which become

$$\mathsf{weak_a^{seq}}\ \frac{\mathsf{Cut}\ \dfrac{\Rightarrow B \qquad B \Rightarrow A_1}{\Rightarrow A_1} \qquad C \Rightarrow}{\Gamma, \Box A_1, \Box A_2, ..., \Box A_n, \Diamond C \Rightarrow D}$$

• ($\mathsf{E_\Box C^{seq}}$; $\mathsf{neg_bC^{seq}}$). Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_k$. We have:

$$\mathsf{Cut}\ \frac{\mathsf{E_\Box C^{seq}}\ \dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow \quad \neg B \Rightarrow D \quad \neg C_1 \Rightarrow D \ \ ... \ \ \neg C_k \Rightarrow D}{\Gamma, \Box\Gamma_1, \Box B, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{neg_bC^{seq}}}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}$$

which is converted as follows: First we derive sequent $\Gamma_1, \Gamma_2, D \Rightarrow$ and, for all $1 \leq i \leq n$, sequent $\neg A_i \Rightarrow D$ as follows:

$$\mathsf{Cut}\ \frac{\mathsf{wk}\ \dfrac{\Gamma_1 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow}{B, \Gamma_1, \Gamma_2, D \Rightarrow}\ \mathsf{wk}}{\Gamma_1, \Gamma_2, D \Rightarrow} \qquad \frac{\dfrac{B \Rightarrow A_i}{\neg A_i \Rightarrow \neg B} \qquad \dfrac{\neg B \Rightarrow D}{\neg A_i, \neg B \Rightarrow D}\ \mathsf{wk}}{\neg A_i \Rightarrow D}\ \mathsf{Cut}$$

Then we can apply $\mathsf{neg_b C^{seq}}$:

$$\dfrac{\Gamma_1, \Gamma_2, D \Rightarrow \quad \neg A_1 \Rightarrow D \ \ldots \ \neg A_n \Rightarrow D \quad \neg C_1 \Rightarrow D \ \ldots \ \neg C_k \Rightarrow D}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{neg_b C^{seq}}$$

- ($\mathsf{E_\Box C^{seq}}$; $\mathsf{strC^{seq}}$). Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_k$. We have:

$$\dfrac{\mathsf{E_\Box C^{seq}}\ \dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \ldots \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow}{\Gamma, \Box\Gamma_1, \Box B, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{strC^{seq}}}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{Cut}$$

which become

$$\dfrac{\dfrac{\mathsf{wk}\ \dfrac{\Gamma_1 \Rightarrow B}{\Gamma_1, \Gamma_2, D \Rightarrow B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow}{\Gamma_1, B, \Gamma_2, D \Rightarrow}\ \mathsf{wk}}{\Gamma_1, \Gamma_2, D \Rightarrow}\ \mathsf{Cut}}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow E}\ \mathsf{strC^{seq}}$$

- ($\mathsf{M_\Box C^{seq}}$; $\mathsf{strC^{seq}}$) is similar to the previous case. □

Notably, the cut-free calculi in Theorem 4.4 are the $\mathsf{C_\Box}$-versions of the cut-free calculi in Theorem 4.4. This means that, once the interaction rules are opportunely modified, the generalisation of the rules to n principal formulas doesn't give problems with respect to cut elimination.

We have also seen that rule $\mathsf{weak_a^{seq}}$ doesn't need to be changed. Instead, if we don't modify the other interaction rules we obtain calculi in which Cut is not eliminable, as it is shown by the following example.

Example 4.2. Sequent $\Box p, \Box\neg p, \Diamond\top \Rightarrow$ is derivable by $\mathsf{M_\Box C^{seq}} + \mathsf{weak_b^{seq}} + \mathsf{Cut}$, but is not derivable by $\mathsf{M_\Box C^{seq}} + \mathsf{weak_b^{seq}}$ without Cut. The derivation is as follows:

$$\dfrac{\mathsf{M_\Box C^{seq}}\ \dfrac{p, \neg p \Rightarrow \bot}{\Diamond\top, \Box p, \Box\neg p \Rightarrow \Box\bot} \qquad \dfrac{\bot \Rightarrow \qquad \Rightarrow \top}{\Box p, \Box\neg p, \Box\bot, \Diamond\top \Rightarrow}\ \mathsf{weak_b^{seq}}}{\Box p, \Box\neg p, \Diamond\top \Rightarrow}\ \mathsf{Cut}$$

Without Cut the sequent is instead not derivable, as the only applicable rule would be $\mathsf{weak_b^{seq}}$, but neither p nor $\neg p$ is a contradiction.

4.2 Hilbert systems

For each sequent calculus we now define an equivalent Hilbert system. To this purpose, in addition the formulas in Figure 1, we also consider the axioms and rules in Figure 8. As before, the Hilbert systems are defined by the set of modal axioms and rules that are added to IPL. The systems are axiomatised as follows.

$\mathsf{IE_1} := \mathsf{E_\Box} + \mathsf{E_\Diamond} + \mathsf{weak_a} + \mathsf{weak_b}$
$\mathsf{IE_2} := \mathsf{E_\Box} + \mathsf{E_\Diamond} + \mathsf{neg_a} + \mathsf{neg_b}$
$\mathsf{IE_3} := \mathsf{E_\Box} + \mathsf{E_\Diamond} + \mathsf{str}$
$\mathsf{IM} := \mathsf{E_\Box} + \mathsf{E_\Diamond} + \mathsf{M_\Box} + \mathsf{M_\Diamond} + \mathsf{str}$

Moreover, letting H^* be any of the four systems listed above, we have the following additional systems:

weak$_a$	$\neg(\Box\top \wedge \Diamond\bot)$	weak$_b$	$\neg(\Diamond\top \wedge \Box\bot)$	str	$\dfrac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$
neg$_a$	$\neg(\Box A \wedge \Diamond\neg A)$	neg$_b$	$\neg(\Box\neg A \wedge \Diamond A)$		

Figure 8: Hilbert axioms and rules for interactions between $\Box$ and $\Diamond$.

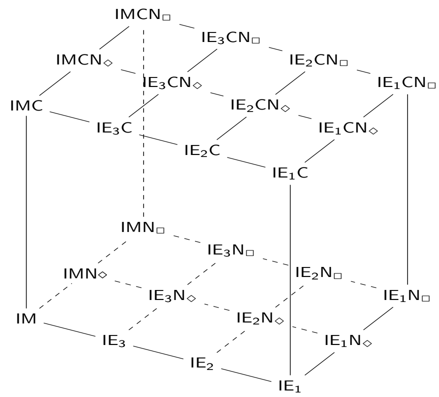

Figure 9: The lattice of intuitionistic non-normal bimodal logics.

$$
\begin{array}{lcl}
\mathsf{H^*C} & := & \mathsf{H^*} + \mathsf{C}_\Box \\
\mathsf{H^*N}_\Diamond & := & \mathsf{H^*} + \mathsf{N}_\Diamond \\
\mathsf{H^*N}_\Box & := & \mathsf{H^*} + \mathsf{N}_\Box \\
\mathsf{H^*CN}_\Diamond & := & \mathsf{H^*} + \mathsf{C}_\Box + \mathsf{N}_\Diamond \\
\mathsf{H^*CN}_\Box & := & \mathsf{H^*} + \mathsf{C}_\Box + \mathsf{N}_\Box
\end{array}
$$

Proposition 4.5. Let G.L be any sequent calculus for intuitionistic non-normal bimodal logics. Then G.L is equivalent to system L.

Proof. Any axiom and rule of L is derivable in G.L. Here we only consider the interactions between the modalities, as the derivations of the other axioms have been already given in Proposition 3.3.

$$
\cfrac{\cfrac{\cfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Box\top, \Diamond\bot \Rightarrow}\,\mathsf{weak}_a^{seq}}{\Box\top \wedge \Diamond\bot \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box\top \wedge \Diamond\bot)}\,\mathsf{R}\neg
\qquad\qquad
\cfrac{\cfrac{\cfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Diamond\top, \Box\bot \Rightarrow}\,\mathsf{weak}_b^{seq}}{\Diamond\top \wedge \Box\bot \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Diamond\top \wedge \Box\bot)}\,\mathsf{R}\neg
$$

$$
\cfrac{\cfrac{\cfrac{\mathsf{Cut}\,\cfrac{\mathsf{wk}\,\cfrac{\Rightarrow \neg(A \wedge B)}{A, B \Rightarrow \neg(A \wedge B)} \qquad A, B, \neg(A \wedge B) \Rightarrow}{A, B \Rightarrow}}{\Box A, \Diamond B \Rightarrow}\,\mathsf{str}^{seq}}{\Box A \wedge \Diamond B \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box A \wedge \Diamond B)}\,\mathsf{R}\neg
$$

$$
\cfrac{\cfrac{\cfrac{A, \neg A \Rightarrow \qquad \neg A \Rightarrow \neg A}{\Box A, \Diamond\neg A \Rightarrow}\,\mathsf{neg}_a^{seq}}{\Box A \wedge \Diamond\neg A \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box A \wedge \Diamond\neg A)}\,\mathsf{R}\neg
\qquad\qquad
\cfrac{\cfrac{\cfrac{A, \neg A \Rightarrow \qquad \neg A \Rightarrow \neg A}{\Box\neg A, \Diamond A \Rightarrow}\,\mathsf{neg}_b^{seq}}{\Box\neg A \wedge \Diamond A \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box\neg A \wedge \Diamond A)}\,\mathsf{R}\neg
$$

Moreover, any rule of $\mathsf{G.L}$ is derivable in L. As before we only need to consider the interaction rules. The derivations are immediate, we show as example the following.

- If L contains axiom $\mathsf{weak_a}$, then rule $\mathsf{weak_a^{seq}}$ is derivable. Assume $\vdash_{\mathsf{L}} A$ and $\vdash_{\mathsf{L}} B \supset \bot$. Then $\vdash_{\mathsf{L}} \top \supset A$, and, since $\vdash_{\mathsf{L}} A \supset \top$, by $\mathsf{E}_\Box$ we have $\vdash_{\mathsf{L}} \Box A \supset \Box\top$. Moreover, since $\vdash_{\mathsf{L}} \bot \supset B$, by $\mathsf{E}_\Diamond$ we have $\vdash_{\mathsf{L}} \Diamond B \supset \Diamond\bot$, hence $\vdash_{\mathsf{L}} \neg\Diamond\bot \supset \neg\Diamond B$. By $\mathsf{weak_a}$ we also have $\vdash_{\mathsf{L}} \Box\top \supset \neg\Diamond\bot$. Thus $\vdash_{\mathsf{L}} \Box A \supset \neg\Diamond B$, which gives $\vdash_{\mathsf{L}} \neg(\Box A \wedge \Diamond B)$.
- If L contains axiom $\mathsf{neg_b}$, then rule $\mathsf{neg_b^{seq}}$ is derivable. Assume $\vdash_{\mathsf{L}} A \supset \neg B$ and $\vdash_{\mathsf{L}} \neg B \supset A$. Then by $\mathsf{E}_\Box$, $\vdash_{\mathsf{L}} \Box A \supset \Box\neg B$. By $\mathsf{neg_b}$ we have $\vdash_{\mathsf{L}} \Box\neg B \supset \neg\Diamond B$. Thus $\vdash_{\mathsf{L}} \Box A \supset \neg\Diamond B$, which gives $\vdash_{\mathsf{L}} \neg(\Box A \wedge \Diamond B)$. □

5 Decidability and other consequences of cut elimination

Analytic cut-free sequent calculi are a very powerful tool for proof analysis. In this section we take advantage of the fact that Cut is admissible in all sequent calculi defined in Sections 3 and 4 in order to prove some additional properties of the corresponding logics. By looking at the form of the rules, we first observe that all calculi satisfy the requirements on intuitionistic non-normal modal logics that we have initially made, *i.e.* that they are conservative over IPL (R1); that they satisfy the disjunction property (R2); and that the duality principles $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$ are not derivable (R3). In a similar way we show that all calculi are pairwise non-equivalent, hence the lattices of intuitionistic non-normal modal logics contain, respectively, 8 distinct monomodal $\Box$-logics, 4 distinct monomodal $\Diamond$-logics, and 24 distinct bimodal logics.

Some form of subformula property often follows from cut elimination. For calculi containing rules $\mathsf{neg_a^{seq}}$ and $\mathsf{neg_b^{seq}}$ we must consider a property that is slightly different to the usual one, as $\neg A$ can appear in a premiss of a rule where $\Box A$ or $\Diamond A$ appears in the conclusion. As we shall see, the considered property is strong enough to provide, together with the admissibility of contraction, a standard proof of decidability for $\mathsf{G3}$ calculi.

We conclude the section with some further remarks about the logics that we have defined, that in particular concern the relations between intuitionistic and classical modal logics.

Fact 5.1. Any intuitionistic non-normal modal logic defined in Section 3 and Section 4 satisfies requirements R1, R2 and R3 (the latter one being relevant only for bimodal logics).

Proof. (R1) Any logic is conservative over IPL. In fact, the non-modal rules of each sequent calculus are exactly the rules of $\mathsf{G3ip}$.

(R2) Any logic satisfies the disjunction property. In fact, given a derivable sequent of the form $\Rightarrow A \vee B$, since no modal rule has such a conclusion, the last rule applied in its derivation is necessarily R$\vee$. This has premiss $\Rightarrow A$ or $\Rightarrow B$, which in turn is derivable.

(R3) Axioms $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$ are not derivable in L for any system L. In particular, neither $\neg\Box\neg A \supset \Diamond A$, nor $\neg\Diamond\neg A \supset \Box A$ (*i.e.* the right-to-left implications of $\mathsf{Dual}_\Box$ and $\mathsf{Dual}_\Diamond$) is derivable. For instance, if we try to derive bottom-up the sequent $\neg\Box\neg A \Rightarrow \Diamond A$ in $\mathsf{G.L}$, the only applicaple rule would be $\mathsf{L}\supset$. This has premiss $\neg\Box\neg A \Rightarrow \Box\neg A$. Again, $\mathsf{L}\supset$ is the only applicable rule, with the same sequent as premiss. Since this is not an initial sequent, we have that $\neg\Box\neg A \Rightarrow \Diamond A$ is not derivable. The situation is analogous for $\neg\Diamond\neg A \Rightarrow \Box A$. □

Theorem 5.2. The lattice of intuitionistic non-normal bimodal logics contains 24 distinct systems.

Proof. We leave to the reader to check that taken two logics L_1 and L_2 of the lattice, we can always find some formulas (or rules) that are derivable in L_1 and not in L_2, or *vice versa*. This can be easily done by considering the corresponding calculi $\mathsf{G.L}_1$ and $\mathsf{G.L}_2$. In particular, if L_1 is stronger than L_2, then the characteristic axiom of L_1 is not derivable in L_2. If instead L_1 and L_2 are incomparable, then they both have some characteristic axioms (or rules) that are not derivable in the other. For rule str we can consider the counterexample to cut elimination in Example 4.1 □

Definition 5.1 (Strict subformula and negated subformula)**.** For any formulas A and B, we say that A is a *strict subformula* of B if A is a subformula of B and $A \not\equiv B$. Moreover, we say that A is a *negated subformula* of B if there is a formula C such that C is a strict subformula of B and $A \equiv \neg C$.

Definition 5.2 (Subformula property and negated subformula property)**.** We say that a sequent calculus $\mathsf{G.L}$ enjoys the *subformula property* if all formulas in any derivation are subformulas of the endsequent. We say that $\mathsf{G.L}$ enjoys the *negated subformula property* if all formulas in any derivation are either subformulas or negative subformulas of the endsequent.

As an immediate consequence of cut elimination we have the following result.

Theorem 5.3. Any sequent calculus different from $\mathsf{G.IE_2(C,N_\Diamond,N_\Box)}$ enjoys the subformula property. Moreover, calculi $\mathsf{G.IE_2(C,N_\Diamond,N_\Box)}$ enjoy the negated subformula property.

Having that the calculi enjoy the above subformula properties we can easily adapt the proof of decidability for G3ip in Troelstra and Schwichtenberg [26] and obtain thereby a proof of decidability for our calculi.

Theorem 5.4 (Decidability)**.** For any intuitionistic non-normal modal logic defined in Section 3 and Section 4 it is decidable whether a given formula is derivable.

We conclude this section with some remarks about the logics we that have defined. Firstly, we notice that there are three different systems (that is $\mathsf{IE_1}$, $\mathsf{IE_2}$, $\mathsf{IE_3}$) that we can make correspond to the same classical logic (that is logic E), and the same holds for some of their extensions. This is essentially due to the lost of duality between $\Box$ and $\Diamond$, that permits us to consider interactions of different strengths that are equally derivable in classical logic but are not intuitionistically equivalent. We see therefore that the picture of systems that emerge from a certain set of logic principles is much richer in the intuitionistic case than in the classical one.

Furthermore, logic IE_3 (as well as its non-monotonic extensions) leads us to the following consideration. It is normally expected that an intuitionistic modal logic is strictly weaker than the corresponding classical modal logic, mainly because IPL is weaker than CPL. However, if we make correspond IE_3 to classical E, this is not the case anymore. In fact, rule str is classically equivalent to $\mathsf{Mon}_\Box$, and hence not derivable in E. At the same time, however, it would be odd to consider IE_3 as corresponding to classical M, as neither $\mathsf{M}_\Box$ nor $\mathsf{M}_\Diamond$ is derivable.

As a consequence, this particular case suggests that assuming an intuitionistic base not only allows us to make subtle distinctions between principles that are not distinguishable in classical logic, but also gives us the possibility to investigate systems that in a sense lie between two different classical logics, and don't correspond essentially to any of the two.

6 Semantics

In this section we present a semantics for all systems defined in Sections 3 and 4. As we shall see, the present semantics represents a general framework for intuitionistic modal logics, that is able to capture modularly further intuitionistic non-normal modal logics as CK and $\mathsf{CCDL}^{\mathsf{p}}$. The models are obtained by combining intuitionistic Kripke models and neighbourhood models (Definition 2.1) in the following way:

Definition 6.1. A *coupled intuitionistic neighbourhood model* (CINM) is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$, where $\mathcal{W}$ is a non-empty set, $\preceq$ is a preorder over $\mathcal{W}$, $\mathcal{V}$ is a hereditary valuation function $\mathcal{W} \longrightarrow Atm$ (*i.e.* $w \preceq v$ implies $\mathcal{V}(w) \subseteq \mathcal{V}(v)$), and $\mathcal{N}_\Box$, $\mathcal{N}_\Diamond$ are two neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ such that:

$$w \preceq v \text{ implies } \mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v) \text{ and } \mathcal{N}_\Diamond(w) \supseteq \mathcal{N}_\Diamond(v) \quad (hp).$$

Functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ can be *supplemented*, *closed under intersection*, or *contain the unit* (cf. properties in Definition 2.1). Moreover, letting $-\alpha$ denote the set $\{w \in \mathcal{W} \mid \text{for all } v \succeq w, v \notin \alpha\}$, $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ can be related in the following ways:

For all $w \in \mathcal{W}$, $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$	Weak interaction ($weakInt$);
If $\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus -\alpha \in \mathcal{N}_\Diamond(w)$	Negation closure int_a ($negInt_a$);
If $-\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus \alpha \in \mathcal{N}_\Diamond(w)$	Negation closure int_b ($negInt_b$);
If $\alpha \in \mathcal{N}_\Box(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\Diamond(w)$	Strong interaction ($strInt$).

The forcing relation $w \Vdash A$ associated to CINMs is defined inductively as follows:

$w \Vdash p$	iff	$p \in \mathcal{V}(w)$;
$w \nVdash \bot$;		
$w \Vdash B \wedge C$	iff	$w \Vdash A$ and $w \Vdash B$;
$w \Vdash B \vee C$	iff	$w \Vdash A$ or $w \Vdash B$;
$w \Vdash B \supset C$	iff	for all $v \succeq w$, $v \Vdash B$ implies $v \Vdash C$;
$w \Vdash \Box B$	iff	$[B] \in \mathcal{N}_\Box(w)$;
$w \Vdash \Diamond B$	iff	$\mathcal{W} \setminus [B] \notin \mathcal{N}_\Diamond(w)$.

CINMs for monomodal logics $\Box$-IE^* and $\Diamond$-IE^* are defined by removing, respectively, $\mathcal{N}_\Diamond$ or $\mathcal{N}_\Box$ from the above definition (as well as the forcing condition for the lacking modality), and are called $\Box$-INMs and $\Diamond$-INMs.

As usual, given a class $\mathcal{C}$ of CINMs, we say that a formula A is *satisfiable* in $\mathcal{C}$ if there are $\mathcal{M} \in \mathcal{C}$ and $w \in \mathcal{M}$ such that $w \Vdash A$, and that A is *valid* in $\mathcal{C}$ if for all $\mathcal{M} \in \mathcal{C}$ and $w \in \mathcal{M}$, $w \Vdash A$.

Observe that we are taking for $\supset$ the satisfaction clause of intuitionistic Kripke models, while for $\Box$ and $\Diamond$ we are taking the satisfaction clauses of classical neighbourhood models. Differently from classical neighbourhood models, however, we have here two neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ (instead of one). This allows us to consider different relations between the two functions (*i.e.* the interaction conditions in Definition 6.1) that we make correspond to interaction axioms (and rules) with different strength.

The way functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are related to the order $\preceq$ by condition (hp) guarantees that CINMs preserve the hereditary property of intuitionistic Kripke models:

Proposition 6.1. CINMs satisfy the *hereditary* property: for all $A \in \mathcal{L}$, if $w \Vdash A$ and $w \preceq v$, then $v \Vdash A$.

Proof. By induction on A. For the non-modal cases the proof is standard. For $A \equiv \Box B, \Diamond B$ it is immediate by (hp). $\Box$

Depending on its axioms, to each system are associated models with specific properties, as summarised in the following table:

$\mathsf{M}_\Box$	$\mathcal{N}_\Box$ is supplemented
$\mathsf{N}_\Box$	$\mathcal{N}_\Box$ contains the unit
$\mathsf{C}_\Box$	$\mathcal{N}_\Box$ is closed under $\cap$
$\mathsf{M}_\Diamond$	$\mathcal{N}_\Diamond$ is supplemented
$\mathsf{N}_\Diamond$	$\mathcal{N}_\Diamond$ contains the unit

$\mathsf{weak_a} + \mathsf{weak_b}$	$weakInt$
$\mathsf{neg_a} + \mathsf{neg_b}$	$negInt$
str	$strInt$

Conditions $negInt_a$ and $negInt_b$ are always considered together and summarised as $negInt$. In case of supplemented models (*i.e.* when both $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented) it suffices to consider $weakInt$ as the semantic condition corresponding to any interaction axiom (or rule). In fact, it is immediate to verify that whenever a model $\mathcal{M}$ is $weakInt$, and $\mathcal{N}_\Box$ or $\mathcal{N}_\Diamond$ is supplemented, then $\mathcal{M}$ also satisfies $negInt$ and $strInt$.

Given the semantic properties in the above table, we have that $\Box$-INMs coincide essentially with the neighbourhood spaces by Goldblatt [11] (although there the property of containing the unit is not considered). The only difference is that in Goldblatt's spaces the neighbourhoods are assumed to be closed with respect to the order, that is:

$$\text{If } \alpha \in \mathcal{N}_\Box(w),\ v \in \alpha \text{ and } v \preceq u, \text{ then } u \in \alpha \quad (\preceq\text{-closure}).$$

As already observed by Goldblatt, however, this property is irrelevant from the point of view of the validity of formulas, as a formula A is valid in $\Box$-INMs (that are supplemented, closed under intersection, contain the unit) if and only of it is valid in the corresponding $\Box$-INMs that satisfy also the $\preceq$-closure. It is easy to verify that the same equivalence holds if we consider CINMs for bimodal logics, provided that the $\preceq$-closure is demanded only for the neighbourhoods in $\mathcal{N}_\Box$, and not for those in $\mathcal{N}_\Diamond$.

It is immediate to prove soundness of intuitionistic non-normal modal logics with respect to the corresponding CINMs.

Theorem 6.2 (Soundness)**.** Any intuitionistic non-normal modal logic is sound with respect to the corresponding CINMs.

Proof. It is immediate to prove that a given axiom is valid whenever the corresponding property is satisfied. For $\mathsf{neg_a}$ and $\mathsf{neg_b}$ notice that $-[A] = [\neg A]$. $\square$

We now prove completeness by the canonical model construction. In the following, let L be any intuitionistic non-normal modal logic and $\mathcal{L}$ be the corresponding language. We call L-*prime* any set X of formulas of $\mathcal{L}$ which is consistent ($X \not\vdash_{\mathsf{L}} \bot$), closed under derivation ($X \vdash_{\mathsf{L}} A$ implies $A \in X$) and such that if $(A \vee B) \in X$, then $A \in X$ or $B \in X$. For all $A \in \mathcal{L}$, we denote with $\uparrow_{pr} A$ the class of prime sets X such that $A \in X$. The standard properties of prime sets hold, in particular:

Lemma 6.3. (a) If $X \not\vdash_{\mathsf{L}} A \supset B$, then there is a L-prime set Y such that $X \cup \{A\} \subseteq Y$ and $B \notin Y$. (b) For any $A, B \in \mathcal{L}$, $\uparrow_{pr} A \subseteq \uparrow_{pr} B$ implies $\vdash_{\mathsf{L}} A \supset B$.

Lemma 6.4. Let L be any logic non containing axioms $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$. The *canonical model* $\mathcal{M}^c$ for L is defined as the tuple $\langle \mathcal{W}^c, \preceq^c, \mathcal{N}_\Box^c, \mathcal{N}_\Diamond^c, \mathcal{V}^c \rangle$, where:

- $\mathcal{W}^c$ is the class of L-prime sets;
- for all $X, Y \in \mathcal{W}^c$, $X \preceq^c Y$ if and only if $X \subseteq Y$;
- $\mathcal{N}_\Box^c(X) = \{\uparrow_{pr} A \mid \Box A \in X\}$;
- $\mathcal{N}_\Diamond^c(X) = \mathcal{P}(\mathcal{W}^c) \setminus \{\mathcal{W}^c \setminus \uparrow_{pr} A \mid \Diamond A \in X\}$;
- $\mathcal{V}^c(X) = \{p \in \mathcal{L} \mid p \in X\}$.

Then for all $X \in \mathcal{W}^c$ and all $A \in \mathcal{L}$ we have

$$X \Vdash A \quad \text{iff} \quad A \in X.$$

Moreover: (i) If L contains $\mathsf{N}_\Box$, then $\mathcal{N}_\Box^c$ contains the unit;
(ii) If L contains $\mathsf{C}_\Box$, then $\mathcal{N}_\Box^c$ is closed under intersection;
(iii) If L contains $\mathsf{N}_\Diamond$, then $\mathcal{N}_\Diamond^c$ contains the unit;
(iv) If L contains $\mathsf{weak_a}$ and $\mathsf{weak_b}$, then $\mathcal{M}^c$ is *weakInt*;
(v) If L contains $\mathsf{neg_a}$, then $\mathcal{M}^c$ is $negInt_a$;
(vi) If L contains $\mathsf{neg_b}$, then $\mathcal{M}^c$ is $negInt_b$;
(vii) If L contains str, then $\mathcal{M}^c$ is *strInt*.

Proof. By induction on A we prove that $X \Vdash A$ if and only if $A \in X$. If $A \equiv p, \bot, B \wedge C, B \vee C, B \supset C$ the proof is immediate. If $A \equiv \Box B$: From right to left, assume $\Box B \in X$. Then by definition $\uparrow_{pr} B \in \mathcal{N}_\Box^c(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c}$, therefore $X \Vdash \Box B$. From left to right, assume $X \Vdash \Box B$. Then we have $[B]_{\mathcal{M}^c} \in \mathcal{N}_\Box^c(X)$, and, by inductive hypothesis, $[B]_{\mathcal{M}^c} = \uparrow_{pr} B$. By definition, this means that there is $C \in \mathcal{L}$ such that $\Box C \in X$ and $\uparrow_{pr} C = \uparrow_{pr} B$. Then by Lemma 6.3, $\vdash_{\mathsf{L}} C \supset B$ and $\vdash_{\mathsf{L}} B \supset C$. Thus by $\mathsf{E}_\Box$, $\vdash_{\mathsf{L}} \Box C \supset \Box B$, and, by closure under derivation, $\Box B \in X$. If $A \equiv \Diamond B$: From right to left, assume $\Diamond B \in X$. Then by definition $\mathcal{W}^c \setminus \uparrow_{pr} B \notin \mathcal{N}_\Diamond^c(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c}$, therefore $X \Vdash \Diamond B$. From left to right, assume $X \Vdash \Diamond B$. Then we have $\mathcal{W}^c \setminus [B]_{\mathcal{M}^c} \notin \mathcal{N}_\Diamond^c(X)$, and, by inductive hypothesis, $\mathcal{W}^c \setminus \uparrow_{pr} B \notin \mathcal{N}_\Diamond^c(X)$. This means that there is $C \in \mathcal{L}$ such that $\Diamond C \in X$ and $\uparrow_{pr} C = \uparrow_{pr} B$. Thus, $\vdash_{\mathsf{L}} C \supset B$ and $\vdash_{\mathsf{L}} B \supset C$, therefore by $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Diamond C \supset \Diamond B$. By closure under derivation we then have $\Diamond B \in X$.

Notice also that $\mathcal{M}^c$ is well defined: It follows immediately by the definition that $X \preceq^c Y$ implies both $\mathcal{N}_\Box^c(X) \subseteq \mathcal{N}_\Box^c(Y)$ and $\mathcal{N}_\Diamond^c(X) \supseteq \mathcal{N}_\Diamond^c(Y)$.

Points (i)–(vi) are proved as follows: (i) $\Box\top \in X$ for all $X \in \mathcal{W}^c$. Then by definition $\mathcal{W}^c = \uparrow_{pr}\top \in \mathcal{N}^c_\Box(X)$. (ii) Assume $\alpha, \beta \in \mathcal{N}^c(X)$. Then there are $A, B \in \mathcal{L}$ such that $\Box A, \Box B \in X$, $\alpha = \uparrow_{pr}A$ and $\beta = \uparrow_{pr}B$. By closure under derivation we have $\Box(A \wedge B) \in X$, and, by definition, $\uparrow_{pr}(A \wedge B) \in \mathcal{N}^c_\Box(X)$, where $\uparrow_{pr}(A \wedge B) = \uparrow_{pr}A \cap \uparrow_{pr}B = \alpha \cap \beta$. (iii) $\neg\Diamond\bot \in X$ for all $X \in \mathcal{W}^c$, thus by consistency, $\Diamond\bot \notin X$. If $\mathcal{W}^c \setminus \uparrow_{pr}\bot \notin \mathcal{N}^c_\Box(X)$, then there is $A \in \mathcal{L}$ such that $\uparrow_{pr}A = \uparrow_{pr}\bot$ and $\Diamond A \in X$, that implies $\Diamond\bot \in X$. Therefore $\mathcal{W}^c = \mathcal{W}^c \setminus \uparrow_{pr}\bot \in \mathcal{N}^c_\Box(X)$.

(iv) Assume by contradiction that $\alpha \in \mathcal{N}^c_\Box(X)$ and $\alpha \notin \mathcal{N}^c_\Diamond(X)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \uparrow_{pr}A$, $\alpha = \mathcal{W}^c \setminus \uparrow_{pr}B$, and $\Box A, \Diamond B \in X$, therefore $\uparrow_{pr}A = \mathcal{W}^c \setminus \uparrow_{pr}B$. By the properties of prime sets, this implies $\vdash_{\mathsf{L}} \neg(A \wedge B)$ and $\vdash_{\mathsf{L}} A \vee B$, and by the disjunction property, $\vdash_{\mathsf{L}} A$ or $\vdash_{\mathsf{L}} B$. If we assume $\vdash_{\mathsf{L}} A$, then $\vdash_{\mathsf{L}} A \supset\subset \top$ and $\vdash_{\mathsf{L}} B \supset\subset \bot$. Therefore by $\mathsf{E}_\Box$ and $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Box A \supset \Box\top$ and $\vdash_{\mathsf{L}} \Diamond B \supset \Diamond\bot$, thus by closure under derivation, $\Box\top, \Diamond\bot \in X$. But $\neg(\Box\top \wedge \Diamond\bot) \in X$, against the consistency of prime sets. If we now assume $\vdash_{\mathsf{L}} B$, then $\vdash_{\mathsf{L}} B \supset\subset \top$ and $\vdash_{\mathsf{L}} A \supset\subset \bot$. We obtain an analogous contradiction by $\neg(\Diamond\top \wedge \Box\bot)$.

(v) Assume $\alpha \in \mathcal{N}^c_\Box(X)$. Then there is $A \in \mathcal{L}$ such that $\alpha = \uparrow_{pr}A$ and $\Box A \in X$. Thus, by $\mathsf{neg_a}$ and consistency of X, $\Diamond\neg A \notin X$. Therefore $\mathcal{W}^c \setminus \uparrow_{pr}\neg A \in \mathcal{N}^c_\Diamond(X)$ (otherwise there would be $B \in \mathcal{L}$ such that $\uparrow_{pr}B = \uparrow_{pr}\neg A$ and $\Diamond B \in X$, which implies $\Diamond\neg A \in X$). Since $\uparrow_{pr}\neg A = -\uparrow_{pr}A$ ($\uparrow_{pr}\neg A = [\neg A]_{\mathcal{M}^c} = -[A]_{\mathcal{M}^c} = -\uparrow_{pr}A$) and $-\uparrow_{pr}A = -\alpha$, we have the claim.

(vi) By contraposition, assume $\mathcal{W}^c \setminus \alpha \notin \mathcal{N}^c_\Diamond(X)$. Then there is $A \in \mathcal{L}$ such that $\mathcal{W}^c \setminus \alpha = \mathcal{W}^c \setminus \uparrow_{pr}A$ and $\Diamond A \in X$. Thus $\alpha = \uparrow_{pr}A$, and by $\mathsf{neg_b}$, $\Box\neg A \notin X$. Therefore $\uparrow_{pr}\neg A \notin \mathcal{N}^c_\Box(X)$ (otherwise there would be $\Box B \in X$ such that $\uparrow_{pr}\neg A = \uparrow_{pr}B$, which implies $\Box\neg A \in X$). Since $\uparrow_{pr}\neg A = -\uparrow_{pr}A = -\alpha$, we have the claim.

(vii) Assume by contradiction that $\alpha \in \mathcal{N}^c_\Box(X)$, $\alpha \subseteq \beta$, and $\beta \notin \mathcal{N}^c_\Diamond(X)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \uparrow_{pr}A$, $\beta = \mathcal{W}^c \setminus \uparrow_{pr}B$ and $\Box A, \Diamond B \in X$. Moreover, $\uparrow_{pr}A \subseteq \mathcal{W}^c \setminus \uparrow_{pr}B$, which implies $\uparrow_{pr}A \cap \uparrow_{pr}B = \emptyset$. Thus $\vdash_{\mathsf{L}} \neg(A \wedge B)$; and by str we have $\vdash_{\mathsf{L}} \neg(\Box A \wedge \Diamond B)$, against the consistency of X. ☐

For logics containing $\mathsf{M}_\Box$ or $\mathsf{M}_\Diamond$ we slightly change the definition of canonical model. We shorten the proof by considering, instead of $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$, the syntactically equivalent rules $\mathsf{Mon}_\Box$ and $\mathsf{Mon}_\Diamond$.

Lemma 6.5. Let L be any logic containing axioms $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$. The *canonical model* $\mathcal{M}^c_+$ for L is the tuple $\langle \mathcal{W}^c, \preceq^c, \mathcal{N}^+_\Box, \mathcal{N}^+_\Diamond, \mathcal{V}^c \rangle$, where $\mathcal{W}^c, \preceq^c, \mathcal{V}^c$ are defined as in Lemma 6.4, and:

$\mathcal{N}^+_\Box(X) = \{\alpha \subseteq \mathcal{W}^c \mid \text{there is } A \in \mathcal{L} \text{ s.t. } \Box A \in X \text{ and } \uparrow_{pr}A \subseteq \alpha\}$;

$\mathcal{N}^+_\Diamond(X) = \mathcal{P}(\mathcal{W}^c) \setminus \{\alpha \subseteq \mathcal{W}^c \mid \text{there is } A \in \mathcal{L} \text{ s.t. } \Diamond A \in X \text{ and } \alpha \subseteq \mathcal{W}^c \setminus \uparrow_{pr}A\}$.

Then we have that $X \Vdash A$ if and only if $A \in X$. Moreover, points (i)–(iii) of Lemma 6.4 still hold. Finally, (iv) if L contains str, then $\mathcal{M}^c_+$ is *weakInt*.

Proof. It is immediate to verify that both $\mathcal{N}^+_\Box$ and $\mathcal{N}^+_\Diamond$ are supplemented. As before, the proof is by induction on A. We only show the modal cases. If $A \equiv \Box B$: From right to left, assume $\Box B \in X$. Then by definition $\uparrow_{pr}B \in \mathcal{N}^+_\Box(X)$, and by inductive hypothesis, $\uparrow_{pr}B = [B]_{\mathcal{M}^c_+}$, therefore $X \Vdash \Box B$. From left to right, assume $X \Vdash \Box B$. Then we have $[B]_{\mathcal{M}^c_+} \in \mathcal{N}^+_\Box(X)$, and, by inductive hypothesis, $[B]_{\mathcal{M}^c_+} = \uparrow_{pr} B$. By definition, this means that there is $C \in \mathcal{L}$ such that $\Box C \in X$ and $\uparrow_{pr} C \subseteq \uparrow_{pr} B$, which implies $\vdash_{\mathsf{L}} C \supset B$. Thus by $\mathsf{Mon}_\Box$, $\vdash_{\mathsf{L}} \Box C \supset \Box B$, and, by closure under derivation, $\Box B \in X$. If $A \equiv \Diamond B$: From right to left, assume $\Diamond B \in X$. Then by definition $\mathcal{W}^c \setminus \uparrow_{pr} B \notin \mathcal{N}^+_\Diamond(X)$, and by inductive hypothesis, $\uparrow_{pr} B = [B]_{\mathcal{M}^c_+}$, therefore $X \Vdash \Diamond B$. From left to right, assume $X \Vdash \Diamond B$. Then we have $\mathcal{W}^c \setminus [B]_{\mathcal{M}^c} \notin \mathcal{N}^+_\Diamond(X)$, and, by inductive hypothesis, $\mathcal{W}^c \setminus \uparrow_{pr} B \notin \mathcal{N}^+_\Diamond(X)$. This means that there is $C \in \mathcal{L}$ such that $\Diamond C \in X$ and $\mathcal{W}^c \setminus B \subseteq \mathcal{W}^c \setminus C$, that is $\uparrow_{pr} C \subseteq \uparrow_{pr} B$. Thus, $\vdash_{\mathsf{L}} C \supset B$, therefore by $\mathsf{E}_\Diamond$, $\vdash_{\mathsf{L}} \Diamond C \supset \Diamond B$. By closure under derivation we then have $\Diamond B \in X$.

Points (i)–(iii) are very similar to points (i)–(iii) in Lemma 6.4. (iv) By contradiction, assume $\alpha \in \mathcal{N}^+_\Box(X)$ and $\alpha \notin \mathcal{N}^+_\Diamond(X)$. Then there are $A, B \in \mathcal{L}$ such that $\uparrow_{pr} A \subseteq \alpha$, $\alpha \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$, and $\Box A, \Diamond B \in X$. Therefore $\uparrow_{pr} A \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$, which implies $\vdash_{\mathsf{L}} \neg(A \wedge B)$. By str we then have $\neg(\Box A \wedge \Diamond B) \in X$, against the consistency of X. ☐

Theorem 6.6 (Completeness)**.** Any intuitionistic non-normal bimodal logic is complete with respect to the corresponding CINMs.

Proof. Assume $\not\vdash_{\mathsf{L}} A$. Then $\not\vdash_{\mathsf{L}} \top \supset A$, thus, by Lemma 6.3, there is a L-prime set Π such that $A \notin \Pi$. By definition, $\Pi \in \mathcal{M}^c_{(+)}$, and by Lemma 6.4, $\mathcal{M}^c_{(+)}, \Pi \not\Vdash A$. By the properties of $\mathcal{M}^c_{(+)}$ we obtain completeness with respect to the corresponding models. ☐

It is immediate to verify that by removing $\mathcal{N}^c_\Diamond$ ($\mathcal{N}^+_\Diamond$) or $\mathcal{N}^c_\Box$ ($\mathcal{N}^+_\Box$) from the definition of $\mathcal{M}^c$ ($\mathcal{M}^c_+$), we obtain analogous results for monomodal logics.

Theorem 6.7. Any intuitionistic non-normal monomodal logic is complete with respect to the corresponding CINMs.

6.1 Finite model property and decidability

We have seen that all intuitionistic non-normal modal logics defined in Section 3 and 4 are sound and complete with respect to a certain class of CINMs. By applying the technique of filtrations to this kind of models, we now show that most of them have also the finite model property, thus providing an alternative proof of decidability. The proofs are given explicitly for bimodal logics, while the simpler proofs for monomodal logics can be easily extracted.

Given a CINM $\mathcal{M}$ and a set Φ of formulas of $\mathcal{L}$ that is closed under subformulas, we define the equivalence relation $\sim$ on $\mathcal{W}$ as follows:

$$w \sim v \quad \text{iff} \quad \text{for all } A \in \Phi,\ w \Vdash A \text{ iff } v \Vdash A.$$

For any $w \in \mathcal{W}$ and $\alpha \subseteq \mathcal{W}$, we denote with $w_\sim$ the equivalence class containing w, and with $\alpha_\sim$ the set $\{w_\sim \mid w \in \alpha\}$ (thus in particular $[A]^\sim_{\mathcal{M}}$ is the set $\{w_\sim \mid w \in [A]_{\mathcal{M}}\}$).

Definition 6.2. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be any CINM and Φ be a set of formulas of $\mathcal{L}$ closed under subformulas. A *filtration* of $\mathcal{M}$ through Φ (or Φ-filtration) is any model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}^*_\Box, \mathcal{N}^*_\Diamond, \mathcal{V}^* \rangle$ such that:

- $\mathcal{W}^* = \{w_\sim \mid w \in \mathcal{W}\}$;
- $w_\sim \preceq^* v_\sim$ iff for all $A \in \Phi$, $\mathcal{M}, w \Vdash A$ implies $\mathcal{M}, v \Vdash A$;

- for all $\Box A \in \Phi$, $\quad [A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Box}(w_\sim)$ iff $[A]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$;
- for all $\Diamond A \in \Phi$, $\quad \mathcal{W}^* \setminus [A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Diamond}(w_\sim)$ iff $\mathcal{W} \setminus [A]_{\mathcal{M}} \in \mathcal{N}_{\Diamond}(w)$;
- for all $p \in \Phi$, $\quad p \in \mathcal{V}^*(w_\sim)$ iff $p \in \mathcal{V}(w)$.

Observe that model $\mathcal{M}^*$ is well-defined, as for all $\Box A, \Diamond B, p \in \Phi$ we have that $w \sim v$ implies (i) $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Box}(w_\sim)$ iff $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Box}(v_\sim)$; (ii) $\mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Diamond}(w_\sim)$ iff $\mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Diamond}(v_\sim)$; and (iii) $p \in \mathcal{V}^*(w_\sim)$ iff $p \in \mathcal{V}^*(v_\sim)$. Moreover, it is immediate to verify that (iv) $\preceq^*$ is a preorder; (v) $\mathcal{V}^*$ is hereditary; (vi) if $w_\sim \preceq^* v_\sim$ and $\Box A \in \Phi$, then $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Box}(w_\sim)$ implies $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Box}(v_\sim)$; (vii) if $w_\sim \preceq^* v_\sim$ and $\Diamond B \in \Phi$, then $\mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Diamond}(v_\sim)$ implies $\mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Diamond}(w_\sim)$; and (viii) for all $\alpha \subseteq \mathcal{W}^*$, $\alpha \in \mathcal{N}^*_{\Box}(w_\sim)$ implies $\alpha \in \mathcal{N}^*_{\Diamond}(w_\sim)$. Thus $\mathcal{M}^*$ is a CINM.

Lemma 6.8 (Filtrations lemma)**.** For any formula $A \in \Phi$,

$$\mathcal{M}^*, w_\sim \Vdash A \;\text{ iff }\; \mathcal{M}, w \Vdash A.$$

Proof. Notice that this is equivalent to prove that $[A]_{\mathcal{M}^*} = [A]^{\sim}_{\mathcal{M}}$. The proof is by induction on A. For $A \equiv p, \bot, B \wedge C, B \vee C$ the proof is immediate.

$A \equiv B \supset C$. Assume $\mathcal{M}, w \nVdash B \supset C$. Then there is $v \succeq w$ such that $\mathcal{M}, v \Vdash B$ and $\mathcal{M}, v \nVdash C$. By inductive hypothesis $\mathcal{M}^*, v_\sim \Vdash B$ and $\mathcal{M}^*, v_\sim \nVdash C$. Moreover, by definition of $\preceq^*$ (and monotonicity of $\mathcal{M}$), $w_\sim \preceq^* v_\sim$. Therefore $\mathcal{M}^*, w_\sim \nVdash B \supset C$. Now assume $\mathcal{M}^*, w_\sim \nVdash B \supset C$. Then there is $v_\sim \in \mathcal{W}^*$ such that $w_\sim \preceq^* v_\sim$, $\mathcal{M}^*, v_\sim \Vdash B$ and $\mathcal{M}^*, v_\sim \nVdash C$. By inductive hypothesis $\mathcal{M}, v \Vdash B$ and $\mathcal{M}, v \nVdash C$, thus $\mathcal{M}, v \nVdash B \supset C$. By definition of $\preceq^*$ we then have $\mathcal{M}, w \nVdash B \supset C$.

$A \equiv \Box B$. $\mathcal{M}^*, w_\sim \Vdash \Box B$ iff $[B]_{\mathcal{M}^*} \in \mathcal{N}^*_{\Box}(w_\sim)$ iff (i.h.) $[B]^{\sim}_{\mathcal{M}} \in \mathcal{N}^*_{\Box}(w_\sim)$ iff $[B]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$ iff $\mathcal{M}, w \Vdash \Box B$.

$A \equiv \Diamond B$. $\mathcal{M}^*, w_\sim \Vdash \Diamond B$ iff $\mathcal{W}^* \setminus [B]_{\mathcal{M}^*} \notin \mathcal{N}^*_{\Diamond}(w_\sim)$ iff (i.h.) $\mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}} \notin \mathcal{N}^*_{\Diamond}(w_\sim)$ iff $\mathcal{W} \setminus [B]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$ iff $\mathcal{M}, w \Vdash \Diamond B$. ☐

Lemma 6.9. Let $\mathcal{M}^*$ be a Φ-filtration of $\mathcal{M}$. (*i*) If $\mathcal{N}_{\Box}(w)$ contains the unit and $\Box\top \in \Phi$, then $\mathcal{N}^*_{\Box}(w_\sim)$ contains the unit. (*ii*) If $\mathcal{N}_{\Diamond}(w)$ contains the unit and $\Diamond\bot \in \Phi$, then $\mathcal{N}^*_{\Diamond}(w_\sim)$ contains the unit.

Proof. Immediate by Definition 6.2 and Lemma 6.8. ☐

Definition 6.3. We call *finest* Φ-filtration (cf. Chellas [2]) any Φ-filtration $\mathcal{M}^*$ of $\mathcal{M}$ such that:

$\mathcal{N}^*_{\Box}(w_\sim) = \{[A]^{\sim}_{\mathcal{M}} \mid \Box A \in \Phi \text{ and } [A]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)\}$; and
$\mathcal{N}^*_{\Diamond}(w_\sim) = \mathcal{P}(\mathcal{W}^*) \setminus \{\mathcal{W}^* \setminus [A]^{\sim}_{\mathcal{M}} \mid \Diamond A \in \Phi \text{ and } \mathcal{W} \setminus [A]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)\}$.

Moreover, let $\mathcal{M}^\circ = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}^\circ_{\Box}, \mathcal{N}^\circ_{\Diamond}, \mathcal{V}^* \rangle$ be a CINM where $\mathcal{W}^*$, $\preceq^*$ and $\mathcal{V}^*$ are as in $\mathcal{M}^*$. We say that:

- $\mathcal{M}^\circ$ is the *supplementation* of $\mathcal{M}^*$ if:
  $\alpha \in \mathcal{N}^\circ_{\Box}(w_\sim)$ iff there is $\beta \in \mathcal{N}^*_{\Box}(w_\sim)$ s.t. $\beta \subseteq \alpha$;
  $\alpha \notin \mathcal{N}^\circ_{\Diamond}(w_\sim)$ iff there is $\beta \notin \mathcal{N}^*_{\Diamond}(w_\sim)$ s.t. $\alpha \subseteq \beta$.
- $\mathcal{M}^\circ$ is the *intersection closure* of $\mathcal{M}^*$ if $\mathcal{N}^\circ_{\Diamond}(w_\sim) = \mathcal{N}^*_{\Diamond}(w_\sim)$, and
  $\alpha \in \mathcal{N}^\circ_{\Box}(w_\sim)$ iff there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_{\Box}(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n = \alpha$.
- $\mathcal{M}^\circ$ is the *quasi-filtering* of $\mathcal{M}^*$ if:

$\alpha \in \mathcal{N}^{\circ}_{\Box}(w_{\sim})$ iff there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^{*}_{\Box}(w_{\sim})$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$;
$\alpha \notin \mathcal{N}^{\circ}_{\Diamond}(w_{\sim})$ iff there is $\beta \notin \mathcal{N}^{*}_{\Diamond}(w_{\sim})$ s.t. $\alpha \subseteq \beta$.

It is immediate to verify that the supplementation of a model $\mathcal{M}$ is supplemented, its intersection closure is closed under intersection, and its quasi-filtering is both supplemented and closed under intersection.

Lemma 6.10. Let $\mathcal{M}^*$ be a finest Φ-filtration of $\mathcal{M}$. (i) If $\mathcal{M}$ is $weakInt$, then $\mathcal{M}^*$ is $weakInt$. (ii) If $\mathcal{M}$ is $strInt$, then $\mathcal{M}^*$ is $strInt$.

Proof. (i) Assume by contradiction that $\alpha \in \mathcal{N}^{*}_{\Box}(w_{\sim})$ and $\alpha \notin \mathcal{N}^{*}_{\Diamond}(w_{\sim})$. Then $\alpha = [A]^{\sim}_{\mathcal{M}}$ for a $A \in \mathcal{L}$ such that $\Box A \in \Phi$ and $[A]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$. Moreover $\alpha = \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$ for a $B \in \mathcal{L}$ such that $\Diamond B \in \Phi$ and $\mathcal{W} \setminus [B]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$. Thus $[A]^{\sim}_{\mathcal{M}} = \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$, which implies $[A]_{\mathcal{M}} = \mathcal{W} \setminus [B]_{\mathcal{M}}$ ($w \in [A]_{\mathcal{M}}$ iff $w_{\sim} \in [A]^{\sim}_{\mathcal{M}}$ iff $w_{\sim} \in \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$ iff $w \in \mathcal{W} \setminus [B]_{\mathcal{M}}$). Then, since $\mathcal{M}$ is $weakInt$, $\mathcal{W} \setminus [B]_{\mathcal{M}} \in \mathcal{N}_{\Diamond}(w)$, which gives a contradiction.

(ii) Assume by contradiction that $\alpha \in \mathcal{N}^{*}_{\Box}(w_{\sim})$, $\alpha \subseteq \beta$ and $\beta \notin \mathcal{N}^{*}_{\Diamond}(w_{\sim})$. Then $\alpha = [A]^{\sim}_{\mathcal{M}}$ for a $A \in \mathcal{L}$ such that $\Box A \in \Phi$ and $[A]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$. Moreover $\beta = \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$ for a $B \in \mathcal{L}$ such that $\Diamond B \in \Phi$ and $\mathcal{W} \setminus [B]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$. Thus $[A]^{\sim}_{\mathcal{M}} \subseteq \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$, which implies $[A]_{\mathcal{M}} \subseteq \mathcal{W} \setminus [B]_{\mathcal{M}}$. Then, since $\mathcal{M}$ is $strInt$, $\mathcal{W} \setminus [B]_{\mathcal{M}} \in \mathcal{N}_{\Diamond}(w)$, which gives a contradiction. □

Lemma 6.11. Let $\mathcal{M}$, $\mathcal{M}^*$ and $\mathcal{M}^{\circ}$ be CINMs, where $\mathcal{M}^*$ is a finest Φ-filtration of $\mathcal{M}$ for a set Φ of formulas that is closed under subformulas. We have:

(i) If $\mathcal{M}$ is supplemented and $weakInt$, and $\mathcal{M}^{\circ}$ is the supplementation of $\mathcal{M}^*$, then $\mathcal{M}^{\circ}$ is $weakInt$ and is a Φ-filtration of $\mathcal{M}$.

(ii) If $\mathcal{M}$ is closed under intersection and $weakInt$, and $\mathcal{M}^{\circ}$ is the closure under intersection of $\mathcal{M}^*$, then $\mathcal{M}^{\circ}$ is $weakInt$ and is a Φ-filtration of $\mathcal{M}$.

(iii) If $\mathcal{M}$ is supplemented, closed under intersection, and $weakInt$, and $\mathcal{M}^{\circ}$ is the quasi-filtering of $\mathcal{M}^*$, then $\mathcal{M}^{\circ}$ is $weakInt$ and is a Φ-filtration of $\mathcal{M}$.

(iv) If $\mathcal{M}$ is closed under intersection and $strInt$, and $\mathcal{M}^{\circ}$ is the closure under intersection of $\mathcal{M}^*$, then $\mathcal{M}^{\circ}$ is $strInt$ and is a Φ-filtration of $\mathcal{M}$.

Proof. Points (i)–(iv) are proved similarly. We show as example the proof of point (iii). Firstly we prove by contradiction that $\mathcal{M}^{\circ}$ is $weakInt$. Assume $\alpha \in \mathcal{N}^{\circ}_{\Box}(w_{\sim})$ and $\alpha \notin \mathcal{N}^{\circ}_{\Diamond}(w_{\sim})$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^{*}_{\Box}(w_{\sim})$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$; and there is $\beta \notin \mathcal{N}^{*}_{\Diamond}(w_{\sim})$ s.t. $\alpha \subseteq \beta$. By definition, this means that there are $\Box A_1, ..., \Box A_n \in \Phi$ s.t. $\alpha_1 = [A_1]^{\sim}_{\mathcal{M}}$, ..., $\alpha_n = [A_n]^{\sim}_{\mathcal{M}}$, and $[A_1]_{\mathcal{M}}, ..., [A_n]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$. Moreover, there is $\Diamond B \in \Phi$ s.t. $\beta = \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$ and $\mathcal{W} \setminus [B]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$. As a consequence, we also have $[A_1]^{\sim}_{\mathcal{M}} \cap ... \cap [A_n]^{\sim}_{\mathcal{M}} \subseteq \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, by the filtration lemma this implies $[A_1]_{\mathcal{M}} \cap ... \cap [A_n]_{\mathcal{M}} \subseteq \mathcal{W} \setminus [B]_{\mathcal{M}}$. Then by intersection closure of $\mathcal{N}_{\Box}$, $[A_1]_{\mathcal{M}} \cap ... \cap [A_n]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$, and by its supplementation, $\mathcal{W} \setminus [B]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$. Finally, since $\mathcal{M}$ is $weakInt$, $\mathcal{W} \setminus [B]_{\mathcal{M}} \in \mathcal{N}_{\Diamond}(w)$, which gives a contradiction.

We now prove that $\mathcal{M}^{\circ}$ is a Φ-filtration of $\mathcal{M}$. Let $\Box A \in \Phi$. If $[A]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$, then $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^{*}_{\Box}(w_{\sim})$, and also $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^{\circ}_{\Box}(w_{\sim})$. Now assume $[A]^{\sim}_{\mathcal{M}} \in \mathcal{N}^{\circ}_{\Box}(w_{\sim})$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^{*}_{\Box}(w_{\sim})$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq [A]^{\sim}_{\mathcal{M}}$.

By definition, this means that there are $\Box A_1, ..., \Box A_n \in \Phi$ s.t. $\alpha_1 = [A_1]^{\sim}_{\mathcal{M}}$, ..., $\alpha_n = [A_n]^{\sim}_{\mathcal{M}}$, and $[A_1]_{\mathcal{M}}, ..., [A_n]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$. Then, since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, $[A_1]_{\mathcal{M}} \cap ... \cap [A_n]_{\mathcal{M}} \subseteq [A]_{\mathcal{M}}$. By intersection closure of $\mathcal{N}_{\Box}$, $[A_1]_{\mathcal{M}} \cap ... \cap [A_n]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$, then by supplementation, $[A]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$.

Now let $\Diamond A \in \Phi$. If $\mathcal{W} \setminus [A]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$, then $\mathcal{W}^* \setminus [A]^{\sim}_{\mathcal{M}} \notin \mathcal{N}^*_{\Diamond}(w_{\sim})$, and also $\mathcal{W}^* \setminus [A]^{\sim}_{\mathcal{M}} \notin \mathcal{N}^{\circ}_{\Diamond}(w_{\sim})$. Now assume $\mathcal{W}^* \setminus [A]^{\sim}_{\mathcal{M}} \notin \mathcal{N}^{\circ}_{\Diamond}(w_{\sim})$. Then there is $\beta \notin \mathcal{N}^*_{\Diamond}(w_{\sim})$ s.t. $\mathcal{W}^* \setminus [A]^{\sim}_{\mathcal{M}} \subseteq \beta$. By definition, $\beta = \mathcal{W}^* \setminus [B]^{\sim}_{\mathcal{M}}$ for a $\Diamond B \in \Phi$ s.t. $\mathcal{W} \setminus [B]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, we have $\mathcal{W} \setminus [A]_{\mathcal{M}} \subseteq \mathcal{W} \setminus [B]_{\mathcal{M}}$. Then by supplementation, $\mathcal{W} \setminus [A]_{\mathcal{M}} \notin \mathcal{N}_{\Diamond}(w)$. ☐

Theorem 6.12. If a formula A is satisfiable in a CINM $\mathcal{M}$ that is *weakInt* or *strInt*, then A is satisfiable in a CINM $\mathcal{M}'$ with the same properties of $\mathcal{M}$ and in addition is finite.

Proof. Standard, by taking $\Phi = Sbf(A) \cup \{\Box\top, \Diamond\bot, \top, \bot\}$ and, depending on the properties of $\mathcal{M}$, the right transformation $\mathcal{M}'$ of $\mathcal{M}$. Observe that whenever Φ is finite, any Φ-filtration $\mathcal{M}'$ of $\mathcal{M}$ is finite as well. ☐

Corollary 6.13. Any intuitionistic non-normal bimodal logic different from $\mathsf{IE_2}(\mathsf{C},\mathsf{N}_{\Diamond},\mathsf{N}_{\Box})$ enjoys the finite model property. Moreover, any intuitionistic non-normal monomodal logic enjoys the finite model property.

7 Constructive K and propositional CCDL

We have seen in Section 6 that $\Box$-INMs coincide essentially with Goldblatt's neighbourhood spaces. In Fairtlough and Mendler [4], Goldblatt's spaces are considered in order to provide a semantics for Propositional Lax Logic (PLL), an intuitionistic monomodal logic for hardware verification that fails to validate the rule of necessitation.

We show in this section that the framework of CINMs is also adapted to cover two additional well-studied intuitionistic non-normal bimodal logics, namely CK (for "constructive K") by Bellin *et al.* [1], and $\mathsf{CCDL^p}$, as we call the propositional fragment of Wijesekera's first-order logic CCDL (Wijesekera [27]). In particular, we show that the two systems can be included in our framework by considering a very simple additional property.

Different possible worlds semantics have already been given for the two logics. In particular, logic $\mathsf{CCDL^p}$ has both a relational semantics (Wijesekera [27]) and a neighbourhood semantics (Kojima [13]), while a relational semantics for CK has been given in Mendler and de Paiva [19] by adding inconsistent worlds to the relational models for $\mathsf{CCDL^p}$. However, if compared to the existing ones, our semantics has the advantage of including CK and $\mathsf{CCDL^p}$ in a more general framework, that shows how the two systems can be obtained as extensions of weaker logics in a modular way. In addition, two further benefits concern specifically CK. In particular, to the best of our knowledge we are presenting the first neighbourhood semantics for this system. Moreover, and most notably, this kind of models don't make use of inconsistent worlds.

In the following we first present logics CK and $\mathsf{CCDL^p}$ by giving both the axiomatisations and the sequent calculi. After that we define their CINMs and prove soundness and completeness. Finally, we present their pre-existing possible worlds semantics and prove directly their equivalence with CINMs.

7.1 Hilbert systems and sequent calculi

Logic CK (Bellin *et al.* [1]) is Hilbert-style defined by adding to IPL the following axioms and rules:

$$\mathsf{K}_\Box \quad \Box(A \supset B) \supset (\Box A \supset \Box B), \qquad \mathsf{K}_\Diamond \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B), \qquad \mathsf{Nec}\ \frac{A}{\Box A}\ .$$

Logic $\mathsf{CCDL^p}$ is the extension of CK with axiom $\mathsf{N}_\Diamond$ $(\neg\Diamond\bot)$.[3] It is worth noticing that, given the syntactical equivalences that we have recalled in Section 3, an equivalent axiomatisation for CK is obtained by extending IPL with rules $\mathsf{E}_\Box$ and $\mathsf{E}_\Diamond$, and axioms $\mathsf{M}_\Box$, $\mathsf{N}_\Box$, $\mathsf{C}_\Box$, and $\mathsf{K}_\Diamond$ (as before, by adding also $\mathsf{N}_\Diamond$ we obtain logic $\mathsf{CCDL^p}$; notice that axiom $\mathsf{M}_\Diamond$ is derivable in both systems, *e.g.* from Nec and $\mathsf{K}_\Diamond$).

Logics CK and $\mathsf{CCDL^p}$ are non-normal as they reject some form of distributivity of $\Diamond$ over $\vee$. In particular, $\mathsf{CCDL^p}$ rejects binary distributivity ($\mathsf{C}_\Diamond$), while CK rejects both binary and nullary distributivity ($\mathsf{C}_\Diamond$, $\mathsf{N}_\Diamond$). The modality $\Box$ is instead normal as the systems contain axiom $\mathsf{K}_\Box$ and the rule of necessitation.

Sequent calculi for CK and $\mathsf{CCDL^p}$ (denoted here as $\mathsf{G.CK}$ and $\mathsf{G.CCDL^p}$) are defined, respectively, in Bellin *et al.* [1] and Wijesekera [27]. In order to present the calculi we consider the following rule, that we call $\mathsf{W^{seq}}$ (for "Wijesekera"):

$$\mathsf{W^{seq}}\ \frac{A_1, ..., A_n, B \Rightarrow C}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Diamond C} \quad (n \geq 1)\ .$$

Both [1] and [27] allow the set $\{A_1, ..., A_n\}$ in $\mathsf{W^{seq}}$ to be empty, thus including implicitly $\mathsf{M^{seq}_\Diamond}$. By uniformity with the formulation of the other rules, we require it to contain at least one formula. Then, given the present formulation, $\mathsf{G.CK}$ and $\mathsf{G.CCDL^p}$ are defined by extending $\mathsf{G3ip}$ as follows:

$$\begin{array}{lll} \mathsf{G.CK} & := & \mathsf{M_\Box C^{seq}} + \mathsf{M^{seq}_\Diamond} + \mathsf{N^{seq}_\Box} + \mathsf{W^{seq}} \\ \mathsf{G.CCDL^p} & := & \mathsf{M_\Box C^{seq}} + \mathsf{M^{seq}_\Diamond} + \mathsf{N^{seq}_\Box} + \mathsf{W^{seq}} + \mathsf{strC^{seq}} + \mathsf{N^{seq}_\Diamond} \end{array}$$

Observe that $\mathsf{G.CCDL^p}$ can be seen as an extension of our top calculus $\mathsf{G.IMCN_\Box}$, as it is $\mathsf{G.IMCN_\Box}$ + $\mathsf{W^{seq}}$. Instead, $\mathsf{G.CK}$ is not comparable with any our bimodal calculus, as it contains rule $\mathsf{N^{seq}_\Box}$ and doesn't contain $\mathsf{N^{seq}_\Diamond}$, what is never the case in the calculi of our cube.

Theorem 7.1 ([1] for $\mathsf{G.CK}$, [27] for $\mathsf{G.CCDL^p}$)**.** Cut is admissible in $\mathsf{G.CK}$ and $\mathsf{G.CCDL^p}$. Moreover, $\mathsf{G.CK}$ and $\mathsf{G.CCDL^p}$ are equivalent with the corresponding axiomatisations.

Notice that having $\mathsf{W^{seq}}$ istead of our "weak interaction" rules, allows us to take $\mathsf{N^{seq}_\Box}$ and not $\mathsf{N^{seq}_\Diamond}$ (as in $\mathsf{G.CK}$), and still obtain a cut-free calculus. If instead we take both $\mathsf{W^{seq}}$ and $\mathsf{N^{seq}_\Diamond}$ (an in $\mathsf{G.CCDL^p}$), we need to take also $\mathsf{strC^{seq}}$ in order to have the Cut rule admissible, as it is shown by the following derivation:

$$\dfrac{\mathsf{W^{seq}}\ \dfrac{p, \neg p \Rightarrow \bot}{\Box p, \Diamond\neg p \Rightarrow \Diamond\bot} \qquad \dfrac{\bot \Rightarrow}{\Box p, \Diamond\neg p, \Diamond\bot \Rightarrow}\ \mathsf{N^{seq}_\Diamond}}{\Box p, \Diamond\neg p \Rightarrow}\ \mathsf{Cut}$$

[3] The axiomatisation given by Wijesekera [27] includes also $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$; however this formula is derivable from the other axioms (cf. e.g. Simpson [23], p. 48).

It is immediate to verify that the endsequent $\Box p, \Diamond\neg p \Rightarrow$ is derivable in $\mathsf{G.CCDL^p} \setminus \{\mathsf{strC^{seq}}\}$ if and only if the Cut rule is applied, but it has a cut-free derivation in $\mathsf{G.CCDL^p}$ by applying $\mathsf{strC^{seq}}$ to $p, \neg p \Rightarrow$. Notice also that adding $\mathsf{strC^{seq}}$ to the calculus preserve the equivalence with the axiomatisation, as str is derivable from $\mathsf{K}_\Diamond$, $\mathsf{Mon}_\Diamond$ and $\mathsf{N}_\Diamond$.

7.2 Intuitionistic neighbourhood models for CK and CCDL$^\mathsf{p}$

We now define CINMs for CK and $\mathsf{CCDL^p}$, and prove soundness and completeness of the two systems.

Definition 7.1 (Intuitionistic neighbourhood models for CK and $\mathsf{CCDL^p}$)**.** A CINM for CK (CK-model in the following) is any CINM in which $\mathcal{N}_\Box$ is supplemented, closed under intersection and contains the unit; $\mathcal{N}_\Diamond$ is supplemented; and such that:

$$\text{If } \alpha \in \mathcal{N}_\Box(w) \text{ and } \beta \in \mathcal{N}_\Diamond(w), \text{ then } \alpha \cap \beta \in \mathcal{N}_\Diamond(w) \qquad (WInt).$$

A CINM for $\mathsf{CCDL^p}$ ($\mathsf{CCDL^p}$-model in the following) is any CINM for CK satisfying also the condition of *weakInt* ($\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$).

Notice that, as a consequence, function $\mathcal{N}_\Diamond$ in $\mathsf{CCDL^p}$-models contains the unit. We now show that logics CK and $\mathsf{CCDL^p}$ are sound and complete with respect to the corresponding models.

Theorem 7.2 (Soundness)**.** Logics CK and $\mathsf{CCDL^p}$ are sound with respect to CK- and $\mathsf{CCDL^p}$-models, respectively.

Proof. We just consider axiom $\mathsf{K}_\Diamond$. Assume $w \Vdash \Box(A \supset B)$ and $w \nVdash \Diamond B$. Then $[A \supset B] \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [B] \in \mathcal{N}_\Diamond(w)$. By $WInt$, $[A \supset B] \cap (\mathcal{W} \setminus [B]) \in \mathcal{N}_\Diamond(w)$. Since $[A \supset B] \cap (\mathcal{W} \setminus [B]) \subseteq (\mathcal{W} \setminus [A])$, by supplementation $\mathcal{W} \setminus [A] \in \mathcal{N}_\Diamond(w)$; therefore $w \nVdash \Diamond A$. □

Completeness is proved as before by the canonical model construction.

Lemma 7.3. Let the canonical models $\mathcal{M}^c_{\mathsf{CK}}$ for CK, and $\mathcal{M}^c_{\mathsf{CCDL^p}}$ for $\mathsf{CCDL^p}$, be defined as in Lemma 6.5. Then $\mathcal{M}^c_{\mathsf{CK}}$ and $\mathcal{M}^c_{\mathsf{CCDL^p}}$ are, respectively, a CK-model and a $\mathsf{CCDL^p}$-model.

Proof. We show that both $\mathcal{M}^c_{\mathsf{CK}}$ and $\mathcal{M}^c_{\mathsf{CCDL^p}}$ satisfy the condition of $WInt$: Assume $\alpha \in \mathcal{N}^+_\Box(X)$ and $\alpha \cap \beta \notin \mathcal{N}^+_\Diamond(X)$. Then there are $A, B \in \mathcal{L}$ such that $\uparrow_{pr} A \subseteq \alpha$, $\alpha \cap \beta \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$ and $\Box A, \Diamond B \in X$. As a consequence, $\uparrow_{pr} A \cap \beta \subseteq \mathcal{W}^c \setminus \uparrow_{pr} B$, that by standard properties of set inclusion implies $\beta \subseteq (\mathcal{W}^c \setminus \uparrow_{pr} A) \cup (\mathcal{W}^c \setminus \uparrow_{pr} B) = \mathcal{W}^c \setminus \uparrow_{pr} (A \wedge B)$. Moreover, since $(\Box A \wedge \Diamond B) \supset \Diamond(A \wedge B)$ is derivable (from $A \supset (B \supset A \wedge B)$, by $\mathsf{Mon}_\Box$ and $\mathsf{K}_\Diamond$), we have $\Diamond(A \wedge B) \in X$. Thus by definition, $\beta \notin \mathcal{N}^+_\Diamond(X)$. In addition, by Lemma 6.5 (iv) $\mathcal{M}^c_{\mathsf{CCDL^p}}$ is also *weakInt*, as str is derivable in $\mathsf{CCDL^p}$. □

Theorem 7.4 (Completeness)**.** Logics CK and $\mathsf{CCDL^p}$ are complete with respect to CK- and $\mathsf{CCDL^p}$-models, respectively.

Proof. Same proof of Theorem 6.6, using Lemma 7.3. □

7.3 Pre-existing semantics and direct proofs of equivalence

7.3.1 Semantic equivalence for CCDL^p

We now consider pre-existing possible worlds semantics for systems CK and CCDL^p, and prove directly their equivalence with CINMs. We begin with system CCDL^p, and consider the relational models by Wijesekera [27] as well as the neighbourhood models by Kojima [13].

Definition 7.2 (Relational models for CCDL^p (Wijesekera [27])). A relational model for CCDL^p is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in Definition 6.1, and $\mathcal{R}$ is any binary relation on $\mathcal{W}$. The forcing relation $w \Vdash_r A$ is defined as $w \Vdash A$ (Definition 6.1) for $A \equiv p, B \wedge C, B \vee C, B \supset C$; and in the following way for modal formulas:

$w \Vdash_r \Box B$ iff for all $v \succeq w$, for all $u \in \mathcal{W}$, $v\mathcal{R}u$ implies $u \Vdash_r B$;
$w \Vdash_r \Diamond B$ iff for all $v \succeq w$, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $u \Vdash_r B$.

Definition 7.3 (Kojima's neighbourhood models for CCDL^p (Kojima [13])). Kojima's neighbourhood models for CCDL^p are tuples $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V} \rangle$, where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are, respectively, a non-empty set, a preorder on $\mathcal{W}$ and a hereditary valuation function; and $\mathcal{N}_k$ is a neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ such that:

- $w \preceq v$ implies $\mathcal{N}_k(v) \subseteq \mathcal{N}_k(w)$;
- $\mathcal{N}_k(w) \neq \emptyset$ for all $w \in \mathcal{W}$.

The forcing relation $w \Vdash_k A$ is defined as usual for $A \equiv p, \bot, B \wedge C, B \vee C, B \supset C$; and for modal formulas it is defined as follows:

$w \Vdash_k \Box B$ iff for all $\alpha \in \mathcal{N}_k(w)$, for all $v \in \alpha$, $v \Vdash_k B$;
$w \Vdash_k \Diamond B$ iff for all $\alpha \in \mathcal{N}_k(w)$, there is $v \in \alpha$ s.t. $v \Vdash_k B$.

Theorem 7.5 (Wijesekera [27], Kojima [13]). Logic CCDL^p is sound and complete w.r.t. relational models for CCDL^p, as well as w.r.t. Kojima's models for CCDL^p.

That relational, Kojima's and CINMs for CCDL^p are equivalent is a corollary of the respective completeness theorems. It is instructive, however, to prove the equivalence directly. A proof of equivalence of Kojima's and relational models is given in Kojima [13]. Here we prove directly the equivalence of Kojima's and CINMs for CCDL^p. By combining the two proofs we then obtain direct transformations between relational and CINMs.

The following property will be considered in the proof of some of the next lemmas:

For all $\alpha \in \mathcal{N}_\Diamond(w)$, there is $\beta \in \mathcal{N}_\Diamond(w)$ s.t. $\beta \subseteq \alpha$ and $\beta \subseteq \bigcap \mathcal{N}_\Box(w)$ $(WInt')$.

This property is satisfied by all CINMs for CCDL^p and for CK, as it follows from the intersection closure of $\mathcal{N}_\Box$ and the $WInt$.

Lemma 7.6. Let $\mathcal{M}_k = \langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V} \rangle$ be a Kojima's model for CCDL^p, and let $\mathcal{M}_n$ be the model $\langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in $\mathcal{M}_k$, and:

$\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \bigcup \mathcal{N}_k(w) \subseteq \alpha\}$;
$\mathcal{N}_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \beta \in \mathcal{N}_k(w) \text{ s.t. } \beta \subseteq \alpha\}$.

Then $\mathcal{M}_n$ is a CINM for $\mathsf{CCDL^p}$ and is pointwise equivalent to $\mathcal{M}_k$.

Proof. It is immediate to verify that $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented and contain the unit; that $\mathcal{N}_\Box$ is closed under intersection; and that $w \preceq v$ implies $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v)$ and $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$. We show that $\mathcal{M}_n$ satisfies the other properties of $\mathsf{CCDL^p}$-models.

($weakInt$) Assume $\alpha \in \mathcal{N}_\Box(w)$. Then $\bigcup \mathcal{N}_k(w) \subseteq \alpha$, and, since $\mathcal{N}_k(w) \neq \emptyset$, there is $\beta \in \mathcal{N}_k(w)$ such that $\beta \subseteq \alpha$. Therefore $\alpha \in \mathcal{N}_\Diamond(w)$.

($WInt$) Assume $\alpha \in \mathcal{N}_\Box(w)$ and $\beta \in \mathcal{N}_\Diamond(w)$. Then $\bigcup \mathcal{N}_k(w) \subseteq \alpha$ and there is $\gamma \in \mathcal{N}_k(w)$ such that $\gamma \subseteq \beta$. Thus $\gamma \subseteq \bigcup \mathcal{N}_k(w)$, which implies $\gamma \subseteq \alpha \cap \beta$. Therefore $\alpha \cap \beta \in \mathcal{N}_\Diamond(w)$.

By induction on A we now prove that for all $A \in \mathcal{L}$ and all $w \in \mathcal{W}$,

$$\mathcal{M}_n, w \Vdash A \text{ iff } \mathcal{M}_k, w \Vdash_k A.$$

We only consider the inductive cases $A \equiv \Box B, \Diamond B$.

$A \equiv \Box B$. $\mathcal{M}_n, w \Vdash \Box B$ iff $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff $\bigcup \mathcal{N}_k(w) \subseteq [B]_{\mathcal{M}_n}$ iff (i.h.) $\bigcup \mathcal{N}_k(w) \subseteq [B]_{\mathcal{M}_k}$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \subseteq [B]_{\mathcal{M}_k}$ iff $\mathcal{M}_k, w \Vdash_k \Box B$.

$A \equiv \Diamond B$. $\mathcal{M}_n, w \Vdash \Diamond B$ iff $\mathcal{W} \setminus [B]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$ iff (i.h.) for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_k} \neq \emptyset$ iff $\mathcal{M}_k, w \Vdash_k \Diamond B$. □

Lemma 7.7. Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM for $\mathsf{CCDL^p}$, and let $\mathcal{M}_k$ be the model $\langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V} \rangle$ where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in $\mathcal{M}_n$, and:

$$\mathcal{N}_k(w) = \{\alpha \in \mathcal{N}_\Diamond(w) \mid \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}.$$

Then $\mathcal{M}_k$ is a Kojima's model for $\mathsf{CCDL^p}$ and is pointwise equivalent to $\mathcal{M}_n$.

Proof. First notice that $\mathcal{M}_k$ is a Kojima's model: By intersection closure, $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Box(w)$, hence by $weakInt$, $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Diamond(w)$. Thus $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_k(w)$, which implies $\mathcal{N}_k(w) \neq \emptyset$. Moreover assume $w \preceq v$ and $\alpha \in \mathcal{N}_k(v)$. So $\alpha \in \mathcal{N}_\Diamond(v)$ and $\alpha \subseteq \bigcap \mathcal{N}_\Box(v)$. Since $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$ and $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v)$, we have both $\alpha \in \mathcal{N}_\Diamond(w)$ and $\alpha \subseteq \bigcap \mathcal{N}_\Box(w)$, therefore $\alpha \in \mathcal{N}_k(w)$.

By induction on A we show that for all $A \in \mathcal{L}$ and all $w \in \mathcal{W}$,

$$\mathcal{M}_n, w \Vdash A \text{ iff } \mathcal{M}_k, w \Vdash_k A.$$

As before we only consider the inductive cases $A \equiv \Box B, \Diamond B$:

$A \equiv \Box B$. $\mathcal{M}_k, w \Vdash_k \Box B$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \subseteq [B]_{\mathcal{M}_k}$ iff (since $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_k(w)$) $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_k}$ iff (i.h.) $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_n}$ iff (by properties of $\mathcal{N}_\Box(w)$) $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff $\mathcal{M}_n, w \Vdash \Box B$.

$A \equiv \Diamond B$. Assume $\mathcal{M}_k, w, \Vdash_k \Diamond B$. Then for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_k} \neq \emptyset$, and, by i.h., $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Thus for all $\alpha \in \mathcal{N}_\Diamond(w)$ s.t. $\alpha \subseteq \bigcap \mathcal{N}_\Box(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Let β be any neighbourhood in $\mathcal{N}_\Diamond(w)$. By $WInt'$, there is $\gamma \subseteq \beta$ s.t. $\gamma \in \mathcal{N}_\Diamond(w)$ and $\gamma \subseteq \bigcap \mathcal{N}_\Box(w)$. Then $\gamma \cap [B]_{\mathcal{M}_n} \neq \emptyset$, which implies $\beta \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Therefore $\mathcal{M}_n, w \Vdash \Diamond B$. Now assume $\mathcal{M}_n, w \Vdash \Diamond B$. Then for all $\alpha \in \mathcal{N}_\Diamond(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Thus for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [B]_{\mathcal{M}_n} \neq \emptyset$, and, by i.h., $\alpha \cap [B]_{\mathcal{M}_k} \neq \emptyset$. Therefore $\mathcal{M}_k, w \Vdash_k \Diamond B$. □

Theorem 7.8. A formula A is valid in Kojima's models for $\mathsf{CCDL^p}$ if and only if it is valid in CINMs for $\mathsf{CCDL^p}$.

Proof. By Lemmas 7.6 and 7.7. If a Kojima's model for $\mathsf{CCDL^p}$ falsifies A, then there is a CINM for $\mathsf{CCDL^p}$ that falsifies A; and *vice versa* if a CINM for $\mathsf{CCDL^p}$ falsifies A, then there is a Kojima's model for $\mathsf{CCDL^p}$ that falsifies A. □

Given the previous lemmas and Theorems 4.3 and 4.7 in Kojima [13], we can also see how to obtain an equivalent relational model starting from a CINM for $\mathsf{CCDL^p}$, and *vice versa*.

Lemma 7.9. Let $\mathcal{M}_r = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for $\mathsf{CCDL^p}$, and let $\mathscr{R}(w) = \{v \mid w\mathcal{R}v\}$. We define the neighbourhood model $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ by taking $\mathcal{W}$, $\preceq$, $\mathcal{V}$ as in $\mathcal{M}_r$, and the following neighbourhood functions:

$\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \text{for all } v \succeq w, \mathscr{R}(v) \subseteq \alpha\}$;
$\mathcal{N}_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } v \succeq w \text{ s.t. } \mathscr{R}(v) \subseteq \alpha\}$.

Then $\mathcal{M}_n$ is a CINM for $\mathsf{CCDL^p}$, and it is pointwise equivalent to $\mathcal{M}_r$.

Lemma 7.10. Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM for $\mathsf{CCDL^p}$. The relational model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ is defined as follows:

- $\mathcal{W}^* = \{(w, \alpha) \mid w \in \mathcal{W}, \alpha \in \mathcal{N}_\Diamond(w), \text{ and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$;
- $(w, \alpha) \preceq^* (v, \beta)$ iff $w \preceq v$;
- $(w, \alpha)\mathcal{R}^*(v, \beta)$ iff $v \in \alpha$;
- $\mathcal{V}^*((w, \alpha)) = \{p \mid p \in \mathcal{V}(w)\}$ for all $w \in \mathcal{W}$.

Then $\mathcal{M}^*$ is a relational model for $\mathsf{CCDL^p}$. Moreover, for all $A \in \mathcal{L}$ and $w \in \mathcal{W}$, the following claims are equivalent:

1) $\mathcal{M}_n, w \Vdash A$.
2) For all $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.
3) There is $(w, \alpha) \in \mathcal{W}^*$ such that $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.

Theorem 7.11. A formula A is valid in relational models for $\mathsf{CCDL^p}$ if and only if it is valid in CINMs for $\mathsf{CCDL^p}$.

Proof. By Lemma 7.9 and Lemma 7.10. A direct proof of the two lemmas is left to the reader. □

7.3.2 Semantic equivalence for CK

We now present the relational models for CK by Mendler and de Paiva [19], and prove directly their equivalence with CINMs. Relational models for CK are defined by enriching Wijesekera's models for $\mathsf{CCDL^p}$ with inconsistent (or "fallible") worlds (*i.e.* worlds satisfying $\bot$) as follows.

Definition 7.4 (Relational models for CK)**.** Relational models for CK are defined exactly as relational models for $\mathsf{CCDL^p}$ (Definition 7.2), except that the standard forcing relation for $\bot$ ($w \not\Vdash_r \bot$) is replaced by the following ones:

If $w \Vdash_r \bot$, then for all v, $w \preceq v$ or $w\mathcal{R}v$ implies $v \Vdash_r \bot$;
If $w \Vdash_r \bot$, then $w \Vdash_r p$ for all propositional variables $p \in \mathcal{L}$.

Observe that fallible worlds are related through $\preceq$ and $\mathcal{R}$ only to other fallible worlds. Moreover, the above definition preserves the validity of $\top$ and $\bot \supset A$, for all A.

Theorem 7.12 (Mendler and de Paiva [19]). Logic CK is sound and complete w.r.t. relational models for CK.

In order to prove the equivalence between relational and CINMs for CK, we consider transformations of models that are relatively similar to those in Lemmas 7.9 and 7.10. However, the transformations are now a bit more complicated due to the presence of inconsistent worlds.

Lemma 7.13. Let $\mathcal{M}_r = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for CK. Moreover, for all $w \in \mathcal{W}$, let $\mathscr{R}(w) = \{v \mid w\mathcal{R}v\}$. We denote with $\mathcal{W}^+$ the set $\{w \in \mathcal{W} \mid \mathcal{M}_r, w \not\Vdash_r \bot\}$ (*i.e.* the set of consistent worlds of $\mathcal{M}_r$), and for all $\alpha \subseteq \mathcal{W}$, we denote with α^+ the set $\alpha \cap \mathcal{W}^+$.

We define the neighbourhood model $\mathcal{M}_n = \langle \mathcal{W}^+, \preceq^+, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}^+ \rangle$, where $\preceq^+$ and $\mathcal{V}^+$ are the restrictions to $\mathcal{W}^+$ of $\preceq$ and $\mathcal{V}$, and $\mathcal{N}_\Box, \mathcal{N}_\Diamond$ are the following neighbourhood functions:

$\mathcal{N}_\Box(w) = \{\alpha^+ \subseteq \mathcal{W} \mid \text{for all } v \succeq w, \mathscr{R}(v) \subseteq \alpha\}$;
$\mathcal{N}_\Diamond(w) = \{\alpha^+ \subseteq \mathcal{W} \mid \text{there is } v \succeq w \text{ s.t. } \mathscr{R}(v) \subseteq \alpha^+\}$.

Then $\mathcal{M}_n$ is a CINM for CK. Moreover, for all $A \in \mathcal{L}$ and $w \in \mathcal{W}^+$,

$$\mathcal{M}_n, w \Vdash A \text{ iff } \mathcal{M}_r, w \Vdash_r A.$$

Proof. It is imediate to verify that $\mathcal{M}_n$ is a CINM for CK. In particular, for the *WInt*, assume $\alpha^+ \in \mathcal{N}_\Box(w)$ and $\beta^+ \in \mathcal{N}_\Diamond(w)$. Then there is $v \succeq w$ s.t. $\mathscr{R}(v) \subseteq \beta^+$; thus $\mathscr{R}(v) \subseteq \alpha$. Then $\mathscr{R}(v) \subseteq \alpha \cap \beta^+ = (\alpha \cap \beta)^+$. Therefore $(\alpha \cap \beta)^+ = \alpha^+ \cap \beta^+ \in \mathcal{N}_\Diamond(w)$.

We now prove that for all $w \in \mathcal{W}^+$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_r, w \Vdash_r A$. This is equivalent to say that $[A]_{\mathcal{M}_n} = [A]^+_{\mathcal{M}_r}$. As usual we only consider the modal cases.

$A \equiv \Box B$. Let $w \in \mathcal{W}^+$. $\mathcal{M}_n, w \Vdash \Box B$ iff $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff (i.h.) $[B]^+_{\mathcal{M}_r} \in \mathcal{N}_\Box(w)$ iff for all $v \succeq w$, $\mathscr{R}(v) \subseteq [B]_{\mathcal{M}_r}$ iff $\mathcal{M}_r, w \Vdash_r \Box B$.

$A \equiv \Diamond B$. Assume $\mathcal{M}_r, w \Vdash_r \Diamond B$ and $w \in \mathcal{W}^+$. Then for all $v \succeq w$, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $\mathcal{M}_r, u \Vdash_r B$. Thus for all $v \succeq w$, $\mathcal{R}(v) \not\subseteq \mathcal{W} \setminus [B]_{\mathcal{M}_r}$, which in particular implies $\mathcal{R}(v) \not\subseteq (\mathcal{W} \setminus [B]_{\mathcal{M}_r})^+$. Moreover, $(\mathcal{W} \setminus [B]_{\mathcal{M}_r})^+ = \mathcal{W}^+ \setminus [B]^+_{\mathcal{M}_r} =$ (i.h.) $\mathcal{W}^+ \setminus [B]_{\mathcal{M}_n}$. Then $\mathcal{W}^+ \setminus [B]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$, therefore $\mathcal{M}_n, w \Vdash \Diamond B$. Now assume $\mathcal{M}_n, w \Vdash \Diamond B$. Then $\mathcal{W}^+ \setminus [B]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$. This implies that for all $v \succeq w$, $\mathcal{R}(v) \not\subseteq \mathcal{W}^+ \setminus [B]_{\mathcal{M}_n}$; that is, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $u \notin \mathcal{W}^+ \setminus [B]_{\mathcal{M}_n}$. Thus $u \notin \mathcal{W}^+$ or $u \in [B]_{\mathcal{M}_n}$. If $u \notin \mathcal{W}^+$, then $\mathcal{M}_r, u \Vdash_r \bot$, hence $\mathcal{M}_r, u \Vdash_r B$. If $u \in [B]_{\mathcal{M}_n}$, by i.h. $u \in [B]^+_{\mathcal{M}_r}$, thus $\mathcal{M}_r, u \Vdash_r B$. Therefore $\mathcal{M}_r, w \Vdash_r \Diamond B$. $\Box$

Lemma 7.14. Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM for CK, and take $\mathbf{f} \notin \mathcal{W}$. The relational model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ is defined as follows:

- $\mathcal{W}^* = \{(w, \alpha) \mid w \in \mathcal{W}, \mathcal{N}_\Diamond(w) \neq \emptyset, \alpha \in \mathcal{N}_\Diamond(w), \text{ and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$
$\cup\ \{(v, \bigcap \mathcal{N}_\Box(v) \cup \{\mathbf{f}\}) \mid v \in \mathcal{W} \text{ and } \mathcal{N}_\Diamond(v) = \emptyset\}$
$\cup\ \{(\mathbf{f}, \{\mathbf{f}\})\}$;

- $(w, \alpha) \preceq^* (v, \beta)$ iff $w \preceq v$ or $w, v = \mathbf{f}$;
- $(w, \alpha)\mathcal{R}^*(v, \beta)$ iff $v \in \alpha$;
- $\mathcal{V}^*((w, \alpha)) = \{p \mid p \in \mathcal{V}(w)\}$ for all $w \in \mathcal{W}$; and $\mathcal{V}^*((\mathbf{f}, \{\mathbf{f}\})) = Atm$;
- $\mathcal{M}^*, (\mathbf{f}, \{\mathbf{f}\}) \Vdash_r \bot$.

Then $\mathcal{M}^*$ is a relational model for CK. Moreover, for all $A \in \mathcal{L}$ and $w \in \mathcal{W}$, the following claims are equivalent:

1) $\mathcal{M}_n, w \Vdash A$.
2) For all $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.
3) There is $(w, \alpha) \in \mathcal{W}^*$ such that $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.

Proof. It is immediate to show $\mathcal{M}^*$ is a relational model for CK, in particular the conditions on inconsistent worlds are satisfied. We prove by induction on A that points 1), 2) and 3) are equivalent. As usual we only consider the inductive cases $A \equiv \Box B, \Diamond B$.

- $A \equiv \Box B$.
  - 1) implies 2). Assume $\mathcal{M}_n, w \Vdash \Box B$. Then $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$, that implies $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_n}$. Let $(w, \alpha) \in \mathcal{W}^*$, and $(w, \alpha) \preceq^* (v, \beta)$. Then $w \preceq v$, so $\bigcap \mathcal{N}_\Box(v) \subseteq \bigcap \mathcal{N}_\Box(w)$. We distinguish two cases:

    (*a*) $\mathbf{f} \in \beta$. Then $(v, \beta)\mathcal{R}^*(u, \gamma)$ implies $u \in \bigcap \mathcal{N}_\Box(v)$ or $u = \mathbf{f}$.
    If $u = \mathbf{f}$, then $(u, \gamma) = (\mathbf{f}, \{\mathbf{f}\})$, so $\mathcal{M}^*, (u, \gamma) \Vdash_r B$.
    If $u \in \bigcap \mathcal{N}_\Box(v)$, then $u \in [B]_{\mathcal{M}_n}$. By i.h. we have $\mathcal{M}^*, (u, \gamma) \Vdash_r B$ for all γ s.t. $(u, \gamma) \in \mathcal{W}^*$.

    (*b*) $\mathbf{f} \notin \beta$. Then $\beta \subseteq \bigcap \mathcal{N}_\Box(v)$, thus $\beta \subseteq [B]_{\mathcal{M}_n}$. Let $(v, \beta)\mathcal{R}^*(u, \gamma)$. Then $u \in \beta$, so $\mathcal{M}_n, u \Vdash B$. By i.h. we have $\mathcal{M}^*, (u, \gamma) \Vdash_r B$.

    By (*a*) and (*b*) we have that for all $(v, \beta) \succeq^* (w, \alpha)$ and all (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$, $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Therefore for all α s.t. $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r \Box B$.
  - 2) implies 3). Immediate because for all $w \in \mathcal{W}$ there is α s.t. $(w, \alpha) \in \mathcal{W}^*$.
  - 3) implies 1). Assume $\mathcal{M}^*, (w, \alpha) \Vdash_r \Box B$ for an α s.t. $(w, \alpha) \in \mathcal{W}^*$. Then for all $(v, \beta) \succeq^* (w, \alpha)$ and all (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$, $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Thus in particular, for all δ s.t. $(w, \delta) \in \mathcal{W}^*$, for all (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$, $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Take any world $z \in \bigcap \mathcal{N}_\Box(w)$. There exists γ s.t. $(z, \gamma) \in \mathcal{W}^*$. Then $(w, \bigcap \mathcal{N}_\Box(w))\mathcal{R}^*(z, \gamma)$ or $(w, \bigcap \mathcal{N}_\Box(w) \cup \{f\})\mathcal{R}^*(z, \gamma)$ (depending on whether $\mathcal{N}_\Diamond(w) \neq \emptyset$ or $\mathcal{N}_\Diamond(w) = \emptyset$; in the first case $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Diamond(w)$). Thus $\mathcal{M}^*, (z, \gamma) \Vdash_r B$; and by i.h., $\mathcal{M}_n, z \Vdash B$. So $\bigcap \mathcal{N}_\Box(w) \subseteq [B]_{\mathcal{M}_n}$, which implies $[B]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$. Therefore $\mathcal{M}_n, w \Vdash \Box B$.
- $A \equiv \Diamond B$.
  - 1) implies 2). Assume $\mathcal{M}_n, w \Vdash \Diamond B$, and let $(w, \alpha) \in \mathcal{W}^*$ and $(w, \alpha) \preceq^* (v, \beta)$. We distinguish two cases:

(a) $\mathbf{f} \in \beta$. Then $(y, \beta)\mathcal{R}^*(\mathbf{f}, \{\mathbf{f}\})$, and $\mathcal{M}^*, (\mathbf{f}, \{\mathbf{f}\}) \Vdash_r B$.

(b) $\mathbf{f} \notin \beta$. Then $\beta \in \mathcal{N}_\Diamond(y)$, so $\beta \in \mathcal{N}_\Diamond(y)$. By $\mathcal{M}_n, w \Vdash \Diamond B$, we have that for all $\gamma \in \mathcal{N}_\Diamond(w)$, $\gamma \cap [B]_{\mathcal{M}_n} \neq \emptyset$; thus $\beta \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Then there is $u \in \beta$ s.t. $\mathcal{M}_n, u \Vdash B$. By i.h., for all δ s.t. $(u, \delta) \in \mathcal{W}^*$, $\mathcal{M}^*, (u, \delta) \Vdash_r B$. Moreover, there is ϵ s.t. $(u, \epsilon) \in \mathcal{W}^*$. Thus $(v, \beta)\mathcal{R}^*(u, \epsilon)$ and $\mathcal{M}^*, (u, \epsilon) \Vdash_r B$.

By (a) and (b) we have that for all $(v, \beta) \succeq^* (w, \alpha)$, there is (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Therefore, for all α s.t. $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r \Diamond B$.

– 2) implies 3). Immediate because for all $w \in \mathcal{W}$ there is α s.t. $(w, \alpha) \in \mathcal{W}^*$.

– 3) implies 1). Assume $\mathcal{M}^*, (w, \alpha) \Vdash_r \Diamond B$ for a α s.t. $(w, \alpha) \in \mathcal{W}^*$. Then for all $(v, \beta) \succeq^* (w, \alpha)$, there is (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Thus in particular, for all δ s.t. $(w, \delta) \in \mathcal{W}^*$, there is (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. We distinguish two cases:

(a) $\mathbf{f} \in \delta$ for a $(w, \delta) \in \mathcal{W}^*$. Then $\mathcal{N}_\Diamond(w) = \emptyset$, so $\mathcal{M}_n, w \Vdash \Diamond B$.

(b) $\mathbf{f} \notin \delta$ for all $(w, \delta) \in \mathcal{W}^*$. Then by i.h. we have that for all $(w, \delta) \in \mathcal{W}^*$, there is (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}_n, u \Vdash B$. So $u \in \delta$. This means that for all $\delta \in \mathcal{N}_\Diamond(w)$ s.t. $\delta \subseteq \bigcap \mathcal{N}_\Box(w)$, $\delta \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Then by $WInt'$, we have that for all $\epsilon \in \mathcal{N}_\Diamond(w)$, $\epsilon \cap [B]_{\mathcal{M}_n} \neq \emptyset$. Therefore $\mathcal{M}_n, w \Vdash \Diamond B$.

□

Theorem 7.15. A formula A is valid in relational models for CK if and only if it is valid in CINMs for CK.

Proof. Assume A not valid in relational models for CK. Then there are a relational model $\mathcal{M}_r$ and a world w such that $\mathcal{M}_r, w \not\Vdash_r A$. World w is consistent (*i.e.* $\mathcal{M}_r, w \not\Vdash_r \bot$) as inconsistent worlds satisfy all formulas. Then by Lemma 7.13, there is a CINM $\mathcal{M}_n$ for CK such that $\mathcal{M}_n, w \not\Vdash A$.

Now assume A not valid in CINMs for CK. Then there are $\mathcal{M}_n$ and w such that $\mathcal{M}_n, w \not\Vdash A$. By Lemma 7.14, there are a relational model $\mathcal{M}^*$ and a world (w, α) such that $\mathcal{M}^*, (w, \alpha) \not\Vdash_r A$. □

8 Conclusion and further work

This work represents the initial step towards a general investigation of non-normal modalities with an intuitionistic base. We have defined a new family of intuitionistic non-normal modal logics that can be seen as intuitionistic counterparts of classical non-normal modal logics. In particular, we have defined 12 monomodal logics – 8 logics with $\Box$ modality and 4 logics with $\Diamond$ modality – and 24 bimodal logics. For each of them we have provided both a Hilbert axiomatisation and a cut-free sequent calculus. All logics are decidable and contain some of the modal axioms characterising the classical cube. In addition, bimodal logics contain interactions between the modalities that can be seen as "weak duality principles", and express under which conditions two formulas $\Box A$

and $\Diamond B$ are jointly inconsistent. On the basis of the different strength of such interactions we identify different intuitionistic counterparts of a given classical logic.

Subsequently, we have given a modular semantic characterisation of the logics by means of so-called coupled intuitionistic neighbourhood models. The models contain an order relation and two neighbourhood functions handling the modalities separately. For the two functions we consider the standard properties of neighbourhood models, moreover they can be combined in different ways reflecting the possible interactions between $\Box$ and $\Diamond$. Through a filtration argument we have also proved that most of the logics enjoy the finite model property. Our semantics turned out to be a versatile tool to analyse intuitionistic non-normal modal logics, which is capable of capturing further well-known intuitionistic non-normal bimodal logics as Constructive K and the propositional fragment of Wijesekera's CCDL.

Our results can be extended in several directions. First of all we can study further extensions of the cube by axioms analogous to the standard modal ones such as T, D, $\mathsf{4}$, $\mathsf{5}$, *etc.* (some cases have already been considered by Witczak [29]). Furthermore, we can study computational and proof-theoretical properties such as complexity bounds and interpolation. To this regard we plan to develop sequent calculi with invertible rules and that allow for direct countermodel extraction.

From the semantical side we intend to investigate whether it can be given a semantic characterisation of axiom $\mathsf{C}_\Diamond$, that to our knowledge has not been captured yet.

Finally, it would be interesting to see whether these logics, similarly to CK, can be given a type-theoretical interpretation by a suitable extension of the typed lambda-calculus. All of this will be part of our future research.

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