

# COMPLETE REDUCIBILITY AND SEPARABILITY

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ABSTRACT. Let  $G$  be a reductive linear algebraic group over an algebraically closed field of characteristic  $p > 0$ . A subgroup of  $G$  is said to be separable in  $G$  if its global and infinitesimal centralizers have the same dimension. We study the interaction between the notion of separability and Serre’s concept of  $G$ -complete reducibility for subgroups of  $G$ . The separability hypothesis appears in many general theorems concerning  $G$ -complete reducibility. We demonstrate that many of these results fail without this hypothesis. On the other hand, we prove that if  $G$  is a connected reductive group and  $p$  is very good for  $G$ , then any subgroup of  $G$  is separable; we deduce that under these hypotheses on  $G$ , a subgroup  $H$  of  $G$  is  $G$ -completely reducible provided  $\mathrm{Lie} G$  is semisimple as an  $H$ -module.

Recently, Guralnick has proved that if  $H$  is a reductive subgroup of  $G$  and  $C$  is a conjugacy class of  $G$ , then  $C \cap G$  is a finite union of  $H$ -conjugacy classes. For generic  $p$  — when certain extra hypotheses hold, including separability — this follows from a well-known tangent space argument due to Richardson, but in general, it rests on Lusztig’s deep result that a connected reductive group has only finitely many unipotent conjugacy classes. We show that the analogue of Guralnick’s result is false if one considers conjugacy classes of  $n$ -tuples of elements from  $H$  for  $n > 1$ .

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## 1. INTRODUCTION

Let  $G$  be a reductive algebraic group over an algebraically closed field  $k$ . A subgroup  $H$  of  $G$  is said to be  $G$ -completely reducible ( $G$ -cr) if whenever  $H$  is contained in a parabolic subgroup  $P$  of  $G$ ,  $H$  is contained in some Levi subgroup of  $P$ . (If  $G$  is non-connected, then we have to modify this definition slightly; see Subsection 2.2.) In the special case

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$G = \mathrm{GL}_n(k)$ ,  $H$  is  $G$ -completely reducible if and only if  $H$  acts completely reducibly on  $k^n$ . Thus  $G$ -complete reducibility generalizes the notion of semisimplicity from representation theory, [26, 27, 28].

Much effort has been made to study  $G$ -completely reducible subgroups of reductive groups. There are applications to finite groups of Lie type [15]. Ideas from the theory of  $G$ -complete reducibility play an important part in the study by Liebeck and Seitz [16] of the reductive subgroups of the exceptional simple groups.

In characteristic zero a subgroup  $H$  of  $G$  is  $G$ -completely reducible if and only if  $H$  is reductive (cf. [2, Lem. 2.6]). In this case, it is trivial to show that the class of  $G$ -completely reducible subgroups of  $G$  is closed under certain natural constructions: for instance, if  $H, N$  are  $G$ -completely reducible subgroups of  $G$  such that  $H$  normalizes  $N$ , then  $H$  and  $N$  are reductive, so  $HN$  is reductive, and thus  $HN$  is  $G$ -completely reducible. The analogous result need not hold in positive characteristic [3, Examples 5.1 and 5.3]. Nonetheless, results concerning  $G$ -completely reducible subgroups which hold in characteristic zero tend to hold “generically” in positive characteristic, that is, if the characteristic is large enough, or under certain extra natural restrictions on the subgroups considered. We give some other examples below: see Theorem 1.7 and Theorem 4.4.

The purpose of this paper is to explore the bounds of these generic results, proving that they hold without the extra hypotheses or finding counterexamples when the extra hypotheses are removed. Characteristic 2 is a particularly fertile ground for counterexamples. To prove positive results, we use geometric techniques from our earlier works [2] and [3]. Especially important is the property of *separability* (see [2, Def. 3.27]): the hypothesis that certain subgroups are separable is required for several results involving  $G$ -complete reducibility, e.g., [2, Thm. 3.35, Thm. 3.46].

Let  $\mathfrak{g} = \mathrm{Lie} G$  be the Lie algebra of  $G$ .

**Definition 1.1.** A subgroup  $H$  of  $G$  is said to be *separable in  $G$*  if  $\mathrm{Lie} C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$ .

Here  $C_G(H)$  and  $\mathfrak{c}_{\mathfrak{g}}(H)$  denote the centralizer of  $H$  in  $G$  and in  $\mathfrak{g}$  respectively. Note that we always have  $\mathrm{Lie} C_G(H) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$ .

The motivation for this terminology is as follows. Given  $n \in \mathbb{N}$ , we let  $G$  act on  $G^n$  by simultaneous conjugation:

$$g \cdot (g_1, g_2, \dots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \dots, gg_ng^{-1}).$$

Suppose  $H$  is the algebraic subgroup of  $G$  generated by elements  $g_1, \dots, g_n \in G$ . Then we say  $H$  is (*topologically*) *finitely generated* by  $g_1, \dots, g_n$ . The orbit map  $G \rightarrow G \cdot (g_1, \dots, g_n)$ ,  $g \mapsto g \cdot (g_1, g_2, \dots, g_n)$  is a separable morphism of varieties if and only if  $H$  is separable in  $G$  (see [2, Sec. 3.5]).

There is an analogous notion of separability for subalgebras of  $\mathfrak{g}$  — see Definition 2.9 — which similarly is related to the separability of orbit maps  $G \rightarrow G \cdot (x_1, \dots, x_n)$ , where  $x_i \in \mathfrak{g}$  and  $G$  acts diagonally on  $\mathfrak{g}^n$  by simultaneous adjoint action.

Our first main result concerning this notion of separability is as follows.

**Theorem 1.2.** *Let  $G$  be connected reductive and suppose that  $\mathrm{char} k$  is very good for  $G$ . Then any subgroup of  $G$  is separable in  $G$  and any subalgebra of  $\mathfrak{g}$  is separable in  $\mathfrak{g}$ .*

The proof (see Section 3) depends on Theorem 1.4 below and the existence of reductive pairs under the given characteristic restrictions, where for a reductive subgroup  $H$  of a

reductive group  $G$ , we say that  $(G, H)$  is a *reductive pair* if the Lie algebra  $\mathfrak{h} = \text{Lie } H$  is an  $H$ -module direct summand of  $\mathfrak{g}$  (of course, this is automatically satisfied in characteristic zero). Special cases of Theorem 1.2 are known due to work by Liebeck–Seitz [16, Thm. 3] and Lawther–Testerman [14, Thm. 2].

A prototype of the results we consider is the following fundamental result of P. Slodowy [30, Thm. 1], which is obtained by applying a standard tangent space argument due to R.W. Richardson (cf. [22, Sec. 3] and [23, Lem. 3.1]). Theorem 1.3 has many applications: see [2], [30], [35, Sec. 3], for example. Some of the results that follow from Theorem 1.3 are still valid even when the hypotheses of the theorem are not met, but one often has to work much harder to obtain them (cf. [20, Sec. 1]).

**Theorem 1.3.** *Let  $H$  be a reductive subgroup of a reductive algebraic group  $G$ . Let  $n \in \mathbb{N}$ , let  $(h_1, \dots, h_n) \in H^n$  and let  $K$  be the algebraic subgroup of  $H$  generated by  $h_1, \dots, h_n$ . Suppose that  $(G, H)$  is a reductive pair and that  $K$  is separable in  $G$ . Then*

- (a) *for all  $(g_1, \dots, g_n) \in G \cdot (h_1, \dots, h_n) \cap H^n$ , the  $H$ -orbit map  $H \rightarrow H \cdot (g_1, \dots, g_n)$  is separable. In particular,  $K$  is a separable subgroup of  $H$ ;*
- (b) *the intersection  $G \cdot (h_1, \dots, h_n) \cap H^n$  is a finite union of  $H$ -conjugacy classes;*
- (c) *each  $H$ -conjugacy classes in  $G \cdot (h_1, \dots, h_n) \cap H^n$  is closed in  $G \cdot (h_1, \dots, h_n) \cap H^n$ .*

The following consequence [2, Thm. 3.35] of Theorem 1.3 gives an application to  $G$ -complete reducibility. The second assertion follows from Theorem 1.3(c) and the geometric characterization of  $G$ -complete reducibility given in [2] (see Theorem 2.8 below); the first does not appear explicitly in [2, Thm. 3.35], but it follows immediately from the proof together with Theorem 1.3(a).

**Theorem 1.4.** *Suppose that  $(G, H)$  is a reductive pair. Let  $K$  be a subgroup of  $H$  such that  $K$  is a separable subgroup of  $G$ . Then  $K$  is separable in  $H$ . Moreover, if  $K$  is  $G$ -completely reducible, then it is also  $H$ -completely reducible.*

Theorem 1.3(b) can be used to give a simple proof that a connected reductive group  $G$  has only finitely many conjugacy classes of unipotent elements in generic characteristic: one takes an embedding of  $G$  in some  $\text{GL}(V)$  such that  $(\text{GL}(V), G)$  is a reductive pair then applies Theorem 1.3(b) (taking  $n = 1$ ) to deduce the result for  $G$  from the result for  $\text{GL}(V)$ . For small values of  $p$ , there need not exist any such reductive pair  $(\text{GL}(V), G)$  (see Subsection 2.3). Nonetheless Lusztig proved without any restriction on the characteristic, using some very deep mathematics, that  $G$  has only finitely many unipotent conjugacy classes [18]. Guralnick recently extended this result to non-connected reductive groups [9, Thm. 3.3]; he then used it to prove that Theorem 1.3(b) still holds for an arbitrary reductive subgroup  $H$  of  $G$  if  $n = 1$  [9, Thm. 1.2], even though Richardson’s proof no longer goes through.

In view of this, it is natural to ask whether Theorem 1.3(b) holds in more generality. In Example 7.15, we prove the following result which shows that Guralnick’s conjugacy result [9, Thm. 1.2] does not extend to conjugacy classes of  $n$ -tuples for  $n > 1$ .

**Theorem 1.5.** *There exists a reductive group  $G$ , a reductive subgroup  $M$  of  $G$  and a triple  $(m_1, m_2, m_3) \in M^3$  such that  $(G, M)$  is a reductive pair but  $G \cdot (m_1, m_2, m_3) \cap M^3$  is **not** a finite union of  $M$ -conjugacy classes.*

Thus Theorem 1.3(b) is false in general without the separability hypothesis. We give an example (Proposition 7.16) showing that the second assertion of Theorem 1.4 also fails

without the separability hypothesis. This implies that Theorem 1.3(c) fails as well without the separability hypothesis (see Remark 7.17).

The paper is split into several sections, as we now outline. In Section 2 we introduce the notation and preliminary results needed for the rest of the exposition. In particular, we recall the formalism of  $\mathbb{R}$ -parabolic subgroups from [2], which allows us to consider reductive groups which are not connected; this is very important for many of our results.

In Section 3 we discuss the question of separability in connection with reductive pairs. In Proposition 3.7 we give a construction for certain reductive pairs, where the reductive subgroup of  $G$  is of the form  $C_G(S)$ ; here  $S$  is a reductive group acting suitably on  $G$ .

In Section 4 we investigate the connection between the  $G$ -complete reducibility of a subgroup  $H$  and the semisimplicity of the adjoint representation of  $H$  on  $\mathfrak{g}$ . The following result is the basis of our discussion [2, Thm. 3.46].

**Theorem 1.6.** *Let  $H$  be a separable subgroup of  $G$ . If  $\mathfrak{g}$  is semisimple as an  $H$ -module, then  $H$  is  $G$ -completely reducible.*

One way of removing the separability hypothesis from Theorem 1.6 is to combine Theorems 1.2 and 1.6. This immediately gives the following result.

**Theorem 1.7.** *Let  $G$  be connected reductive and suppose that  $\text{char } k$  is very good for  $G$ . Let  $H$  be a subgroup of  $G$  such that  $\mathfrak{g}$  is a semisimple  $H$ -module. Then  $H$  is  $G$ -completely reducible.*

The assumption in Theorem 1.7 that  $\text{char } k$  is very good for  $G$  is rather restrictive. In Section 4 we discuss to what extent we can remove the separability hypothesis from Theorem 1.6 with a weaker assumption on  $\text{char } k$ ; see Theorems 4.1 and 4.4.

In Section 5 we extend two general results of Richardson concerning orbits of reductive groups on affine varieties [23, Thm. A, Thm. C], see Theorem 5.4. We apply these extensions in turn to questions of  $G$ -complete reducibility; in particular, we discuss some results which examine the relationship between  $G$ -complete reducibility and  $H$ -complete reducibility of a subgroup  $K$  of  $H$ , where  $H$  is a subgroup of  $G$  of the form  $H = C_G(S)$ , with  $S$  a reductive group acting suitably on  $G$ : see Proposition 5.7. The case when the group  $S$  considered above is a subgroup of  $G$  is discussed in Section 6.

In Section 7, we provide an important collection of closely related constructions. They give counterexamples to several of our results, including Theorems 1.3 and 1.4, under weakened hypotheses. In particular, here we also prove Theorem 1.5.

## 2. PRELIMINARIES

**2.1. Notation.** Throughout, we work over an algebraically closed field  $k$ ; we let  $k^*$  denote the multiplicative group of  $k$ . All algebraic groups are assumed to be linear. By a subgroup of an algebraic group we mean a closed subgroup and by a homomorphism of algebraic groups we mean a homomorphism of abstract groups that is also a morphism of algebraic varieties. Let  $H$  be a linear algebraic group. We denote by  $\overline{\langle S \rangle}$  the algebraic subgroup of  $H$  generated by a subset  $S$ . We let  $DH$  denote the derived group  $[H, H]$ ,  $Z(H)$  the centre of  $H$ , and  $H^0$  the connected component of  $H$  that contains 1. If  $S$  is a subset of  $H$ , then  $C_H(S)$  is the centralizer of  $S$  in  $H$  and  $N_H(S)$  is the normalizer of  $S$  in  $H$ . In general we use an upper-case roman letter  $G, H, K$ , etc., to denote an algebraic group and the corresponding

lower-case gothic letter  $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ , etc., to denote its Lie algebra. If  $\mathfrak{h}$  is a Lie algebra and  $S$  is a subset of  $\mathfrak{h}$ , then  $\mathfrak{c}_{\mathfrak{h}}(S)$  is the centralizer of  $S$  in  $\mathfrak{h}$ .

Let  $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{h})$  denote the adjoint representation; then we let  $H_{\text{ad}}$  denote the image of  $H$  under this map and  $\mathfrak{h}_{\text{ad}}$  denote  $\text{Lie } H_{\text{ad}}$ . Note that  $(H_{\text{ad}})^0$  is the adjoint form of  $D(H^0)$  [5, V.24.1].

For the set of cocharacters (one-parameter subgroups) of  $H$  we write  $Y(H)$ ; the elements of  $Y(H)$  are the homomorphisms from  $k^*$  to  $H$ .

The *unipotent radical* of  $H$  is denoted  $R_u(H)$ ; it is the maximal connected normal unipotent subgroup of  $H$ . The algebraic group  $H$  is called *reductive* if  $R_u(H) = \{1\}$ ; note that we do not insist that a reductive group is connected. In particular,  $H$  is reductive if it is simple as an algebraic group ( $H$  is said to be *simple* if  $H$  is connected and all proper normal subgroups of  $H$  are finite). If  $N$  is a normal subgroup of  $H$ , then  $H$  is reductive if and only if  $N$  and  $H/N$  are. The algebraic group  $H$  is called *linearly reductive* if all rational representations of  $H$  are semisimple.

If  $H$  acts on the affine variety  $X$ , then we denote by  $X^H$  the fixed point subvariety of  $X$ : that is,  $X^H = \{x \in X \mid h \cdot x = x \ \forall h \in H\}$ . If  $S$  is a subset of  $X$ , then we denote the pointwise stabilizer of  $S$  in  $H$  by  $C_H(S)$ ; we write  $C_H(x)$  instead of  $C_H(\{x\})$  for  $x$  in  $X$ . If  $X = K$  is an algebraic group and  $H$  acts on  $K$  by automorphisms, then we write  $C_K(H)$  instead of  $K^H$ . Then we also have an induced linear action of  $H$  on  $\mathfrak{k} = \text{Lie } K$ ; we write  $\mathfrak{c}_{\mathfrak{k}}(H)$  instead of  $\mathfrak{k}^H$ .

Throughout the paper  $G$  denotes a reductive algebraic group, possibly non-connected, with Lie algebra  $\mathfrak{g}$ . A subgroup of  $G$  normalized by some maximal torus  $T$  of  $G$  is called a *regular* subgroup of  $G$  (connected reductive regular subgroups of connected reductive groups are often also referred to as *subsystem subgroups*, e.g., see [17]).

Fix a maximal torus  $T$  of  $G$ . We write  $X(T)$  for the character group of  $T$ . Let  $\Psi = \Psi(G, T) \subseteq X(T)$  denote the set of roots of  $G$  with respect to  $T$ . We write  $\mathfrak{t} = \text{Lie } T$  for the Lie algebra of  $T$ . If  $\alpha \in \Psi$ , then  $U_{\alpha}$  denotes the corresponding root subgroup and  $\mathfrak{u}_{\alpha}$  denotes the root space  $\text{Lie } U_{\alpha}$ . Thus the root space decomposition of  $\mathfrak{g}$  is given by

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{u}_{\alpha}.$$

We denote by  $G_{\gamma}$  the simple rank 1 subgroup  $\langle U_{\gamma} \cup U_{-\gamma} \rangle$  of  $G$  and by  $\mathfrak{g}_{\gamma}$  the Lie algebra of  $G_{\gamma}$ . Fix a Borel subgroup  $B$  of  $G$  containing  $T$  and let  $\Sigma = \Sigma(G, T)$  be the set of simple roots of  $\Psi$  defined by  $B$ . Then  $\Psi^+ = \Psi(B, T)$  is the set of positive roots of  $G$ . For  $\beta \in \Psi^+$  write  $\beta = \sum_{\alpha \in \Sigma} c_{\alpha\beta} \alpha$  with  $c_{\alpha\beta} \in \mathbb{N}_0$ . A prime  $p$  is said to be *good* for  $G$  if it does not divide any non-zero  $c_{\alpha\beta}$ , and *bad* otherwise. A prime  $p$  is good for  $G$  if and only if it is good for every simple factor of  $G^0$  [32]; the bad primes for the simple groups are 2 for all groups except type  $A_n$ , 3 for the exceptional groups and 5 for type  $E_8$ . A prime  $p$  is said to be *very good* for  $G$ , if  $p$  is good for  $G$  and  $p$  does not divide  $n + 1$  for any simple component of  $G$  of type  $A_n$ . If  $G$  is simple and  $\text{char } k$  is very good for  $G$ , then the Lie algebra  $\mathfrak{g}$  is simple [33].

*Remark 2.1.* Separability of subgroups of  $G$  and of subalgebras of  $\mathfrak{g}$  (see Definition 2.9) is automatic in characteristic zero (cf. [11, Thm. 13.4]). Likewise, the notion of  $G$ -complete reducibility is not interesting in characteristic zero, as a subgroup of  $G$  is  $G$ -completely reducible if and only if it is reductive (cf. [2, Lem. 2.6]); most of our results and proofs become

trivial in characteristic zero. In the remainder of the paper  $p$  denotes the characteristic char  $k$  of  $k$  in case char  $k > 0$ .

Let  $\gamma \in \Psi$ . We denote by  $\gamma^\vee \in Y(G)$  the corresponding coroot. Then  $\gamma^\vee$  is a homomorphism from  $k^*$  to  $G_\gamma$ . If  $\alpha, \beta \in \Sigma$ , then we have  $s_\alpha \cdot \beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha$  [31, Lem. 7.1.8], where  $s_\alpha$  is the reflection corresponding to  $\alpha$  in the Weyl group of  $G$ .

In Section 7 we need the following well-known result (which is implicit for instance in [12, 6.5]). For convenience we include a proof.

**Lemma 2.2.** *Assume that  $G$  is connected. If the derived group  $DG$  of  $G$  is simply connected, then the same holds for any Levi subgroup of any parabolic subgroup of  $G$ .*

*Proof.* We use the following characterization of simply connectedness of  $DG$ : Let  $T$  be a maximal torus of  $G$  and let  $\alpha_1, \dots, \alpha_n \in X(T)$  be a choice of simple roots. Then  $DG$  is simply connected if and only if there exist characters  $\chi_1, \dots, \chi_n \in X(T)$  such that  $\langle \chi_i, \alpha_j^\vee \rangle = \delta_{ij}$  for all  $i, j \in \{1, \dots, n\}$ . For  $G$  semisimple this is clear. The general case follows from the fact that an integer  $\langle \chi, \alpha^\vee \rangle$  only depends on the restriction of  $\chi$  to  $T \cap DG$  and the fact that any character of  $T \cap DG$  can be lifted to a character of  $T$  ([5, Prop. III.8.2(c)]).

The result now follows immediately from the following well-known description of Levi subgroups  $L$  containing  $T$  [5, Prop. IV.14.18]: the simple roots of  $L$  can be chosen from the set  $\{\alpha_1, \dots, \alpha_n\}$ .  $\square$

**2.2.  $G$ -Complete Reducibility.** In [2, Sec. 6], Serre's original notion of  $G$ -complete reducibility is extended to include the case when  $G$  is reductive but not necessarily connected (so that  $G^0$  is a connected reductive group). The crucial ingredient of this extension is the introduction of so-called *Richardson-parabolic subgroups* (*R-parabolic subgroups*) of a reductive group  $G$ . We briefly recall the main definitions here; for more details on this formalism, see [2, Sec. 6].

Let  $G$  be a reductive algebraic group. For a cocharacter  $\lambda \in Y(G)$ , the *R-parabolic subgroup* corresponding to  $\lambda$  is defined by  $P_\lambda := \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$ . Then  $P_\lambda$  admits a Levi decomposition  $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$ , where  $L_\lambda = \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} = g\} = C_G(\lambda(k^*))$ . We call  $L_\lambda$  an *R-Levi subgroup* of  $P_\lambda$ . For an R-parabolic subgroup  $P$  of  $G$ , the different R-Levi subgroups of  $P$  correspond in this way to different choices of  $\lambda \in Y(G)$  such that  $P = P_\lambda$ ; moreover, the R-Levi subgroups of  $P$  are all conjugate under the action of  $R_u(P)$ . An R-parabolic subgroup  $P$  is a parabolic subgroup in the sense that  $G/P$  is a complete variety; the converse is true when  $G$  is connected, but not in general ([20, Rem. 5.3]). The map  $c_\lambda : P_\lambda \rightarrow L_\lambda$  given by  $c_\lambda(g) = \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1}$  is a surjective homomorphism of algebraic groups with kernel  $R_u(P_\lambda)$ ; it coincides with the usual projection  $P_\lambda \rightarrow L_\lambda$ .

*Remark 2.3.* For a subgroup  $H$  of  $G$ , there is a natural inclusion  $Y(H) \subseteq Y(G)$ . If  $\lambda \in Y(H)$ , and  $H$  is reductive, we can therefore associate to  $\lambda$  an R-parabolic subgroup of  $H$  as well as an R-parabolic subgroup of  $G$ . To avoid confusion, we reserve the notation  $P_\lambda$  for R-parabolic subgroups of  $G$ , and distinguish the R-parabolic subgroups of  $H$  by writing  $P_\lambda(H)$  for  $\lambda \in Y(H)$ . The notation  $L_\lambda(H)$  has the obvious meaning. Note that  $P_\lambda(H) = P_\lambda \cap H$ ,  $L_\lambda(H) = L_\lambda \cap H$  and  $R_u(P_\lambda(H)) = R_u(P_\lambda) \cap H$  for  $\lambda \in Y(H)$ .

*Remark 2.4.* If  $\lambda \in Y(G)$  and  $P_\lambda^0 = G^0$ , then  $P_\lambda$  is an R-Levi subgroup of itself: for  $R_u(P_\lambda)$ , being connected, is contained in  $G^0$ , so  $R_u(P_\lambda) = R_u(P_\lambda) \cap G^0 = R_u(P_\lambda(G^0)) = R_u(P_\lambda \cap G^0) = R_u(G^0) = \{1\}$ .

**Definition 2.5.** Suppose  $H$  is a subgroup of  $G$ . We say  $H$  is  *$G$ -completely reducible* ( $G$ -cr for short) if whenever  $H$  is contained in an R-parabolic subgroup  $P$  of  $G$ , then there exists an R-Levi subgroup  $L$  of  $P$  with  $H \subseteq L$ .

*Remark 2.6.* If  $H$  is a  $G$ -completely reducible subgroup of  $G$ , then  $H$  is reductive [27, Property 4].

Since all parabolic subgroups (respectively all Levi subgroups of parabolic subgroups) of a connected reductive group are R-parabolic subgroups (respectively R-Levi subgroups of R-parabolic subgroups), Definition 2.5 coincides with Serre's original definition for connected groups [28]. Sometimes we come across subgroups of  $G$  which are not contained in any proper R-parabolic subgroup of  $G$ ; these subgroups are trivially  $G$ -completely reducible. Following Serre again, we call these subgroups  *$G$ -irreducible* ( $G$ -ir).

*Remark 2.7.* Since R-Levi subgroups of R-parabolic subgroups play an important rôle in many of our proofs, for brevity we sometimes abuse language and refer to an *R-Levi subgroup of  $G$* ; by this we mean an R-Levi subgroup of some R-parabolic subgroup of  $G$ . Similarly, when  $G$  is connected, we may refer to a *Levi subgroup of  $G$* ; this means a Levi subgroup of some parabolic subgroup of  $G$ .

A key result is the following [2, Cor. 3.7], which gives a geometric criterion for  $G$ -complete reducibility.

**Theorem 2.8.** *Let  $g_1, \dots, g_n \in G$  and let  $H = \overline{\langle \{g_1, \dots, g_n\} \rangle}$ . Then  $H$  is  $G$ -completely reducible if and only if the conjugacy class  $G \cdot (g_1, \dots, g_n)$  is closed in  $G^n$ .*

We frequently require results from [2, Sec. 6.3] for non-connected  $G$ , though we usually simply cite the relevant result in [2] for connected  $G$ .

**2.3. Separability.** In Section 3 we require the following analogue of Definition 1.1 for subalgebras of  $\mathfrak{g}$ . Recall that  $C_G(\mathfrak{h}) = \{g \in G \mid \text{Ad } g(x) = x \text{ for all } x \in \mathfrak{h}\}$ .

**Definition 2.9.** A subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is said to be *separable in  $\mathfrak{g}$*  if  $\text{Lie } C_G(\mathfrak{h}) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ .

The above definition has the same motivation as in the group case. Given  $n \in \mathbb{N}$ , we let  $G$  act on  $\mathfrak{g}^n$  by diagonal adjoint action. Suppose  $\mathfrak{h}$  is the subalgebra of  $\mathfrak{g}$  generated by elements  $x_1, \dots, x_n \in \mathfrak{g}$ . Then the orbit map  $G \rightarrow G \cdot (x_1, \dots, x_n)$ ,  $g \mapsto g \cdot (x_1, \dots, x_n)$  is a separable morphism of varieties if and only if  $\mathfrak{h}$  is separable in  $\mathfrak{g}$  (see [5, II.6.7]).

As in the group case ([30, Thm. 1]), it is straightforward to generalize Richardson's tangent space arguments [22, Sec. 3] and [23, Lem. 3.1] to the action of  $G$  on  $\mathfrak{g}^n$ . We then obtain the following analogue of Theorem 1.3.

**Theorem 2.10.** *Let  $H$  be a reductive subgroup of a reductive algebraic group  $G$ . Let  $n \in \mathbb{N}$ , let  $(x_1, \dots, x_n) \in \mathfrak{h}^n$  and let  $\mathfrak{k}$  be the subalgebra of  $\mathfrak{h}$  generated by  $x_1, \dots, x_n$ . Suppose that  $(G, H)$  is a reductive pair and that  $\mathfrak{k}$  is separable in  $\mathfrak{g}$ . Then*

- (a) *all  $H$ -orbit maps in  $G \cdot (x_1, \dots, x_n) \cap \mathfrak{h}^n$  are separable. In particular,  $\mathfrak{k}$  is separable in  $\mathfrak{h}$ ;*

- (b) the intersection  $G \cdot (x_1, \dots, x_n) \cap \mathfrak{h}^n$  is a finite union of  $H$ -conjugacy classes;
- (c) each  $H$ -conjugacy class in  $G \cdot (x_1, \dots, x_n) \cap \mathfrak{h}^n$  is closed in  $G \cdot (x_1, \dots, x_n) \cap \mathfrak{h}^n$ .

We immediately obtain the following analogue of the first assertion of Theorem 1.4.

**Corollary 2.11.** *Suppose that  $(G, H)$  is a reductive pair. Let  $\mathfrak{k}$  be a subalgebra of  $\mathfrak{h}$  such that  $\mathfrak{k}$  is separable in  $\mathfrak{g}$ . Then  $\mathfrak{k}$  is separable in  $\mathfrak{h}$ .*

Because every subgroup of  $\mathrm{GL}(V)$  is separable in  $\mathrm{GL}(V)$  and every subalgebra of  $\mathfrak{gl}(V)$  is separable in  $\mathfrak{gl}(V)$  (e.g., see [2, Ex. 3.28]), Theorem 1.4 and Corollary 2.11 imply the following.

**Corollary 2.12.** *If  $(\mathrm{GL}(V), G)$  is a reductive pair, then every subgroup of  $G$  is separable in  $G$  and every subalgebra of  $\mathfrak{g}$  is separable in  $\mathfrak{g}$ .*

We deduce from this that not every  $G$  can be embedded in some  $\mathrm{GL}(V)$  in such a way that  $(\mathrm{GL}(V), G)$  is a reductive pair: this applies, for instance, to  $G = \mathrm{SL}_2(k)$  when  $p = 2$ , because  $H = G$  is not a separable subgroup of  $G$ . However, if  $G$  is of a given Dynkin type, then generically — that is, for almost all values of  $p$  — the conclusion of Corollary 2.12 holds; for example, if  $G$  is an exceptional simple group of adjoint type and  $p$  is good for  $G$ , then  $(\mathrm{GL}(\mathfrak{g}), G)$  is a reductive pair (cf. Example 4.7).

The final result of this section shows that a non-separable  $G$ -cr subgroup  $K$  of  $G$  is, up to isogeny, a separable subgroup of a regular subgroup of  $G$ . Given a reductive group  $M$ , we let  $\pi_M : M \rightarrow M_{\mathrm{ad}}$  denote the natural morphism.

**Proposition 2.13.** *Let  $K$  be a  $G$ -completely reducible subgroup of  $G$ . Then there exists a subgroup  $M$  of  $G$  containing a maximal torus of  $G$  such that  $K \subseteq M$ ,  $K$  is  $M$ -irreducible and  $\pi_M(K)$  is separable in  $M_{\mathrm{ad}}$ .*

*Proof.* Since a regular reductive subgroup of an R-Levi subgroup of  $G$  is a regular reductive subgroup of  $G$  and  $K$  is  $G$ -cr, we may assume by [2, Cor. 3.5] that  $K$  is  $G$ -ir after replacing  $G$  by an R-Levi subgroup of  $G$  that is minimal with respect to containing  $K$ . By [2, Cor. 2.7(i)],  $K$  is  $M$ -ir in any reductive subgroup  $M$  of  $G$  containing  $K$ . If  $\pi_G(K)$  is separable in  $G_{\mathrm{ad}}$ , then we can take  $M = G$ , so suppose not. By [2, Prop. 3.39], there exists a subgroup  $M'$  of  $G_{\mathrm{ad}}$  containing a maximal torus of  $G_{\mathrm{ad}}$  such that  $\pi_G(K) \subseteq M'$  and  $M'$  is not separable in  $G_{\mathrm{ad}}$ . As  $(G_{\mathrm{ad}})^0$  is of adjoint type, its Lie algebra has trivial centre, so any overgroup of  $(G_{\mathrm{ad}})^0$  is separable in  $G_{\mathrm{ad}}$ . This forces  $M'$  to be of strictly smaller dimension than  $G_{\mathrm{ad}}$ . Let  $M = \pi_G^{-1}(M')$ , a subgroup of  $G$  that is of strictly smaller dimension than  $G$  and contains a maximal torus of  $G$ . Since  $M$  is an overgroup of the  $G$ -ir group  $K$ ,  $M$  is reductive. The result now follows by induction on  $\dim G$ .  $\square$

Proposition 2.13 is false if we do not assume that  $K$  is  $G$ -completely reducible: see Example 7.18 below.

### 3. REDUCTIVE PAIRS AND SEPARABILITY

The notion of separability is central to many of the results in this paper. Theorems 1.3 and 1.4 both illustrate the importance of reductive pairs in this context. In this section we elaborate on this theme. For examples and constructions of reductive pairs, we refer to [30, Sec. I.3] and [2, Sec. 3.5].



**Lemma 3.1.** *Let  $\varphi : G \rightarrow G'$  be a separable isogeny of connected reductive groups, let  $H$  be a subgroup of  $G$  and let  $\mathfrak{k}$  be a subalgebra of  $\mathfrak{g}$ . Then  $H$  is separable in  $G$  if and only if  $\varphi(H)$  is separable in  $G'$  and  $\mathfrak{k}$  is separable in  $\mathfrak{g}$  if and only if  $d\varphi(\mathfrak{k})$  is separable in  $\mathfrak{g}'$ .*

*Proof.* Since the differential  $d\varphi$  of  $\varphi$  is an isomorphism, it is clear that  $d\varphi(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{k})) = \mathfrak{c}_{\mathfrak{g}'}(d\varphi(\mathfrak{k}))$ . Furthermore, we have

$$(3.2) \quad \text{Ad}(\varphi(g)) \circ d\varphi = d\varphi \circ \text{Ad}(g) \quad \text{for all } g \in G,$$

from which it follows that  $d\varphi(\mathfrak{c}_{\mathfrak{g}}(H)) = \mathfrak{c}_{\mathfrak{g}'}(\varphi(H))$ .

So to prove the first statement it suffices to show that  $\varphi(C_G(H))^0 = C_{G'}(\varphi(H))^0$ . For this, in turn, it suffices to show that  $\varphi^{-1}(C_{G'}(\varphi(H))^0) \subseteq C_G(H)$ . If  $y \in H$ , then the image of  $\varphi^{-1}(C_{G'}(\varphi(H))^0)$  under the morphism  $x \mapsto xyx^{-1}y^{-1}$  is an irreducible subset of  $\ker \varphi$  which contains 1, so it must equal  $\{1\}$ .

To prove the second statement, it suffices to show that  $\varphi(C_G(\mathfrak{k})) = C_{G'}(d\varphi(\mathfrak{k}))$ . This follows easily from Eqn. (3.2).  $\square$

**Lemma 3.3.** *Let  $G$  be connected reductive and suppose that  $p$  is very good for  $G$ . Then there exists a separable isogeny  $S \times H \rightarrow G$ , where  $S$  is a torus and  $H$  is a product of simply connected simple groups.*

*Proof.* Let  $G_1, \dots, G_r$  be the simple factors of  $DG$  and let  $\tilde{G}_i$  be the simply connected cover of  $G_i$  for each  $i$ . Then  $\text{Lie } \tilde{G}_i$  is simple for each  $i$ , by our hypothesis on  $p$ . Set  $S = Z(G)^0$  and  $H = \tilde{G}_1 \times \dots \times \tilde{G}_r$ . It is easily checked that the multiplication map  $S \times H \rightarrow G$  is a separable isogeny.  $\square$

We are now in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* By Lemmas 3.1 and 3.3, we may assume that  $G = S \times H_1 \times \dots \times H_r$ , where  $S$  is a torus and each  $H_i$  is a simply connected simple group. It suffices to prove that there exists a reductive group  $G'$ , a separable isogeny  $\eta : G \rightarrow G'$  and an embedding of  $G'$  in some  $\text{GL}(V')$  such that  $(\text{GL}(V'), G')$  is a reductive pair: for then every subgroup of  $G$  is separable in  $G$  and every subalgebra of  $\mathfrak{g}$  is separable in  $\mathfrak{g}$ , by Corollary 2.12 and Lemma 3.1. To do this, we find for each  $i$  a simple group  $H'_i$  of the same Dynkin type as  $H_i$  and an embedding of  $H'_i$  in some  $\text{GL}(V'_i)$  such that  $(\text{GL}(V'_i), H'_i)$  is a reductive pair. As  $H_i$  is simply connected, there is a natural isogeny  $\eta_i : H_i \rightarrow H'_i$ , and  $\eta_i$  is separable as  $p$  is very good for  $H_i$ . We set  $G' = S \times H'_1 \times \dots \times H'_r$  and  $\eta = \text{id}_S \times \eta_1 \times \dots \times \eta_r$ . Since each  $(\text{GL}(V'_i), H'_i)$  is a reductive pair, the existence of a reductive pair  $(\text{GL}(V'), G')$  follows from [30, I.3(4),(5),(7)] (note that for any embedding of  $S$  in some  $\text{GL}(W)$ ,  $(\text{GL}(W), S)$  is a reductive pair because  $S$  is linearly reductive).

To complete the proof, therefore, it is enough to show that for every Dynkin type, there exists a simple group  $K$  of this Dynkin type and an embedding of  $K$  in some  $\text{GL}(V)$  such that  $(\text{GL}(V), K)$  is a reductive pair. If  $K = \text{SO}(V)$  or  $K = \text{Sp}(V)$  and  $p \neq 2$ , then  $(\text{GL}(V), K)$  is a reductive pair [22, Lem. 5.1]. This deals with types  $B_n, C_n$  and  $D_n$ . If  $K = \text{SL}(V)$ , then, since  $p$  is very good for  $G$ , it follows that  $(\text{GL}(V), K)$  is a reductive pair: the scalar matrices form a  $K$ -stable direct complement to  $\mathfrak{sl}(V)$  in  $\mathfrak{gl}(V)$ . This deals with type  $A_n$ . If  $K$  is an adjoint simple group of exceptional type and  $p$  is good for  $K$ , then  $(\text{GL}(\mathfrak{k}), K)$  is a reductive pair, thanks to [22, §5]. This completes the proof.  $\square$

**Corollary 3.4.** *Let  $G$  be connected reductive and suppose that  $p$  is very good for  $G$ . Let  $g_1, \dots, g_n \in G$  and let  $x_1, \dots, x_m \in \mathfrak{g}$ . Then the orbit maps  $G \rightarrow G \cdot (g_1, \dots, g_n)$  and  $G \rightarrow G \cdot (x_1, \dots, x_m)$  are separable.*

*Proof.* The orbit map  $G \rightarrow G \cdot (g_1, \dots, g_n)$  is separable if and only if the algebraic subgroup of  $G$  generated by  $g_1, \dots, g_n$  is separable in  $G$  and the orbit map  $G \rightarrow G \cdot (x_1, \dots, x_m)$  is separable if and only if the subalgebra of  $\mathfrak{g}$  generated by  $x_1, \dots, x_m$  is separable in  $\mathfrak{g}$  (see the comment after Definition 1.1 and Subsection 2.3), so the corollary follows immediately from Theorem 1.2.  $\square$

*Remarks 3.5.* (i). For  $G$  simple of exceptional type and for simple subgroups of  $G$  and  $p > 7$ , Theorem 1.2 is due to case-by-case checks of Liebeck–Seitz [16, Thm. 3] and Lawther–Testerman [14, Thm. 2].

(ii). The restriction on  $p$  in Theorem 1.2 is necessary. For instance, for  $G = \mathrm{SL}(V)$  with  $\dim V = p$ , the group  $G$  is not separable in itself. Also for  $G$  simple of exceptional type and  $p$  a bad prime for  $G$ , the pair  $(\mathrm{GL}(\mathfrak{g}), G)$  need no longer be a reductive pair, so the proof breaks down: for instance, in Proposition 7.11 we provide an example for  $G$  of type  $G_2$  and  $p = 2$  of a non-separable subgroup of  $G$  (cf. Corollary 2.12).

(iii). The requirement in Theorem 1.2 that  $G$  be connected is also necessary. For instance, if  $G = k^* \rtimes C_2$ , where the non-trivial element  $c$  of the cyclic group  $C_2$  acts on  $k^*$  by  $c \cdot a = a^{-1}$ , then  $C_2$  is a non-separable subgroup of  $G$ .

(iv). The case  $n = 1$  in Corollary 3.4 is a well-known fundamental result due to P. Slodowy, [29, p38].

We finish the section with some further useful results on separability and reductive pairs.

**Lemma 3.6.** *Suppose  $G$  is connected. Let  $S$  be an algebraic group acting faithfully on  $G$  by automorphisms. Then the corresponding representation of  $S$  on  $\mathfrak{g}$  has finite kernel.*

*Proof.* It is enough to prove that  $S = \{1\}$  under the extra hypotheses that  $S$  is connected and  $S$  acts trivially on  $\mathfrak{g}$ , so we assume this. Let  $s \in S$ . Let  $B, B^-$  be any pair of opposite Borel subgroups of  $G$ . Since  $s \cdot B$  is also a Borel subgroup of  $B$ , there exists  $g \in G$  such that  $s \cdot B = gBg^{-1}$ . Let  $\mathfrak{b} = \mathrm{Lie} B$ . Then  $\mathfrak{b} = s \cdot \mathfrak{b} = \mathrm{Ad} g(\mathfrak{b})$ , so  $g \in B$ , by [5, IV.14.1 Cor. 2], so  $s$  normalizes  $B$ . Similarly  $s$  normalizes  $B^-$ , so  $s$  normalizes the maximal torus  $T = B \cap B^-$  of  $G$ . Hence  $s$  centralizes  $T$ , by the rigidity of tori [5, Prop. III.8.10]. Since  $B$  and  $B^-$  were arbitrary,  $s$  centralizes the set of semisimple elements of  $G$  which is dense in  $G$  (e.g., see [5, Thm. 12.3(2)]). Thus  $s$  centralizes  $G$ . But the  $S$ -action on  $G$  is faithful, so  $S = \{1\}$  as required.  $\square$

**Proposition 3.7.** *Let  $S$  be an algebraic group acting on  $G$  by automorphisms. Suppose that  $S$  acts semisimply on  $\mathfrak{g}$  and  $\mathrm{Lie} C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$ . Then*

- (a)  $C_G(S)$  is  $G$ -completely reducible and  $(G, C_G(S))$  is a reductive pair.
- (b) If  $S$  is a subgroup of  $G$ , then  $N_G(S)$  is  $G$ -completely reducible and if further  $S^0$  is central in  $S$ , then  $(G, N_G(S))$  is a reductive pair.

*Proof.* (a). Let  $H$  be the union of the components of  $G$  that meet  $C_G(S)$ . Then  $H$  is a finite-index subgroup of  $G$ , so  $C_G(S)$  is  $G$ -cr if and only if it is  $H$ -cr, by [3, Prop. 2.12]. Hence we can assume that  $G = H$ . Replacing  $S$  by  $S/C_S(G)$ , we can also assume that  $S$  acts faithfully on  $G$ . Since  $C_G(S)$  meets every component of  $G$ , it follows that  $S$  acts

faithfully on  $G^0$ . The completely reducible representation  $S \rightarrow \mathrm{GL}(\mathfrak{g})$  has finite kernel by Lemma 3.6, so  $S$  is reductive.

If  $\mathrm{char} k = 0$ , then, as  $S$  is reductive, it is linearly reductive. So in this case all we have to show is that  $C_G(S)$  is reductive. Since  $C_G(S)^0 = C_{G^0}(S)^0$ , this follows immediately from [23, Prop. 10.1.5].

Now assume that  $\mathrm{char} k = p > 0$ . Since  $S$  is reductive,  $S$  has an  $\overline{\mathbb{F}}_p$ -structure, thanks to [20, Prop. 3.2]. As pointed out in [20, Notation 3.3], this implies that  $S$  has an increasing chain of finite subgroups  $S_1 \subseteq S_2 \subseteq \cdots$  such that  $\bigcup_{i \geq 1} S_i$  is Zariski dense in  $S$ . From this it follows that for some  $i \geq 1$ ,  $\mathfrak{g}$  is a semisimple  $S_i$ -module (cf. [2, Lem. 2.10]), and we have  $C_G(S_i) = C_G(S)$  and  $\mathfrak{c}_{\mathfrak{g}}(S_i) = \mathfrak{c}_{\mathfrak{g}}(S)$ . So, after replacing  $S$  by  $S_i$ , we may assume that  $S$  is finite. Put  $G' = G \rtimes S$ . Clearly,  $G'$  is reductive. Furthermore,  $\mathfrak{g}' = \mathrm{Lie} G' = \mathfrak{g}$  is a semisimple  $S$ -module, and  $\mathrm{Lie} C_{G'}(S) = \mathfrak{c}_{\mathfrak{g}'}(S)$  by construction. Therefore,  $S$  is  $G'$ -cr, by Theorem 1.6. Since  $S$  is  $G'$ -cr,  $C_{G'}(S)$  is  $G'$ -cr, by [2, Cor. 3.17]. So the normal subgroup  $C_G(S)$  of  $C_{G'}(S)$  is also  $G'$ -cr, [2, Thm. 3.10]. Now  $C_G(S)$  is  $G$ -cr, by [3, Prop. 2.8].

Since  $\mathfrak{g}$  is a semisimple  $S$ -module, it has a direct sum decomposition into  $S$ -isotypic summands. By hypothesis,  $\mathrm{Lie} C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$ , so  $\mathrm{Lie} C_G(S)$  is the trivial  $S$ -isotypic summand. Hence there is a unique  $S$ -stable complement to  $\mathrm{Lie} C_G(S)$  in  $\mathfrak{g}$ , namely, the sum  $\mathfrak{m}$  of the non-trivial  $S$ -isotypic summands. The uniqueness of  $\mathfrak{m}$  implies that it is also  $C_G(S)$ -stable, so it follows that  $(G, C_G(S))$  is a reductive pair.

(b). Now assume that  $S$  is a subgroup of  $G$ . Then  $S$  is  $G$ -cr by Theorem 1.6, so  $N_G(S)$  is  $G$ -cr [2, Thm. 3.14]. Moreover,  $N_G(S)/SC_G(S)$  is finite by [20, Lem. 6.8]. Since  $S^0 \subseteq C_G(S)$  by assumption,  $N_G(S)/C_G(S)$  is finite. It follows that  $\mathrm{Lie} N_G(S) = \mathrm{Lie} C_G(S)$ . By the uniqueness of the subspace  $\mathfrak{m}$  of  $\mathfrak{g}$  from the proof of part (a) above,  $\mathfrak{m}$  is also  $N_G(S)$ -stable. Hence  $(G, N_G(S))$  is a reductive pair.  $\square$

The following is immediate by Theorem 1.4 and Proposition 3.7(a). Note that it applies in particular when  $S$  is a torus, so that  $C_G(S)$  is an R-Levi subgroup of  $G$  [2, Cor. 6.10].

**Corollary 3.8.** *Let  $G$  and  $S$  be as in the statement of Proposition 3.7. Let  $H$  be a subgroup of  $C_G(S)$ . If  $H$  is separable in  $G$ , then  $H$  is separable in  $C_G(S)$ .*

For a regular reductive subgroup  $H$  of  $G$  the next lemma gives a useful criterion for  $(G, H)$  to be a reductive pair.

**Lemma 3.9.** *Let  $T$  be a maximal torus of  $G$  and let  $H$  be a reductive subgroup of  $G$  containing  $T$ . Assume that  $\Psi(H) = \Psi(H, T)$  is a closed subsystem of  $\Psi = \Psi(G, T)$ . Then  $(G, H)$  is a reductive pair.*

*Proof.* Let  $\mathfrak{m}$  be the sum of the root spaces  $\mathfrak{u}_{\alpha}$  with  $\alpha \notin \Psi(H)$ . Then  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . By the conjugacy of the maximal tori in  $H^0$ , we have  $H = H^0 N_H(T)$ . Since  $N_H(T)$  permutes the root spaces in  $\mathfrak{g}$  and also those in  $\mathfrak{h}$ ,  $N_H(T)$  stabilizes  $\mathfrak{m}$ . So all we have to show is that  $\mathfrak{m}$  is stable under the  $U_{\beta}$ ,  $\beta \in \Psi(H)$ .

Let  $\alpha \in \Psi \setminus \Psi(H)$  and  $\beta \in \Psi(H)$ . If  $\gamma = \alpha + i\beta$  is a root for some integer  $i \geq 0$ , then we must have  $\gamma \notin \Psi(H)$ , as otherwise  $\alpha = \gamma - i\beta \in \Psi(H)$ , because  $\Psi(H)$  is closed and symmetric; see [7, Ch. 6 Prop. 23(iii)]. Now let  $u \in U_{\beta}$ . Then, by [4, Lem. 5.2],  $\mathrm{Ad} u(\mathfrak{u}_{\alpha}) \subseteq \bigoplus \mathfrak{u}_{\alpha+i\beta}$ , where the sum is over all integers  $i \geq 0$  such that  $\alpha + i\beta$  is a root. By the above, this sum is contained in  $\mathfrak{m}$ .  $\square$

*Remarks 3.10.* (i). Note that if  $S$  is a linearly reductive group acting on  $G$  by automorphisms, then the conditions of Proposition 3.7(a) are satisfied, [23, Lem. 4.1].

(ii). We can apply Proposition 3.7(b) and Theorem 1.4 to prove that if  $S$  is a linearly reductive subgroup of  $G$  with  $S^0$  central in  $S$  and  $H$  is a  $G$ -cr subgroup of  $N_G(S)$  such that  $H$  is separable in  $G$ , then  $H$  is  $N_G(S)$ -cr. Here the separability condition cannot be removed; see Example 7.19.

(iii). Note that the hypothesis on  $S^0$  in Proposition 3.7(b) cannot be removed. For example, suppose  $p = 3$  and let  $G = (k^* \times k^* \times k^*) \rtimes (C_3 \times C_2)$ , where the cyclic group  $C_3$  acts on  $k^* \times k^* \times k^*$  by a cyclic permutation of the factors and the cyclic group  $C_2 = \{1, a\}$  acts on  $k^* \times k^* \times k^*$  by  $a \cdot (x, y, z) = (x^{-1}, y^{-1}, z^{-1})$ . Let  $S$  be the linearly reductive subgroup  $\Delta C_2$ , where  $\Delta$  is the diagonal inside  $k^* \times k^* \times k^*$ . Then  $N_G(S)^0 = \Delta$  and  $C_3 \subseteq N_G(S)$ , so  $(G, N_G(S))$  is not a reductive pair because  $\text{Lie } \Delta$  does not admit a  $C_3$ -stable complement in  $\text{Lie}(k^* \times k^* \times k^*)$ .

(iv). We observe that Slodowy's example [30, I.3(7)] is a special case of Proposition 3.7(a), namely when  $G = \text{GL}(V)$  and  $C_G(S)$  is a Levi subgroup of  $G$ .

(v). In the special case of Proposition 3.7 when  $S$  is linearly reductive and  $G$  is connected, Richardson showed in [23, Prop. 10.1.5] that both  $C_G(S)$  and  $N_G(S)$  are reductive.

(vi). The converse of Corollary 3.8 is false: see Proposition 7.11.

We give another application of Proposition 3.7.

**Example 3.11.** Suppose that  $p = 2$ . Let  $G$  be simple of type  $D_4$  and let  $S$  be the group of order 3 generated by the triality graph automorphism of  $G$ . Then  $K = C_G(S)^0$  is of type  $G_2$ . Since  $S$  is linearly reductive, Proposition 3.7(a) implies that  $(G, K)$  is a reductive pair. In Section 7, we construct a subgroup  $H$  of  $K$  isomorphic to  $S_3$  which is not separable in  $K$ , see Proposition 7.11. It follows from Theorem 1.4 that  $H$  is also non-separable in  $G$ . In addition, by Lemma 7.10(a),  $H$  is  $K$ -cr and thus, thanks to [2, Cor. 3.21],  $H$  is also  $G$ -cr.

This example also gives rise to a non-separable subgroup in the exceptional group  $G'$  say of type  $F_4$  as follows. Since  $D_4$  is a closed subsystem of the root system of type  $F_4$  (the  $D_4$ -subsystem consists of the long roots in the  $F_4$  system), Lemma 3.9 implies that  $(G', G)$  is a reductive pair. It follows from Theorem 1.4 and [30, I.3(4)] that  $H$  is also non-separable in the group  $G'$ .

#### 4. THE ADJOINT MODULE AND COMPLETE REDUCIBILITY

In the proof of Theorem 1.6, the hypothesis of separability is only used for a rather coarse dimension-counting argument, so it is natural to ask whether it can be removed. This is a more subtle problem than it at first appears.

Our first result shows that we can remove the separability assumption from  $H$  in Theorem 1.6 under extra hypotheses on  $H$ .

**Theorem 4.1.** *Suppose that  $H$  is a subgroup of  $G$  such that  $H$  acts semisimply on  $\mathfrak{m}_{\text{ad}}$  for every reductive subgroup  $M$  of  $G$  that contains  $H$  and a maximal torus of  $G$  (this includes the case  $M = G$ ). Then  $H$  is  $G$ -completely reducible.*

*Proof.* Suppose  $P$  is an  $R$ -parabolic subgroup of  $G$  containing  $H$ , let  $T$  be a maximal torus of  $P$  and let  $\lambda \in Y(T)$  such that  $P = P_\lambda$ . By Remark 2.4, we can assume that  $P^0$  is proper in  $G^0$ . This implies that  $\lambda$  is non-central in  $G^0$  [2, Lem. 2.4].

First assume that  $H$  acts semisimply on  $\mathfrak{g}$  and that  $\mathfrak{z}(\mathfrak{g}) = \{0\}$ . We then show that there exists a reductive subgroup  $M$  of  $G$  such that  $\dim M < \dim G$  and  $M$  contains  $H$ ,  $\lambda(k^*)$  and a maximal torus of  $G$ . After that we prove the statement of the theorem using induction on the dimension of  $G$ .

Put  $\mathfrak{n} = \text{Lie } R_u(P)$ ,  $\mathfrak{l}_\lambda = \text{Lie } L_\lambda$  and  $\mathfrak{s} = \text{Lie } \lambda(k^*)$ . Since  $L_\lambda$  centralizes  $\lambda(k^*)$  and normalizes  $R_u(P)$ ,  $\lambda(k^*)R_u(P)$  is normalized by  $L_\lambda$ . Therefore,  $\lambda(k^*)R_u(P)$  is a normal subgroup of  $P$ . So  $\text{Lie}(\lambda(k^*)R_u(P)) = \mathfrak{n} \oplus \mathfrak{s}$  is an  $H$ -submodule of  $\mathfrak{g}$ , since  $H$  is contained in  $P$ . Now  $\mathfrak{n}$  is an  $H$ -submodule of  $\mathfrak{n} \oplus \mathfrak{s}$ , so, since  $\mathfrak{g}$  is a semisimple  $H$ -module, there exists a 1-dimensional  $H$ -submodule  $\mathfrak{s}_1$  of  $\mathfrak{n} + \mathfrak{s}$  which is a direct  $H$ -complement to  $\mathfrak{n}$ .

As  $L_\lambda$  acts trivially on  $\mathfrak{z}(\mathfrak{l}_\lambda)$ ,  $(\mathfrak{n} + \mathfrak{z}(\mathfrak{l}_\lambda))/\mathfrak{n}$  is a trivial  $L_\lambda$ -module. By [5, Prop. I.3.17], we have  $\text{Ad}(u)(x) - x \in [\mathfrak{l}, \mathfrak{n}] \subseteq \mathfrak{n}$  for all  $u \in R_u(P)$  and all  $x \in \mathfrak{p}$ , so  $(\mathfrak{n} + \mathfrak{z}(\mathfrak{l}_\lambda))/\mathfrak{n}$  is also a trivial  $R_u(P)$ -module and therefore a trivial  $P$ -module. So  $\mathfrak{s}_1$  must be a trivial  $H$ -module. Let  $x \in \mathfrak{s}_1$  be non-zero. Clearly,  $H$  fixes the nilpotent and semisimple parts of  $x$ . By [5, Thm. III.10.6],  $\mathfrak{n}$  is the set of nilpotent elements of  $\mathfrak{n} + \mathfrak{s}$ . So  $x$  has non-zero semisimple part and we may assume that  $x$  is semisimple. Thus we have found a non-zero semisimple element of  $\text{Lie } P$  that is fixed by  $H$  and  $\lambda(k^*)$ . After conjugating  $\lambda$  and  $T$  by the same element of  $P$ , we may assume that  $x \in \text{Lie } T$ .

Put  $M := C_G(x)$ . This is a reductive subgroup of  $G$  which contains  $T$ ,  $H$  and  $\lambda(k^*)$ . Furthermore,  $\dim M < \dim G$ , since otherwise  $G^0 \subseteq M$  which is impossible, because  $\mathfrak{c}_{\mathfrak{g}}(G^0) = \mathfrak{z}(\mathfrak{g}) = \{0\}$ . So  $M$  has the required properties.

To prove the assertion of the theorem, we pass to the adjoint group. Let  $\pi_G : G \rightarrow G_{\text{ad}}$  be the natural morphism and let  $\tilde{\lambda} = \pi_G \circ \lambda \in Y(G_{\text{ad}})$ . By our hypothesis,  $\pi_G(H)$  acts semisimply on  $\mathfrak{g}_{\text{ad}}$ . Note that  $\tilde{\lambda}$  is non-trivial. If we apply the above argument to  $G_{\text{ad}}$ ,  $\pi_G(H)$  and  $\tilde{\lambda}$ , we get a reductive subgroup  $M'$  of  $G_{\text{ad}}$  such that  $\dim M' < \dim G_{\text{ad}}$  and  $M'$  contains  $\pi_G(T)$ ,  $\pi_G(H)$  and  $\tilde{\lambda}(k^*)$ . But then  $M := \pi_G^{-1}(M')$  is a reductive subgroup of  $G$  such that  $\dim M < \dim G$  and  $M$  contains  $T$ ,  $H$  and  $\lambda(k^*)$ . Clearly,  $M$  and  $H$  satisfy the same assumptions as  $G$  and  $H$ . By the induction hypothesis,  $H$  is  $M$ -cr, so there exists  $u \in P_\lambda(M)$  such that  $H \subseteq uL_\lambda(M)u^{-1}$ . But  $R_u(P_\lambda(M)) = R_u(P_\lambda) \cap M$  and similarly for  $L_\lambda(M)$ . So  $H$  is contained in the Levi subgroup  $uL_\lambda u^{-1}$  of  $P$ .  $\square$

The following example indicates the sort of problem that can arise without the assumptions on  $H$  made in Theorem 4.1.

**Example 4.2.** Let  $p$  be a prime, put  $n = p^2$  and let  $G$  be the simple algebraic group of type  $A_{n-1}$  whose character group is the lattice that is strictly between the root lattice and the weight lattice (the quotient group of the latter two lattices is a cyclic group of order  $p^2$ ). Then  $\mathfrak{g}$  is the direct sum of its centre, which is 1-dimensional, and its derived algebra, which is isomorphic to  $\mathfrak{psl}_n$  (see [10, Table 1]). Thus  $\mathfrak{g}$  is the direct sum of two simple  $G$ -modules and is therefore a semisimple  $G$ -module. But the only proper non-zero ideal of  $\mathfrak{g}_{\text{ad}} \cong \mathfrak{pgl}_n$  is its derived subalgebra, which is of dimension  $n^2 - 2$  ([10, Table 1]). Since every  $G$ -submodule of  $\mathfrak{g}_{\text{ad}}$  is an ideal, it follows that  $G$  does not act semisimply on  $\mathfrak{g}_{\text{ad}}$ .

*Remarks 4.3.* (i). Example 4.2 shows that it is possible for a subgroup  $H$  to act semisimply on  $\mathfrak{g}$ , but not on  $\mathfrak{g}_{\text{ad}}$ . Thus the argument in the proof of Theorem 4.1 cannot be used to extend Theorem 1.6 to the non-separable case.

(ii). Suppose  $G$  is semisimple (so  $Z(G)$  is finite), and  $H$  is a subgroup of  $G$  which acts semisimply on  $\mathfrak{g}$ , but  $H$  is not separable in  $G$ . Observe that  $H$  might be “trivially” non-separable in  $G$ , in the sense that  $\mathfrak{g}$  has non-zero centre so that  $G$  is not even separable in itself. One might hope to deal with such possibilities by passing to the adjoint form  $G_{\text{ad}}$  of  $G$ , but Example 4.2 shows that if we do this, the image of  $H$  in  $G_{\text{ad}}$  may fail to act semisimply on  $\mathfrak{g}_{\text{ad}}$ .

Our next result shows that we can also remove the separability assumption in Theorem 1.6 by strengthening the conditions on  $G$ , rather than on  $H$  (as in Theorem 4.1). In order to state the result, we need some terminology. If  $G$  is connected and simple of type  $A_n$ , then either  $\text{Lie } G \cong \mathfrak{sl}_n$ ,  $\text{Lie } G \cong \mathfrak{pgl}_n$ , or  $p^2|(n+1)$  and  $\text{Lie } G$  is of *intermediate type*; see [10, Table 1]. In contrast to Theorem 1.7, this next result does not impose any characteristic restrictions stemming from simple factors of  $G$  of type  $A_n$ .

**Theorem 4.4.** *Assume that  $G$  is connected,  $p$  is good for  $G$ , and no simple factor of type  $A_n$  of the derived group  $DG$  of  $G$  has Lie algebra of intermediate type. Let  $H$  be a subgroup of  $G$  which acts semisimply on  $\text{Lie } DG$ . Then  $H$  is  $G$ -completely reducible.*

*Proof.* Since  $Z(G)^0$  acts trivially on  $\mathfrak{g}$  and since it is contained in any Levi subgroup of any parabolic subgroup of  $G$ , we may assume that  $Z(G)^0 \subseteq H$ . Then  $H = Z(G)^0(H \cap DG)$ . Applying [2, Lem. 2.12] to the isogeny  $Z(G)^0 \times DG \rightarrow G$  and the projection  $Z(G)^0 \times DG \rightarrow DG$ , we see that we may replace  $G$  by  $DG$  and  $H$  by  $H \cap DG$ .

Let  $G_1, G_2, \dots, G_r$  be the simple factors of  $G$  and let  $\mu : \prod_i G_i \rightarrow G$  be the isogeny given by multiplication. Denote the projection  $G \rightarrow G_i$  by  $\pi_i$  and put  $\mathfrak{g}_i = \text{Lie } G_i$ . Note that  $\mathfrak{g}_i$  is  $H$ -semisimple, since it is a  $G$ -submodule of  $\mathfrak{g}$ . It is easily checked that for each  $i$  the set of automorphisms of  $\mathfrak{g}_i$  given by  $H$  is the same as that given by  $\pi_i(\mu^{-1}(H))$ . So  $\pi_i(\mu^{-1}(H))$  acts semisimply on  $\mathfrak{g}_i$  for each  $i$ . Thanks to [2, Lem. 2.12], we may now assume that  $G$  is simple.

First assume that  $G$  is not of type  $A$ . Then  $H$  is  $G$ -cr by Theorem 1.7.

Now assume that  $G$  is of type  $A_{n-1}$ . Since  $\mathfrak{g}$  is not of intermediate type, the isogenies  $\varphi : \text{SL}_n \rightarrow G$  and  $\psi : G \rightarrow \text{PGL}_n$  cannot both be inseparable. So  $\varphi^{-1}(H)$  acts semisimply on  $\mathfrak{sl}_n$  or  $\psi(H)$  acts semisimply on  $\mathfrak{pgl}_n$ . By [2, Lem. 2.12(ii)(b)], we may assume that  $G \cong \text{SL}_n$  or that  $G \cong \text{PGL}_n$ .

First assume that  $G = \text{SL}_n$ . We have  $G_{\text{ad}} = \text{PGL}_n$  and  $\mathfrak{g}_{\text{ad}} = \mathfrak{pgl}_n$ . The trace form on  $\mathfrak{gl}_n$  is non-degenerate and induces a non-degenerate  $\text{GL}_n$ -invariant pairing between  $\mathfrak{sl}_n$  and  $\mathfrak{pgl}_n$ . This shows that  $\mathfrak{pgl}_n \cong \mathfrak{sl}_n^*$  as  $\text{GL}_n$ -modules and therefore that  $H$  acts semisimply on  $\mathfrak{g}_{\text{ad}}$ . More generally, thanks to [6], if  $M$  is any reductive subgroup of  $G$  containing a maximal torus of  $G$ , then  $M^0$  is a Levi subgroup of  $G$  (see also [7, Ex. Ch. VI §4.4]). Regarding  $G$  as a subgroup of  $\text{GL}_n$ , we can write  $M^0 = (\prod_{i=1}^r \text{GL}_{n_i}) \cap \text{SL}_n$  and  $\mathfrak{m} = (\oplus_{i=1}^r \mathfrak{gl}_{n_i}) \cap \mathfrak{sl}_n$  for some  $n_i$ . Since the restriction of the trace form of  $\mathfrak{gl}_n$  to  $\oplus_{i=1}^r \mathfrak{gl}_{n_i}$  is non-degenerate — it is the direct sum of the trace forms of the  $\mathfrak{gl}_{n_i}$  — the orthogonal complement of  $\mathfrak{m}$  in  $\oplus_{i=1}^r \mathfrak{gl}_{n_i}$  is 1-dimensional and is therefore equal to  $k \cdot \text{id}_n$ . So  $\mathfrak{m}^* \cong (\oplus_{i=1}^r \mathfrak{gl}_{n_i}) / (k \cdot \text{id}_n)$  as  $M$ -modules. Now  $\mathfrak{m}_{\text{ad}} = \oplus_{i=1}^r \mathfrak{pgl}_{n_i}$ , which is isomorphic to a quotient of  $\mathfrak{m}^* = (\oplus_{i=1}^r \mathfrak{gl}_{n_i}) / (k \cdot \text{id}_n)$ . Thus if  $H$  acts semisimply on  $\mathfrak{g}$ , then  $H$  acts semisimply on  $\mathfrak{m}_{\text{ad}}$  for any regular reductive subgroup  $M$  which contains  $H$  and a maximal torus of  $G$ . The result now follows from Theorem 4.1.

Finally, assume that  $G$  is isomorphic to  $\mathrm{PGL}_n$ . Let  $\varphi : \mathrm{SL}_n \rightarrow \mathrm{PGL}_n$  be the canonical projection. Since  $\mathfrak{pgl}_n \cong \mathfrak{sl}_n^*$  as  $\mathrm{GL}_n$ -modules,  $\varphi^{-1}(H)$  acts semisimply on  $\mathfrak{sl}_n$ . The desired result follows from the previous case and [2, Lem. 2.12(ii)(b)].  $\square$

We record an important special case of Theorem 4.4.

**Corollary 4.5.** *Assume that  $G$  is connected,  $p$  is good for  $G$ , and  $DG$  is either adjoint or simply connected. Let  $H$  be a subgroup of  $G$  which acts semisimply on  $\mathrm{Lie} DG$ . Then  $H$  is  $G$ -completely reducible.*

*Remarks 4.6.* (i). Note that Theorem 4.4 applies in cases when Theorem 1.6 does not. For example, let  $G = \mathrm{SL}_2(k)$ , where  $k$  has characteristic 2, let  $T$  be a maximal torus of  $G$ , and set  $H = N_G(T)$ . Then  $\mathfrak{g}$  is semisimple as an  $H$ -module, so Theorem 4.4 applies, but, as  $H$  is not separable in  $G$  (cf. [19, Ex. 3.4(b)]), Theorem 1.6 does not apply. We can also take  $G = \mathrm{GL}_2(k)$ ,  $k$  of characteristic 2, and  $H = N_G(T)$ , where  $T$  is a maximal torus of  $G$ . Then  $\mathrm{Lie} DG = \mathfrak{sl}_2$  is a semisimple  $H$ -module, but  $\mathfrak{g} = \mathfrak{gl}_2$  is not semisimple as an  $H$ -module.

(ii). We do not know whether the assumptions other than  $\mathfrak{g}$  being  $H$ -semisimple are necessary in Theorem 4.4. To show that the assumption on the simple type  $A$  factors can be removed, it would suffice to show that a subgroup  $H$  of  $\mathrm{GL}_n$  which acts semisimply on  $\mathfrak{psl}_n$  is  $\mathrm{GL}_n$ -cr — that is, acts semisimply on the natural module  $k^n$ .

(iii). Let  $H$  be a group and let  $\rho : H \rightarrow \mathrm{GL}(V)$  be a finite dimensional representation of  $H$ . We have  $V \otimes V^* \cong \mathfrak{gl}(V)$  as  $\mathrm{GL}(V)$ -modules and therefore also as  $H$ -modules. Furthermore,  $V$  is  $H$ -semisimple if and only if  $\rho(H)$  is  $\mathrm{GL}(V)$ -cr. So, by Theorem 4.4 (or Theorem 1.6),  $V$  is  $H$ -semisimple if  $V \otimes V^*$  is  $H$ -semisimple. This result is a special case of a theorem of Serre, see [25, Thm. 3.3].

We finish this section with an example which illustrates that the converse of Theorem 1.6 is false.

**Example 4.7.** In [28, Cor. 5.5], Serre shows that  $\mathfrak{g}$  is a semisimple  $H$ -module if and only if  $H$  is  $G$ -completely reducible, under the assumption that  $p > 2h - 2$ , where  $h$  is the Coxeter number of  $G$ . Theorem 1.7 improves the bound for the forward implication of this equivalence. However, Serre's bound is sharp for the reverse implication, as the following example (due to some unpublished calculations of Serre) shows, see also [2, Rem. 3.43(iii)].

Let  $H$  be an exceptional simple group of adjoint type and suppose  $p \leq 2h - 2$ , where  $h$  denotes the Coxeter number of  $H$ . Then, with the possible exception of type  $G_2$  in characteristic 5, Serre's construction yields a simple subgroup  $K$  of  $H$  of type  $A_1$  such that  $K$  is  $H$ -irreducible, but  $K$  does not act semisimply on  $\mathfrak{h}$  — that is,  $K$  is not  $G$ -completely reducible, where  $G := \mathrm{GL}(\mathfrak{h})$ .

This example also illustrates that the converse of the second assertion of Theorem 1.4 is not true in general, even when  $K$  is separable in  $G$  and  $(G, H)$  is a reductive pair. For we can choose  $p$  and  $H$  above such that  $p$  is good for  $H$ ; then  $(\mathrm{GL}(\mathfrak{h}), H)$  is a reductive pair (cf. [2, Ex. 3.37]), since the Killing form on  $\mathfrak{h}$  is non-degenerate [22, §5]. It then follows from Theorem 1.4 that  $K$  is separable in  $H$ , since  $K$  is separable in  $\mathrm{GL}(\mathfrak{h})$ .

## 5. CLOSED ORBITS AND SEPARABILITY

In this section we generalize two results of R.W. Richardson [23, Thm. A, Thm. C], which we are then able to apply to  $G$ -complete reducibility. We require two preliminary results,

which allow us to relax the hypotheses in the original theorems. Armed with these lemmas, it is quite straightforward to adapt Richardson's proofs to our setting.

Our first result is an analogue of [23, Prop. 6.1], which Richardson proves for  $S$  linearly reductive and  $G$  connected, using non-commutative cohomology of algebraic groups. It should be noted that, although our result is more general than Richardson's, in order to apply it to linearly reductive groups one needs to know that they are always completely reducible in any ambient reductive group ([2, Lem. 2.6]), and the proof of this depends on the argument that Richardson uses.

**Lemma 5.1.** *Let  $G$  be a normal subgroup of the reductive group  $G'$  and suppose that  $S$  is a  $G'$ -completely reducible subgroup of  $G'$ . Then for every  $R$ -parabolic subgroup  $P$  of  $G$  normalized by  $S$ , there exists an  $R$ -Levi subgroup of  $P$  normalized by  $S$ .*

*Proof.* Note that since  $G$  is normal in  $G'$ ,  $G$  is reductive, and  $G^0$  is a connected reductive normal subgroup of  $G'$ . We first show that it is enough to prove the result when  $G$  is connected. If  $P$  is an  $R$ -parabolic subgroup of  $G$  normalized by  $S$ , then  $P^0$  is a parabolic subgroup of  $G^0$  normalized by  $S$ . If  $L$  is a Levi subgroup of  $P^0$  normalized by  $S$ , then  $L = M^0$  for some  $R$ -Levi subgroup  $M$  of  $P$  [2, Cor. 6.8]. Let  $T$  be a maximal torus of  $M$ ; then  $T \subseteq L$ . Since  $S$  normalizes  $P$ , it follows that  $sMs^{-1}$  is an  $R$ -Levi subgroup of  $P$  for any  $s \in S$ . Moreover, since  $S$  normalizes  $L$ , we see that  $T \subseteq L = sLs^{-1} \subseteq sMs^{-1}$ . This shows that  $M = sMs^{-1}$ , for every  $s \in S$ , by [2, Cor. 6.5], and therefore  $S$  normalizes  $M$ . Thus, if the result is true for  $G^0$ , then it is true for  $G$ .

Now suppose  $G$  is connected, and let  $H = C_{G'}(G)^0$ . Since  $G$  is normal in  $G'$ , we have  $G^0 = GH$ , by [20, Lem. 6.8]. Let  $P$  be a parabolic subgroup of  $G$  normalized by  $S$ . Then  $P \times H$  is a parabolic subgroup of  $G \times H$ , and it follows from [5, Prop. IV.11.14(1)] applied to the multiplication map  $f: G \times H \rightarrow G^0$  that  $PH$  is a parabolic subgroup of  $G^0$ . As  $S$  normalizes  $P$ ,  $S$  normalizes  $PH$ , so  $S \subseteq N_{G'}(PH)$ , which is an  $R$ -parabolic subgroup of  $G'$  by [20, Prop. 5.4(a)]. Now  $S$  is  $G'$ -cr by hypothesis, so  $S$  is contained in some  $R$ -Levi subgroup  $M$  of  $N_{G'}(PH)$ , so  $S$  normalizes  $M^0$ , which is a Levi subgroup of  $N_{G'}(PH)^0 = PH$ . By [2, Lem. 6.15(ii)],  $f^{-1}(PH)$  is a parabolic subgroup of  $G \times H$  with Levi subgroup  $f^{-1}(M^0)$ .

If  $(g, h) \in f^{-1}(PH)$ , then  $gh = g'h'$  for some  $g' \in P$  and some  $h' \in H$ , so  $(g')^{-1}g = h'h^{-1} \in G \cap H \subseteq Z(G) \subseteq P$ , so  $g \in P$ . It follows that  $f^{-1}(PH) = P \times H$ . Hence  $f^{-1}(M^0) = L \times H$  for some Levi subgroup  $L$  of  $P$  and hence  $M^0 = LH$ . A similar calculation shows that  $L = M^0 \cap G$ , so  $S$  normalizes  $L$  and we are done.  $\square$

**Lemma 5.2.** *Let  $S$  be a group and let  $f: V \rightarrow W$  be a surjective homomorphism of finite dimensional  $S$ -modules. If  $V/V^S$  has no trivial composition factor, then  $f(V^S) = W^S$ .*

*Proof.* Since  $f$  is surjective, it induces a surjective homomorphism  $V/V^S \rightarrow W/f(V^S)$  of  $S$ -modules. Since  $V/V^S$  has no trivial composition factor, neither does  $W/f(V^S)$ . In particular,  $W/f(V^S)$  does not have any non-zero  $S$ -fixed points. So  $W^S$  must be equal to  $f(V^S)$ .  $\square$

*Remark 5.3.* If  $V$  is  $S$ -semisimple — in particular, if  $S$  is linearly reductive — then  $V/V^S$  has no trivial composition factor, so Lemma 5.2 holds.

We can now state our generalization of Richardson's two results [23, Thm. A, Thm. C]. Observe that if  $S$  is a subgroup of  $G$ , then the condition on  $S$  in Theorem 5.4(b)(ii) is just that of separability of  $S$  in  $G$ .



**Theorem 5.4.** *Let  $G$  be a normal subgroup of the reductive group  $G'$ . Let  $S$  be a subgroup of  $G'$  such that  $G' = GS$ . Let  $G'$  act on an affine variety  $X$  and let  $x \in X^S$ . Let  $\mathcal{O}$  denote the unique closed  $G$ -orbit in the closure of  $G \cdot x$ .*

- (a) *Suppose that  $S$  is  $G'$ -completely reducible. Then  $C_G(S)$  is  $G$ -completely reducible and there exists  $\lambda \in Y(C_G(S))$  such that  $\lim_{a \rightarrow 0} \lambda(a) \cdot x$  exists and is a point of  $\mathcal{O}$ . In particular,  $G \cdot x$  is closed if  $C_G(S) \cdot x$  is closed.*
- (b) *Suppose that*
- (i)  $\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(S)$  *does not have any trivial  $S$ -composition factors;*
  - (ii)  $\mathfrak{c}_{\mathfrak{g}}(S) = \text{Lie } C_G(S)$ .
- Then each irreducible component of  $G \cdot x \cap X^S$  is a single  $C_G(S)^0$ -orbit. In particular,  $C_G(S) \cdot x$  is closed if  $G \cdot x$  is closed.*

*Proof.* (a). Suppose that  $S$  is  $G'$ -cr. Then  $C_G(S)$  is  $G$ -cr, by the same arguments as in the proof of Proposition 3.7(a). If  $G \cdot x$  is closed, then we can take  $\lambda$  to be the zero cocharacter, so assume that  $G \cdot x$  is not closed. Then  $G^0 \cdot x$  is not closed. Moreover, the unique closed  $G$ -orbit  $\mathcal{O}$  in the closure of  $G \cdot x$  contains the unique closed  $G^0$ -orbit in the closure of  $G^0 \cdot x$ . Thus, since  $Y(G) = Y(G^0)$  and  $Y(C_G(S)) = Y(C_{G^0}(S))$ , we may replace  $G$  by  $G^0$  and assume that  $G$  is connected.

We can now follow the first part of Richardson's original proof in [23, Sec. 8] word for word to give the result, simply replacing references to [23, Prop. 6.1] with our Lemma 5.1.

(b). Now assume that  $\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(S)$  does not have any trivial  $S$ -composition factors and that  $\mathfrak{c}_{\mathfrak{g}}(S) = \text{Lie } C_G(S)$ . We can write  $G \cdot x$  as a finite union  $\mathcal{O}_1 \cup \dots \cup \mathcal{O}_n$  of  $G^0$ -orbits. Each  $\mathcal{O}_i$  is a  $G$ -translate of  $G^0 \cdot x$ , from which we deduce that the  $\mathcal{O}_i$  are the irreducible components of  $G \cdot x$ . We may assume that  $\mathcal{O}_1, \dots, \mathcal{O}_r$  ( $r \leq n$ ), are the  $G^0$ -orbits in  $G \cdot x$  that meet  $X^S$ . Then  $G \cdot x \cap X^S = (\mathcal{O}_1 \cap X^S) \cup \dots \cup (\mathcal{O}_r \cap X^S)$ , a disjoint union of closed subsets of  $G \cdot x$ . This shows that, after replacing  $G'$  by  $G^0S$ , we may assume that  $G$  is connected.

Note that  $G \cdot x$  is  $S$ -stable. Put  $Y = (G \cdot x)^S = G \cdot x \cap X^S$ . By the argument at the end of the proof of [23, Thm. A], we may assume that the orbit map  $G \rightarrow G \cdot x$  of  $x$  is separable. Let  $y \in Y$ . Then the orbit map of  $y$  is separable: that is, its differential  $\psi : \mathfrak{g} \rightarrow T_y(G \cdot y)$  is surjective. Since the orbit map of  $y$  is  $S$ -equivariant,  $\psi$  is  $S$ -equivariant. By Lemma 5.2, we have  $\psi(\mathfrak{c}_{\mathfrak{g}}(S)) = (T_y(G \cdot y))^S$ . Clearly,  $T_y(Y) \subseteq (T_y(G \cdot x))^S$ , so  $T_y(Y) \subseteq \psi(\mathfrak{c}_{\mathfrak{g}}(S))$ . Since  $\mathfrak{c}_{\mathfrak{g}}(S) = \text{Lie } C_G(S)$ , the Tangent Space Lemma [23, Lem. 3.1] gives the desired result.

It now follows that  $C_G(S)^0 \cdot x$ , and therefore  $C_G(S) \cdot x$ , is closed in  $G \cdot x$ . So if  $G \cdot x$  is closed, then  $C_G(S) \cdot x$  is closed.  $\square$

*Remarks 5.5.* (i). Assume that  $S$  is a separable subgroup of  $G$  which acts semisimply on  $\mathfrak{g}$ . Then  $S$  is  $G$ -cr by Theorem 1.6. So parts (a) and (b) of Theorem 5.4 both apply. Thus in this case  $C_G(S) \cdot x$  is closed if and only if  $G \cdot x$  is closed.

(ii). Suppose that  $S$  is linearly reductive. Then conditions (i) and (ii) of Theorem 5.4(b) are satisfied (see [23, Lem. 4.1]), and  $S$  is  $G'$ -cr [2, Lem. 2.6]; the special case of Theorem 5.4 for  $S$  linearly reductive is precisely Richardson's original theorem [23, Thm. C] (clearly, the hypothesis in Theorem 5.4 that  $G' = GS$  is harmless).

(iii). It follows from the proof of Theorem 5.4(b) that [23, Thm. A] holds with the hypothesis that  $S$  acts semisimply on  $\mathfrak{g}$  replaced by the weaker hypothesis of Theorem 5.4(b)(ii).

(iv). Theorem 5.4(b) does not hold for an arbitrary  $G'$ -completely reducible subgroup  $S$ . In fact it does not even hold when  $S$  is a  $G$ -completely reducible subgroup of  $G$ : in [3, Prop. 3.9] it is shown that if  $S$  is a  $G$ -completely reducible subgroup of  $G$  and  $K$  is a subgroup of  $C_G(S)$ , then  $K$  is  $C_G(S)$ -completely reducible if and only if  $KS$  is  $G$ -completely reducible. Now [3, Examples 5.1, 5.3, 5.5] give instances of commuting  $G$ -completely reducible subgroups  $K$  and  $S$  such that  $KS$  is not  $G$ -completely reducible, whence  $K$  is not  $C_G(S)$ -completely reducible. If the field  $k$  is large enough (cf. [2, Rem. 2.9]), we can pick an  $n$ -tuple  $(k_1, \dots, k_n) \in G^n$  topologically generating  $K$  for some  $n$ . Then by Theorem 2.8,  $G \cdot (k_1, \dots, k_n)$  is closed in  $G^n$ , but  $C_G(S) \cdot (k_1, \dots, k_n)$  is not.

**Corollary 5.6.** *Let  $G, G', S, X$  and  $x \in X^S$  be as in Theorem 5.4. Suppose that*

- (i) *the adjoint representation of  $S$  on  $\mathfrak{g}$  is semisimple;*
- (ii)  *$\mathfrak{c}_{\mathfrak{g}}(S) = \text{Lie } C_G(S)$ .*

*Then  $C_G(S)$  is  $G$ -completely reducible and  $G \cdot x$  is closed if and only if  $C_G(S) \cdot x$  is closed.*

*Proof.* Since  $GS = G'$ , the quotient  $S/(S \cap G^0)$  is isomorphic to a finite-index subgroup of  $G'/G^0$ , and hence is reductive. Now  $S \cap G^0$  is normal in  $S$ , so  $S \cap G^0$  acts semisimply on  $\mathfrak{g}$  by Clifford's Theorem. The kernel  $N$  of the  $(S \cap G^0)$ -action on  $\mathfrak{g}$  is a subgroup of the diagonalizable group  $Z(G^0)$  and is therefore reductive. Since  $N$  and  $(S \cap G^0)/N$  are reductive,  $S \cap G^0$  is reductive, and it follows that  $S$  is reductive.

Suppose  $\text{char } k = 0$ . Note that, in this case, properties (i) and (ii) and the  $G'$ -complete reducibility of  $S$  follow from the fact that  $S$  is reductive. Therefore, the hypotheses of Theorem 5.4(a) and (b) are satisfied.

Now suppose  $\text{char } k = p > 0$ . By the same arguments as in the proof of Proposition 3.7(a), we may assume that  $S$  is finite and deduce that then  $S$  is  $G'$ -cr. Thus, the hypotheses of Theorem 5.4(a) and (b) are again satisfied.  $\square$

Now we translate our results in terms of  $G$ -complete reducibility.

**Proposition 5.7.** *Let  $G$  be a normal subgroup of the reductive group  $G'$ . Let  $S$  be a reductive subgroup of  $G'$  such that  $G' = GS$ . Let  $K \subseteq H$  be subgroups of  $G$  such that  $S$  normalizes  $K$  and  $H^0 = C_G(S)^0$ .*

- (a) *Suppose that  $S$  is  $G'$ -completely reducible. Then  $H$  is reductive and  $K$  is  $G$ -completely reducible if it is  $H$ -completely reducible.*
- (b) *Suppose that*
  - (i)  *$\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(S)$  does not have any trivial  $S$ -composition factors;*
  - (ii)  *$\mathfrak{c}_{\mathfrak{g}}(S) = \text{Lie } C_G(S)$ ;*
  - (iii)  *$C_G(S)$  is reductive.*

*Then  $K$  is  $H$ -completely reducible if it is  $G$ -completely reducible.*

*Proof.* In case (a),  $C_G(S)$  is reductive by Theorem 5.4(a) and in case (b), this is true by assumption. Since  $H^0 = C_G(S)^0$ , it follows that  $H$  is reductive.

Clearly, we can now assume that  $\text{char } k = p > 0$ . By the argument in the proof of Proposition 3.7(a), we may also assume that  $S$  is finite. As in [2, Lem. 2.10], we can replace  $K$  by a subgroup  $K'$  that is topologically generated by some  $k_1, \dots, k_n$  with the property that for any  $\lambda \in Y(G)$ , we have  $K \subseteq P_\lambda$  if and only if  $K' \subseteq P_\lambda$ , and  $K \subseteq L_\lambda$  if and only if  $K' \subseteq L_\lambda$ . Thus  $K$  is  $G$ -cr (respectively  $H$ -cr) if and only if  $K'$  is  $G$ -cr (respectively  $H$ -cr).

Since  $S$  is finite, we may assume, by replacing  $(k_1, \dots, k_n)$  with a larger tuple if necessary, that  $S$  permutes the  $k_i$  and therefore that  $S$  also normalizes  $K'$ . Since  $H^0 = C_G(S)^0$ , we have that  $H \cdot (k_1, \dots, k_n)$  is closed if and only if  $C_G(S) \cdot (k_1, \dots, k_n)$  is closed.

Let  $G'$  act on  $G^m$  by simultaneous conjugation. The symmetric group  $S_n$  acts naturally on  $G^m$ , and the  $G'$ -action commutes with this action. Set  $X = G^m/S_n$  and let  $\pi : G^m \rightarrow X$  be the natural map; the fibres of  $\pi$  are precisely the  $S_n$ -orbits (see [1, Sec. 2], for example). For any subgroup  $M$  of  $G'$  and any  $(g_1, \dots, g_n) \in G^m$ , we have that  $M \cdot (g_1, \dots, g_n)$  is closed in  $G^m$  if and only if  $M \cdot \pi((g_1, \dots, g_n))$  is closed in  $X$ . Put  $x = \pi((k_1, \dots, k_n))$ . Then  $x \in X^S$ . Assertions (a) and (b) now follow from Theorem 2.8 and from Theorem 5.4(a) and (b), respectively.  $\square$

*Remark 5.8.* The hypothesis that  $S$  is reductive in Proposition 5.7 is unnecessary. This is clear in characteristic 0 — all that is needed is that  $C_G(S)$  is reductive. In characteristic  $p > 0$ , it can be shown that **any** algebraic group  $S$  admits an ascending chain of finite subgroups  $S_1 \subseteq S_2 \subseteq \dots$  such that  $\bigcup_{i \geq 1} S_i$  is Zariski dense in  $S$ , so the argument of Proposition 3.7(a) holds for non-reductive  $S$  as well.

The next corollary is a generalization of [2, Cor. 3.21]. Note that the hypotheses on  $S$  in Corollary 5.9 are satisfied if  $S$  is linearly reductive.

**Corollary 5.9.** *Let  $S$  be an algebraic group acting on  $G$  by automorphisms. Suppose that  $S$  acts semisimply on  $\mathfrak{g}$  and  $\mathfrak{c}_{\mathfrak{g}}(S) = \text{Lie } C_G(S)$ . Let  $K \subseteq H$  be subgroups of  $G$  such that  $S$  normalizes  $K$  and  $H^0 = C_G(S)^0$ . Then  $H$  is reductive and  $K$  is  $G$ -completely reducible if and only if  $K$  is  $H$ -completely reducible.*

*Proof.* By Proposition 3.7(a),  $C_G(S)$  is reductive. So  $H$  is also reductive. We can now assume that  $\text{char } k = p > 0$ . By the argument in the proof of Proposition 3.7(a), we can assume that  $S$  is finite. Then we put  $G' = G \rtimes S$  and obtain, as in that proof, that  $S$  is  $G'$ -cr and that  $C_G(S)$  is  $G$ -cr. Now the assumptions of Proposition 5.7(a) and (b) are satisfied.  $\square$

## 6. CENTRALIZERS AND NORMALIZERS

In this section we continue the theme of Section 5, looking at the special case when  $S$  is a subgroup of  $G$  acting on  $G$  by inner automorphisms. The extra restriction on  $S$  allows us to consider subgroups  $H$  sitting between  $C_G(S)^0$  and  $N_G(S)$ . The following result gives a criterion for  $K$  to be  $H$ -cr; it generalizes [3, Prop. 3.9] and the argument we use is very similar.

**Proposition 6.1.** *Suppose  $S$  is a  $G$ -completely reducible subgroup of  $G$ , and suppose  $H$  is a subgroup of  $G$  such that  $C_G(S)^0 \subseteq H \subseteq N_G(S)$ . Let  $K$  be a subgroup of  $H$ . Then:*

- (a)  $HS$  is reductive.
- (b) *The following are equivalent:*
  - (i)  $KS$  is  $G$ -completely reducible;
  - (ii)  $KS$  is  $HS$ -completely reducible;
  - (iii)  $KS$  is  $N_G(S)$ -completely reducible;
  - (iv)  $KS/S$  is  $N_G(S)/S$ -completely reducible.
- (c) *Suppose that  $H$  is reductive. Let  $\psi$  be the canonical projection from  $H$  to  $H/C_G(S)^0$ . Then  $K$  is  $H$ -completely reducible if and only if  $KS/S$  is  $N_G(S)/S$ -completely reducible and  $\psi(K)$  is  $H/C_G(S)^0$ -completely reducible.*

*Proof.* (a). We have  $C_G(S)^0 S^0 = N_G(S)^0$ , by [20, Lem. 6.8]. As  $C_G(S)^0 \subseteq H$ , we have  $(HS)^0 = N_G(S)^0$ . Now  $N_G(S)$  is reductive by [2, Prop. 3.12], so  $HS$  is reductive.

(b). By part (a),  $(HS)^0 = N_G(S)^0$ , so  $HS$  is a finite-index subgroup of  $N_G(S)$ . The equivalence of (ii) and (iii) now follows from [3, Prop. 2.12]. The equivalence of (i) and (iii) follows from [3, Cor. 3.3], and the equivalence of (iii) and (iv) from [3, Thm. 3.4].

(c). Now suppose that  $H$  is reductive. The subgroups  $H \cap S$  and  $(H \cap S)^0$  are normal in  $H$ . Let  $\pi$  be the canonical projection from  $H$  to  $H/(H \cap S)^0$ . We have a homomorphism  $\pi \times \psi$  from  $H$  to  $H/(H \cap S)^0 \times H/C_G(S)^0$ . If  $h \in \ker(\pi \times \psi)$  then  $h \in (H \cap S)^0 \cap C_G(S)^0 \subseteq S^0 \cap C_G(S)^0 \subseteq Z(S^0)$ , so  $\ker(\pi \times \psi)^0$  is a torus, so  $\pi \times \psi$  is non-degenerate. By [2, Lem. 2.12(i) and (ii)],  $K$  is  $H$ -cr if and only if  $\pi(K)$  is  $H/(H \cap S)^0$ -cr and  $\psi(K)$  is  $H/C_G(S)^0$ -cr. To finish the proof, it is enough to show that  $\pi(K)$  is  $H/(H \cap S)^0$ -cr if and only if  $KS/S$  is  $N_G(S)/S$ -cr. The natural map from  $H/(H \cap S)^0 \rightarrow HS/S$  is an isogeny, so  $\pi(K)$  is  $H/(H \cap S)^0$ -cr if and only if  $KS/S$  is  $HS/S$ -cr, by [2, Lem. 2.12(ii)]. As  $HS/S$  is a finite-index subgroup of  $N_G(S)/S$ ,  $KS/S$  is  $HS/S$ -cr if and only if  $KS/S$  is  $N_G(S)/S$ -cr [3, Prop. 2.12]. This completes the proof.  $\square$

**Corollary 6.2.** *Let  $G, H, S$  and  $K$  be as in Proposition 6.1. Suppose that  $(H \cap S)^0$  is a torus. Then  $H$  is reductive and the following are equivalent:*

- (i)  $K$  is  $H$ -completely reducible;
- (ii)  $KS$  is  $G$ -completely reducible.
- (iii)  $KS$  is  $HS$ -completely reducible;
- (iv)  $KS$  is  $N_G(S)$ -completely reducible;
- (v)  $KS/S$  is  $N_G(S)/S$ -completely reducible;

*Proof.* Since  $N_G(S)^0 = C_G(S)^0 S^0$ , and  $C_G(S)^0 \subseteq H \subseteq N_G(S)$ , we have  $H^0 = C_G(S)^0 (H \cap S)^0$ , so  $H$  is reductive. Moreover,  $H^0/C_G(S)^0$  is a quotient of  $(H \cap S)^0$ , which is a torus by hypothesis. Hence  $H/C_G(S)^0$  is a finite extension of a torus, which implies that any subgroup of  $H/C_G(S)^0$  is  $H/C_G(S)^0$ -cr. Thus  $K$  is  $H$ -cr if and only if  $KS/S$  is  $N_G(S)/S$ -cr, by Proposition 6.1(c). The result now follows from Proposition 6.1(b).  $\square$

*Remarks 6.3.* (i). Suppose  $H$  is reductive. If  $K$  is  $H$ -cr, then Proposition 6.1(c) says that  $KS/S$  is  $N_G(S)/S$ -cr, whence  $KS$  is  $G$ -cr, by Proposition 6.1(b). The converse, however, is not true in this generality; just take  $S = H = G$  and  $K$  a non- $G$ -cr subgroup of  $G$ .

(ii). Corollary 6.2 holds in particular if  $S$  is linearly reductive, since then the condition that  $(H \cap S)^0$  is a torus is automatic. However, Example 7.19 shows that even when  $S$  is linearly reductive, the situation for subgroups of  $N_G(S)$  is not as straightforward as for subgroups of  $C_G(S)$  (cf. Corollary 5.9).

## 7. AN IMPORTANT EXAMPLE

We consider a collection of important examples, which serve to illustrate many of the points raised in the previous sections. Throughout this section, we suppose that  $p = 2$  and let  $G$  be a simple group of type  $G_2$ . We fix a maximal torus  $T$  and a Borel subgroup  $B$  of  $G$  with  $T \subseteq B$ . Let  $\Psi$  be the set of roots of  $G$  with respect to  $T$ . We fix a base  $\Sigma = \{\alpha, \beta\}$  for the set  $\Psi^+$  of positive roots with respect to  $B$ , where  $\alpha$  is short and  $\beta$  is long. The positive roots are  $\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta$  and  $3\alpha + 2\beta$ . For each root  $\gamma$ , we choose an isomorphism  $\kappa_\gamma: k \rightarrow U_\gamma$  and set  $s_\gamma = \kappa_\gamma(1)\kappa_{-\gamma}(-1)\kappa_\gamma(1)$  (cf. [11, 32.3]). Then  $s_\gamma$  represents

the reflection corresponding to  $\gamma$  in the Weyl group  $N_G(T)/T$  of  $G$ . Since  $p = 2$ , the order of  $s_\gamma$  is 2 for every  $\gamma \in \Psi$ .

We use various equations from [11, 33.5]. Some are reproduced below. For brevity, we do not give the commutation relations between the root subgroups: these are the equations of the form

$$\kappa_\gamma(a)\kappa_{\gamma'}(b) = \kappa_{\gamma'}(b)\kappa_\gamma(a)g,$$

where  $g$  is a product of elements of the form  $\kappa_{\gamma''}(p_{\gamma''}(a, b))$  over certain roots  $\gamma''$ , each  $p_{\gamma''}$  being a monomial in  $a$  and  $b$ . (Recall, however, that  $U_\gamma$  and  $U_{\gamma'}$  commute if no positive integral combination of  $\gamma$  and  $\gamma'$  is a root.) We refer to these equations collectively below as “the CRs”.

Since  $G$  is simply connected, we have  $G_\gamma \cong \mathrm{SL}_2(k)$  for every  $\gamma \in \Psi$ , by Lemma 2.2.

We have

$$(7.1) \quad \langle \alpha, \alpha^\vee \rangle = 2, \quad \langle \beta, \alpha^\vee \rangle = -3, \quad \langle \alpha, \beta^\vee \rangle = -1, \quad \langle \beta, \beta^\vee \rangle = 2$$

(see [11, 32.3]; note that  $\langle \beta, \alpha \rangle$  in Humphreys’ notation coincides with  $\langle \beta, \alpha^\vee \rangle$ ). This yields

$$(7.2) \quad \langle \alpha + \beta, \alpha^\vee \rangle = -1, \quad \langle 2\alpha + \beta, \alpha^\vee \rangle = 1, \quad \langle 3\alpha + \beta, \alpha^\vee \rangle = 3, \quad \langle 3\alpha + 2\beta, \alpha^\vee \rangle = 0.$$

We have

$$s_\alpha \cdot \alpha = -\alpha, \quad s_\alpha \cdot \beta = 3\alpha + \beta,$$

and it follows that

$$(7.3) \quad s_\alpha \cdot \alpha^\vee = -\alpha^\vee, \quad s_\alpha \cdot \beta^\vee = \alpha^\vee + \beta^\vee.$$

We need to know how  $s_\alpha$  acts on the  $U_\gamma$ . We can choose the homomorphisms  $\kappa_\gamma$  so that  $s_\alpha$  maps each  $U_\gamma$  to  $U_{s_\alpha \cdot \gamma}$  by conjugation in the following way (see [11, 33.1 and 33.5]):

$$(7.4) \quad s_\alpha \kappa_\beta(a) s_\alpha = \kappa_{3\alpha+\beta}(a), \quad s_\alpha \kappa_{\alpha+\beta}(a) s_\alpha = \kappa_{2\alpha+\beta}(a), \quad s_\alpha \kappa_{2\alpha+\beta}(a) s_\alpha = \kappa_{\alpha+\beta}(a), \\ s_\alpha \kappa_{3\alpha+\beta}(a) s_\alpha = \kappa_\beta(a), \quad s_\alpha \kappa_{3\alpha+2\beta}(a) s_\alpha = \kappa_{3\alpha+2\beta}(a),$$

and we can choose  $0 \neq e_\gamma \in \mathfrak{u}_\gamma$  for each positive root  $\gamma$  such that the adjoint action of  $s_\alpha$  on the  $e_\gamma$  is given by

$$(7.5) \quad \mathrm{Ad} s_\alpha(e_\beta) = e_{3\alpha+\beta}, \quad \mathrm{Ad} s_\alpha(e_{\alpha+\beta}) = e_{2\alpha+\beta}, \quad \mathrm{Ad} s_\alpha(e_{2\alpha+\beta}) = e_{\alpha+\beta}, \\ \mathrm{Ad} s_\alpha(e_{3\alpha+\beta}) = e_\beta, \quad \mathrm{Ad} s_\alpha(e_{3\alpha+2\beta}) = e_{3\alpha+2\beta}.$$

Using Eqn. (7.4) and the CRs, we get

$$(7.6) \quad s_\alpha \kappa_\beta(a) \kappa_{\alpha+\beta}(a') \kappa_{2\alpha+\beta}(b) \kappa_{3\alpha+\beta}(b') \kappa_{3\alpha+2\beta}(c) s_\alpha \\ = \kappa_\beta(b') \kappa_{\alpha+\beta}(b) \kappa_{2\alpha+\beta}(a') \kappa_{3\alpha+\beta}(a) \kappa_{3\alpha+2\beta}(ab' + a'b + c).$$

Let

$$L := \langle G_\alpha \cup T \rangle.$$

Let  $P$  be the parabolic subgroup of  $G$  that contains  $B$  and has  $L$  as a Levi subgroup. The roots of  $R_u(P)$  with respect to  $T$  are  $\beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$ .

Let  $S$  be the torus  $\alpha^\vee(k^*)$ ; we have  $S = G_\alpha \cap T$ . Fix  $t \in S$  such that  $t$  has order 3. By Eqns. (7.1) and (7.2),

$$(7.7) \quad t \text{ acts trivially on } U_\beta, U_{3\alpha+\beta}, U_{3\alpha+2\beta}, \mathfrak{u}_\beta, \mathfrak{u}_{3\alpha+\beta}, \mathfrak{u}_{3\alpha+2\beta}, \text{ and} \\ t \text{ acts non-trivially on } U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}, \mathfrak{u}_\alpha, \mathfrak{u}_{\alpha+\beta}, \mathfrak{u}_{2\alpha+\beta}.$$

Set

$$H := \langle \{s_\alpha, t\} \rangle \subseteq G_\alpha.$$

Note that  $H \cong S_3$ . Since  $G_\alpha \cong \mathrm{SL}_2(k)$  (Lemma 2.2),  $\alpha^\vee$  is an isomorphism from  $k^*$  onto  $G_\alpha \cap T$ . Set

$$z := d\alpha^\vee(1) \in \mathfrak{g}_\alpha = \mathrm{Lie} G_\alpha$$

(where we regard 1 as an element of the tangent space  $T_1(k^*) \cong k$ ); then  $z \neq 0$  and  $k \cdot z$  is the centre of  $\mathfrak{g}_\alpha$ . In particular,  $H$  centralizes  $z$ . If  $\gamma \in \Psi$ , then

$$(7.8) \quad U_\gamma \text{ centralizes } z \iff 2 \text{ divides } \langle \gamma, \alpha^\vee \rangle.$$

By the CRs,  $G_\alpha$  commutes with  $G_{3\alpha+2\beta}$ . The multiplication map  $G_\alpha \times G_{3\alpha+2\beta} \rightarrow G_\alpha G_{3\alpha+2\beta}$  is bijective because  $G_\alpha \cap G_{3\alpha+2\beta}$ , being a proper normal subgroup of  $G_\alpha$ , must be trivial, but this map is not an isomorphism. For  $\mathfrak{z}(\mathfrak{g}_{3\alpha+2\beta}) \subseteq \mathrm{Lie}(G_{3\alpha+2\beta} \cap T)$ , so we have  $\mathfrak{z}(\mathfrak{g}_{3\alpha+2\beta}) \subseteq \mathfrak{c}_\mathfrak{t}(G_\alpha) \subseteq \mathfrak{c}_\mathfrak{t}(H)$ , which equals  $k \cdot z$ , so  $\mathfrak{g}_\alpha \cap \mathfrak{g}_{3\alpha+2\beta}$  is non-empty. In fact  $\mathfrak{g}_\alpha \cap \mathfrak{g}_{3\alpha+2\beta} = k \cdot z$ , because  $k \cdot z$  is the unique proper non-zero normal subalgebra of  $\mathrm{Lie} G_\alpha = \mathfrak{sl}_2(k)$ .

Now set

$$M := G_\alpha G_{3\alpha+2\beta}.$$

Note that  $T \subseteq M$  and  $M$  is a semisimple maximal rank subgroup of  $G$  of type  $\tilde{A}_1 A_1$ . Consequently,  $M$  is  $G$ -ir. Moreover, we have

$$C_G(M) = \ker(\alpha) \cap \ker(3\alpha + 2\beta) = \ker(\alpha) \cap \ker(2\beta) = \ker(\alpha) \cap \ker(\beta) = Z(G) = \{1\}.$$

It is easy to see that  $\mathfrak{z}(\mathfrak{m}) = k \cdot z$ .

**Lemma 7.9.** *With the notation as above, we have*

- (a)  $(G, M)$  is a reductive pair;
- (b)  $N_G(M) = M$ ;
- (c)  $M = C_G(z)$ .

*Proof.* (a). Since  $\Psi(M)$  is a closed subsystem of  $\Psi$ , this follows immediately from Lemma 3.9.

(b). Let  $g \in N_G(M)$ . Without loss of generality, we can assume that  $g$  normalizes  $T$ , so  $g$  permutes  $\Psi(M) = \{\pm\alpha, \pm(3\alpha+2\beta)\}$ . As  $\alpha$  is short and  $3\alpha+2\beta$  is long,  $g$  must normalize  $G_\alpha$  and  $G_{3\alpha+2\beta}$ . But  $G_\alpha$  and  $G_{3\alpha+2\beta}$  are rank one groups, so they have no outer automorphisms. Since  $C_G(M) = \{1\}$ , we therefore have that  $g \in M$ , as required.

(c). Since  $C_G(z)^0 \supseteq M$  and  $M$  is  $G$ -ir,  $C_G(z)^0$  is  $G$ -ir, so  $C_G(z)^0$  is reductive. This implies that  $C_G(z)^0$  is generated by the root subgroups that it contains together with  $T$ . It follows from Eqns. (7.1), (7.2) and (7.8) that  $C_G(z)^0 = M$ . Hence  $C_G(z) \subseteq N_G(C_G(z)^0) = N_G(M) = M$ , by part (b). Thus  $C_G(z) = M$ .  $\square$

**Lemma 7.10.** *With the notation as above, we have*

- (a)  $H$  is  $G$ -completely reducible and  $M$ -completely reducible;
- (b)  $C_G(H) = G_{3\alpha+2\beta}$  and  $N_G(H) = H G_{3\alpha+2\beta}$ .

*Proof.* (a). It is easily checked that  $H$  is not contained in any Borel subgroup of  $L$ , so  $H$  is  $L$ -ir. Now  $L$  is a Levi subgroup both of  $G$  and of  $M$ , so  $H$  is both  $G$ -cr and  $M$ -cr by [2, Cor. 3.22].

(b). Since  $H$  is  $G$ -cr,  $C_G(H)$  is  $G$ -cr by [2, Cor. 3.17], so  $C_G(H)^0$  is reductive. Now  $C_G(H)^0$  cannot have rank 2, because  $H$  is not centralized by any maximal torus of  $G$ . Hence  $C_G(H)^0$  must be equal to its rank 1 subgroup  $G_{3\alpha+2\beta}$ . By a similar argument,  $C_G(G_{3\alpha+2\beta})^0 = G_\alpha$ .

By [2, Thm. 3.10],  $G_{3\alpha+2\beta} = C_G(H)^0$  is  $G$ -cr. Let  $g \in N_G(H)$ . Then  $g$  normalizes  $C_G(H)^0 = G_{3\alpha+2\beta}$ , so  $g$  normalizes  $C_G(G_{3\alpha+2\beta})^0 = G_\alpha$ , so  $g$  normalizes  $M$ . Lemma 7.9(b) now implies that  $N_G(H) = N_M(H) = HG_{3\alpha+2\beta}$  and  $C_G(H) = C_M(H) = G_{3\alpha+2\beta}$ .  $\square$

The following example shows that the converse of Corollary 3.8 can fail, even when  $H$  is  $L$ -ir. It also gives a counterexample to the converse of the first assertion of Theorem 1.4.

**Proposition 7.11.** *The subgroup  $H$  is separable in  $L$ , but not in  $G$ .*

*Proof.* Since  $DL = G_\alpha \cong \mathrm{SL}_2(k)$ ,  $L$  is isomorphic either to  $\mathrm{GL}_2(k)$  or to  $\mathrm{SL}_2(k) \times k^*$ . To rule out the latter case, it's enough to show that  $s_\alpha$  acts non-trivially on  $\mathfrak{t}$ ; this follows from Eqn. (7.3). It now follows easily that  $\mathfrak{c}_\mathfrak{l}(H) = k \cdot z$ . Now  $z$  is tangent to  $\mathrm{Lie} Z(L)$  as  $L \cong \mathrm{GL}_2(k)$ . We deduce that  $H$  is separable in  $L$ .

It follows from Eqns. (7.5) and (7.7) that  $H$  centralizes  $e_\beta + e_{3\alpha+\beta}$ . But  $C_G(H) = G_{3\alpha+2\beta}$  by Lemma 7.10(b), so  $e_\beta + e_{3\alpha+\beta}$  is not tangent to  $\mathrm{Lie} C_G(H) = \mathfrak{g}_{3\alpha+2\beta}$ . Hence  $H$  is not separable in  $G$ .  $\square$

*Remark 7.12.* It is easily checked that if  $x \in \mathfrak{c}_\mathfrak{g}(H)$  is semisimple, then  $x \in \mathrm{Lie} C_G(H)$ . This cannot happen for a  $G$ -irreducible and non-separable subgroup of  $G$ ; cf. the proof of [2, Thm. 3.39].

For  $a \in k$ , set

$$u(a) = \kappa_\beta(a)\kappa_{3\alpha+\beta}(a).$$

The CRs yield

$$(7.13) \quad u(a)u(b) = u(a+b)\kappa_{3\alpha+2\beta}(ab)$$

and

$$(7.14) \quad u(a)^{-1} = u(a)\kappa_{3\alpha+2\beta}(a^2).$$

Define

$$H_a = u(a)Hu(a)^{-1}.$$

By Eqn. (7.7),  $u(a)$  centralizes  $t$ ; by Eqn. (7.6), we have  $u(a)s_\alpha u(a)^{-1} = s_\alpha \kappa_{3\alpha+2\beta}(a^2)$ . Hence  $H_a = \langle \{t, s_\alpha \kappa_{3\alpha+2\beta}(a^2)\} \rangle \subseteq M$  for all  $a \in k$ .

The following example shows that Theorem 1.3(b) can fail if we do not require  $K$  to be separable in  $G$  (recall that  $(G, M)$  is a reductive pair by Lemma 7.9(a)). This example also proves Theorem 1.5.

**Example 7.15.** Let  $(m_1, m_2, m_3) = (s_\alpha, t, \kappa_{3\alpha+2\beta}(1))$ . For  $a \in k$ , set

$$(m_1(a), m_2(a), m_3(a)) = u(a) \cdot (m_1, m_2, m_3).$$

Note that  $m_3(a) = m_3$  for all  $a \in k$ . By construction,  $(m_1(a), m_2(a), m_3(a))$  is  $G$ -conjugate to  $(m_1(b), m_2(b), m_3(b))$  for all  $a, b \in k$ . We now show that if  $a \neq b$ , then  $(m_1(a), m_2(a), m_3(a))$  is not  $M$ -conjugate to  $(m_1(b), m_2(b), m_3(b))$ . This shows that  $G \cdot (m_1, m_2, m_3)$  is an infinite union of  $M$ -conjugacy classes.

For  $a \in k$  let

$$\widehat{H}_a := \langle \{m_1(a), m_2(a), m_3(a)\} \rangle = \langle H_a \cup \{\kappa_{3\alpha+2\beta}(1)\} \rangle.$$

Note that  $\widehat{H}_0 = \langle \{m_1, m_2, m_3\} \rangle$  and that

$$C_G(\widehat{H}_0) = C_G(H) \cap C_G(\kappa_{3\alpha+2\beta}(1)) = G_{3\alpha+2\beta} \cap C_G(\kappa_{3\alpha+2\beta}(1)) = U_{3\alpha+2\beta},$$

by Lemma 7.10(b), so

$$C_G(\widehat{H}_a) = C_G(u(a)\widehat{H}_0u(a)^{-1}) = u(a)C_G(\widehat{H}_0)u(a)^{-1} = u(a)U_{3\alpha+2\beta}u(a)^{-1} = U_{3\alpha+2\beta}.$$

Suppose  $a \neq b$ . If  $(m_1(a), m_2(a), m_3(a))$  and  $(m_1(b), m_2(b), m_3(b))$  are  $M$ -conjugate, then there exists  $m \in M$  such that  $(mu(a)) \cdot (m_1, m_2, m_3) = u(b) \cdot (m_1, m_2, m_3)$ . We have  $mu(a)u(b)^{-1} \in C_G(\langle\{m_1(b), m_2(b), m_3(b)\}\rangle) = C_G(\widehat{H}_b) = U_{3\alpha+2\beta} \subseteq M$ , so  $u(a)u(b)^{-1} \in M$ . But  $u(a)u(b)^{-1} = u(a+b)\kappa_{3\alpha+2\beta}(ab+b^2)$  by Eqns. (7.13) and (7.14), so  $u(a+b) \in M$ . So we must have  $u(a+b) = 1$ , whence  $a = b$ , as required.

These calculations show that even though  $(G, M)$  is a reductive pair,  $G \cdot (m_1, m_2, m_3) \cap M^3$  consists of an infinite union of  $M$ -conjugacy classes. Observe that this is consistent with Theorem 1.3(b); a similar calculation to the one in the proof of Proposition 7.11 shows that  $\widehat{H}_0$  is not separable in  $G$ .

Our next example shows that the second assertion of Theorem 1.4 is false with the separability assumption on  $K$  removed, even though  $(G, M)$  is a reductive pair by Lemma 7.9(a). Since  $M$  is a regular subgroup of  $G$ , this is also a new example of the failure of [2, Thm. 3.26] in bad characteristic (compare [2, Ex. 3.45], which gives subgroups  $H' \subseteq M' \subseteq G'$ , with  $G'$  connected reductive,  $M'$  regular in  $G'$  such that  $H'$  is  $M'$ -cr, but not  $G'$ -cr).

**Proposition 7.16.** *Let  $a \in k^*$ . Then  $H_a$  is  $G$ -completely reducible but not  $M$ -completely reducible.*

*Proof.* Since  $H_a$  is  $G$ -conjugate to  $H$  and  $H$  is  $G$ -cr (Lemma 7.10(a)),  $H_a$  is  $G$ -cr. Let  $\lambda = \alpha^\vee + 2\beta^\vee \in Y(T)$ : then  $\langle \alpha, \lambda \rangle = 0$  and  $\langle \beta, \lambda \rangle = 1$ . It is clear that  $P = P_\lambda$  and  $L = L_\lambda$ . We have a homomorphism  $c_\lambda: P_\lambda \rightarrow L_\lambda$  as defined in Subsection 2.2. If  $h \in H$  then  $u(a)hu(a)^{-1} = hu$  for some  $u \in U_{3\alpha+2\beta} \subseteq \ker(c_\lambda)$ , so  $c_\lambda(H_a) = H$ . To prove that  $H_a$  is not  $M$ -cr, it suffices by [2, Lem. 2.17 and Thm. 3.1] to show that  $H_a$  and  $H$  are not  $M$ -conjugate. But this is the case since  $u(a) \notin M$  and  $N_G(H) \subseteq M$  (Lemma 7.10(b)), so we are done.  $\square$

*Remark 7.17.* Proposition 7.16 shows that part (c) of Theorem 1.3 fails without the separability hypothesis on  $K$ : for, by Theorem 2.8, the  $G$ -conjugacy class  $G \cdot (m_1(a), m_2(a))$  is closed in  $G \times G$  but  $M \cdot (m_1(a), m_2(a))$  is not.

The next example shows that Proposition 2.13 can fail if we allow  $H$  to be non- $G$ -cr.

**Example 7.18.** Let  $C = \{u(a) \mid a \in k\}$ . Let  $H' = \langle H \cup C \rangle$  and suppose that  $K$  is any reductive subgroup of  $G$  containing  $H'$ . Set  $y := e_\beta + e_{3\alpha+\beta}$ . Then  $y$  belongs to the tangent space  $T_1(C)$ , which is contained in  $\text{Lie } K$ . Now  $\mathfrak{u}_{3\alpha+2\beta}$  is centralized by  $U_\beta$  and  $U_{3\alpha+2\beta}$  and we have  $\text{Ad } \kappa_{3\alpha+\beta}(a)(e_\beta) = e_\beta + ae_{3\alpha+2\beta}$ ,  $\text{Ad } \kappa_\beta(a)(e_{3\alpha+\beta}) = e_{3\alpha+\beta} + ae_{3\alpha+2\beta}$  by the CRs, so  $\text{Ad } u(a)(y) = y$ . Hence  $y \in \mathfrak{c}_\mathfrak{k}(H')$ , but  $y$  is not tangent to  $C_K(H')$  since it is not tangent to  $C_G(H)$  (cf. the proof of Proposition 7.11).

Suppose that  $d\pi_K(y)$  is tangent to  $C_{K_{\text{ad}}}(\pi_K(H'))$ . By [5, Prop. IV.14.26], there exists a connected unipotent subgroup  $V$  of  $C_{K_{\text{ad}}}(\pi_K(H'))$  such that  $d\pi_K(y) \in \text{Lie } V$ . Note that  $V \subseteq (K_{\text{ad}})^0$  and  $\pi_K$  gives an isogeny from  $D(K^0)$  to  $\pi_K(D(K^0)) = (K_{\text{ad}})^0$ . Let  $U = (\pi_K^{-1}(V) \cap D(K^0))^0$ . Any semisimple element of  $U$  must belong to the finite group  $\ker \pi_K \cap D(K^0)$ , so  $U$  has only finitely many semisimple elements. Hence  $U$  is a unipotent group. Since  $\pi_K$  is an isogeny from  $D(K^0)$  to  $(K_{\text{ad}})^0$  and  $\ker d\pi_K$  consists of semisimple elements, it follows that the restriction of  $\pi_K$  is an isomorphism from  $U$  onto  $V$ . Hence there exists  $y' \in \text{Lie } U$



such that  $d\pi_K(y') = d\pi_K(y)$ . As  $\ker d\pi_K \subseteq \mathfrak{z}(\mathfrak{k})$  and  $y, y'$  are both nilpotent, we must have  $y = y'$ , so  $y \in \text{Lie } U$ .

Fix  $h \in H'$ . If  $u \in U$ , then  $\pi_K(h)\pi_K(u)\pi_K(h)^{-1} = \pi_K(u)$ , so we have  $huh^{-1} = cu$  for some  $c \in \ker \pi_K$ . The map  $u \mapsto huh^{-1}u^{-1}$  is a morphism from the connected set  $U$  to the finite set  $\ker \pi_K \cap D(K^0)$ , so  $huh^{-1}u^{-1} = 1$  for all  $h \in H'$  and all  $u \in U$ . Hence  $U \subseteq C_K(H')$ . This implies that  $y \in \text{Lie } C_K(H')$ , a contradiction. We conclude that  $d\pi_K(y)$  is not tangent to  $C_{K_{\text{ad}}}(\pi_K(H'))$ . Since  $d\pi_K(y) \in \mathfrak{c}_{\mathfrak{k}_{\text{ad}}}(\pi_K(H'))$ , it follows that  $\pi_K(H')$  is not separable in  $K_{\text{ad}}$ .

Here is a further example arising from this construction which relates to the discussions in Sections 3 and 6 (see in particular Remarks 3.10(ii) and 6.3(ii)).

**Example 7.19.** Recall that  $S$  is the torus  $\alpha^\vee(k^*)$  and therefore,  $(G, N_G(S))$  is a reductive pair by Proposition 3.7(b). We have that  $H_a \subseteq N_G(S)$ , since  $U_{3\alpha+2\beta}$  centralizes  $S$ . Let  $a \in k^*$ . We will show that  $H_a S$  is not  $G$ -cr, although  $H_a$  is (Proposition 7.16). Let  $\lambda \in Y(T)$  and  $c_\lambda$  be as in the proof of Proposition 7.16. Then

$$c_\lambda(H_a S) = \langle S \cup \{s_\alpha\} \rangle.$$

If  $H_a S$  lies in a Levi subgroup of  $P_\lambda$  then  $uH_a S u^{-1} \subseteq L_\lambda$  for some  $u \in R_u(P_\lambda)$ , so  $c_\lambda(H_a S) = uH_a S u^{-1}$ . Thus to show that  $H_a S$  is not  $G$ -cr, it is enough to show that  $\langle S \cup \{s_\alpha\} \rangle$  is not  $R_u(P_\lambda)$ -conjugate to  $H_a S$ . To see this, note that if  $u \in R_u(P_\lambda)$  with  $u\langle S \cup \{s_\alpha\} \rangle u^{-1} = H_a S$ , then  $uS u^{-1} = S$ , whence  $u$  centralizes  $S$  (since  $S$  normalizes  $R_u(P_\lambda)$ ). Since the centralizer of  $S$  in  $R_u(P_\lambda)$  is  $U_{3\alpha+2\beta}$  by Eqns. (7.1) and (7.2), we have  $u \in U_{3\alpha+2\beta}$ . But  $U_{3\alpha+2\beta}$  centralizes  $H_a S$ , so  $\langle S \cup \{s_\alpha\} \rangle = H_a S$ , a contradiction.

Since  $H_a S$  is not  $G$ -cr,  $H_a S$  is not  $N_G(S)$ -cr (Corollary 6.2). Since the canonical projection  $f: N_G(S) \rightarrow N_G(S)/S$  is non-degenerate and  $f(H_a) = f(H_a S)$ , it follows from [2, Lem. 2.12(ii)] that  $H_a$  is not  $N_G(S)$ -cr. Thus we have an example of a subgroup  $H_a \subseteq N_G(S)$ , with  $S$  linearly reductive — in fact, a torus — such that  $H_a$  is  $G$ -cr, but not  $N_G(S)$ -cr.

Finally, we consider a rationality question. Let  $k_0$  be a subfield of an algebraically closed field  $k_1$ . Suppose that  $G_1$  is a reductive algebraic group defined over  $k_0$ . If  $K_1$  is a subgroup of  $G_1$  defined over  $k_0$ , then we say that  $K_1$  is  $G_1(k_0)$ -completely reducible if whenever  $P_1$  is an R-parabolic subgroup of  $G_1$  such that  $K_1 \subseteq P_1$  and  $P_1$  is defined over  $k_0$ , there exists an R-Levi subgroup  $L_1$  of  $P_1$  such that  $K_1 \subseteq L_1$  and  $L_1$  is defined over  $k_0$  (see [2, Sec. 5] for further details). In particular,  $K_1$  is  $G_1$ -cr if and only if  $K_1$  is  $G_1(k_1)$ -cr. An example of McNinch [2, Ex. 5.11] shows that if  $K_1$  is  $G_1(k_0)$ -cr, then  $K_1$  need not be  $G_1(k_1)$ -cr. The next example shows that the converse can also happen.

**Example 7.20.** Suppose that  $k_0$  is a subfield of  $k$  such that  $G$  is defined over  $k_0$  and  $k_0$ -split. We can assume that  $T$  is chosen so that  $T$  is defined over  $k_0$  and  $k_0$ -split and so that for every  $\gamma \in \Psi$ , the homomorphisms  $\gamma: T \rightarrow k^*$ ,  $\gamma^\vee: k^* \rightarrow T$  and  $\kappa_\gamma: k \rightarrow U_\gamma$  are defined over  $k_0$ . Now suppose that  $k/k_0$  is not separable. Then  $k_0$  is not perfect, so there exists  $a \in k \setminus k_0$  such that  $a^2 \in k_0$ . Consider  $H_a = \langle \{t, s_\alpha \kappa_{3\alpha+2\beta}(a^2)\} \rangle$ . Then  $H_a$  is defined over  $k_0$ , since  $H_a$  is a finite subgroup of  $G(k_0)$ . As  $H_a$  is  $G$ -cr (Proposition 7.16),  $H_a$  is  $G(k)$ -cr. We show that  $H_a$  is not  $G(k_0)$ -cr.

Recall that  $P = P_\lambda$  and  $L = L_\lambda$ , where  $\lambda$  is as in the proof of Proposition 7.16. Since  $G$  and  $T$  are split,  $P$  and  $L$  are defined over  $k_0$  (cf. [5, Props. V.20.4 and V.20.5]). Suppose there exists a Levi subgroup  $L'$  of  $P$  such that  $L'$  is defined over  $k_0$  and  $H_a$  is contained in  $L'$ . By [5, Prop. V.20.5], there exists  $u \in R_u(P)(k_0)$  such that  $L' = uL u^{-1}$ . Then  $u^{-1}H_a u$ ,

$u(a)^{-1}H_a u(a) \subseteq L$ , so  $c_\lambda(u^{-1}gu) = u^{-1}gu$  and  $c_\lambda(u(a)^{-1}gu(a)) = u(a)^{-1}gu(a)$  for all  $g \in H_a$ . Since  $c_\lambda(u) = c_\lambda(u(a)) = 1$ , we have

$$u^{-1}gu = c_\lambda(u^{-1}gu) = c_\lambda(g) = c_\lambda(u(a)(u(a)^{-1}gu(a))u(a)^{-1}) = c_\lambda(u(a)^{-1}gu(a)) = u(a)^{-1}gu(a)$$

for all  $g \in H_a$ . Thus  $u = u(a)c$  for some  $c \in C_G(H) \cap R_u(P) = G_{3\alpha+2\beta} \cap R_u(P) = U_{3\alpha+2\beta}$  (using Lemma 7.10(b)): say  $u = \kappa_\beta(a)\kappa_{3\alpha+\beta}(a)\kappa_{3\alpha+2\beta}(y)$  for some  $y \in k$ . But then  $u \notin P(k_0)$  since  $a \notin k_0$ , a contradiction. Thus no such  $L$  can exist, and  $H_a$  is not  $G(k_0)$ -cr.

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