European Financial Management, 2012
doi: 10.1111/j.1468-036X.2012.00642.x

Strategic Asset Allocation and the Role of Alternative Investments

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Abstract

We introduce a framework for strategic asset allocation with alternative investments. Our framework uses a quantifiable risk preference parameter, λ, instead of a utility function. We account for higher moments of the return distributions and approximate best-fit distributions. Thus, we replace the empirical return distributions with two normal distributions. We then use these in the strategic asset allocation. Our framework yields better results than Markowitz's framework. Furthermore, our framework better manages regime switches that occur during crises. To test the robustness of our results, we use a battery of robustness checks and find stable results.

Keywords: *alternative investments*; *higher moments*; *strategic asset allocation*

JEL Classification: *G*2, *G*12, *G*31

We are very grateful to an anonymous referee for many helpful comments and to the editor John Doukas for very useful suggestions. Moreover, the paper has benefitted from comments by Greg N. Gregoriou, Dieter G. Kaiser, Harry M. Kat, Lutz Johanning, Christian Koziol, Rainer Lauterbach, Michael McDonald, Mark Mietzner, Juliane Proelss, and Maximilian Trossbach as well as the participants of the EFM Alternative Investments Conference 2011 (Toronto), International Business Research Conference (8th Annual Meeting, Dubai), European Financial Management Association (17th Annual Meeting, Athena), Campus for Finance 2009 (Vallendar), and Midwest Finance Association (58th Annual Meeting, Chicago) for helpful comments and suggestions. All remaining errors are our own.

1. Introduction

Alternative investment funds, which exceeded US$9 trillion worldwide in 2009, have become increasingly important for the portfolios of institutional investors. This paper proposes a framework for strategic asset allocation that is able to incorporate the special characteristics of alternative investments.

If investors want to build exposure to alternative investments, they must decide on their strategic asset allocation. Because strategic asset allocation explains most of a portfolio's return variability, it is the major determinant of investment performance and the most critical decision in the investment process (Hoernemann *et al.*, 2005[1]). Use of an appropriate strategic asset model is even more important when alternative investments are considered.

Alternative investments typically suffer from data biases due to appraisal smoothing and stale pricing. Furthermore, return distributions of alternative investments have significantly higher moments (skewness and kurtosis) which the standard deviation does not cover. Thus, every standard method for portfolio optimisation employing alternative investments is likely to be inaccurate (see, e.g., Fung and Hsieh, 1997; Fung and Hsieh, 2001; Martin, 2001; Brooks and Kat, 2002; Popova *et al.*, 2003; Agarwal and Naik, 2004; Jondeau and Rockinger, 2006). Furthermore, institutional investors have different objective functions than individual investors (Morton *et al.*, 2006; Cumming and Johan, 2006; Cumming *et al.*, 2011; Groh and von Liechtenstein, 2011; Nielsen, 2011).

Therefore, our framework corrects for data biases in the return time series of some alternative investments (private equity and hedge funds). We use a mixture of normal methods to replace the empirical return distributions, which often exhibit skewness and positive excess kurtosis, with two normal distributions to approximate a best-fit distribution. This approach ensures that the best-fit return distributions exhibit higher moments close to their empirical pendants. We then use the best-fit distributions in the optimisation procedure. To derive the strategic asset allocation, we apply a goal function to examine real investor preferences for risk aversion. Our investors' objective function maximises the probability of outperforming some benchmark return while minimising the probability of underperforming another benchmark.

The previous literature on asset allocation with alternative investments focuses on the effects of adding one alternative investment class to a traditional mixed-asset portfolio. It associates the addition of hedge funds with positive effects on portfolio performance (see, e.g., Amin and Kat, 2002; Lhabitant and Learned, 2002; Amin and Kat, 2003; Gueyie and Amvella, 2006; Kooli, 2007). In addition, findings assign positive effects for private equity (see, e.g., Chen *et al.*, 2002; Schmidt, 2004; Ennis and Sebastian, 2005). The literature also finds that real estate investment trusts (REITs) can increase portfolio performance (see, e.g., National Association of Real Estate Investment Trusts, hereafter NAREIT, 2002; Hudson-Wilson *et al.*, 2004; Chen *et al.*, 2005; Lee and Stevenson, 2005; Chiang and Ming-Long, 2007).

Huang and Zhong (2011) are a notable exception to this literature. Their work, which is the most similar to ours, shows that commodities, REITs, and treasury inflation-protected

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[1] The authors present an alternative to the often-cited studies of Brinson *et al.* (1986, 1991). They use a slightly different framework and cover a longer time horizon. They also include alternative assets and use synthetic portfolios.

securities (TIPS) provide positive diversification benefits to investor portfolios. For the case of commodities, there is no consensus on whether or not adding them to portfolios increases investor value. Gorton and Rouwenhorst (2006) and Conover *et al.* (2010) find positive effects from their addition. In contrast, Erb and Harvey (2006) and Daskalaki and Skiadopoulos (2011) find no such effects.

To the best of our knowledge, this paper is the first that (1) incorporates a variety of alternative investments (e.g., commodities, private equity, hedge funds, and real estate) and traditional investments (stocks and government bonds), (2) adjusts risk–return profiles to account for data biases, (3) uses a strategic asset allocation model that is flexible enough to capture the risk–return profile adequately, and (4) incorporates real investor preferences.

Our general findings are that only defensive portfolios use stocks of large US firms as part of the traditional asset classes. In all portfolios, however, bonds are of great importance and are added up to the maximum allocation restriction, and emerging markets gain in relevance with decreasing risk aversion. For alternative investments, REITs play a major role in portfolios as risk aversion decreases. In contrast, commodities have comparably stable medium allocations in all portfolios. Hedge-fund allocations are comparable to bond allocations because they are integrated into virtually all optimal portfolios with the maximum portfolio allocation. By comparison, private equity plays a very important role, especially in defensive portfolios. Furthermore, we find that our asset allocation consistently outperforms portfolios formed with the standard Markowitz approach in out-of-sample Monte Carlo simulations, independent of the risk-adjusted performance measure used.

Portfolio optimisation inherently requires making several choices that influence the resulting asset allocation, such as the period considered, allocation restrictions, index selection, optimisation parameters, and the objective function. To test the validity of our strategic asset allocation approach, we apply seven robustness checks to identify the influence of these choices on our results. The first robustness check tests the sensitivity of our results against the background of the recent financial crisis. In spite of the financial crisis, the results for alternative investments are even stronger. Allocation restrictions do not alter our results. The cumulative weights for alternative investments remain stable for different values of the risk aversion parameter. We also end up with nearly identical allocations when allowing for time-varying correlations. Furthermore, our results remain stable when using different indices representing the various asset classes. Similarly, using different parameters in the mixture of normal distributions does not affect our results. Finally, using an objective function based on value at risk does not change our results.

In conclusion, we find that alternative investments are important for the strategic asset allocation of institutional investors such as endowments, family offices, pension funds, and high net worth individuals with sufficient time horizons and investment capital. However, not all alternative investment classes are of equal importance. Alternative investments are not appropriate as substitutes for traditional asset classes and may better serve as complements to achieving the desired risk–return profiles.

The rest of this paper proceeds as follows: Section 2 describes the data set and the correction of data biases. Section 3 presents the optimisation procedure and the results. Section 4 contains our robustness checks. Section 5 discusses possible extensions to our approach. Section 6 concludes with a summary and discussion of the results.

2. Data Set

Since Markowitz's (1952) seminal paper on portfolio theory, the literature acknowledges that diversification can increase expected portfolio returns while reducing volatility. However, investors should not blindly add another asset class to their portfolios without carefully considering its properties in the context of their portfolios. A naïvely chosen allocation to the newly added asset class may not improve the risk–return profile, and can even worsen it. This raises the questions of whether alternative investments really improve the risk-adjusted performance of a mixed-asset portfolio and whether they should be included in the strategic asset allocation.

This analysis uses the following indices as proxies for each asset class: two traditional asset classes (proxy indices in parentheses) – stocks (S&P500 Total Return Index and MSCI Emerging Markets Total Return Index) and government bonds (JP Morgan US Government Bonds Total Return Index) and four alternative assets: private equity, subdivided into buyouts (US Buyout) and venture capital (US Venture Capital),[2] commodities (S&P GSCI Commodity TR Index), hedge funds (Hedge Fund Research, Inc., or HFRI, Fund of Funds Composite),[3] and REITs (FTSE EPRA/NAREIT Total Return Index).[4] All time series in our investigation are on a monthly basis (except the private equity time series, based on quarterly data) with a January 1999 inception date, because all indices report data from this date on. The end date of the time series is December 2009.

Although the previously described indices are the most common for their asset classes, there exist other indices for different asset classes. Several indices can be used to represent private equity. These indices can be classified as listed or transaction-based private equity indices (for a discussion of the various index concepts, see Cumming *et al*., 2011). The LPX50 is the main representative of listed private equity indices, and CepreX Venture Capital is the main representative of transaction-based indices. For hedge funds, the Dow Jones Credit Suisse Hedge Fund Index is the main competitor of the HFRI Funds

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[2] Both indices are based on the Thomson Reuters VentureXpert database. We follow the approach of Cumming *et al.* (2011). For related work on venture capital, see Metrick and Yasuda (2011), Cumming and Johan, (2006, 2011), Groh and von Liechtenstein (2011), Nielsen (2011), Caselli *et al.* (2009), Hartmann-Wendels *et al*. (2011) and Ernst *et al.* (2012).

[3] We use an investable fund of hedge funds index as our proxy index, in contrast to standalone hedge funds, which have historically higher performances, for the following reasons. For the choice of all of our 'representative' asset class benchmarks we look for a 'market portfolio' that best describes the respective risk and return characteristics. In this context, we follow the argument by Fung and Hsieh (2000) that a fund of hedge funds represents typical investors in portfolios of hedge funds, generally with an available net-of-fees performance history. We strongly believe that if we want to estimate the investment experience of hedge funds, it is natural to examine the experience of hedge fund investors. Instead, when focusing on (non-investable) index data, we may suffer from such biases as liquidation bias, survivorship bias, attrition rate bias, and selection bias. For example, estimates for survivorship bias vary from 0.16% (Ackermann *et al*., 1999) to 6.22% (Liang, 2002) across different hedge fund styles and data vendors. For related work on venture capital, see Li and Kazemi (2007), Ding and Shawky (2007), Goltz *et al.* (2007) and Eling (2009).

[4] Table A-1 in Appendix A provides detailed descriptions of the proxy indices.

Table 1

Autocorrelation structure of the appraisal value-based private equity indices

This table shows the autocorrelation coefficients for the quarterly distribution of returns for the appraisal value-based private equity indices (US Buyout and US Venture Capital) based on Thomson Reuters VentureXpert database from January 1999 to December 2009 for lags 1 to 4. The boldface represents significance at the 95% level.

	Lag 1	Lag 2	Lag 3	Lag 4
US Buyout	**0.3561**	**0.2945**	0.2178	0.1903
US Venture Capital	**0.6153**	**0.4988**	**0.3897**	0.0559

Table 2

Autocorrelation structure of the monthly return distribution of all asset classes

This table shows the autocorrelation coefficients for the monthly return distributions of the S&P 500, MSCI Emerging Markets, JPM US Government Bonds, FTSE EPRA/NAREIT, S&P GSCI Commodity, HFRI Fund of Funds, US Buyout, and US Venture Capital from January 1999 to December 2009 for the monthly lags 1 to 12. The bold formatting represents significance at the 95% level.

Lags	S&P 500	MSCI Emerging Markets	JPM US Government Bonds	FTSE EPRA/ NAREIT	S&P GSCI Commodity	HFRI Fund of Funds
Lag 1	0.1008	0.2096	0.1236	0.0039	0.1762	0.0854
Lag 2	−0.0160	0.1845	0.0567	**−0.3224**	0.0963	0.1219
Lag 3	0.0195	0.0489	0.0858	0.1381	0.1258	0.0997
Lag 4	0.0260	−0.0176	−0.1206	**0.3031**	0.0171	−0.1228
Lag 6	0.0241	−0.0603	0.0542	−0.0707	0.0239	0.0451
Lag 7	−0.1282	−0.1060	−0.0681	**−0.2712**	−0.0079	0.0791
Lag 8	0.0900	0.0513	−0.0067	0.0636	−0.0608	0.0813
Lag 9	0.1304	0.0125	−0.1007	0.1748	−0.0189	**0.1839**
Lag 10	0.1732	0.0950	−0.0395	0.0012	−0.0385	**0.2078**
Lag 11	0.0184	0.0160	0.0989	**−0.2226**	0.0374	0.1185
Lag 12	−0.0435	−0.0097	0.0517	0.1047	0.1719	0.0352

of Funds Index. Finally, for commodities, the Rogers International Commodity Index is the main alternative to the S&P GSCI Commodities Index.[5]

Before we introduce the descriptive statistics of the asset classes considered, we need to discuss several potential biases that could distort the inherent risk–return profile. The sources of distortion are manifold. For instance, appraisal-based private equity indices exhibit distortion through smoothed returns resulting from deformation. Deformation can be to appraisal smoothing, quarterly data availability, and/or stale pricing and statistically cause a positive autocorrelation (see Table 1). These relations are common among illiquid investments such as private equity and individual hedge fund strategies (see Table 2 and Avramov *et al*., 2008). They typically arise due to irregular price determination, long periods between price determinations, and the use of book

[5] The asset allocation results for different indices are discussed in Section 4.

value instead of market prices (see, e.g., Geltner, 1991; Gompers and Lerner, 1997). The resulting positive autocorrelation causes a significant underestimation of risk and market exposure (Asness *et al*., 2001) due to the smoothed returns when naïvely using raw data.

Table 1 shows that private equity exhibits a significantly positive autocorrelation of 0.6153 in the first of four lags for US venture capital. In contrast, hedge funds do not show any significant autocorrelation in the first lags, since they are represented by a fund of funds index rather than by single hedge fund strategies. This highly positive autocorrelation makes it necessary to correct the private equity time series to adequately capture the risk–return profile of this asset class.

To adjust for appraisal smoothing, stale pricing, and illiquidity to obtain an unbiased data set, we desmooth the private equity time series by using the method of Getmansky *et al.* (2004) that incorporates the whole autocorrelation structure of the return distribution (the reasoning behind this method is given in Appendix C).[6] Thereafter, we rescale the private equity return series from quarterly to monthly data (see Cumming *et al*., 2011 for further details).

Furthermore, some scholars emphasise that hedge fund time series are subject to a considerable survivorship bias.[7] Because we use an investable fund of hedge funds index, its performance is not affected by any survivorship bias. Therefore, we do not conduct any adjustments.

Table 3 provides the descriptive statistics after adjusting for the aforementioned distortions of the risk–return profile.

The descriptive statistics presented in Table 3 show that risk, measured by standard deviation, of both private equity segments increases after the desmoothing of the returns. For US Buyout (US Venture Capital), the standard deviation increases by a factor of 1.79 (1.45). Note that emerging markets have the highest mean return (1.21%) but only the third highest standard deviation (6.96%), followed by REITs, with a mean return of 0.81% and a highest standard deviation of 7.30%.

The higher moments (skewness and kurtosis) are additional potential sources of risk. Hedge funds exhibit the lowest skewness, -0.519 (kurtosis 6.728), whereas REITs show the highest kurtosis, 13.162 (skewness -0.300), among all asset classes. Therefore, hedge funds and REITs show the most unfavorable higher-moment properties, because negative skewness and positive excess kurtosis indicate that the outliers are on the left side of the return distribution and occur more often than expected under the normal distribution (known as tail risk). The excess kurtosis for most asset classes is close to zero (except for venture capital).

Analysing the higher moments of the return distribution for the asset classes shows that some return distributions do not follow a normal distribution. The Jarque–Bera (1980) test rejects the null hypothesis of a normally distributed return distribution for REITs and venture capital at the 1% level. Thus, relying on a simple mean–variance framework and ignoring the higher moments does not adequately capture the risk–return profile. Failure to consider higher moments increases the probability of

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[6] This method improves Geltner's (1991) approach because the entire lag structure is considered simultaneously. In addition, there is no need for a desmoothing parameter (see (Byrne and Lee, 1995) for the problematic determination of the desmoothing parameter).

[7] However, most scholars usually estimate survivorship bias at 2–3% (see, e.g., Brown *et al*., 1999; Fung and Hsieh, 2000; Anson, 2006).

Table 3

Descriptive statistics from the monthly return distribution of all asset classes

This table shows the mean, monthly standard deviation, skewness, kurtosis, minimum, maximum, median, 25th percentile, 75th percentile, square root of lower partial moment 2 with threshold 0 (LPM), Conditional Value at Risk (CVaR) at the 95% confidence level, and the Maximum Drawdown (MaxDD) of the monthly return distributions of the S&P 500, MSCI Emerging Markets, JPM US Government Bonds, FTSE EPRA/NAREIT, S&P GSCI Commodity, HFRI Fund of Funds, US Buyout (original and desmoothed), and US Venture Capital (original and desmoothed) from January 1999 to December 2009. Private equity (US Buyout and US Venture Capital) return time series with significant autocorrelation are considered after an autocorrelation adjustment (using the method of Getmansky *et al.* (2004)). All indices are total return indices or earnings are retained. All discrete returns are converted into logarithmical returns. Finally, the assumption of a normal return distribution is proved via Jarque–Bera (1980) tests.

	S&P 500	MSCI Emerging Markets	JPM US Govern-ment Bonds	FTSE EPRA/ NAREIT	S&P GSCI Com-modity	HFRI Fund of Funds	US Buyout	US Buyout (de-smoothed)	US Venture Capital	US Venture Capital (de-smoothed)
Mean	0.05 %	1.21 %	0.33 %	0.81 %	0.73 %	0.33 %	0.31 %	0.32 %	0.42 %	0.43 %
Standard Deviation	5.11 %	6.96 %	2.99 %	7.30 %	7.07 %	3.14 %	1.83 %	3.27 %	3,70 %	5.37 %
Kurtosis	4.478	2.976	4.791	13.162	4.252	6.728	3.24	2.834	6,91	7.183
Skewness	−0.462	−0.315	−0.001	−0.300	−0.510	−0.519	−0.19	−0.135	1,63	1.441
Minimum	−14.14 %	−19.53 %	−8.24 %	−32.87 %	−22.66 %	−10.74 %	−4.49 %	−7.89 %	−5,33 %	−12.42 %
Maximum	10.62 %	16.88 %	9.46 %	28.93 %	18.03 %	9.32 %	4.48 %	8.32 %	14,83 %	23.01 %
Median	0.48 %	1.83 %	0.35 %	1.37 %	1.29 %	0.28 %	0.20 %	0.35 %	−0,09 %	−0.16 %
25^{th} Percentile	−2.94 %	−3.26 %	−1.76 %	−2.81 %	−3.63 %	−1.74 %	−0.70 %	−1.92 %	−1.38 %	−2.25 %
75^{th} Percentile 75 %	3.15 %	6.07 %	2.10 %	4.69 %	5.79 %	1.96 %	1.55 %	2.49 %	1.38 %	2.37 %
LPM	1.96 %	2.24 %	1.01 %	2.08 %	2.44 %	1.04 %	0.56 %	1.14 %	0.97 %	1.58 %
CVaR	−11.03 %	−14.06 %	−5.48 %	−17.96 %	−14.94 %	−6.02 %	−3.83 %	−6.77 %	−5.07 %	−8.78 %
MaxDD	61.58 %	56.08 %	24.51 %	69.36 %	62.39 %	24.18 %	34.33 %	43.83 %	63.29 %	69.85 %
Jarque–Bera	16.707***	2.189	17.646***	569.887***	14.337***	82.390***	1.085	0.556	142.293***	141.932***

The superscripts ***, **, and * indicate statistical significance at the 1 %, 5 %, and 10 % levels, respectively, based on monthly returns.

maintaining biased and suboptimal weight estimations, as well as underestimating tail losses.

Table 4 provides insight into the diversification potential of each asset class. Hedge funds have a high diversification potential because the correlation to all other asset classes is statistically not different from zero (except for private equity). Similar diversification potential applies to government bonds, which also have a correlation to all other asset classes statistically not different from zero (except for venture capital). It is worth noting that there is no significantly negative correlation between asset classes.

After reviewing the descriptive statistics of the return distributions, we cannot determine *a priori* that one asset class is a substitute for another. Therefore, we consider all the asset classes for the portfolio construction. To create optimal investor portfolios, our model considers the characteristics of the asset classes.

3. Methodology and Results

We have discussed the descriptive characteristics of the different alternative asset classes as well as potential biases. We also concentrated on correcting these biases from the raw return series and discussed their statistical properties. Some of the resulting return distributions are not normally distributed and exhibit skewness and excess kurtosis. For this reason, and assuming that investors do not have quadratic utility functions (therefore ignoring higher moments of the return distribution), a simple Markowitz (1952) mean–variance framework will likely end up with an inefficient portfolio composition and underestimation of tail risk.

To capture higher moments, the literature offers a number of alternative distributions to the normal distribution. The multivariate Student *t*-distribution is well suited for fat-tailed data, but it does not allow for asymmetry. The non-central multivariate t-distribution also has fat tails and is skewed; however, the skewness is linked directly to the location parameter, making it somewhat inflexible. The lognormal distribution has been used to model asset returns, but its skewness is a function of its mean and variance, not a separate parameter.

Thus, to capture higher moments of not normally distributed returns, we need a distribution that is flexible enough to fit the skewness and the kurtosis. We use a combination of two different geometric Brownian motions to generate a mixture of normal diffusions. The normal mixture distribution is an extension of the normal distribution and has successfully been applied in different research fields, and is used nowadays in the finance literature.

The idea of 'mixing' two distributions to approximate empirical distributions is not new. In statistics, a mixture model is a probabilistic model for representing the presence of subpopulations within an overall population, without requiring that an observed data set identify the subpopulation to which an individual observation belongs. Financial applications constantly used mixture models but, with the introduction of alternative ways to model jumps to incorporate crises in catastrophe models, their popularity has increased. They have been applied to such problems as modelling complex financial risks (Brigo and Mercurio, 2000, 2001, 2002; Alexander, 2001, 2004; Alexander and Scourse, 2003; Buckley *et al*., 2004; McWilliam and Loh, 2008; Tashman and Frey, 2008). For instance, Venkataraman (1997) applies this concept to risk management; López de Prado and Peijan (2004), Venkatramanan (2005), and Kaiser *et al.* (2010) use the normal mixture distribution in asset allocation problems; Brigo *et al.* (2002) apply

Table 4

Correlation matrix

This table shows the correlations between the asset classes from Table 2. We use the S&P 500, MSCI Emerging Markets, JPM US Government Bonds, FTSE EPRA/NAREIT, S&P GSCI Commodity, HFRI Fund of Funds, US Buyout, and US Venture Capital from January 1999 to December 2009. Values in boldface are significantly different from zero at the 5 % level.

	S&P 500	MSCI Emerging Markets	JPM US Government Bonds	FTSE EPRA/ NAREIT	S&P GSCI Commodity	HFRI Fund of Funds	US Buyout	US Venture Capital
S&P 500	**1.000**							
MSCI Emerging Markets	**0.275**	**1.000**						
JPM US Government Bonds	−0.183	−0.044	**1.000**					
FTSE EPRA/ NAREIT	**0.648**	0.153	−0.067	**1.000**				
S&P GSCI Commodity	**0.305**	−0.020	−0.102	0.189	**1.000**			
HFRI Fund of Funds	0.157	0.172	−0.176	0.161	0.184	**1.000**		
US Buyout	0.103	**0.292**	**−0.241**	−0.061	−0.082	0.088	**1.000**	
US Venture Capital	0.077	**0.337**	−0.144	−0.127	−0.043	0.049	**0.720**	**1.000**

it in stochastic processes; and Bekaert and Engstrom (2011) use a mixture of gamma distributions to explain asset returns during crises.

We choose the normal mixture distribution primarily for its flexibility and its tractability to capture asymmetric return distributions – especially important for alternative investments (see, e.g., Ding and Shawky, 2007; Metrick and Yasuda, 2010, 2011).[8] In particular, let $f_1(x, \mu_1, \sigma_1)$ denote the probability density function of the first normal distribution with mean μ_1 and standard deviation σ_1, and let $f_2(x, \mu_2, \sigma_2)$ denote the probability density function of the second normal distribution. We can then approximate the empirical distribution of hedge fund returns by a new distribution with the following probability density function:

$$\begin{aligned} \mathrm{f}(\mathrm{x}, \mu_1, \sigma_1, \mu_2, \sigma_2) &= 0.2 \cdot \mathrm{f}_1(\mathrm{x}, \mu_1, \sigma_1) + 0.8 \cdot \mathrm{f}_2(\mathrm{x}, \mu_2, \sigma_2) \\ &= 0.2 \cdot \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(\mathrm{x}-\mu_1)^2}{\sigma_1^2}\right) \\ &\quad + 0.8 \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(\mathrm{x}-\mu_2)^2}{\sigma_2^2}\right) \end{aligned} \tag{1}$$

Our economic justification is as follows. Consider a regime-switching model with two economic states: the usual and the unusual. The usual state exists 80% of the time, when the hedge fund can achieve a return with the distribution given by the second normal density; the unusual state exists 20% of the time, when the return is given by the other normal distribution (see Graflund and Nisson, 2003, for further regime-switching models).[9]

Note that we do not specify whether the unusual return is better than the usual return in terms of having a higher mean and/or lower volatility. Indeed, the unusual return could be better, worse, or even the same. The latter case harks back to the classic assumption that returns are unconditionally normal. In general, our setting allows for conditional normal returns, but unconditional returns need not be normal.

This specification offers many advantages. First, we have four free parameters: $\mu_1, \sigma_1, \mu_2, \sigma_2$; so we can match the first through fourth moments of the empirical distribution exactly. We can also capture the skewness and excess kurtosis. Second, with the normal density function, the new approximating distribution is still tractable. Third, as noted earlier, this specification treats the traditional normal approximation as a special case. Figure 1 provides an illustration of this method.

Because we cannot solve the approximating parameters $\mu_1, \sigma_1, \mu_2, \sigma_2$ analytically, we must solve for them numerically. In particular, we look for means and standard deviations for the two normal distributions that can approximate the first four moments of the empirical distribution as closely as possible. Mean, variance, skewness, and kurtosis generally have different dimensions, so we minimise the weighted relative deviation rather than the absolute deviation.

[8] Our approach is similar to that of Popova *et al.* (2007).

[9] The assumed breakdown of 80% and 20% may seem restrictive, but we tested different pairs for robustness (see Section 4).

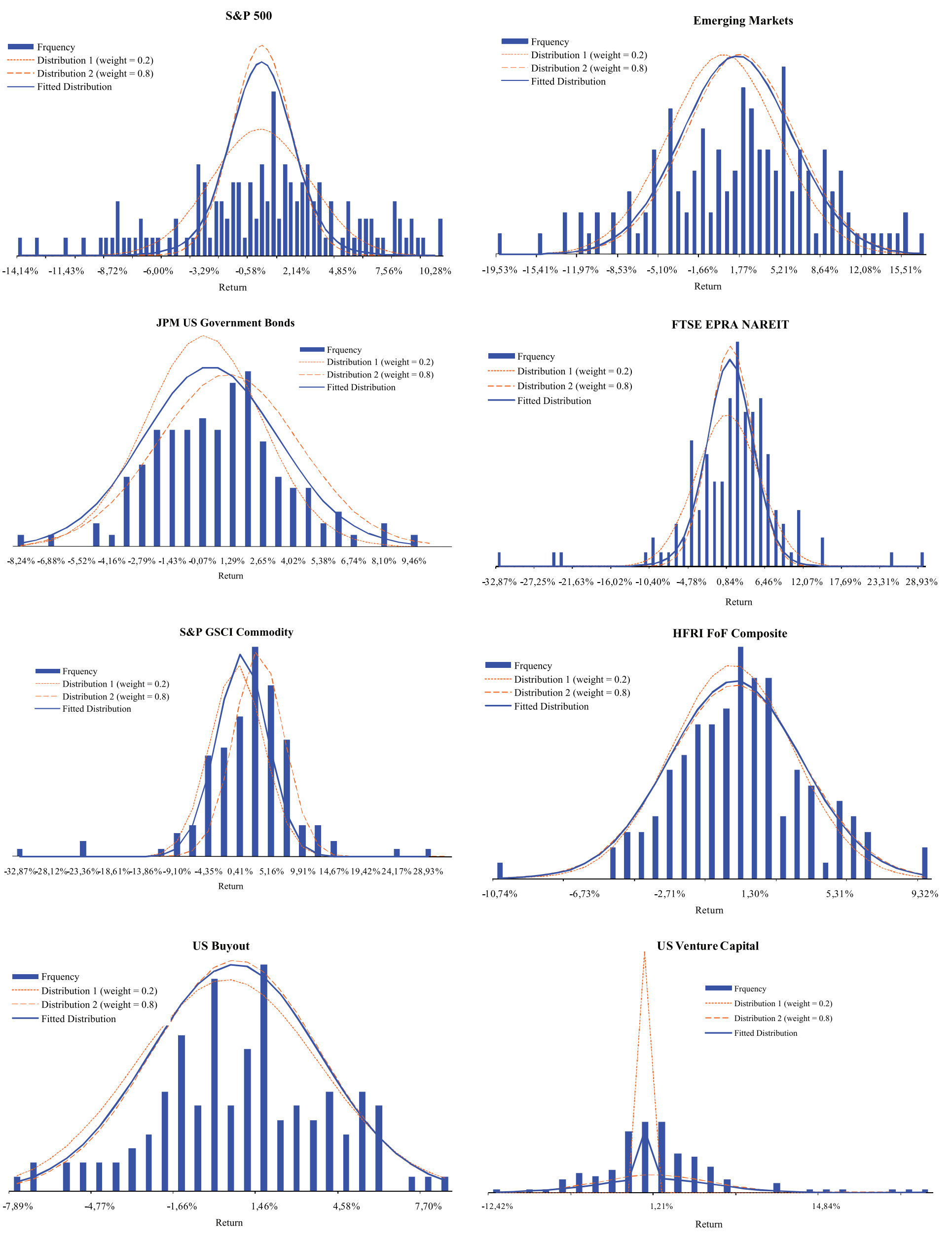

Fig. 1. Histograms and fitted distributions for all asset classes

This figure shows the monthly return histograms of the eight asset classes and the corresponding fitted return distribution for each asset class from January 1999 to December 2009. The fitted return distribution is composed of two auxiliary distributions—distributions 1 and 2—that are weighted with factors 0.2 and 0.8, respectively.

Let $w = (w_1, w_2, w_3, w_4)$ be a vector of strictly positive constants that serve as weights for the four moments we want to match. Our objective function is then

$$\begin{aligned} \min w_1 &\times \left| \frac{\text{theoretical mean} - \text{empirical mean}}{\text{empirical mean}} \right| + w_2 \\ &\times \left| \frac{\text{theoretical variance} - \text{empirical variance}}{\text{empirical variance}} \right| \\ &+ w_3 \times \left| \frac{\text{theoretical skewness} - \text{empirical skewness}}{\text{empirical skewness}} \right| + w_4 \\ &\times \left| \frac{\text{theoretical kurtosis} - \text{empirical kurtosis}}{\text{empirical kurtosis}} \right| \end{aligned} \tag{2}$$

The objective function takes a value of zero if all four moments can be matched exactly, and positive values otherwise. Our investigation uses equal weights for all moments, that is, each moment has the same importance in the objective function.[10, 11] The approximating parameters we obtain for the hedge fund strategies are provided in Table 5. Table 6 shows the first four moments of the empirical return distributions and compares them with the moments obtained from the mixture of normal methods. Obviously, the moments are close and thus the fit is good (see Figure 1).

Our next step is to construct a strategic asset allocation with the broad variety of asset classes. Because the mean–variance approach does not work, we must find an appropriate objective function. Real-world investors looking to incorporate alternative investments into their portfolios are typically family offices, corporations, pension funds, high net worth individuals, and large endowments. These investors are typically judged and compensated in comparison to a prespecified benchmark (see, e.g., Grinold and Kahn, 1999). Standard objective functions are not able to capture this relative aspect but, rather, rely solely on absolute terms. Additionally, these investors generally seek higher expected returns than in a money market, but are risk averse and therefore pay special attention to downside risk because they must often make regular distributions.

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[10] Hence, it is unlikely to obtain a perfect match since the moment dependencies are not linear.

[11] The idea behind the moment weight vector is that we integrate in our model more flexibility for investors. Due to differences in background, institutional investors cannot be regarded as a homogeneous group (we thank the referee for pointing this out); instead, they have diverging risk preferences (see Proelss and Schweizer, 2011). For example, insurance companies must meet future contractual obligations incurred from their sold contracts and consequently cannot bear the same risk as, say, a defined contribution pension plan. They are thus limited in their choice of asset allocation. Asset allocations for insurance companies must focus on strategies that will further reduce their risk profiles, especially those arising from higher moments. Therefore, one can assume that those investors put higher weights on the third and fourth moment of the return distribution to ensure a better fit and avoid being surprised by unexpected tail risk ([1,1,2,2] exemplary specification). In contrast, endowments and foundations must provide regular cash flows to their beneficiaries. Consequently, their concern is regular and sufficient cash flows from their investments. Endowments, however, are subject to fewer restrictions on minimum distribution standards and thus face less risk than pension funds or insurance companies, which have rigid contractual obligations (National Association of College and University Business Officers, 2006). Therefore, they may put higher weights on the first two moments ([2,2,1,1] exemplary specification).

Table 5

Moments of the normally distributed auxiliary distributions

This table shows the mean and the standard deviation of the two auxiliary distributions, as well as the weighting factor for the S&P 500, MSCI Emerging Markets, JPM US Government Bonds, FTSE EPRA/NAREIT, S&P GSCI Commodity, HFRI Fund of Funds, US Buyout, and US Venture Capital from January 1999 to December 2009. The values in the w-vector are all equal to one.

	Distribution 1 0.2		Distribution 2 0.8	
	Mean	Standard Deviation	Mean	Standard Deviation
S&P 500	0 %	10 %	1 %	6 %
MSCI Emerging Markets	1 %	16 %	18 %	16 %
JPM US Government Bonds	0 %	9 %	5 %	11 %
FTSE EPRA/NAREIT	5 %	16 %	11 %	11 %
S&P GSCI Commodity	0 %	14 %	11 %	13 %
HFRI Fund of Funds	3 %	10 %	5 %	11 %
US Buyout	1 %	12 %	5 %	11 %
US Venture Capital	0 %	1 %	7 %	16 %

Table 6

Comparison of the moments of empirical and approximated distributions

This table shows the first four moments (annualised) of the empirical and approximated distributions for the asset classes from Table 2 (see Appendix B for the rescaling from monthly to annual return distributions). The numbers on the left are the theoretical moments in the approximated distributions; the numbers in parentheses are the empirical moments.

	Mean	Standard Deviation	Skewness	Kurtosis
S&P 500	0.80 % (0.62 %)	7.00 % (17.70 %)	−0.14 (−0.13)	3.44 (3.12)
MSCI Emerging Markets	14.60 % (14.53 %)	17.39 % (24.12 %)	−0.09 (−0.09)	3.01 (3.00)
JPM US Government Bonds	4.00 % (4.02 %)	10.81 % (10.36 %)	0.00 (0.00)	3.00 (3.15)
FTSE EPRA/NAREIT	9.80 % (9.74 %)	12.40 % (25.27 %)	−0.09 (−0.09)	3.05 (3.85)
S&P GSCI Commodity	8.80 % (8.82 %)	13.92 % (24.50 %)	−0.15 (−0.15)	3.13 (3.10)
HFRI Fund of Funds	4.60 % (4.00 %)	10.84 % (10.86 %)	−0.16 (−0.15)	3.15 (3.31)
US Buyout	4.20 % (3.82 %)	11.32 % (11.31 %)	−0.03 (−0.04)	3.03 (3.00)
US Venture Capital	5.60 % (5.16 %)	14.59 % (18.61 %)	0.27 (0.42)	3.53 (3.35)

Therefore, it is crucial for them to achieve a certain minimum return to be able to pay out their obligations.

We can thus specify the objective function of our investor as follows (See also Morton *et al.* (2006). Let r denote the random return of the portfolio, and r_1 and r_2 some benchmark returns, which can be constants or random variables. Our investor's objective is to maximise the function

$$\Pr(r > r_1) - \lambda \Pr(r < r_2) \quad (3)$$

In other words, our investor wants to maximise the probability of outperforming some benchmark return while minimising the probability of underperforming the other one. Thus, the first benchmark could be some constant, for example, 10% per annum, or a random return of some other indices such as the S&P 500 as the market return. The second benchmark is usually chosen as 0%, the risk-free rate, or a government bond yield. Our analysis defines the first benchmark as a constant 8% per annum, and the second as 0%.[12]

The term λ is a positive constant and represents the trade-off between the two objectives. The λ depends on the investors' risk aversion. The higher λ, the less aggressive the investors (the higher their risk aversion), since they weight the second objective more highly and are more concerned about losses than gains. Similar to the relative risk aversion coefficient in canonical utility functions, plausible values of λ lie between one and ten. We also consider two constraints when optimising our portfolios numerically: We do not allow short-selling and we restrict the maximum asset class allocation (MAA) to 20%.[13] Using these constraints and the objective function stated above, we numerically calculate the optimal hedge fund portfolio for different values of λ. For different λvalues, all asset classes are at least incorporated into one optimal portfolio, but of course the allocations vary by strategy and are not all of equal importance (see Figure 2).

The first interesting result for the traditional asset classes is that stocks of large US firms (S&P 500 as a proxy) are considered only in the optimal portfolios for defensive risk-concerned investors ($\lambda = 1$). In comparison, stock investments in emerging markets gain in importance with a decrease in risk aversion, up to the MAA of 20% for λ greater than 3.5. Bonds are highly important and are added up to the MAA of 20% in all portfolios, since bonds provide downside protection for institutional investors, that is, to achieve a higher return than their minimum return. For REITs, the first analysed alternative investment, the allocation in the optimal portfolios increases up to 20% with decreasing risk aversion. It is not surprising that allocations to REITs vary in defensive portfolios, because REITs show the highest historical standard deviation and the most unfavorable higher moment properties among all considered asset classes. In contrast, commodities have a comparably stable allocation between 6% and 15% in all portfolios. Hedge fund allocations are comparable to those of bonds because they are integrated into all optimal portfolios at a 20% allocation (except $\lambda = 1$). Private equity plays a very important role, especially in defensive portfolios, and is allocated to a maximum of 40% in a portfolio (buyout and venture capital) until a λ of 4.5. From this point on, the allocation decreases and, for a value of 2.5, venture capital drops out of the portfolio. When summing up the allocations for alternative investments, we find that they have a cumulative weight of about 60% in offensive and performance-orientated portfolios ($\lambda = 1$), and about 77% in defensive portfolios ($\lambda = 6$).

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[12] For reasons of robustness, we also assume two stochastic benchmarks instead: the T-bill rate and the Barclays Capital Aggregate Bond Index for the second benchmark. The results remained qualitatively stable. Tables and figures are available from the authors upon request.

[13] This maximum allocation restriction aims to avoid having the portfolio dominated by a single asset class. When the minimum diversification restriction is imposed, the results are not as prone to optimisations without such a restriction, because optimal portfolio allocations do not comparably rely on the past performance of the respective assets.

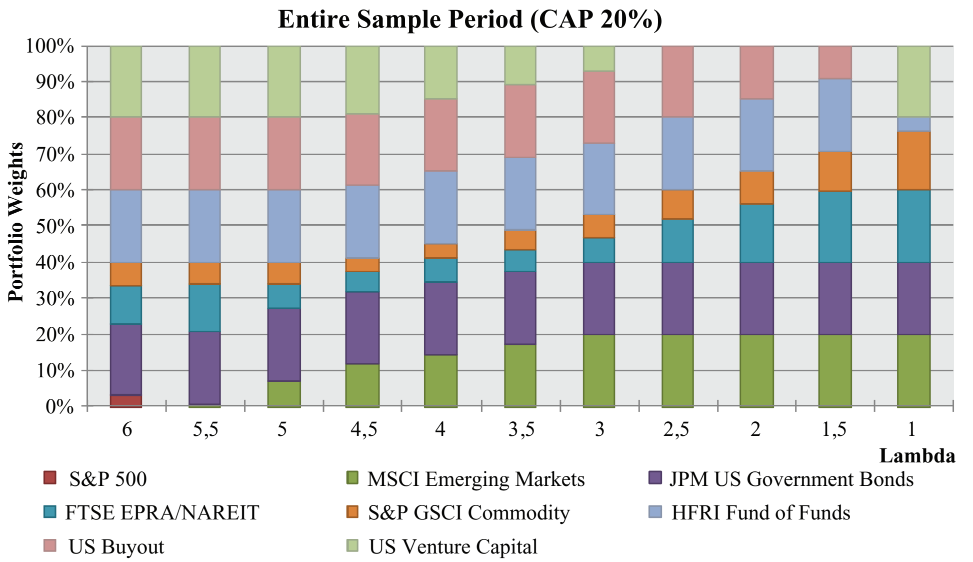

Fig. 2. Optimal portfolio allocations

This figure shows the relation between the risk aversion factor λ and the corresponding optimal portfolio allocations for the asset classes with a maximum weight restriction per asset class of 20% (CAP). The sample period is January 1999 to December 2009.

Our results show that traditional and alternative investments form substantial portions of investor portfolios. This result holds independent of the considered investor's risk aversion. We see that only a combination of both asset class categories leads to the highest investor utility. Therefore, traditional and alternative investments are not substitutes but, rather, complements.

However, to show that our approach dominates over the standard Markowitz approach, we need to examine the out-of-sample performance. Therefore, we conduct an out-of-sample Monte Carlo analysis according to Jobson and Korkie (1981) and Ledoit and Wolf (2008). Specifically, we use historical returns from January 1999 through June 2004 to construct Markowitz's efficient portfolios and equal expected return portfolios using our method for $\lambda = 1, 3$, and 6 respectively. Subsequently, we use historical returns from July 2004 through December 2009 to construct 1,000 time series of future returns using a bootstrap approach according to Efron and Tibshirani (1994). We then use the future return time series to calculate portfolio returns.

To assess how beneficial our optimisation technique is, we calculate the risk-adjusted performance for every risk measure separately as follows: the Sharpe ratio for standard deviation, the Sortino ratio for lower partial moments the return on conditional value at risk for conditional value at risk, and the Sterling ratio for maximum drawdown.

We note from Table 7 that our optimisation technique outperforms the Markowitz approach significantly for the Sharpe ratio[14] and the other risk-adjusted performance

[14] The test for statistical significance is applied for the Sharpe ratio following Jobson and Korkie (1981) and Ledoit and Wolf (2008) only because, for the other risk-adjusted performance measures, no test statistic can be found in the literature. Admittedly, we expect

Table 7
Out-of-sample analyses

This table shows the difference in the risk-adjusted portfolio performance and expected return of allocations for investor objective function maximisation (l = 1, 3, 6) compared to benchmark allocations (determined by the Markowitz portfolio selection process, where an efficient frontier portfolio with an equal return is selected) for a one-year holding period. Calculations are based on a standard block bootstrap Monte Carlo simulation with 1,000 runs, following Efron and Tibshirani (1994). For the out-of-sample analysis, we use the period January 1999 to June 2004 to construct the benchmark portfolio, and July 2004 to December 2009 to construct the time series of future returns. We calculate a corresponding risk-adjusted performance measure for each risk measure. For the standard deviation, we calculate the Sharpe ratio (SR); for the LPM 2 with threshold 0, we calculate the Sortino ratio (SoR); for the VaR with a 95 % confidence level, we calculate the return on value at risk (RoVaR); for the conditional value at risk with a 95 % confidence level, we calculate the return on conditional value at risk (RoCVaR); and for the MaxDD, we calculate the Sterling ratio (StR). All risk-adjusted performance measures are calculated using the same arithmetic equation: (portfolio return – risk-free return)/risk measure. For this analysis, the risk-free return is set to 3 %. Results remain stable when using zero or the historical risk-free return. The superscripts ***,**, and * denote that the assumption that an equal risk-adjusted performance measure is rejected at the 1 %, 5 %, and 10 % significance levels, respectively. Equivalent test statistics for other risk measures are not available.

λ	Expected Return	SR	SoR	RoVaR	RoCVaR	StR
1	0.68 %	0.095***	0.246	0.220	0.265	0.089
3	0.56 %	0.063***	0.164	0.151	0.194	0.068
6	−0.08 %	−0.006	−0.016	−0.015	−0.020	−0.007

measures. It performs especially well when the risk measures capture downside risk and for lower levels of risk aversion. Figure 3 shows the differences in portfolio returns.

4. Robustness Checks

To approve our earlier results, this section conducts a series of robustness checks:[15] different time periods, different maximum allocation restrictions, time-varying correlations, alternative indices representing asset classes, different weightings in the probability density functions, and value at risk as the objective function.

Our first robustness check analyses the influence of the recent financial crisis on the optimal portfolio allocations for alternative investments. First, we find that the importance of alternative investments for risk diversification in a defensive portfolio was underestimated before the financial crisis, because the cumulative weight was only about 54%, which is clearly below the 77% for the entire sample period. This finding can mainly be attributed to private equity that was underrepresented in defensive portfolios and that did not suffer as much as other asset classes from market overreactions during

it is most difficult to outperform, given our optimisation procedure, the Sharpe ratio, because it is directly linked to the Markowitz approach. Because we outperform the Sharpe ratio significantly and find more favourable risk-adjusted performance measures, compared to the Markowitz approach, we are confident that the results also hold for the other risk measures.

[15] The tables for all robustness checks are available from the authors upon request.

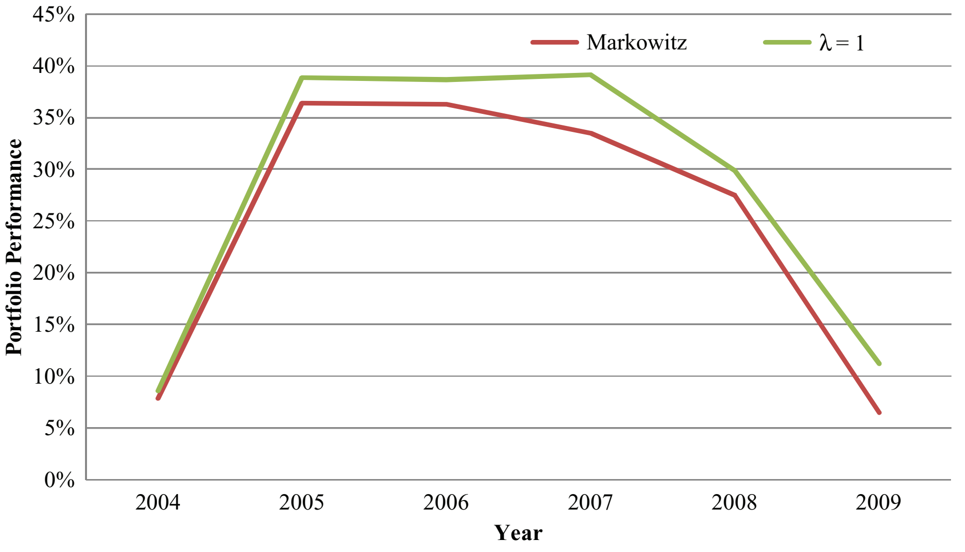

Fig. 3. Out-of-sample portfolio performance

This figure shows the portfolio performance of the allocations for investor objective function maximisation (l = 1) and a maximum allocation restriction per asset class of 20% compared to benchmark allocations (determined by the Markowitz portfolio selection process where an efficient frontier portfolio with an equal return is selected). Calculations are based on a standard block bootstrap Monte Carlo simulation with 1,000 runs, following Efron and Tibshirani (1994). For the out-of-sample analysis, we use the period from January 1999 to December 2003 to construct the Markowitz and the l = 1 portfolio. The out-of-sample portfolio performance is calculated as the cumulated return over the period January 2004 to December 2009 to construct the time series of future returns.

the financial crisis. They suffered less because interim changes in private equity portfolio values are driven by appraisal changes (see, e.g., DeBondt and Thaler, 1985; Chopra *et al*., 1992). In contrast, the cumulative portfolio allocations for offensive portfolios are about 20% higher when ignoring the financial crisis.

When conducting the second robustness check to study the effect of the maximum allocation restriction, we find that the cumulative portfolio allocations for alternative investments do not differ substantially for the less restrictive MAA of 25%, compared to the stricter one. Allocations to private equity as an asset class are reduced even when for some portfolios the allocation of buyout reaches the higher MAA. Furthermore, hedge funds have larger allocations (25%) in defensive portfolios and slightly lower ones in offensive portfolios when considering the entire sample period – the allocation is constantly at 25%, regardless of the risk aversion parameter, when the financial crisis is ignored.

Our approach so far has been based on the assumption that the correlations between all these assets remain constant over time. In reality, however, correlations are time-varying and stochastic. They are difficult to include when planning portfolios because their dynamic nature can greatly complicate the numerical optimisations. Therefore, instead of directly integrating constant correlations into our portfolio selection problem, we conduct a third robustness check to test whether our portfolio remains robust against time-varying correlations.

To do this, we draw from the Wishart distribution ten times for three different λ's and simulate the new correlation matrix.[16] Then we run the same optimisation procedure as before to determine the new optimal portfolio for three risk aversion classes, with $\lambda = 1$, 3, and 6. If the new portfolio does not deviate substantially from the initial portfolio, we can conclude that our initial portfolio will remain stable and robust against time-varying correlations.

The results show that our initial portfolios are quite stable and are only slightly affected by changes in the correlation matrix, especially the high risk aversion portfolio. In some cases, the newly optimised portfolio is exactly the same; in others, some asset allocations undergo minor changes. For investors with high risk aversion ($\lambda = 6$), four of the ten portfolios are identical to the initial one. For investors with low risk aversion ($\lambda = 1$), three of the ten portfolios are identical.

The fourth robustness check is an out-of-sample analysis without adding the financial crisis, according to Jobson and Korkie (1981) and Ledoit and Wolf (2008). Therefore, we use historical returns from January 1999 through December 2003 to construct the benchmark portfolios and historical returns from January 2004 through December 2006 to generate future return time series. We find that Markowitz is outperformed by all the risk-adjusted performance measures we study here, regardless of the level of risk aversion, also when the financial crisis is omitted. Hence, our approach is more suitable for capturing regime switches, which were particularly prevalent during the financial crisis.

A fifth robustness check considers various alternative indices that represent the different asset classes, especially for alternative investment classes, as discussed in Section 2. We find that all alternative indices exhibit significantly higher moments in their return distributions, and thus our proposed asset allocation approach seems promising for these indices as well. Indeed, we find that our method outperforms the standard Markowitz approach in out-of-sample analyses for all risk-adjusted performance measures.[17]

Sixth, we check whether our results remain stable if we assume different weightings of the probability density functions in the normal mixture distributions. Therefore, we conduct our analysis of alternative weighting schemes. This analysis uses a weighting scheme with a probability of 70% for regime one and a probability of only 30% for regime two. First, we calculate the optimal portfolio allocations for the aforementioned weighting schemes and find that the maximum difference in the allocations for the asset classes is only three percentage points. We then determine the resulting optimal asset allocations for different risk aversion parameters and check whether our asset allocation still achieves better risk-adjusted performance measures in out-of-sample analyses than standard Markowitz asset allocation. We find that our proposed asset allocation approach outperforms the standard Markowitz approach for all risk-adjusted performance measures.[18] Our approach is hence not limited to the specific parameters of the normal mixture distribution but, rather, is flexible to deal with different weighting schemes.

The seventh and last robustness check considers value at risk as an alternative investor objective function. Although the chosen objective function is commonly used by institutional investors, there is also increased interest in using it as the relevant risk

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[16] For further details see, for instance, Zhang (2006).

[17] Tables are available upon request from the authors.

[18] Tables are available upon request from the authors.

measure (Cassar and Gerakos, 2011). Therefore, we alternatively consider the objective function

$$\max \Pr(r > r_1) - \lambda \mathrm{VaR}_{99\%}(r) \tag{4}$$

where r denotes the stochastic return of our portfolio and r_1 is some benchmark return. In other words, our investor wants to maximise the probability of outperforming some benchmark return while minimising the value at risk for a one-year holding period at the 99% level.[19] We then determine the optimal asset allocation for different risk aversion parameters and compare our approach to the Markowitz approach. Again, we find that our approach outperforms the standard Markowitz approach for all risk-adjusted performance measures.

Summarising, this section finds that our asset allocation approach is superior to the classical Markowitz approach for all considered scenarios.

This paper introduces a new asset allocation approach especially suited to incorporate alternative investments. We use commonly used indices to represent the different asset classes. A natural extension would be to consider not only indices but the different hedge fund styles and types of commodities comprising the indices.

The use of indices has several advantages over individual assets (single hedge funds or hedge fund styles/different commodities). First, the use of indices enables one to not have to account for differences in liquidity. Furthermore, trading costs at the index level are comparable. Portfolio allocation models at the individual asset level have to account for liquidity and trading costs because they are not comparable across different types of alternative investments. Second, the use of indices is attractive because the indices are calculated net of fees and taxes. Portfolio allocation based on specific underlying assets, by contrast, requires accounting for differential fee and tax structures specific to the particular asset. Despite the additional complexity when using individual assets in our asset allocation approach, there are also promising advantages. There are many different styles of hedge funds (and to a lesser extent different types of private equity funds) with very different strategies and hence very different risk–return profiles. In addition, there are many different commodities with very different risk–return profiles. Using aggregated indices for these asset classes means losing the possibility of combining individual assets to achieve the best investor-specific risk–return profile. This is especially severe, since individual assets in alternative investments exhibit higher moments that can be used in our suggested approach to achieve superior portfolio diversification. This is also true for derivative securities, which could be worth considering in future extensions of our approach.

Another auspicious possibility for extension is the introduction of dynamics in our approach. The incorporation of higher moments in dynamic asset allocation models as well as using a dynamic objective function seems promising.

6. Conclusion

Markowitz's (1952) classic mean–variance approach is widely used for tactical asset allocation, but it fails to include further risk factors such as skewness and kurtosis, which

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[19] We also control for different value at risk levels (90% and 95%) and different holding periods (three and five years). The results remain qualitatively stable. Tables are available upon request from the authors.

is important when considering alternative investments because the return distributions of different hedge fund strategies are usually not normally distributed. This can lead to non-optimal strategic allocation suggestions.

This paper introduces a more flexible method, a mixture of normal methods, to individually incorporate the higher moments of different alternative investment return distributions. We use these distributions to determine strategic asset allocations for investors with different degrees of risk aversion and preferences. We are also able to incorporate stochastic and static benchmarks.

In our method, investors choose one benchmark they wish to outperform while simultaneously choosing a second benchmark for minimum acceptable performance. After defining the goal function, we solve the optimisation problem for a set of risk parameters and obtain very stable portfolio allocations, regardless of the level of λ. Finally, we perform seven robustness checks on our obtained portfolios with respect to the financial crisis, the maximum allocation restriction, time-varying correlations, as well as out-of-sample tests, and find robust results.

Our approach incorporates the heterogeneity of different asset classes and individual investor preferences to deliver robust results for institutional investors' strategic asset allocation. Our results are, in most cases, superior to Markowitz's (1952) classic mean–variance approach, particularly when markets face regime switches, such as during the recent financial crisis. At these times, a robust and reliable strategic asset allocation method is crucial.

Appendix A.

Table A-1
Data descriptions

This table reports the proxy indices for each asset class. The frequencies, inception dates, end dates, and additional information sources are given for the proxy time series.

Asset class	Proxy index	Frequency	Inception date	End date	Additional information
US Stocks	S&P 500 Composite - Total Return Index	Monthly	Jan 99	Dec 09	standardandpoors.com
Emerging Markets	MSCI Emerging Markets - Total Return Index	Monthly	Jan 99	Dec 09	datastream.com
US Government Bonds	JPM United States Govt. Bond - Total Return Index	Monthly	Jan 99	Dec 09	datastream.com
Real Estate Investment Trusts	FTSE EPRA NAREIT - Total Return Index	Monthly	Jan 99	Dec 09	nareit.com
Commodities	S&P GSCI Commodity - Total Return Index	Monthly	Jan 99	Dec 09	datastream.com
Hedge Funds	HFRI Fund of Hedge-fund Composite Index	Monthly	Jan 99	Dec 09	hedgefundresearch.com
Buyout	Thomson Reuters VentureXpert	Quarterly	Jan 99	Dec 09	thomsonreuters.com
Venture Capital	Thomson Reuters VentureXpert	Quarterly	Jan 99	Dec 09	thomsonreuters.com
Private Equity	LPX 50 Index	Monthly	Jan 99	Dec 09	lpx-group.com
Buyout	CepreX US Buyout	Monthly	Jan 99	Dec 09	cepres.com
Venture Capital	CepreX US Venture Capital	Monthly	Jan 99	Dec 09	cepres.com
Hedge Funds	Dow Jones Credit Suisse Core Hedge-fund Index	Monthly	Jan 99	Dec 09	hedgeindex.com
Commodities	Rogers International Commodities Index	Monthly	Jan 99	Dec 09	rogersrawmaterials.com

Appendix B. Rescaling of Moments

The moments of a monthly return distribution can be rescaled to an annual return distribution as follows. Let r_i denote the monthly return, and i and R denote the annual return. Therefore,

$$R = \sum_{i=1}^{12} r_i$$

Assume r_1s are independent and identically distributed. Let $E[r_i] = \bar{r}$, $Var(r_i) = \sigma_r^2$, $E[R_i] = \bar{R}$, and $Var(R_i) = \sigma_R^2$. It is well known that

$$\bar{R} = 12\bar{r}$$

and

$$\sigma_R = \sqrt{12}\sigma_r$$

The skewness of the annual return is defined as

$$\begin{aligned} Skew\,(R) &= \frac{E(R-\bar{R})^3}{\sigma_R^3} \\ &= \frac{E\left(\sum_{i=1}^{12} r_i - 12\bar{r}\right)^3}{12\sqrt{12}\sigma_r^3} \\ &= \frac{E\left[\sum_{i=1}^{12} (r_i - \bar{r})\right]^3}{12\sqrt{12}\sigma_r^3} \\ &= \frac{E\left[\sum_{i=1}^{12} \sum_{j=1}^{12} \sum_{k=1}^{12} (r_i - \bar{r})(r_j - \bar{r})(r_k - \bar{r})\right]}{12\sqrt{12}\sigma_r^3} \\ &= \frac{\sum_{i=1}^{12} \sum_{j=1}^{12} \sum_{k=1}^{12} E\left[(r_i - \bar{r})(r_j - \bar{r})(r_k - \bar{r})\right]}{12\sqrt{12}\sigma_r^3} \end{aligned}$$

So,

$$\begin{aligned} &E\left[(r_i - \bar{r})(r_j - \bar{r})(r_k - \bar{r})\right] \\ &\quad = \begin{cases} E\left[(r_i - \bar{r})^3\right] = Skew(r_i)\sigma_r^3, & \text{if} \quad i = j = k; \\ 0 & \text{if} \quad i, j, k \text{ are not the same} \end{cases} \end{aligned}$$

The equation above can be written as

$$\begin{aligned} Skew\,(R) &= \frac{\left(\sum_{i=1}^{12} Skew\,(r_i)\,\sigma_r^3\right)}{12\sqrt{12}\sigma_r^3} \\ &= \frac{12 Skew\,(r_i)\,\sigma_r^3}{12\sqrt{12}\sigma_r^3} \\ &= \frac{Skew\,(r_i)}{\sqrt{12}} \end{aligned}$$

The kurtosis of the annual return is defined as

$$\begin{aligned}
Kurt\,(R) &= \frac{E\left(R-\bar{R}\right)^4}{\sigma_R^4} \\
&= \frac{E\left(\sum_{i=1}^{12} r_i - \bar{r}\right)^4}{144\sigma_r^4} \\
&= \frac{E\left[\sum_{i=1}^{12}(r_i-\bar{r})\right]^4}{144\sigma_r^4} \\
&= \frac{E\left[\sum_{i=1}^{12}\sum_{j=1}^{12}\sum_{k=1}^{12}\sum_{l=1}^{12}(r_i-\bar{r})(r_j-\bar{r})(r_k-\bar{r})(r_l-\bar{r})\right]}{144\sigma_r^4} \\
&= \frac{\sum_{i=1}^{12}\sum_{j=1}^{12}\sum_{k=1}^{12}\sum_{l=1}^{12}E[(r_i-\bar{r})(r_j-\bar{r})(r_k-\bar{r})(r_l-\bar{r})]}{144\sigma_r^4}
\end{aligned}$$

Now since

$$\begin{aligned}
&E[(r_i-\bar{r})(r_j-\bar{r})(r_k-\bar{r})(r_l-\bar{r})] \\
&= \begin{cases} E[(r_i-\bar{r})^4] = Kurt(r_i)\sigma_r^4, & \text{if } \quad i=j=k=l; \\ E\left[(r_i-\bar{r})^2\left(r_j-\bar{r}\right)^2\right] = \sigma_r^4 & \text{if respective two of } i,j,k,l \text{ are the same;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned}
Kurt\,(R) &= \frac{\left(\sum_{i=1}^{12} Kurt\,(r_i)\,\sigma_r^4\right) + \frac{12\cdot 11}{2}\cdot\frac{4\cdot 3}{2}\sigma_r^4}{144\sigma_r^4} \\
&= \frac{12 Kurt\,(r_i)\,\sigma_r^4 + 396\sigma_r^4}{144\sigma_r^4} \\
&= \frac{Kurt\,(r_i)}{12} + \frac{11}{4}
\end{aligned}$$

Appendix C. The Method of Getmansky *et al.* (2004) Method

The basis for the procedure of Getmansky et al. (2004) is the idea that the observable return does not equal the real return. The observable return R_t^o is, rather, composed of the real returns R_t of the previous periods. Therefore

$$R_t^0 = \theta_o R_t + \theta_1 R_{t-1} + \cdots + \theta_k R_{t-k}$$

$$\theta_k \in [0,1], \quad j = 0 \ldots k, \quad \text{and}$$

$$1 = \theta_0 + \theta_1 + \cdots + \theta_k$$

The observable return is therefore the weighted sum of real returns of the previous periods. It follows that the mean of the observable returns is equal to the mean of the real returns. However, the volatility of the observable returns is smaller than the volatility of the actual returns. More precisely, the following is valid for the volatility of the observable returns:

$$\mathrm{Std}\left[R_t^o\right] = \frac{1}{\sqrt{\theta_o^2 + \theta_1^2 + \ldots + \theta_k^2}}\sigma \leq \sigma$$

where σ is the volatility of the real returns.

To calculate the real returns, the weighting factors must first be determined. Thereby we take advantage of the fact that the observable return can be written as the moving average process, whereas the weighting factors stay the same. The weighting factors for this moving average process can be estimated via maximum likelihood. Finally, the real returns can be calculated with the estimated weighting factors.

References

Ackermann, C., McEnally, R. and Ravenscraft, D., ‘The performance of hedge funds: risk, return, and incentives’, *Journal of Finance*, Vol. 54, 1999, pp. 833–74.

Agarwal, V. and Naik, N.Y., ‘Risk and portfolio decisions involving hedge funds’, *Review of Financial Studies*, Vol. 17, 2004, pp. 63–98.

Alexander, C., ‘Option pricing with normal mixture returns’, ISMA Centre research paper (2001).

Alexander, C., ‘Normal mixture diffusion with uncertain volatility: modeling short- and long-term smile effects’, *Journal of Banking & Finance*, Vol. 28, 2004, pp. 2957–80.

Alexander, C. and Scourse, A., ‘Bivariate normal mixture spread option valuation’, *Quantitative Finance*, Vol. 4, 2003, pp. 1–12.

Amin, G.S. and Kat, H.M., ‘Diversification and yield enhancement with hedge funds’, *Journal of Alternative Investments*, Vol. 5, 2002, pp. 50–8.

Amin, G.S. and Kat, H.M., ‘Hedge fund performance 1990–2000: Do the ‘money machines’ really add value?’ *Journal of Financial & Quantitative Analysis*, Vol. 38, 2003, pp. 251–74.

Anson, M. J. P., *Handbook of Alternative Assets* (New Jersey: John Wiley, 2006).

Asness, C.S., Krail, R. and Liew, J.M., ‘Do hedge funds hedge?’ Available at http://ssrn.com/abstract = 252810, 2001.

Avramov, D., Kosowski, R., Naik, N.Y. and Teo, M., ‘Investing in hedge funds when returns are predictable’, AFA New Orleans Meetings Paper, 2008.

Bekaert, G. and Engstrom. E.C., ‘Asset return dynamics under bad environment–good environment fundamentals’. Available at http://ssrn.com/abstract = 1440226, 2011.

Brigo D. and Mercurio, F., ‘A mixed-up smile’, *Risk*, Vol. 13, 2000, pp. 123–6.

Brigo D. and Mercurio, F., *Interest-Rate Models: Theory and Practice* (Berlin: Springer, 2001).

Brigo D. and Mercurio, F., ‘Lognormal-mixture dynamics and calibration to market volatility smiles’, *International Journal of Theoretical and Applied Finance*, Vol. 5, 2002, pp. 427–46.

Brigo, D., Mercurio, F. and Sartorelli, G., ‘Lognormal-mixture dynamics under different means’, Working Paper, 2002.

Brinson, G.P., Hood, L.R. and Beebower, G.L., ‘Determinants of portfolio performance’, *Financial Analysts Journal*, Vol. 42, 1986, pp. 39–48.

Brinson, G.P., Hood, L.R. and Beebower, G.L., ‘Determinants of portfolio performance II: an update’, *Financial Analysts Journal*, Vol. 47, 1991, pp. 40–8.

Brooks, C. and Kat, H.M., ‘The statistical properties of hedge fund index returns and their implications for investors’, *Journal of Alternative Investments*, Vol. 5, 2002, pp. 26–44.

Brown, S.J., Goetzmann, W.N. and Ibbotson, R.G., ‘Offshore hedge funds: survival and performance, 1989–95’, *Journal of Business*, Vol. 72, 1999, pp. 91–117.

Buckley, I., G. Comezana, B. Djerroud and L. Seco, ‘Portfolio optimization when asset returns have the Gaussian mixture distribution’, Working Paper (Imperial College, 2004).

Byrne, P. and Lee, S., 'Is there a place for property in the multi-asset portfolio?' *Journal of Property Finance*, Vol. 6, 1995, pp. 60–83.

Cassar, G. and Gerakos, J., 'How do hedge funds manage portfolio risks'. Available at http://ssrn.com/abstract = 1722250, 2011.

Caselli, S., Gatti, S., and Perrini, F., 'Are venture capitalists a catalyst for innovation?' *European Financial Management*, Vol. 15, 2009, pp. 92–111.

Chen, P., Baierl, G.T. and Kaplan, P.D., 'Venture capital and its role in strategic asset allocation', *Journal of Portfolio Management*, Vol. 28, 2002, pp. 83–9.

Chen, H., Ho, K., Lu, C. and Wu, C., 'Real estate investment trusts: an asset allocation perspective', *Journal of Portfolio Management*, Vol. 31, 2005, pp. 46–55.

Chiang, K.C.H. and Ming-Long, L., 'Spanning tests on public and private real estate', *Journal of Real Estate Portfolio Management*, Vol. 13, 2007, pp. 7–15.

Chopra, N., Lakonishok, J. and Ritter,J.R., 'Measuring abnormal performance: do stocks overreact?' *Journal of Financial Economics*, Vol. 31, 1992, pp. 235–68.

Conover, C.M., Jensen, G.R., Johnson, R.R. and Mercer, J.M., 'Is now the time to add commodities to your portfolio?' *Journal of Investing*, Vol. 19, 2010, pp. 10–9.

Cumming, D.J., Haß, L.H. and Schweizer, D., 'Private equity benchmarks and portfolio optimization'. Available at http://ssrn.com/abstract = 1687380, 2011.

Cumming, D.J., Fleming, G., and Johan, S.A., 'Institutional investment in listed private equity,' *European Financial Management*, Vol. 17, 2011, pp. 594–618.

Cumming, D.J., and Johan, S.A., 'Is it the law or the lawyers? Investment covenants around the world', *European Financial Management*, Vol. 12, 2006, pp. 553–74.

Daskalaki, C. and Skiadopoulos, G., 'Should investors include commodities into their portfolios after all? New evidence', *Journal of Banking & Finance*, Vol. 35, 2011, pp. 2606–26.

DeBondt, W.F.M. and Thaler, R.H., 'Does the stock market overreact?' *Journal of Finance*, Vol. 40, 1985, pp. 793–805.

Ding, B. and Shawky, H.A., 'The performance of hedge fund strategies and the asymmetry of return distributions', *European Financial Management*, Vol., 13, 2007, pp. 309–31.

Efron, B. and Tibshirani, R.J., *An Introduction to the Bootstrap* (New York: Chapman & Hall, 1994).

Eling, M., 'Does hedge fund performance persist? Overview and new empirical evidence', *European Financial Management*, Vol. 15, 2009, 362–401.

Ennis, R.M. and Sebastian, M.D., 'Asset allocation with private equity', *Journal of Private Equity*, Vol. 8, 2005, pp. 81–7.

Erb, C.B. and Harvey, C.R., 'The strategic and tactical value of commodity futures', *Financial Analysts Journal*, Vol. 62, 2006, pp. 69–97.

Ernst, S., Koziol, C. and Schweizer, D., 'Are private equity investors boon or bane for an economy? – A theoretical analysis, forthcoming *European Financial Management*.

Fung, W. and Hsieh, D.A., 'Empirical characteristics of dynamic trading strategies: the case of hedge funds', *Review of Financial Studies*, Vol. 10, 1997, pp. 275–302.

Fung, W. and Hsieh, D.A., Performance characteristics of hedge funds and commodity funds: natural vs. spurious biases, *Journal of Financial & Quantitative Analysis*, Vol. 35, 2000, pp. 291–308.

Fung, W. and Hsieh, D.A., 'The risk in hedge fund strategies: theory and evidence from trend followers', *Review of Financial Studies*, Vol. 14, 2001, pp. 313–41.

Geltner, D.M., 'Smoothing in appraisal-based returns', *Journal of Real Estate Finance & Economics*, Vol. 4, 1991, pp. 327–45.

Getmansky, M., Lo, A.W. and Makarov, I., 'An econometric model of serial correlation and illiquidity in hedge fund returns', *Journal of Financial Economics*, Vol. 74, 2004, pp. 529–609.

Goltz, F., Martellini, L. and Vaissié, M., 'Hedge fund indices: reconciling investability and representativity', *European Financial Management*, Vol. 13, 2007, 257–86.

Gompers, P.A. and Lerner, J., 'Risk and reward in private equity investments: the challenge of performance assessment', *Journal of Private Equity*, Vol. 1, 1997, pp. 5–12.

Gorton, G.B. and Rouwenhorst, G.K., 'Facts and fantasies about commodity futures', *Financial Analysts Journal*, Vol. 62, 2006, pp. 47–68.

Graflund, A. and Nilsson, B., 'Dynamic portfolio selection: the relevance of switching regimes and investment horizon', *European Financial Management*, Vol. 9, 2003, pp. 179–200.
Grinold, R.C., and Kahn, R.N., *Active Portfolio Management: A Quantitative Approach for Producing Superior Returns and Controlling Risk* (New York: McGraw-Hill, 1999).
Groh, A.P., and von Liechtenstein, H. 'The first step of the capital flow from institutions to entrepreneurs: the criteria for sorting venture capital funds,' *European Financial Management* Vol. 17, 2011, pp. 532–59.
Gueyie, J. and Amvella, S.P., 'Optimal portfolio allocation using funds of hedge funds', *Journal of Private Wealth Management*, Vol. 9, 2006, pp. 85–95.
Hartmann-Wendels, T., Keienburg, G., and Sievers, S., 'Adverse selection, investor experience and security choice in venture capital finance: evidence from Germany', *European Financial Management*, Vol. 17, 2011, pp. 464–99.
Hoernemann, J.T., Junkans, D.A. and Zarate, C.M., 'Strategic asset allocation and other determinants of portfolio returns', *Journal of Private Wealth Management*, Fall, 2005, pp. 26–38.
Huang, J.-Z. and Zhong, Z., 'Time−variation in diversification benefits of commodity, REITs, and TIPS', *Journal of Real Estate Finance and Economics*, 2011, forthcoming.
Hudson-Wilson, S., Fabozzi, F.J., Gordon, J.N. and Giliberto, S.M., 'Why real estate?' *Journal of Portfolio Management*, Vol. 30, 2004, pp. 12–25.
Jarque, C.M. and Bera, A.K., 'Efficient tests for normality, homoscedasticity and serial independence of regression residuals', *Economics Letters*, Vol. 6, 1980, pp. 255–9.
Jobson, J.D. and Korkie, B.M., 'Performance hypothesis testing with the Sharpe and Treynor measures', *Journal of Finance*, Vol. 36, 1981, pp. 889–908.
Jondeau, E. and Rockinger, M., 'Optimal portfolio allocation under higher moments', *European Financial Management*, Vol. 12, 2006, pp. 29–55.
Kaiser, D.G., Schweizer, D. and Lue, W., 'Efficient hedge fund strategy allocations – A systematic framework for investors that incorporates higher moments'. Available at http://ssrn.com/abstract = 1080509, 2010.
Kooli, M., 'The diversification benefits of hedge funds and funds of hedge funds', *Derivatives Use, Trading & Regulation*, Vol. 12, 2007, pp. 290–300.
Ledoit, O. and Wolf, M., 'Robust performance hypothesis tests with the Sharpe ratio', *Journal of Empirical Finance*, Vol. 15, 2008, pp. 850–9.
Lee, S. and Stevenson, S., 'The case for REITs in the mixed-asset portfolio in the short and long run', *Journal of Real Estate Portfolio Management*, Vol. 11, 2005, pp. 55–80.
Lhabitant, F. and Learned, M., 'Hedge fund diversification: how much is enough?' *Journal of Alternative Investments*, Vol. 5, 2002, pp. 23–49.
Li, Y. and Kazemi, H., 'Conditional properties of hedge funds: evidence from daily returns' *European Financial Management*, Vol. 13, 2007, pp. 211–38.
Liang, B., 'Hedge funds, fund of funds, and commodity trading advisors', *Working Paper* (Case Western Reserve University, 2002).
López de Prado, M.M. and Peijan, A., 'Measuring loss potential of hedge fund strategies', *Journal of Alternative Investments*, Vol. 7, 2004, pp. 7–31.
Markowitz, H.M., 'Portfolio selection', *Journal of Finance*, Vol. 7, 1952, pp. 77–91.
Martin, G., 'Making sense of hedge fund returns: a new approach', in E. Acar, ed., *Added Value in Financial Institutions: Risk or Return* (New York: Prentice Hall, 2001), pp. 165–82.
McWilliam, N. and Loh, K., 'Incorporating multidimensional tail-dependencies in the valuation of credit derivatives', *Working Paper* (2008).
Metrick, A. and Yasuda, A.A., 'The economics of private equity funds', *Review of Financial Studies*, Vol. 23, 2010, pp. 2303–41.
Metrick, A., and Yasuda, A.A., 'Venture capital and other private equity: a survey,' *European Financial Management*, Vol. 17, 2011, pp. 619–64.
Morton, D., Popova, E. and Popova, I., 'Efficient fund of hedge funds construction under downside risk measures', *Journal of Banking & Finance*, Vol. 30, 2006, pp. 503–18.

NACUBO, 'NACUBO Endowment Study', Annual Report, 2006, National Association of College and University Business Officers, Washington, DC.

NAREIT, 'Diversification benefits of REITs', an analysis by Ibbotson Associates'. Available at http://www.nareit.com, 2002, National Association of Real Estate Investment Trusts.

Nielsen, K.M., 'The return to direct investment in private firms: new evidence on the private equity premium puzzle,' *European Financial Management*, Vol. 17, 2011, pp. 436–63.

Popova, I., Morton, D., Popova, E. and Yau, J., 'Optimizing benchmark-based portfolios with hedge funds', *Journal of Alternative Investments*, Vol. 10, 2007, pp. 35–55.

Popova, I., Popova, E., Morton, D. and Yau, J., 'Optimal hedge fund allocation with asymmetric preferences and distributions'. Available at http://ssrn.com/abstract = 900012, 2003.

Proelss, J. and Schweizer, D., 'Polynomial goal programming and the implicit higher moment preferences of U.S. institutional investors in hedge funds'. Available at http://ssrn.com/abstract = 1360248, 2011.

Schmidt, D.M., 'Private equity-, stock- and mixed-asset portfolios: a bootstrap approach to determine performance characteristics, diversification benefits and optimal portfolio allocations', *Center for Financial Studies*, No. 2004/12, 2004.

Tashman, A. and Frey, R., 'Modeling risk in arbitrage strategies using finite mixtures', *Quantitative Finance*, Vol. 9, 2008, pp. 495–503.

Venkataraman, S., 'Value at risk for a mixture of normal distributions: the use of quasi-Bayesian estimation techniques,' *Economic Perspectives*, Vol. 3, 1997, pp. 2–13.

Venkatramanan, A., 'American spread option pricing', *Working Paper* (University of Reading, 2005).

Zhang, X., 'Specification tests of international asset pricing models, *Journal of International Money and Finance*, Vol. 25, 2006, pp. 275–307.