MR2551190 (2011c:12005) 12H20 (03C60 14L10)

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The theory of the exponential differential equations of semiabelian varieties. (English summary)

Selecta Math. (N.S.) **15** (2009), *no. 3,* 445–486.

For any differential field $\langle F;+,\cdot,D\rangle$ of characteristic zero and for any semiabelian variety S defined over the field of constants C, one defines the *exponential differential equation of* S to be $lD_{LS}(x)=lD_S(y)$, where $lD_S\colon S(F)\to LS(F), lD_{LS}\colon LS(F)\to LLS(F)\cong LS(F)$ are *logarithmic derivative maps*; here LS is the tangent space at the identity of S. The equation defines a differential algebraic subgroup $\Gamma_S=\{(x,y)\in LS\times S\,|\,lD_{LS}(x)=lD_S(y)\}$ of $TS\cong LS\times S$. In particular, the usual exponential map satisfies the exponential differential equation for $\mathbf{G}_m$.

Given a collection $\mathcal{S}$ of semiabelian varieties defined over a fixed countable field C_0 of characteristic zero, we denote by $\mathcal{L}_{\mathcal{S}}$ the field language augmented by relation symbols for the field of constants C and for each solution set Γ_S, $S\in\mathcal{S}$ (of appropriate arity to be interpreted as a subset of TS), and by constant symbols for each element of C_0. Let $T_{\mathcal{S}}$ be the complete first-order $\mathcal{L}_{\mathcal{S}}$-theory of the reduct $\langle F;+,\cdot,C,(\Gamma_S)_{S\in\mathcal{S}},(\widehat{c})_{c\in C_0}\rangle$, in the case where $\langle F;+,\cdot,D\rangle$ is a differentially closed field whose constant field C contains the fixed countable field C_0. The main goal of this very interesting and well-written paper is to provide an axiomatization of the first-order theory $T_{\mathcal{S}}$.

The author proves that such a theory $T_{\mathcal{S}}$ also arises from a category-theoretic version of Hrushovski's *amalgamation with predimension* technique. In particular, he obtains a pregeometry with its associated notion of dimension, and the definition of the rotund subvarieties of the tangent bundles TS which are basic ingredients in axiomatization. The axioms for $T_{\mathcal{S}}$ consist of a description of the algebraic structure of the solution sets Γ_S for $S\in\mathcal{S}$ together with necessary and sufficient conditions for a system of equations to have solutions. The necessary conditions generalize Ax's differential fields version of Schanuel's conjecture [J. Ax, Ann. of Math. (2) **93** (1971), 252–268; MR0277482 (43 #3215)] to semiabelian varieties, while the sufficient conditions generalize work of C. Crampin for the multiplicative group ["Reducts of differentially closed fields to fields with a relation for exponentiation", D.Phil. thesis, Univ. Oxford, Oxford, 2006]. The author also obtains a purely algebraic result, the "Weak CIT", for semiabelian varieties, which concerns the intersections of algebraic subgroups with algebraic varieties.

Much of the work of the paper under review was done as part of the author's D.Phil. thesis ["The theory of exponential differential equations", Univ. Oxford, Oxford, 2006], under the supervision of Boris Zilber.

Reviewed by *Ş. A. Basarab*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.