Statistical Distribution for Generalized Ideal Gas of Fractional-Statistics Particles

Yong-Shi Wu
Department of Physics, University of Utah, Salt Lake City, Utah 84112
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We derive the occupation-number distribution in a generalized ideal gas of particles obeying fractional statistics, including mutual statistics, by adopting a state-counting definition. When there is no mutual statistics, the statistical distribution interpolates between bosons and fermions, and respects a fractional exclusion principle (except for bosons). Anyons in a strong magnetic field at low temperatures constitute such a physical system. Applications to the thermodynamic properties of quasiparticle excitations in the Laughlin quantum Hall fluid are discussed.

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Statistics is the distinctive property of a particle (or elementary excitation) that plays a fundamental role in determining macroscopic or thermodynamic properties of a quantum many-body system. In recent years, it has been recognized that particles with "fractional statistics" intermediate between bosons and fermions can exist in two-dimensional [1] or in one-dimensional [2,3] systems. Most of the study has been done in the context of many-body quantum mechanics. There have been calculations of thermodynamic properties of certain systems [2,4–6] with the help of exact solutions, but general formulation of quantum statistical mechanics (QSM) for ideal gas with an occupation-number distribution that interpolates between bosons and fermions is still lacking.

A single-particle quantum state can accommodate an arbitrary number of identical bosons, while no two identical fermions can occupy one and the same quantum state (Pauli's exclusion principle). In QSM [7], this difference gives rise to different counting of many-body states, or different statistical weight W. For bosons or fermions, the number of quantum states of N identical particles occupying a group of G states is, respectively, given by

$$W_b = \frac{(G+N-1)!}{N!\,(G-1)!} \quad \text{or} \quad W_f = \frac{G!}{N!\,(G-N)!}. \tag{1}$$

A simple interpolation implying fractional exclusion is

$$W = \frac{[G+(N-1)(1-\alpha)]!}{N!\,[G-\alpha N-(1-\alpha)]!}, \tag{2}$$

with $\alpha = 0$ corresponding to bosons and $\alpha = 1$ fermions. Such an expression can be the starting point of QSM for *intermediate* statistics with $0 < \alpha < 1$. Let us first clarify its precise meaning in connection with "occupation of single-particle states," and justify it as a new definition of quantum statistics, *à la* Haldane [3].

Following Ref. [3], we consider the situations in which the number G_i of independent states of a single particle (elementary excitation) of species i, confined to a finite region of matter, is *finite and extensive*, i.e., proportional to the size of the matter region in which the particle exists. Now let us add more particles *with the boundary conditions and size of the condensed-matter region fixed.* The N-particle wave function, when the coordinates of $N-1$ particles and their species are held fixed, can be expanded in a basis of wave functions of the remaining particle. The crucial point is that in the presence of other particles, the number d_i of available single-particle states in this basis for a particle of species i generally is not a constant, as given by G_i; rather it may depend on the particle numbers $\{N_i\}$ of all species. This happens, for example, when localized particle states are *nonorthogonal*; as a result, the number of available single-particle states changes as particles are added at fixed size and boundary conditions. Haldane [3] defined the statistical interactions α_{ij} through the *linear* relation

$$\Delta d_i = -\sum_j \alpha_{ij}\,\Delta N_j, \tag{3}$$

where $\{\Delta N_j\}$ is a set of allowed changes of the particle numbers. In the same spirit, but more directly to the purposes of QSM, we prefer defining the statistics by counting [8] the number of many-body states at fixed $\{N_i\}$,

$$W = \prod_i \frac{\left[G_i + N_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})\right]!}{(N_i)!\left[G_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})\right]!}. \tag{4}$$

The parameters α_{ij} must be *rational*, in order that a thermodynamic limit can be achieved through a sequence of systems with different sizes and particle numbers. We call α_{ij} for $i \neq j$ mutual statistics. We note that Eq. (4) applies to the usual Bose or Fermi ideal gas with i labeling single-particle energy levels and $\alpha_{ij} = \alpha\delta_{ij}$ ($\alpha = 0, 1$). So with an extension of the meaning of species, *this definition allows different species indices to refer to particles of the same kind but with different quantum numbers* [9]. Note there is no periodicity in so-defined statistics, and it makes sense to consider the cases with $\alpha > 1$ or 2.

As usual in QSM, we start with the ideal situations in which total energy (eigenvalue) is always a simple sum,

$$E = \sum_i N_i \varepsilon_i, \tag{5}$$

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with ε_i identified as the energy of a particle of species i. We call such a system a generalized [10] ideal gas if Eq. (4) also applies. Postponing the discussions about when Eqs. (4) and (5) are obeyed, let us first apply them to study QSM of generalized ideal gas. Following the standard procedure [7], one may consider a grand canonical ensemble at temperature T and with chemical potential μ_i for species i. According to the fundamental principles of QSM, the grand partition function is given by (with k the Boltzmann constant)

$$Z = \sum_{\{N_i\}} W(\{N_i\}) \exp\left\{\sum_i N_i(\mu_i - \varepsilon_i)/kT\right\}. \tag{6}$$

As usual, we expect that for very large G_i and N_i, the summand has a very sharp peak around the set of most probable (or mean) particle numbers $\{N_i\}$. Using the Stirling formula $\ln N! = N\ln(N/e)$, and introducing the average "occupation number" defined by $n_i \equiv N_i/G_i$, we express $\ln W$ as (with $\beta_{ij} \equiv \alpha_{ij}G_j/G_i$)

$$\sum_i G_i\left\{-n_i\ln n_i - \left(1-\sum_j \beta_{ij}n_j\right)\ln\left(1-\sum_j\beta_{ij}n_j\right) + \left[1+\sum_j(\delta_{ij}-\beta_{ij})n_j\right]\ln\left[1+\sum_j(\delta_{ij}-\beta_{ij})n_j\right]\right\}. \tag{7}$$

The most probable distribution of n_i is determined by

$$\frac{\partial}{\partial n_i}\left[\ln W + \sum_i G_i n_i(\mu_i - \varepsilon_i)/kT\right] = 0. \tag{8}$$

It follows that

$$n_i e^{(\varepsilon_i-\mu_i)/kT} = \left[1+\sum_k(\delta_{ik}-\beta_{ik})n_k\right] \times \prod_j\left[\frac{1-\sum_k\beta_{jk}n_k}{1+\sum_k(\delta_{jk}-\beta_{jk})n_k}\right]^{\alpha_{ji}}. \tag{9}$$

Setting $w_i = n_i^{-1} - \sum_k \beta_{ik}n_k/n_i$, we have

$$(1+w_i)\prod_j\left(\frac{w_j}{1+w_j}\right)^{\alpha_{ji}} = e^{(\varepsilon_i-\mu_i)/kT}. \tag{10}$$

Therefore the most probable average occupation numbers n_i $(i = 1,2,\ldots)$ can be obtained by solving

$$\sum_j(\delta_{ij}w_j + \beta_{ij})n_j = 1, \tag{11}$$

with w_i determined by the functional equations (10). The thermodynamic potential $\Omega = -kT\ln Z$ is given by

$$\Omega \equiv -PV = -kT\sum_i G_i \ln\frac{1+n_i-\sum_j\beta_{ij}n_j}{1-\sum_j\beta_{ij}n_j}, \tag{12}$$

and the entropy, $S = (E - \sum_i \mu_i N_i - \Omega)/T$, is

$$\frac{S}{k} = \sum_i G_i\left\{n_i\frac{\varepsilon_i-\mu_i}{kT} + \ln\frac{1+n_i-\sum_j\beta_{ij}n_j}{1-\sum_j\beta_{ij}n_j}\right\}. \tag{13}$$

Other thermodynamic functions follow straightforwardly. As usual, one can easily verify that the fluctuations, $(\overline{N_i^2} - \bar{N}_i^2)/\bar{N}_i^2$, of the occupation numbers are negligible, which justifies the validity of the above approach.

The simplest example is the ideal gas, i.e., identical particles with no mutual statistics, for which we set $\alpha_{ij} = \alpha\delta_{ij}$ and $\mu_i = \mu$. Then the average occupation number n_i satisfies

$$(1-\alpha n_i)^\alpha[1+(1-\alpha)n_i]^{1-\alpha} = n_i e^{(\varepsilon_i-\mu)/kT}, \tag{14}$$

and we have the statistical distribution

$$n_i = \frac{1}{w(e^{(\varepsilon_i-\mu)/kT}) + \alpha}, \tag{15}$$

where the function $w(\zeta)$ satisfies the functional equation

$$w(\zeta)^\alpha[1+w(\zeta)]^{1-\alpha} = \zeta \equiv e^{(\varepsilon-\mu)/kT}. \tag{16}$$

Note that $w(\zeta) = \zeta - 1$ for $\alpha = 0$, and $w(\zeta) = \zeta$ for $\alpha = 1$. Thus, Eq. (15) recovers the familiar Bose and Fermi distributions, respectively, with $\alpha = 0$ and $\alpha = 1$. For semions with $\alpha = 1/2$, Eq. (14) becomes a quadratic equation, which can be easily solved to give

$$n_i = \frac{1}{\sqrt{1/4 + \exp[2(\varepsilon_i - \mu)/kT]}}. \tag{17}$$

For intermediate statistics $0 < \alpha < 1$, it is not hard to select the solution $w(\zeta)$ of Eq. (16) that interpolates between bosonic and fermionic distributions. In particular, when ζ is very large, we have $w(\zeta) \approx \zeta$ and, neglecting α compared to $w(\zeta)$, we recover the Boltzmann distribution

$$n_i = e^{-(\varepsilon_i-\mu)/kT}, \tag{18}$$

at sufficiently low densities for any statistics.

Furthermore, we note that ζ is always non-negative, so is w; it follows from Eq. (15) that

$$n_i \le 1/\alpha. \tag{19}$$

This expresses the generalized exclusion principle for fractional statistics. In particular, at absolute zero, $\zeta = 0$ if $\varepsilon_i < \mu$, and $\zeta = +\infty$ if $\varepsilon_i > \mu$. From Eq. (16), we have $w = 0$ and ∞, respectively. Thus, we see that at $T = 0$, for statistics $\alpha \neq 0$, the average occupation numbers for single-particle states with continuous energy spectrum obey a step distribution like fermions:

$$n_i = \begin{cases} 0, & \text{if } \varepsilon_i > E_F, \\ 1/\alpha, & \text{if } \varepsilon_i < E_F. \end{cases} \tag{20}$$

The Fermi surface $\varepsilon = E_F$ is determined by the requirement $\sum_{\varepsilon_i < E_F} G_i = \alpha N$. Below the Fermi surface, the average occupation number is $1/\alpha$ for each single-particle state, in agreement with fractional exclusion.

One may be tempted to consider, in parallel to the usual Bose and Fermi ideal gas, the case with

$$\varepsilon_i = \frac{\hbar^2 k_i^2}{2m}, \quad G_i = \frac{V(\Delta k)^2}{(2\pi)^2}, \tag{21}$$

say in two dimensions with V the area. Then treating momentum as continuous, one has the Fermi momentum

$$k_F^2 = (4\pi\alpha)N/V. \tag{22}$$

Moreover, at finite temperatures, by using (15) and (16), the sum $\sum_i G_i n_i = N$ can be performed to give

$$\frac{\mu}{kT} = \alpha \frac{2\pi\hbar^2}{mkT}\frac{N}{V} + \ln\left[1 - \exp\left(-\frac{2\pi\hbar^2}{mkT}\frac{N}{V}\right)\right]. \tag{23}$$

Using the identity derived by integration by parts,

$$\int_0^\infty d\varepsilon_i \ln \frac{1 - \alpha n_i}{1 + (1 - \alpha)n_i} = \int_0^\infty d\varepsilon_i\, \varepsilon_i n_i, \tag{24}$$

we have the statistics-independent relation $PV = E$. In the Boltzmann limit $[\exp(\mu/kT) << 1]$, $w(\zeta) = \zeta + \alpha - 1$,

$$PV = NkT\left[1 + (2\alpha - 1)N\lambda^2/4V\right], \tag{25}$$

where $\lambda = \sqrt{2\pi\hbar^2/mkT}$. So the "statistical interactions" are attractive or repulsive depending on whether $\alpha < 1/2$ or $\alpha > 1/2$.

Whether a given system satisfies the seemingly harmless conditions (5) or (21) together with (4) is highly nontrivial. It turns out [11] that statistical transmutation happens in 1D Bethe-ansatz solvable gas, so that both of these conditions apply if particles of different pseudomomenta can be viewed as belonging to different species. On the other hand, while it was claimed [3] that the definition (4) for fractional statistics does not apply to free anyons, i.e., Newtonian particles carrying flux tubes [12,13] in two spatial dimensions, anyons in a magnetic field are known to satisfy both (4) and (5) if all anyons are in the lowest Landau level (LLL) [14], as is the case at very low temperatures. The Jastrow-type prefactor $\Pi_{a<b}(z_a - z_b)^{\theta/\pi}$, with $0 \le \theta < 2\pi$, in the anyon wave function has the effect of increasing the flux through the system by $(\theta/\pi)(N - 1)$. Thus, with fixed size and number of flux, the dimension of the effective boson Fock space [15,16] is given by $d = N_\phi - (\theta/\pi)(N - 1)$, where $N_\phi = qBV/hc \equiv V/V_0$, with q the anyon charge. Equation (2) applies with the single-anyon degeneracy $G = N_\phi$ and the statistics $\alpha = \theta/\pi$. Applying Eqs. (15) and (16) with only one energy $\varepsilon = \hbar\omega_c/2$, we have

$$n \equiv \frac{N}{G} \equiv \frac{\rho}{\rho_0} = \frac{1}{w(e^{(\varepsilon-\mu)/kT}) + \alpha}, \tag{26}$$

where $w(\zeta)$ is the positive solution of Eq. (16). Here $\rho \equiv N/V$ is the areal density, and $\rho_0 \equiv 1/V_0$. Equation (26), together with (16), determines the chemical potential μ in terms of ρ/ρ_0 and T. Thermodynamic quantities can also be expressed as functions of the ratio ρ/ρ_0. In particular, the thermodynamic potential is

$$\Omega = -kT\frac{V}{V_0}\ln\frac{1 + w}{w} = -kT\frac{V}{V_0}\ln\frac{1 + (1 - \alpha)n}{1 - \alpha n}. \tag{27}$$

The equation of state is

$$\frac{PV}{NkT} = \left(\frac{\rho}{\rho_0}\right)^{-1}\ln\frac{1 + (1 - \alpha)\rho/\rho_0}{1 - \alpha(\rho/\rho_0)}. \tag{28}$$

The pressure P is linear in T for fixed ρ. It diverges at the critical density $\rho_c = (1/\alpha)\rho_0$, which corresponds to the complete filling of the LLL. The emergence of an incompressible state at filling fraction $1/\alpha$ is a consequence of the generalized exclusion principle (20) [17]. The magnetization per unit area is

$$\mathcal{M} = -\mu_0\rho + \frac{2\mu_0}{\lambda^2}\ln\frac{1 + (1 - \alpha)\rho/\rho_0}{1 - \alpha(\rho/\rho_0)}, \tag{29}$$

where $\mu_0 = q\hbar/2mc$ is the Bohr magneton. Note the first (de Haas–van Alphen) term is statistics independent. At low temperatures, $kT \ll \hbar\omega_c$, the second term can be neglected except for ρ very close to $(1/\alpha)\rho_0$, where it gives rise to a nonvanishing, α-dependent susceptibility

$$\chi = kT\frac{q}{2\pi\hbar c}\left(-\frac{1}{B}\right)\frac{\rho/\rho_0}{(1 - \alpha\rho/\rho_0)[1 + (1 - \alpha)\rho/\rho_0]}. \tag{30}$$

The entropy per particle is also α dependent:

$$\frac{S}{N} = k\left(1 - \alpha + \frac{\rho_0}{\rho}\right)\ln\left[1 + (1 - \alpha)\frac{\rho}{\rho_0}\right] - k\ln\frac{\rho}{\rho_0} - k\left(\frac{\rho_0}{\rho} - \alpha\right)\ln\left(1 - \alpha\frac{\rho}{\rho_0}\right). \tag{31}$$

Equations (27), (28), and (29) have been derived in Ref. [6] from the known exact many-anyon solutions in the LLL.

Vortexlike quasiparticle excitations in Laughlin's incompressible $1/m$ fluid (m being odd) [18] are known to be fractionally charged anyons [18–21], and their wave functions are as if they are in the LLL (with electrons acting as quantized sources of "flux") [15]. The existence of two species of excitations, quasiholes (labeled by $-$) and quasielectrons (labeled by $+$), dictates nontrivial mutual statistics. In this case, fixing the boundary conditions means fixing the total magnetic flux N_ϕ passing through the system. The latter is related to the electron number N_e and the excitation numbers $N_\pm$ by

$$N_\phi \equiv eBV/hc = mN_e + N_- - N_+ . \quad (32)$$

The Hilbert-space dimension $d_\pm$ for both quasielectrons and quasiholes is $N_e + 1$ [15]. It follows that

$$\alpha_{--} = -\alpha_{+-} = \alpha_{-+} = 1/m, \quad \alpha_{++} = 2 - 1/m \quad (33)$$

(appropriate for *hard-core* quasielectrons [16,22]). The single excitation degeneracy in the thermodynamic limit is $G_+ = G_- = (1/m)N_\phi$. Ignoring the interaction energies and assuming the system is pure, we apply the formulas (10)–(13) to this case with the values (33) for α_{ij}. The densities $\rho_\pm$ of the excitations are given by

$$n_\pm = \frac{\rho_\pm}{\rho_0} = \frac{w_\mp + \alpha_{\mp\mp} - \alpha_{\pm\mp}}{(w_+ + \alpha_{++})(w_- + \alpha_{--}) - \alpha_{+-}\alpha_{-+}}, \quad (34)$$

where $\rho_0 \equiv G_\pm/V$; $w_\pm$ satisfy the functional equations

$$w_\pm^{\alpha_{\pm\pm}}(1 + w_\pm)^{1-\alpha_{\pm\pm}}\left(\frac{w_\mp}{1 + w_\mp}\right)^{\alpha_{\mp\pm}} = e^{(\varepsilon_\pm - \mu_\pm)/kT}. \quad (35)$$

Here $\varepsilon_\pm$ is the creation energy of a single excitation. At $T = 0$ or very close to it, thermal activation is negligible; there is only one species of excitations behaving like anyons in the LLL. If, say, $N_\phi < mN_e$, then there are only quasielectrons: $n_- = 0$, $n_+ = (mN_e - N_\phi)/G_+ = 1/(w_+ + \alpha_{++})$. One can apply the above Eqs. (26)–(31). At higher temperatures, thermal activation of quasiparticle pairs, satisfying $\mu_+ + \mu_- = 0$, becomes important and the effects of mutual statistics become manifest with increasing density of activated pairs. The thermodynamic properties at different sides of electron filling $\nu \equiv N_e/N_\phi = 1/m$ are not symmetric due to asymmetry in quasielectrons ($\alpha_{++} = 2 - 1/m$) and quasiholes ($\alpha_{--} = 1/m$). The general equation of state is

$$\frac{P}{kT} = \rho_0 \sum_{i=+,-} \ln \frac{1 + \rho_i/\rho_0 - \sum_j \alpha_{ij}(\rho_j/\rho_0)}{1 - \sum_j \alpha_{ij}(\rho_j/\rho_0)}. \quad (36)$$

When the excitation densities satisfy

$$\sum_{j=+,-} \alpha_{ij}\rho_j = \rho_0, \quad (37)$$

the pressure diverges and a new incompressible state is formed, as a result of the generalized exclusion principle obeyed by the excitations upon completely filling the LLL. At $T = 0$, e.g., for $i = +$ (quasiparticles) the limit (37) is reached when electron filling is $\nu = 2/(2m - 1)$, giving rise to the well-known hierarchical state [15,16,23]. At finite T, it may happen at somewhat different filling, because of the additional quasihole contribution in Eq. (37). Moreover, the magnetization per unit area is

$$\mathcal{M} = \sum_i \left(-\mu_i\rho_i + \frac{ekT}{mhc} \ln \frac{\rho_0 + \rho_i - \sum_j \alpha_{ij}\rho_j}{\rho_0 - \sum_j \alpha_{ij}\rho_j}\right), \quad (38)$$

with $\mu_\pm = \partial\varepsilon_\pm/\partial B$. A detailed analysis of the properties of thermally activated quasiparticles at finite temperatures will be given elsewhere [24].

To conclude, we have formulated the QSM of generalized ideal gas (with no interaction energies) for particles of fractional (mutual) statistics, in Haldane's sense. For identical particles with no mutual statistics, the statistical distribution interpolates between bosons and fermions, exhibiting fractional exclusion, which makes the particles (except bosons) more or less like fermions. Theoretical examples of generalized ideal gas include 1D Bethe-ansatz solvable gases and 2D anyons in the LLL. For real physics, our formalism applies to thermodynamics of quasiparticle excitations in pure Laughlin liquids, and may shed new light on 1D quantum systems.

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[1] See, e.g., *Fractional Statistics and Anyon Superconductivity*, edited by F. Wilczek (World Scientific, Singapore, 1989), and references therein.

[2] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).

[3] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[4] D. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. **B251** [FS13] , 117 (1985).

[5] F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).

[6] A. Dasnières de Veigy and S. Ouvry, Phys. Rev. Lett. **72**, 600 (1994).

[7] See, e.g., K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963).

[8] At fixed $\{N_i\}$, the particles of species i can be regarded as either bosons with a Fock space dimension $d_B = d_i$ or fermions with a Fock space dimension $d_F = d_i + N_i - 1$. Moreover, $d_i = G_i - \sum_j \alpha_{ij}(N_j - \delta_{ji})$ solves (3) under the condition that $d_i = G_i$ when $N_j = \delta_{ji}$. See also [22].

[9] For relevant i's the statistics α_{ii} should equal each other, and so do the chemical potentials μ_i.

[10] In the sense of incorporating mutual statistics.

[11] D. Bernard and Y.-S. Wu (to be published).

[12] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).

[13] Y. S. Wu, Phys. Rev. Lett. **52**, 2103 (1984).

[14] M. D. Johnson and G. S. Canright, Phys. Rev. B **41**, 6870 (1990); G. V. Dunne *et al.*, Nucl. Phys. **B370**, 601 (1992).

[15] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[16] S. He, X. C. Xie, and F. C. Zhang, Phys. Rev. Lett. **68**, 3460 (1992).

[17] We note that the Laughlin wave function for the $1/m$ state looks the same as if the LLL is completely filled by "superelectrons" with statistics $\alpha = m$.

[18] R. B. Laughlin, Phys. Rev. Lett. **50**, 1359 (1983).

[19] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[20] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[21] J. Yang and W. P. Su, Phys. Rev. Lett. **70**, 1163 (1993).

[22] M. D. Johnson and G. S. Canright, Phys. Rev. B **49**, 2947 (1994).

[23] M. Ma and F. C. Zhang, Phys. Rev. Lett. **66**, 1769 (1991).

[24] Y. Hatsugai, M. Kohmoto, and Y.-S. Wu (to be published).