EXTENSIONS OF GRADED AFFINE HECKE ALGEBRA MODULES

by

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ABSTRACT

In this dissertation, we study extensions of graded affine Hecke algebra modules. In particular, based on an explicit projective resolution on graded affine Hecke algebra modules, we prove a duality result for Ext-groups. This duality result with analysis on some parabolically induced modules gives a new proof of the fact that all higher Ext-groups between discrete series vanish. Finally, we study a twisted Euler-Poincaré pairing and show the pairing depends on the Weyl group structure of graded affine Hecke algebra modules.

CONTENTS

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CHAPTER 1

INTRODUCTION

1.1 Background

Graded affine Hecke algebras were defined by Lusztig [Lu1] for the study of the representations of affine Hecke algebras and p-adic groups. The relation between affine Hecke algebras and their graded ones can be thought of as an analogue of the relation between Lie groups and Lie algebras, and so graded affine Hecke algebras are simpler in certain aspects.

The classification of irreducible graded Hecke algebra modules has been studied extensively in the literature. A notable result is the Kazhdan-Lusztig geometric classification [KL] for equal parameter cases. In arbitrary parameters, a general result is the Langlands classification [Ev] which states that every irreducible module can be built from a tempered module. The classification of tempered modules can be understood from the classification of discrete series by Opdam-Solleveld [OS2] and the decomposition of relevant parabolically induced modules for tempered modules by Delorme-Opdam [DO]. There are also some other classification results [CK, Ka, Kr, KR].

The main goal of this dissertation is to study the extensions of graded affine Hecke algebra modules, that is, to understand some reducible modules rather than irreducible ones. Since we ultimately want to apply our study to understand the extensions of smooth representations in p-adic group, we also give connections of extensions in the relevant categories in Chapters 2 to 4.

There are some studies and applications of the extension problem in the literature [AP, BW, Me, OS1, OS3, Or, Pr, SS, So]. In the p-adic group setting, Adler-Prasad [AP] recently computed extensions of certain smooth representations and Prasad [Pr] also studies an Ext-analogue of the branching problem. Motivated by the study of the étale cohomology of p-adic domains, Orlik [Or] also computed the Ext-groups of generalized Steinberg representations. In the affine Hecke algebra setting, Opdam-Solleveld [OS1, OS3] computed the extensions of tempered modules of an affine Hecke algebra and applied the result to study an Arthur's formula and Kazhdan's orthogonality conjecture.

While our work is motivated from some results in the setting of p-adic groups and affine Hecke algebras, our approach is self-contained in the theory of graded affine Hecke algebras. Moreover, because of the algebraic nature of our approach, it applies to the graded Hecke algebra of noncrystallographic types (see [Kr] and [KR]). It is also possible to extend some results to some similar algebraic structures such as degenerate affine Hecke-Clifford algebras [Na] and graded Hecke algebras for complex reflection groups [SW].

1.2 Main results

Let $\mathbb{H}$ be a graded affine Hecke algebra (see Definition 3.3.25). Our study of the extension problem begins with an explicit construction of a projective resolution on $\mathbb{H}$-modules. This projective resolution is an analogue of the classical Koszul resolution for relative Lie algebra cohomology for $(\mathfrak{g}, K)$-modules.

The first application of the projective resolution is to prove a duality result for Ext-groups of $\mathbb{H}$-modules. To state the result, we need to introduce three operations $*$, $\bullet$, ι on $\mathbb{H}$ (see Section 6.2 and Section 6.5 for the detailed definitions).

The first anti-involution $*$ arises naturally from the study of unitary duals for the Hecke algebra of a p-adic group (see [BM1]), but we remove the complex conjugation from the original definition for the purpose of our study on complex parameters. The second anti-involution $\bullet$ is studied in a recent paper of Barbasch-Ciubotaru [BC2] as an Hecke algebra analogue of the compact-star operation for $(\mathfrak{g}, K)$-modules in [ALTV, Ye], but again we remove the complex conjugation from their definition. The $\bullet$-operation is also studied by Opdam [Op] in the Macdonald theory for affine Hecke algebras. The last operation ι on $\mathbb{H}$ is the Iwahori-Masumoto involution, which plays the role of tensoring with the sign representation on the level of Weyl groups and is shown by Evens-Mirković [EM] to have close connection with the geometric Fourier-Deligne transform.

For each of the operations $*, \bullet, \iota$, it induces a map from the set of $\mathbb{H}$-modules to the set of $\mathbb{H}$-modules. For an $\mathbb{H}$-module X, we denote by X^*, $X^\bullet$ and $\iota(X)$ (see Section 6.2 and Section 6.5) for the corresponding dual $\mathbb{H}$-modules, respectively.

Our first main theorem of the dissertation is the following duality on the Ext-groups:

Theorem 1.2.1. *(Theorem 6.6.63) Let Π be a based root datum $(\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ (see Section 3.1). Let $\mathbb{H}$ be the graded affine Hecke algebra associated to a based root datum Π and an arbitrary parameter function (Definition 3.3.25). Let $V = \mathbb{C}\otimes_{\mathbb{Z}}\mathcal{X}$ and let $n = \dim V$. Let X and Y be finite dimensional $\mathbb{H}$-modules. Then there exists a natural nondegenerate pairing*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \times \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*, \iota(Y)^\bullet) \to \mathbb{C}.$$

Here, the $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}$*-groups are taken in the category* $\mathfrak{R}(\mathbb{H})$ *of* $\mathbb{H}$*-modules.*

(For some comments on the formulation and the proof of Theorem 1.2.1, see Remark 6.6.64.) Theorem 1.2.1 is an analogue of the Poincaré duality for real reductive groups ([BW, Ch.I Proposition 2.9], [Kn2, Theorem 6.10]). One may also compare with the duality result in [SS, pg 133] by Schneider-Stuhler.

The second result of this dissertation is about the extensions of discrete series. Those discrete series are defined algebraically in terms of weights (Definition 7.1.73) and correspond to discrete series of p-adic groups when the parameter function is positive and equal. Since discrete series are basic building blocks of irreducible $\mathbb{H}$-modules, it may be important to first understand the extensions among them. Our second main result states that:

Theorem 1.2.2. *(Theorem 8.1.93) Let* Π *be a based root datum* $(\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ *(see Section 3.1). Let* $\mathbb{H}$ *be the graded affine Hecke algebra associated to a based root datum* Π *and an arbitrary parameter function (Definition 3.3.25). Let* $V = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$*. Assume* R *spans* V*. Let* X *be an irreducible tempered module and let* Y *be an irreducible discrete series (Definition 7.1.73). Then*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) = \begin{cases} \mathbb{C} & \text{if } X \cong Y \text{ and } i = 0 \\ 0 & \text{otherwise .} \end{cases}$$

Since a discrete series is also tempered, the statement covers the case for X and Y being discrete series. The statement for affine Hecke algebra setting is proven by Opdam-Solleveld [OS1], and the one for p-adic group setting is proven by Meyer [Me]. Ciubotaru-Trapa deduce the result from [OS1] for the graded setting using results of Solleveld. The method we prove Theorem 1.2.2 with is different from theirs and essentially makes use of Theorem 1.2.1. For the outline of the proof, see the beginning of Chapter 8.

To state the last result of the dissertation, we need some more notations. Let δ be an automorphism on the root system with $\delta(\Delta) = \Delta$. We assume δ is either the identity map or the map arising from the longest element in the Weyl group. (In the latter case, that is the map θ in Section 6.1.) The action of δ can be extended to the Weyl group, and then extended to $\mathbb{H}$. For $\mathbb{H} \rtimes \langle\delta\rangle$-modules X and Y, we define the δ-twisted Euler-Poincaré pairing on X and Y (regarded as $\mathbb{H}$-modules):

$$\mathrm{EP}^\delta_{\mathbb{H}}(X,Y) = \sum_i (-1)^i \mathrm{trace}(\delta^* : \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \to \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y)).$$

Here, δ^* is a natural map induced from the action of δ on X and Y. We now state our last theorem, which was previously proven by Reeder [Re] and independently by Odpam-Solleveld [OS1] in other settings:

Theorem 1.2.3. *(Proposition 9.1.97, Theorem 9.3.104) Suppose $\delta : \mathbb{H} \to \mathbb{H}$ is either the identity map or the map θ in Section 6.1. For any finite dimensional $\mathbb{H} \rtimes \langle \delta \rangle$-modules X and Y, $\mathrm{EP}^{\delta}_{\mathbb{H}}(X, Y)$ depends only on the Weyl group structure of X and Y. Furthermore, $\mathrm{EP}^{\delta}_{\mathbb{H}}(X, Y)$ can be expressed in terms of the characters of the Weyl group (see Theorem 9.3.104 for the explicit formula).*

Theorem 1.2.3 suggests that $\mathrm{EP}^{\delta}_{\mathbb{H}}$ is simpler for the study than each individual $\mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}$-group. Some consequences of Theorem 1.2.3 are given at the end of Chapter 9.

1.3 Future direction

We list some possible research projects built from this dissertation:

1. Extensions among discrete series have been studied in Chapter 7. Thus, it is natural to next study the extensions among tempered modules. Indeed, Opdam-Solleveld [OS3] have shown that the extensions of tempered modules can be reduced to calculations involving only finite group representations. Their proof is again in the affine Hecke algebra setting and so it is desirable if we can have a self-contained theory in the graded affine Hecke algebra. We believe intertwining operators will be the main tool for such calculations (as what we see in [OS3, So]).

2. After computing the extensions of tempered modules (or simply applying the results of Opdam-Solleveld [OS3]), we may try to look at more modules other than tempered modules. From the calculations from Chapters 7 and 8, the known information, such as composition factors in standard modules [Ci2, Lu3, CKK] and the duality result (Theorem 1.2.1), may be useful. Along the way, it is also natural to think about a homological interpretation of the Kazhdan-Lusztig type polynomials (for graded Hecke algebras) analogous to classical ones (see [Hu1, Section 8.11]).

3. In Chapter 4, we will see the relations of the extensions of smooth representations of p-adic groups and extensions of graded affine Hecke algebra modules. However, to translate the results to the level of p-adic groups, one may have to go through explicitly the equivalence of categories. It may be interesting to see how the results may be formulated more explicitly in some p-adic groups of small ranks or in $GL(n, F)$. Some examples of Hecke algebra isomorphisms are in [Ki, Lu2, Mo].

1.4 Outline of the dissertation

This dissertation is mainly divided into two parts. Each chapter begins with an introduction giving a more detailed guide.

The first part includes Chapters 2 to 4, which explain the connections between p-adic groups, affine Hecke algebras and graded affine Hecke algebras. The central ideas of the connection are around the theory of types (Section 2.5), Hecke algebra isomorphisms and the Lusztig's reduction theorem (Section 3.4). Emphasis on the connection of Ext-groups is in Chapter 4.

The second part includes Chapters 5 to 9, which study some aspects of extensions of graded affine Hecke algebra modules. Chapter 5 (with some parts in Chapter 4) serves as the groundwork for the study. Chapter 6 proves a duality result for Ext-groups. Chapter 7 reviews the Langlands classification, from which we get some general information on extensions. Chapter 8 computes the extensions of discrete series by using results in Chapters 6 and 7. Chapter 9 studies the Euler-Poincaré pairing and gives applications at the end.

CHAPTER 2

SMOOTH REPRESENTATIONS OF P-ADIC GROUPS

This chapter is devoted to seeing how the study of smooth representations of p-adic groups can be transferred to the study of some related algebras. The main references for this chapter are [BK], [HK], [BH, Chapter 4] and [Ro].

2.1 Representations of p-adic groups

Let G be a reductive p-adic group (over a field of characteristic 0). A (complex) *representation* of G is a pair (π, X) consisting of a complex vector space X and a homomorphism $\pi : G \to GL(X)$ from G to the set $GL(X)$ of invertible linear automorphisms on X. We may sometimes only write π or X for the representation. A subspace Y of a representation X is said to be a *subrepresentation* if Y is G-invariant, meaning that $\pi(g)Y \subset Y$ for all $g \in G$. A representation (π, X) of G is said to be *irreducible* if there is no nontrivial proper G-invariant subspace of X.

Let (π, X) be a representation of G. Let $Z \subset Y$ be subrepresentations (possibly 0 or the entire space) of X. Then, there is a natural G-representation structure $\pi_{Y/Z}$ on the space Y/Z determined by

$$\pi_{Y/Z}(g)(y + Z) = \pi(g)y + Z.$$

Then the G-representation $(\pi_{Y/Z}, Y/Z)$ is said to be a *subquotient* of X.

A representation (π, X) of G is said to be *smooth* if for every vector v in V there is an open compact subgroup of G such that $\pi(g)v = v$ for all $g \in G$. Let $\mathfrak{R}(G)$ be the category of smooth representations of G. A representation (π, X) is said to be *admissible* if for every open compact subgroup K, the K-invariant subspace of X is finite-dimensional.

2.2 The Hecke algebra

The Hecke algebra $\mathcal{H}(G)$ of G is the convolution algebra of locally constant complex-valued functions on G with compact support. Fix a Haar measure. We shall use $*$ to denote the convolution of $\mathcal{H}(G)$. An $\mathcal{H}(G)$-module V is said to be *nondegenerate* if $\mathcal{H}(G)V = V$. Let $\mathfrak{R}(\mathcal{H}(G))$ be the category of nondegenerate $\mathcal{H}(G)$-modules. For each compact open subgroup of K, denote by $\mathcal{H}(G,K)$, the bi-K-invariant functions in $\mathcal{H}(G)$:

$$\mathcal{H}(G,K) = \{ f \in \mathcal{H}(G) : f(kgk') = f(g) \quad \text{for any } k,k' \in K \text{ and } g \in G \}. \tag{2.1}$$

Denote by e_K the normalized characteristic function of K i.e.,

$$e_K(g) = \begin{cases} \mathrm{vol}(K)^{-1} & \text{for } g \in K \\ 0 & \text{for } g \notin K. \end{cases}$$

The element e_K is an idempotent in $\mathcal{H}(G)$ and $\mathcal{H}(G,K) = e_K * \mathcal{H}(G) * e_K$.

Local constancy and compact support of functions in $\mathcal{H}(G)$ implies that for each $f \in \mathcal{H}(G)$, there exists compact open subgroup K_1 and K_2 such that $f(k_1 g) = f(g) = f(gk_2)$ for all $k_1 \in K_1$, $k_2 \in K_2$ and $g \in G$. Hence, since f has compact support, one sees that f is a finite linear combination of characteristic functions of double cosets KgK. Hence,

$$\mathcal{H}(G) = \bigcup_K \mathcal{H}(G,K) = \bigcup_K e_K * \mathcal{H}(G) * e_K,$$

where K runs over all the compact open subgroups of G. Then for a nondegenerate $\mathcal{H}(G)$-module X,

$$X = \bigcup_K e_K X,$$

where K again runs over all the compact open subgroups of G.

Proposition 2.2.4. *The category $\mathfrak{R}(G)$ of smooth representations of G is equivalent to the category $\mathfrak{R}(\mathcal{H}(G))$ of nondegenerate $\mathcal{H}$-modules. In particular, the set of irreducible smooth representations of G is in natural bijection with the set of nondegenerate simple $\mathcal{H}(G)$-modules.*

Proof. We follow the proof of [BH, Ch. 4 Sec.1 Proposition 1]. Fix a Haar measure on G. Let (π, V) be a smooth representation of G. Any function $f \in \mathcal{H}(G)$ defines an endomorphism on V via the map

$$\pi_{\mathcal{H}} : \mathcal{H} \to \mathrm{End}(V), \quad f \mapsto \int_{g\in G} f(g)\pi(g)dg.$$

Note that

$$\begin{aligned}\pi_{\mathcal{H}}(f_1 * f_2) &= \int_{h\in G}\int_{g\in G} f_1(hg^{-1})f_2(g)\pi(h)dgdh \\ &= \int_{h'\in G}\int_{g\in G} f_1(h')f_2(g)\pi(h'g)dgdh' \\ &= \int_{h'\in G} f_1(h')\int_{g\in G} f_2(g)\pi(h')\circ\pi(g)dgdh' \\ &= \left(\int_{h'\in G} f_1(h')\pi(h')dh'\right)\circ\left(\int_{g\in G} f_2(g)\pi(g)dg\right) \\ &= \pi_{\mathcal{H}}(f_1)\circ\pi_{\mathcal{H}}(f_2).\end{aligned}$$

Hence, $\pi_{\mathcal{H}}$ defines an $\mathcal{H}$-module structure on V. The $\mathcal{H}$-module structure on V is indeed nondegenerate since $V = \bigcup_K V^K$ by the smoothness of V and $\pi_{\mathcal{H}}(e_K)V^K = V^K$.

We now define a smooth G-representation structure for each nondegenerate $\mathcal{H}(G)$-module $(\pi_{\mathcal{H}}, X)$. First there is a canonical map from $\mathcal{H}(G)\otimes_{\mathcal{H}(G)} X$ to X. It is clear that the map is surjective. To see the map is also injective, we pick an element $\sum_{i=1}^m f_i\otimes x_i \in \mathcal{H}(G)\otimes_{\mathcal{H}(G)} X$ in the kernel of the map. Then we pick a compact open subgroup K such that $e_K * f_i * e_K = f_i$ and $\pi_{\mathcal{H}}(e_K)x_i = x_i$ for all i. Then we have

$$\sum_{i=1}^m f_i\otimes x_i = e_K\otimes\sum_{i=1}^m \pi_{\mathcal{H}}(f_i)x_i = 0.$$

Hence, $\mathcal{H}(G)\otimes_{\mathcal{H}(G)} X$ is isomorphic to X. Then the left translation on $\mathcal{H}(G)$ induces an action on X. Explicitly, for $x\in X$, choose a compact open subgroup K such that $e_K * x = x$. Then we define the action of g is by $\pi(g)x = \mathrm{vol}(K)^{-1}\pi_{\mathcal{H}}(\mathrm{ch}_{gK})x$, where ch_{gK} is the characteristics function of the set gK.

With these maps and Lemma 2.2.5 below, one can check the equivalence of categories.

□

Lemma 2.2.5. *Let (π, X) be a smooth representation of G. Let K be a compact open subgroup of G. Then for $x\in X$, $\pi(k)x = x$ for all $k\in K$ if and only if $\pi_{\mathcal{H}}(e_K)x = x$.*

Proof.

$$\begin{aligned}\pi_{\mathcal{H}}(e_K)x &= \int_G e_K(g)\pi(g)x\ dg \\ &= \int_K \pi(g)x\ dg \\ &= \int_K \pi(gk^{-1})\pi(k)x\ dg \qquad \text{for any } k\in K \\ &= \int_G e_K(g)\pi(g)\pi(k)x\ dg \\ &= \pi_{\mathcal{H}}(e_K)\pi(k)x.\end{aligned}$$

The equality implies the if direction. For the only if direction, the second expression is equal to x if $\pi(k)x = x$.

□

Proposition 2.2.4 allows us to reduce the study of smooth representations of p-adic groups to the study of nondegenerate $\mathcal{H}(G)$-modules. The Hecke algebra $\mathcal{H}(G)$ is not a unital algebra, but it has a lot of idempotents. It may be useful for us to have a more general definition here, which may also be applied to some other context later.

Definition 2.2.6. An associative algebra $\mathcal{A}$ over $\mathbb{C}$ is *idempotented* if $\mathcal{A}$ has a countable set of idempotents e such that $\mathcal{A}$ is the union of all the sets $e\mathcal{A}e$. An $\mathcal{A}$-module X is said to be *nondegenerate* if $\mathcal{A}X = X$. Since a submodule of a nondegenerate $\mathcal{A}$-module is still nondegenerate, the category of nondegenerate $\mathcal{A}$-modules is abelian.

Let $\mathcal{A}$ be an idempotented algebra and let e be an idempotent element. Then $e\mathcal{A}e$ is an algebra with an unit. Let $\mathfrak{R}(\mathcal{A})$ be the category of nondegenerate $\mathcal{A}$-modules and let $\mathfrak{R}(e\mathcal{A}e)$ be the category of $e\mathcal{A}e$-modules. Let the restriction functor and induction functor be, respectively, as follows:

$$r : \mathfrak{R}(\mathcal{A}) \to \mathfrak{R}(e\mathcal{A}e), \quad X \mapsto eX,$$

$$i : \mathfrak{R}(e\mathcal{A}e) \to \mathfrak{R}(\mathcal{A}), \quad X \mapsto \mathcal{A} \otimes_{e\mathcal{A}e} X.$$

Note that for $M \in \mathfrak{R}(e\mathcal{A}e)$, $r(i(M)) = M$, which follows from the fact that $ea = (eae)e \in e\mathcal{A}e$ for $a \in \mathcal{A}$. The relation between $\mathcal{A}$-modules and $e\mathcal{A}e$-modules is given below:

Proposition 2.2.7. *Let* $\mathrm{Irr}(\mathcal{A}, e)$ *be the set of irreducible* $\mathcal{A}$*-modules* X *with the property that* $eX \neq 0$*. Let* $\mathrm{Irr}(e\mathcal{A}e)$ *be the set of irreducible* $e\mathcal{A}e$*-modules. Then* r *gives a canonical bijection between* $\mathrm{Irr}(\mathcal{A}, e)$ *and* $\mathrm{Irr}(e\mathcal{A}e)$*.*

Proof. We first see that r gives a well-defined map i.e., for $X \in \mathrm{Irr}(\mathcal{A}, e)$, $r(X)$ is irreducible. Let N be a nonzero $e\mathcal{A}e$-submodule of $r(X)$. Then $\mathcal{A}N = X$ by the irreducibility of X. Hence, $(e\mathcal{A})N = eX = r(X)$. On the other hand, $N = (e\mathcal{A}e)N = (e\mathcal{A})N$. Hence, $N = r(X)$ as desired.

Let M be an irreducible $e\mathcal{A}e$-module. Let N be a maximal submodule of $i(M)$ such that $\mathrm{Hom}_{e\mathcal{A}e}(M, r(N)) = 0$. (The existence of the maximal submodule is guaranteed by Zorn's Lemma.) We first show that N is the maximal submodule of $i(M)$. Let N' be a $e\mathcal{A}e$-

submodule of M properly containing N. Then we have the exact sequence $0 \to N' \to i(M)$ and so

$$0 \to \mathrm{Hom}_{\mathcal{A}}(i(M), N') \to \mathrm{Hom}_{\mathcal{A}}(i(M), i(M)).$$

By Frobenius reciprocity and our choice of N', $\mathrm{Hom}_{\mathcal{A}}(i(M), N') = \mathrm{Hom}_{e\mathcal{A}e}(M, r(N')) \neq 0$. On the other hand, by Frobenius reciprocity again we also have

$$\mathrm{Hom}_{\mathcal{A}}(i(M), i(M)) = \mathrm{Hom}_{e\mathcal{A}e}(M, r(i(M))) = \mathrm{Hom}_{e\mathcal{A}e}(M, M) = \mathbb{C}.$$

Then the above exact sequence implies that there exists a composition of maps $i(M) \to N' \to i(M)$ such that the composition is an isomorphism and the second map is the natural injection. This implies the second map is also surjective and so is an isomorphism. Hence, $N \cong i(M)$. This shows N is the unique maximal submodule of $i(M)$ and defines a map $s : \mathrm{Irr}(e\mathcal{A}e) \to \mathrm{Irr}(\mathcal{A}, e)$.

We then show that $r \circ s$ is an identity. For $M \in \mathrm{Irr}(e\mathcal{A}e)$, $r \circ s(M) \subseteq r(i(M)) = M$. Since $r \circ s(M) \neq 0$ and M is irreducible, we have $r \circ s(M) = M$. To conclude r and s are bijections, it suffices to show that r is injective. By Frobenius reciprocity,

$$\mathrm{Hom}_{e\mathcal{A}e}(r(M_1), r(M_2)) = \mathrm{Hom}_{\mathcal{A}}(i \circ r(M_1), M_2) \neq 0.$$

However, by the uniqueness of the maximal submodule, $M_1 \cong M_2$. This completes the proof.

□

We now formulate the above result in terms of smooth representations of p-adic groups, which follows from Lemma 2.2.5 and Proposition 2.2.7.

Corollary 2.2.8. *Let K be a compact open subgroup of G. The set of irreducible smooth representations of G with K-fixed vectors is bijective to the set of simple nondegenerate $\mathcal{H}(G, K)$-modules.*

It is also natural to ask if Proposition 2.2.7 can be formulated in the level of categories. The answer is likely to be negative in general but we have the following equivalent conditions in Proposition 2.2.9 below. We define one more notation. Let $\mathfrak{R}(\mathcal{A}, e)$ be the full subcategory of $\mathfrak{R}(\mathcal{A})$ whose objects are nondegenerate $\mathcal{A}$-modules generated by e-fixed vectors i.e.,

$$\mathfrak{R}(\mathcal{A}, e) = \{X \in \mathfrak{R}(\mathcal{A}) : X = \mathcal{A}eX\}.$$

Proposition 2.2.9. *The following statements are equivalent:*

(1) $i(\mathfrak{R}(e\mathcal{A}e)) = \mathfrak{R}(\mathcal{A}, e)$ *as full subcategories,*

(2) $\mathfrak{R}(\mathcal{A}, e)$ *is closed under subquotients,*

(3) $\mathfrak{R}(e\mathcal{A}e) \cong \mathfrak{R}(A, e)$*.*

Proof. We first prove (1) implies (3). By Frobenius reciprocity, $\mathrm{Hom}_{\mathcal{A}}(i(X), i(Y)) = \mathrm{Hom}_{e\mathcal{A}e}(X, r(i(Y))) = \mathrm{Hom}_{e\mathcal{A}e}(X, Y)$. Hence, $i(\mathfrak{R}(e\mathcal{A}e)) \cong \mathfrak{R}(e\mathcal{A}e)$. This proves (1) implies (3). That (3) implies (2) is clear as $\mathfrak{R}(e\mathcal{A}e)$ is closed under subquotients.

It remains to consider (2) implies (1). Since for any object M in $\mathfrak{R}(e\mathcal{A}e)$, $r(i(M)) = M$ and M generates the $\mathcal{A}$-module $i(M)$, $i(\mathfrak{R}(e\mathcal{A}e))$ is a subcategory of $\mathfrak{R}(\mathcal{A}, e)$. We now prove $\mathfrak{R}(\mathcal{A}, e)$ is a subcategory of $i(\mathfrak{R}(e\mathcal{A}e))$. Let Y be an object in $\mathfrak{R}(\mathcal{A}, e)$. Now we have seen that $i(r(X))$ is also an object in $\mathfrak{R}(\mathcal{A}, e)$. By Frobenius reciprocity, we have a natural surjective map from $i(r(X))$ to X. Let N be the kernel of the surjective map. By the fact that $\mathfrak{R}(\mathcal{A}, e)$ is closed under subquotient, Y is an object in $\mathfrak{R}(\mathcal{A}, e)$. On the other hand, since r is an exact functor, $r(Y) = 0$ and hence $Y = 0$. This implies $X = i(r(X))$, as desired.

□

2.3 The Bernstein decomposition

The Bernstein decomposition, roughly speaking, expresses the category of smooth representations of G into the product of indecomposable full subcategories. The goal of this section is to describe such a decomposition.

We first review a construction of irreducible smooth representations of G by using parabolic induction. Let $P = M_P U_P$ be a parabolic subgroup of G with a Levi subgroup M_P and a unipotent radical U_P. The Levi subgroup M_P is still a reductive p-adic group. Then define Ind_P^G to be a functor from $\mathfrak{R}(M_P)$ to $\mathfrak{R}(G)$ such that for an object (σ, X_σ) in $\mathfrak{R}(M_P)$,

$$\mathrm{Ind}_P^G X_\sigma = \left\{ f : G \to X_\sigma : \begin{array}{c} f \text{ is locally constant and } f(mug) = \sigma(m) f(g) \\ \text{for all } m \in M_P \text{ and } u \in U_P \end{array} \right\},$$

and G acts on $\mathrm{Ind}_P^G X_\sigma$ by a right translation. Ind_P^G admits a left adjoint functor, usually called *Jacquet functor* with respect to U.

An irreducible smooth representation of G is *supercuspidal* if it is not a subquotient of any proper parabolically induced representations. Equivalently a smooth representation (π, X_π) is supercuspidal if and only if the Jacquet functor on X_π, with respect to any nontrivial unipotent radical U, is zero.

The following result says that each irreducible smooth representation of G appears in the parabolic induction from a supercuspidal representation of some Levi subgroup of G in a unique way (up to conjugation).

Theorem 2.3.10. (Jacquet) (see [Ro, Proposition 1.7.2.1, Corollary 1.10.4.3]) *Let* (π, X_π) *be an irreducible smooth representation of* G*. Then*

(1) there exists a parabolic subgroup $P = M_P U_P$ *and a supercuspidal representation* (σ, V_σ) *of* M *such that* (π, X_π) *is a subquotient of the parabolic induced representation* $\mathrm{Ind}_P^G(\sigma, X_\sigma)$*.*

(2) Suppose $(P_1, (\sigma_1, X_{\sigma_1}))$ *and* $(P_2, (\sigma_2, X_{\sigma_2}))$ *are two pairs of data satisfying the condition in (1). Then there is a* $g \in G$ *such that* $gP_1g^{-1} = P_2$ *and* $({}^g\sigma_1, X_1) \cong (\sigma_2, X_2)$*. Here,* ${}^g\sigma_1$ *is a representation of* M_{P_2} *such that* ${}^g\sigma_1(m) = \sigma_1(g^{-1}mg)$ *for any* $m \in M_{P_2}$*.*

(3) Let P *and* P' *be parabolic subgroups containing the same Levi subgroup* M*. Then* $\mathrm{Ind}_P^G(\sigma, X_\sigma)$ *and* $\mathrm{Ind}_{P'}^G(\sigma, X_\sigma)$ *have the same irreducible composition factors.*

Thus, a classification of irreducible smooth representations of G can be achieved by realizing all the supercuspidal representations and decomposing all the parabolically induced representation $\mathrm{Ind}_P^G(\sigma, X_\sigma)$ for all supercuspidal representations (σ, X_σ) of M_P. This is the Harish Chandra's philosophy of cusp forms.

In order to describe the Bernstein decomposition, we also need the notion of unramified characters for a Levi subgroup M of G. Roughly speaking, an unramified character of M is a character of M that is trivial on a certain normal subgroup of M. Since we will not directly use it except in the description of Bernstein decomposition, we refer readers to [BW, Chapter X Section 2] or [Ro, Section 1.4.1] for the precise definition. We denote the set of those characters by $\mathcal{X}_{\mathrm{nr}}(M)$.

We first describe the set which parametrizes the indecomposable subcategories in the Bernstein decomposition.

Definition 2.3.11. Let $\mathscr{M}$ be a set of pairs (M, σ) consisting of a Levi subgroup M of G and a supercuspidal representation σ of M. Define the following equivalence relation in $\mathscr{M}$: $(M_1, \sigma_1) \sim (M_2, \sigma_2)$ if and only if there is $g \in G$ and some unramified character $\nu \in \mathcal{X}_{\mathrm{nr}}(M_2)$ such that

$$gM_1g^{-1} \quad \text{and} \quad {}^g\sigma_1 \cong \sigma_2\nu.$$

Denote by $[M, \sigma]_G$ the equivalence classes containing the pair (M, σ) in $\mathscr{M}$. Denote by $\mathfrak{B}(G)$ the set of all equivalence classes of $\mathscr{M}$.

Recall from Proposition 2.3.10(1) that each irreducible π is a subquotient of a parabolically induced representation $\mathrm{Ind}_P^G\sigma$ for a parabolic subgroup P and a supercuspidal representation σ of M_P. Then we can associate such irreducible representation π to the equivalence class $[M_P,\sigma]_G$ in $\mathcal{B}(G)$. Proposition 2.3.10 (2) guarantees such association is well-defined. We say that $[M_P,\sigma]_G$ is the *inertia class* of π.

We are now ready to describe the subcategories in the Bernstein decomposition. For each $\mathfrak{s}\in\mathfrak{B}(G)$, let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathfrak{R}(G)$ whose objects are (not necessarily irreducible) representations π for which all irreducible subquotients have inertia support $\mathfrak{s}$. Then the Bernstein decomposition of $\mathfrak{R}(G)$ is as follows:

Theorem 2.3.12. *The category $\mathfrak{R}(G)$ of smooth representations of G can be decomposed into the product of full subcategories as follows:*

$$\mathfrak{R}(G)=\prod_{\mathfrak{s}\in\mathfrak{B}(G)}\mathfrak{R}^{\mathfrak{s}}(G).$$

Moreover, each $\mathfrak{R}^{\mathfrak{s}}(G)$ is an indecomposable full subcategory of $\mathfrak{R}(G)$.

2.4 Representations of Iwahori-fixed vectors

We assume $G=\mathcal{G}(F)$ to be a split p-adic group, which is the F-rational points of a connected split reductive group $\mathcal{G}$ over a p-adic field F. Let $\mathfrak{o}$ be the ring of integers in F and let $\mathfrak{p}$ be the maximal ideal in $\mathfrak{o}$. The residue field $\mathfrak{o}/\mathfrak{p}$ is isomorphic to a finite field $\mathbb{F}_q$ of order q. Let $K=\mathcal{G}(\mathfrak{o})$ be a maximal compact subgroup of G. There is a natural surjective homomorphism from K to $\mathcal{G}(\mathfrak{o}/\mathfrak{p})$. The Iwahori subgroup $\mathcal{I}$ is the inverse image of the Borel subgroup of $\mathcal{G}(\mathfrak{o}/\mathfrak{p})$. Moreover, $\mathcal{I}$ is a compact open subgroup of G.

For a smooth representation (π,X) of G, let $\pi^{\mathcal{I}}$ or $X^{\mathcal{I}}$ be the set of $\mathcal{I}$-fixed vectors of $\mathcal{I}$ i.e.,

$$X^{\mathcal{I}}=\{x\in X:\pi(g)x=x\quad\text{for all }g\in\mathcal{I}\}.$$

We state the following Borel-Casselman equivalence of categories:

Theorem 2.4.13. *[Bo] Let G be a split p-adic group. Let B be a Borel subgroup of G. Let $\mathfrak{t}=[B,\mathrm{trivial}]_G\in\mathcal{B}(G)$. We have the following:*

(1) Let π be an irreducible smooth representation of G. Then $\pi^{\mathcal{I}}\neq 0$ if and only if π is an object in $\mathfrak{R}^{\mathfrak{t}}(G)$.

(2) $\mathfrak{R}^{\mathfrak{t}}(G)$ is equivalent to $\mathfrak{R}(\mathcal{H}(G,\mathcal{I}))$.

For the proof of Theorem 2.4.13, one refers to [Bo] or [HK, Corollary 4.2]. We explain how (2) can be obtained from (1) and the Bernstein decomposition.

Proof for (2) (from (1)). Let $\mathfrak{R}(G,\mathcal{I})$ be the subcategory of $\mathfrak{R}(G)$ whose objects are smooth representations of G generated by $\mathcal{I}$-fixed vectors. We first show that $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}(G,\mathcal{I})$. Let X be an object in $\mathfrak{R}(G,\mathcal{I})$. Then by the Bernstein decomposition, decompose X as

$$X = \bigoplus_{\mathfrak{s}\in\mathcal{B}(G)} X^{\mathfrak{s}}.$$

Then by (1), $(X^{\mathfrak{s}})^{\mathcal{I}} = 0$ for $\mathfrak{s} \neq [B, \text{trivial}]_G$. Hence, $X^{\mathcal{I}} = (X^{\mathfrak{t}})^{\mathcal{I}}$. Then X is generated by $(X^{\mathfrak{t}})^{\mathcal{I}}$ and so is an object in $\mathfrak{R}^{\mathfrak{t}}(G)$. We now let X be an object in $\mathfrak{R}^{\mathfrak{t}}(G)$. Let Y be the representation generated by $X^{\mathcal{I}}$. Then we have a short exact sequence,

$$0 \to Y \to X \to Z \to 0,$$

and we want to show that $Z = 0$. Since taking the $\mathcal{I}$-fixed vectors is an exact functor, we have $Z^{\mathcal{I}} = 0$. Then by (1) and the Bernstein decomposition, $Z \in \bigoplus_{\mathfrak{s}\in\mathcal{B}(G)\setminus\{\mathfrak{t}\}} \mathfrak{R}^{\mathfrak{s}}(G)$. Since $X \in \mathfrak{R}^{\mathfrak{t}}(G)$, we conclude that $Z = 0$ from the surjective map from X to Z. Hence, $Z = Y$ is an object in $\mathcal{R}(G,\mathcal{I})$. This completes the proof for $\mathfrak{R}^{\mathfrak{t}}(G) = \mathfrak{R}(G,\mathcal{I})$. By Proposition 2.2.4 and Lemma 2.2.5, we also have $\mathfrak{R}(G,\mathcal{I}) \cong \mathfrak{R}(\mathcal{H}(G), e_{\mathcal{I}})$. Since we also proved $\mathfrak{R}^{\mathfrak{t}}(G) = \mathfrak{R}(G,\mathcal{I})$, $R(\mathcal{H}(G), e_{\mathcal{I}})$ is closed under subquotients. By Proposition 2.2.9, we have

$$\mathfrak{R}^{\mathfrak{t}}(G) = \mathfrak{R}(G,\mathcal{I}) \cong \mathfrak{R}(\mathcal{H}(G), e_{\mathcal{I}}) = \mathfrak{R}(\mathcal{H}(G,\mathcal{I})).$$

2.5 Spherical function algebras and theory of types

It is natural to extend the result of the previous section to other Bernstein components $\mathfrak{R}^{\mathfrak{s}}(G)$. For this reason, we introduce another algebra, that is the spherical function algebra below. The theory relating those spherical function algebras and the Bernstein components is indeed the theory of types by Bushnell-Kutzko [BK] and others.

Definition 2.5.14. Let K be a compact open subgroup of G. Let $\rho : K \to GL(U)$ be a representation of K. Let $\mathcal{H}(G,\rho)$ be the convolution algebra of the compactly supported functions $f : G \to \text{End}(U)$. Define

$$e_{K,\rho}(g) = \begin{cases} \frac{\rho(g)}{\text{vol}(K)} & \text{for } g \in K \\ 0 & \text{for } g \notin K. \end{cases}$$

$e_{\rho,K}$ is an idempotent. Then define the algebra $\mathcal{H}(G,K,\rho)$ with an unit to be

$$\mathcal{H}(G,K,\rho) = e_{\rho,K} * \mathcal{H}(G,\rho) * e_{K,\rho}.$$

Let $\mathfrak{R}(G,\rho)$ be a full subcategory of $\mathfrak{R}(G)$ whose objects are smooth representations of G generated by the K-isotypic components of $\rho^\vee$. Here, $\rho^\vee$ is the contragradient representation of ρ.

Definition 2.5.15. (c.f Proposition 2.2.9) A pair (K,ρ) as in Definition 2.5.14 is called a *type* in G if $\mathfrak{R}(G,\rho)$ is closed under subquotients.

The following is an extension of the Iwahori subgroup case in Theorem 2.4.13:

Theorem 2.5.16. *[BK, Proposition 3.3, Theorem 4.3] Suppose (K,ρ) is a type in G. Then we have the followings:*

(1) $\mathfrak{R}(G,\rho)$ is equivalent to $\mathfrak{R}(\mathcal{H}(G,K,\rho))$.

(2) There exists a finite subset $\mathfrak{S}$ of $\mathfrak{B}(G)$ such that the categories $\prod_{\mathfrak{s}\in\mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(G)$ and $\mathfrak{R}(\mathcal{H}(G,K,\rho))$ are equivalent.

CHAPTER 3

AFFINE HECKE ALGEBRAS AND THEIR GRADED VERSION

An affine Hecke algebra is, roughly speaking, a deformation of an affine Weyl group. Its importance for the study of p-adic group representations comes from the fact that many spherical function algebras (discussed in Section 2.5) are often closely related to an affine Hecke algebra. Moreover, various important information in the harmonic analysis of p-adic groups such as Plancherel measures (e.g., [HO]) and unitarity (e.g., [BM1], [BC1]) can also be recovered from the representation theory of an affine Hecke algebra.

The graded affine Hecke algebra was introduced by Lusztig [Lu1] from a filtration of the affine Hecke algebra. The graded Hecke algebra can be thought to be the linear counterpart of the affine Hecke algebra as an analogue of the relation between a reductive Lie group and its Lie algebra, and so is simpler for some study compared with the affine Hecke algebra. Lusztig [Lu1] showed that the representation theory of an affine Hecke algebra is essentially equivalent to that of its graded version, and classified all simple modules for geometric parameters.

In this chapter, we review several definitions related to affine Hecke algebras and their graded ones. We will also see how the study of representations of affine Hecke algebras can be reduced to the study of the representations of graded affine ones. The main references for this chapter are [Lu1] and [OS2, Section 2].

3.1 Root data and affine Weyl groups

Let R be a reduced crystallographic root system. Let W be the reflection group associated to R. Let Δ be a fixed choice of simple roots of R. Let V_0 be a real vector space containing R and let $V_0^\vee = \mathrm{Hom}_{\mathbb{R}}(V_0, \mathbb{R})$. Let R^+ be the set of positive roots in R determined by Δ. For $\alpha \in R$, let s_α be the reflection associated to α and let $\alpha^\vee \in V_0^\vee$ such that

$$s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha,$$

where $\langle v, \alpha^\vee \rangle = \alpha^\vee(v)$. Let $R^\vee \subset V_0^\vee$ be the collection of all $\alpha^\vee$. The crystallographic condition on R means that $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for any $\alpha \in R$ and $\beta^\vee \in R^\vee$.

Let $\mathcal{X} \subset V_0$ and $\mathcal{Y} \subset V_0^\vee$ be lattices with the following properties:

(1) $\mathcal{X}$ and $\mathcal{Y}$ span V_0 and $V_0^\vee$, respectively;

(2) the paring $\langle .,. \rangle : \mathcal{X} \times \mathcal{Y} \to \mathbb{Z}$ defined by $\langle x, y^\vee \rangle = y^\vee(x)$ is perfect;

(3) $R \subset \mathcal{X}$ and $R^\vee \subset \mathcal{Y}$.

The quadruple $(\mathcal{X}, R, \mathcal{Y}, R^\vee)$ is called a *root datum* and the collection $(\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ is called a *based root datum*.

Let $V = \mathbb{C} \otimes_{\mathbb{R}} V_0$ and let $V^\vee = \mathbb{C} \otimes_{\mathbb{R}} V_0^\vee$. Extend $\langle , \rangle$ to a bilinear form on $V \times V^\vee$.

Example 3.1.17. Let $\mathcal{Q} = \mathbb{Z}R$ be the root lattice of R and let $\mathcal{P}^\vee$ is the coweight lattice of R i.e.,

$$\mathcal{P}^\vee = \left\{ \lambda^\vee \in V_0^\vee : \langle \alpha, \lambda^\vee \rangle \in \mathbb{Z} \quad \text{for all } \alpha \in R \right\}.$$

Then $(\mathcal{Q}, R, \mathcal{P}^\vee, R^\vee)$ is an example of root data.

The *affine Weyl group* W^{aff} associated to R is the group $\mathcal{Q} \rtimes W$, where $\mathcal{Q}$ is the root lattice. Let $W(\mathcal{X}) = \mathcal{X} \rtimes W$. We may sometimes call $W(\mathcal{X})$ an extended or generalized affine Weyl group. For any element $x \in \mathcal{X}$, we shall write a^x for the corresponding element in $\mathcal{X} \rtimes W$ in order to avoid confusion of the multiplication in $\mathcal{X}$ (i.e., $a^{x_1} a^{x_2} = a^{x_1 + x_2}$). For the element $w \in W$, we keep writing w for the corresponding element in $W(\mathcal{X})$.

Example 3.1.18. Let R be of type A_1. Then the group W^{aff} is generated by two reflections and there is no relation between the two reflections. The affine Weyl group is sometimes called an infinite dihedral group.

Define the affine root system $\widetilde{R}$ associated to R to be the set $R^\vee \times \mathbb{Z}$. For $\alpha \in \Delta$, set $\widetilde{\alpha} = (\alpha^\vee, 0)$, and for the longest element $\alpha_0 \in R$, set $\widetilde{\alpha}_0 = (-\alpha_0^\vee, 1)$ and set $\widetilde{\Delta} = \{\widetilde{\alpha} : \alpha \in \Delta\} \cup \{\widetilde{\alpha}_0\}$. We call the elements in $\widetilde{\Delta}$ simple affine roots. Define the set $R^{\vee,+}$ to be the set of positive coroots in $R^\vee$ and the set $R^{\vee,-}$ to be the set of negative coroots in $R^\vee$. Define the set $\widetilde{R}^+$ of positive affine roots and the set $\widetilde{R}^-$ of negative affine roots:

$$\widetilde{R}^+ = R^{\vee,+} \times \{0\} \cup R^\vee \times \mathbb{Z}^+,$$

$$\widetilde{R}^- = R^{\vee,-} \times \{0\} \cup R^\vee \times \mathbb{Z}^-.$$

The affine roots in $R^\vee \times \mathbb{Z}$ can be viewed as the linear transformations on $\mathcal{X}$ given by the action $(\alpha^\vee, k).\lambda = \alpha^\vee(\lambda) + k$. The action of $W(\mathcal{X})$ on $\widetilde{R}$ is given by $(w(\alpha^\vee, k))(\lambda) =$

$(\alpha^\vee, k)(w^{-1}\lambda)$. If we write w in the form $a^\lambda w'$ for $\lambda \in \mathcal{X}$ and $w' \in W$, then $(w'a^\lambda)(\alpha^\vee, k) = (w'(\alpha^\vee), k - \alpha^\vee(\lambda))$.

Define the length function $l : W(\mathcal{X}) \to \mathbb{N}$ by

$$l(w) = |w(\widetilde{R}^+) \cap \widetilde{R}^-|,$$

which is the number of positive affine roots sent to negative affine roots by w. Define the subgroup

$$\Omega = \{w \in W(\mathcal{X}) : l(w) = 0\}.$$

It is known that Ω is abelian and is isomorphic to the quotient $\mathcal{X}/\mathcal{Q}$. Moreover,

$$W(\mathcal{X}) \cong W^{\text{aff}} \rtimes \Omega.$$

Example 3.1.19. Note that in the case of type A_1, let $\mathcal{P}$ be the weight lattice which is the $\mathbb{Z}$-span of $\frac{1}{2}\alpha$. The groups $W(\mathcal{P})$ and $W(\mathcal{Q})$ are isomorphic, but their length functions are not the same under such isomorphism. Denote by $s_{\alpha,r}$ the reflection along the hyperplane (or simply a point) $\frac{r}{2}\alpha$. The element $s_{\alpha,r}$ can also be written of the form $s_\alpha a^{r\frac{\alpha}{2}}$. Then

$$\Omega(\mathcal{P}) = \langle s_{\alpha,-1} \rangle,$$

and

$$\Omega(\mathcal{Q}) = \text{trivial}.$$

3.2 Affine Hecke algebras

We keep using the notation in the previous section.

Define an equivalence relation on $\widetilde{S}$ such that $s \sim s'$ if and only if s and s' are $W(\mathcal{X})$-conjugate. For each equivalence class $[s]$ in $\widetilde{S}$, let $q([s])$ be an indeterminate. For $s \in \widetilde{S}$, set $q(s) = q([s])$. We sometimes call such q a parameter function. Define $\Lambda = \mathbb{C}[q([s]), q([s])^{-1} : s \in \widetilde{S}]$. The parameter function q can be naturally extended to a multiplicative function from $W(\mathcal{X})$ to $\Lambda^\times$ still denoted q such that $q(ww') = q(w)q(w')$ whenever $l(ww') = l(w) + l(w')$. The parameter q defined on $W(\mathcal{X})$ is also $W(\mathcal{X})$-invariant, which can be checked by an inductive argument on the length of w.

Example 3.2.20. In type A_1, if $\mathcal{X}$ is the root lattice, then Λ is generated by two indeterminates over $\mathbb{C}$. If $\mathcal{X}$ is the weight lattice, then Λ is generated by one indeterminate over $\mathbb{C}$.

We shall later write q_w for $q(w)$ $(w \in W(\mathcal{X}))$ for simplicity.

We first introduce a generic affine Hecke algebra as in [OS2], from which it is more convenient to construct a graded affine Hecke algebra.

Definition 3.2.21. Let $\Pi = (\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ be a based root datum and let $q : \widetilde{S} \to \Lambda^\times$ be a $W(\mathcal{X})$-invariant parameter function as above. The *generic affine Hecke algebra* $\mathcal{H} = \mathcal{H}(\Pi, q)$ associated to Π and q is the complex associative algebra with an unit generated by the symbols $\{T_w : w \in W(\mathcal{X})\}$ and the algebra Λ subject to the following relations:

(1) $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$,

(2) $(T_s - q(s)^2)(T_s + 1) = 0$ for $s \in \widetilde{S}$,

(3) $q([s])T_w = T_w q([s])$ for any $s \in \widetilde{S}$ and $w \in W(\mathcal{X})$.

Here, $\mathbb{C}T_e$ is identified with $\mathbb{C}$ i.e., $1 = T_e$. We may also sometimes regard $\mathcal{H}$ as Λ-algebra.

The set $\{T_w : w \in W(\mathcal{X})\}$ forms a Λ-basis for $\mathcal{H}$ (see [Hu1, Theorem 7.1]).

Definition 3.2.22. Let $\mathcal{H}$ be a generic affine Hecke algebra as in the notation of Definition 3.2.21. Let $q_0 : \widetilde{S} \to \mathbb{C}^\times$ be a W-invariant function. Let $\mathcal{L}$ be the ideal in $\mathcal{H}$ generated by the elements $q([s]) - q_0(s)$ for all $s \in \widetilde{S}$. Then the *affine Hecke algebra* $\mathcal{H}_{q_0} := \mathcal{H}(\Pi, q_0)$ is the algebra $\mathcal{H}/\mathcal{L}$.

We now introduce the Bernstein presentation for $\mathcal{H}$. Let

$$\mathcal{X}_{\mathrm{dom}} = \left\{ x \in \mathcal{X} : \langle x, \alpha^\vee \rangle \geq 0 \quad \text{for all } \alpha \in \Delta \right\}.$$

For $x \in \mathcal{X}_{\mathrm{dom}}$, define $\theta_x = q(x)^{-1} T_x$. For any $x \in \mathcal{X}$, x can be uniquely written as $x = x_1 - x_2$ for $x_1, x_2 \in \mathcal{X}_{\mathrm{dom}}$. Then define $\theta_x = \theta_{x_1}\theta_{x_2}^{-1}$. Note that for $x_1, x_2 \in \mathcal{X}$, $l(a^{x_1}a^{x_2}) = l(a^{x_1}) + l(a^{x_2})$ and so $T_{a^{x_1}}T_{a^{x_2}} = T_{a^{x_1}a^{x_2}} = T_{a^{x_2}}T_{a^{x_1}}$. Hence, $x \in \mathcal{X} \mapsto \theta_x$ defines a group homomorphism.

Denote by $\mathcal{A}$ the algebra generated by θ_x ($x \in \mathcal{X}_{\mathrm{dom}}$) and the algebra Λ in $\mathcal{H}$. Denote by $\mathcal{Z}$ be the center of $\mathcal{H}$.

Theorem 3.2.23. (Bernstein) *(see [Lu1, Sec. 3], [OS2, Theorem 1.3])*

(1) The elements in $\{\theta_x T_w : x \in \mathcal{X}, w \in W\}$ form a Λ-basis for $\mathcal{H}$.

(2) The algebra $\mathcal{A}$ is isomorphic to the group algebra of $\mathcal{X}$ over Λ.

(3) The center $\mathcal{Z}$ is generated by Λ and the elements of the form $\sum_{x \in M} \theta_x$, where M runs over all W orbits in $\mathcal{X}$.

The commutation relation between T_w and θ_x is given as follow:

Proposition 3.2.24. *(see [Lu1, Proposition 3.6]) Let $x \in \mathcal{X}$. For $\alpha \in \Delta$, let $s = s_\alpha \in W \subset W(\mathcal{X})$. We have the following:*

(1) If $\alpha^\vee \notin 2\mathcal{Y}$,

$$\theta_x T_s - T_s \theta_{s(x)} = (q_s^2 - 1)\frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}}.$$

(2) If $\alpha^\vee \in 2\mathcal{Y}$,

$$\theta_x T_s - T_s \theta_{s(x)} = \big((q_s^2 - 1) + \theta_{-\alpha}(q_s q_{\widetilde{s}} - q_s q_{\widetilde{s}}^{-1})\big)\frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}}.$$

Here, $\widetilde{s}$ is defined as in [Lu1, 2.4].

The condition that $\alpha^\vee \in 2\mathcal{Y}$ occurs for the affine Weyl group of type C_n ($n \geq 1$). In that case, $\theta_x - \theta_{s(x)} = \theta_x - \theta_x\theta_{-\langle x, \alpha^\vee\rangle\alpha} = \theta_x(1 - \theta_{-\alpha^\vee}^{2n})$ for $n = \frac{1}{2}\langle x, \alpha^\vee\rangle \in \mathbb{Z}$. Thus, the expression in the left-hand-side is well-defined and is in $\mathcal{A}$. Similarly, (1) is also a well-defined expression in $\mathcal{A}$.

3.3 Graded affine Hecke algebra

We begin with a general definition for the graded affine Hecke algebra and then discuss how to construct a graded affine Hecke algebra from an affine Hecke algebra.

Definition 3.3.25. [Lu1] Let $\Pi = (\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ be a based root datum. Let $V = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$, as in Section 3.1. Let S be the set of simple reflections of W. Let $k : S \to \mathbb{C}$ be a parameter function such that $k(s) = k(s')$ if s and s' are W-conjugate. Write $k_s = k(s)$. The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(\Pi, k)$ is the associative complex algebra with an unit generated by the symbols $\{t_w : w \in W\}$ and $\{f_v : v \in V\}$ satisfying the following relations:

(1) The map $w \mapsto t_w$ from $\mathbb{C}[W] = \oplus_{w \in W}\mathbb{C}w \to \mathbb{H}$ is an algebra injection.

(2) The map $v \mapsto f_v$ from $S(V) \to \mathbb{H}$ is an algebra injection.

For simplicity, we shall simply write v for f_v from now on.

(3) The generators satisfy the following relation:

$$t_{s_\alpha} v - s_\alpha(v) t_{s_\alpha} = k_{s_\alpha}\langle v, \alpha^\vee\rangle \qquad \alpha \in \Delta, v \in V.$$

We say the parameter function k is *equal* if k is a constant function.

From (1) to (3), one also deduces that $\mathbb{H}$ is naturally isomorphic to the skew group ring $S(V) \rtimes W$ as vector spaces. More precisely, if we fix a basis $\{v_1, \ldots v_n\}$ for V, then $\{t_w v_1^{m_1} \ldots v_n^{m_n} : w \in W, \quad m_1, \ldots, m_n \in \mathbb{Z}\}$ is a basis of $\mathbb{H}$.

We now briefly review the Lusztig's construction of the graded affine Hecke algebra from an affine Hecke algebra in [Lu1]. We specify some properties for Lusztig's construction for the simplification of the exposition. (Precisely, we specify $t_0 = 1$ in the notation of [Lu1, Ch. 4].)

Let $\mathcal{H}$ be a generic affine Hecke algebra associated to $\Pi = (\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ and a parameter function q. Recall that $\mathcal{A}$ is an commutative algebra generated by θ_x ($x \in X$) and $q([s]), q([s])^{-1}$ ($s \in \widetilde{S}$). Let $\mathcal{I}$ be the ideal in $\mathcal{A}$ generated by the elements of the $\theta_x - 1$ for $x \in X$ and $q([s]) - 1$ ($s \in \widetilde{S}$). Let $\overline{\mathcal{A}}_i = \mathcal{I}^i/\mathcal{I}^{i+1}$. Then we define the graded algebra $\overline{\mathcal{A}} := \bigoplus_{i\in\mathbb{Z}} \overline{\mathcal{A}}_i$. The graded algebra $\overline{\mathcal{A}}$ has the following properties:

(i) $\mathcal{I}$ is a maximal ideal in $\mathcal{A}$.

(ii) Denote v_x as the image of $\theta_x - 1$ in $\overline{\mathcal{A}}_1 = \mathcal{I}/\mathcal{I}^2$. Then for $x, x' \in X$, since $\theta_{x+x'} - 1 = (\theta_x - 1)(\theta_{x'} - 1) + (\theta_x - 1) + (\theta_{x'} - 1)$, $v_{x+x'} = v_x + v_{x'}$. Thus, there is a natural isomorphism between $\overline{\mathcal{A}}_1$ and $V \times \mathbb{C}^p$ as vector spaces, where $V = \mathbb{C} \otimes_{\mathbb{Z}} X$ and p is the number of equivalence classes $[s]$ in $\widetilde{S}$.

(iii) From (ii), $\overline{\mathcal{A}}$ is naturally isomorphic to the polynomial ring of $V \times \mathbb{C}^p$.

We then have a filtration on $\mathcal{H}$

$$\mathcal{H} \supset \mathcal{I}\mathcal{H} \supset \mathcal{I}^2\mathcal{H} \supset \ldots \supset \mathcal{I}^r\mathcal{H} \supset \ldots$$

This filtration is compatible with the multiplication of $\mathcal{H}$ (i.e., $(\mathcal{I}^i\mathcal{H})(\mathcal{I}^j\mathcal{H}) \subset \mathcal{I}^i\mathcal{I}^j\mathcal{H}$) ([Lu1, Proposition 4.2]), which can be deduced from Proposition 3.2.24. Hence, we have the graded $\mathbb{C}$-algebra

$$\overline{\mathcal{H}} := \bigoplus_{i\in\mathbb{Z}} \overline{\mathcal{H}}_i,$$

where $\overline{\mathcal{H}}_i = \mathcal{I}^i\mathcal{H}/\mathcal{I}^{i+1}\mathcal{H}$. For $w \in W$, let t_w be the image of T_w in $\overline{\mathcal{H}}$. Then we have the following:

(iv) From Proposition 3.2.23, $\mathcal{I}^i \cap \mathcal{I}^{i+1}\mathcal{H} = \mathcal{I}^{i+1}$. Hence, $\overline{\mathcal{A}}$ is a natural subalgebra of $\overline{\mathcal{H}}$.

(v) Fix a $\mathbb{Z}$ basis of X, namely $x_1, \ldots, x_n$. Recall that v_{x_i} is the image of $\theta_{x_i} - 1$ in $\overline{\mathcal{A}}$ and set $v_i = v_{x_i}$. For $s \in \widetilde{S}$, let r_s be the image of $q([s]) - 1$ in $\overline{\mathcal{A}}$. Let $\overline{\Lambda}$ be the subalgebra generated by all r_s. Then $\overline{\mathcal{H}}$ has a $\overline{\Lambda}$-basis

$$\{t_w v_i^{m_1} \ldots v_n^{m_n} : w \in W, \quad m_1, \ldots, m_n, m \in \mathbb{Z}_{\geq 0}\}.$$

(vi) $t_w t_{w'} = t_{ww'}$ for $w, w' \in W$.

(vii) There is a natural W-action on $V = \mathbb{C} \otimes_{\mathbb{Z}} X$. By (i), V is a natural subspace of $\overline{\mathcal{A}}_1$. Then there is also a natural W-action on $\overline{\mathcal{A}}_1$ with $w(r_s) = r_s$ for all $w \in W$ and $s \in \widetilde{S}$. Then for $v \in V \subset \overline{\mathcal{A}}_1$ and $\alpha \in \Delta$, we have the following relations in $\overline{\mathcal{H}}$:

(a) if $\alpha \notin 2\mathcal{Y}$,

$$vt_{s_\alpha} - t_{s_\alpha} s_\alpha(v) = 2r_{s_\alpha}\langle v, \alpha^\vee\rangle;$$

(b) if $\alpha \in 2\mathcal{Y}$,

$$vt_{s_\alpha} - t_{s_\alpha} s_\alpha(v) = (r_{s_\alpha} + r_{a^\alpha s_\alpha})\langle v, \alpha^\vee\rangle.$$

(viii) Denote by $\overline{\mathcal{Z}}$ the center of $\overline{\mathcal{H}}$. The center $\overline{\mathcal{Z}}$ is $\overline{\mathcal{A}}^W$.

The details of (iv)-(viii) are in [Lu1, Section 4.3] by Lusztig.

Now we pick $r_0(s) \in \mathbb{C}$ for each $s \in \widetilde{S}$ such that $r_0(s) = r_0(s')$ if $[s] = [s']$. Let $\overline{\mathcal{L}}$ be the ideal generated by $r_s - r_0(s)$ ($s \in \widetilde{S}$) in $\overline{\mathcal{H}}$. Then the relations (iii)-(vii) determine that $\overline{\mathcal{H}}/\overline{\mathcal{L}}$ is isomorphic to a graded affine Hecke algebra in Definition 3.3.25. The appropriate value $r_0(s)$ we have to pick to make meaningful correspondence will be determined by specifying $q([s])$ to a certain number in $\mathbb{C}^\times$. The next section will see how the study of some affine Hecke algebra modules can be reduced to the study of the graded affine Hecke algebra modules.

3.4 Lusztig's reduction theorem

In this section, we will state a variation of the Lusztig's reduction theorem given in [OS2, Theorem 2.8], whose proof is due to Luszitg [Lu1].

Let $\mathbb{T} = \mathrm{Hom}(\mathcal{X}, \mathbb{C}^\times)$. Fix $t_0 \in \mathbb{T}$. Assume for any $\alpha \in \Delta$,

$$\theta_\alpha(t_0) > 0. \tag{4.1}$$

Let $q_0 : \widetilde{S} \to \mathbb{C}^\times$ be a parameter function such that $q_0(s) = q_0(s')$ if $[s] = [s']$. The space of parameter functions can be identified with $(\mathbb{C}^\times)^p$. (Recall that p is the number of equivalence classes in $\widetilde{S}$.) Here, we will fix a parameter function q_0 and assume

$$q_0(s) > 0 \quad \text{for all } s \in \widetilde{S}. \tag{4.2}$$

We sometimes call q_0 is a *positive real* parameter function.

We shall naturally identify $\mathcal{A}$ with the coordinate ring of $\mathbb{T} \times (\mathbb{C}^\times)^p$ and consider (t_0, u_0) as a point in $\mathbb{T} \times (\mathbb{C}^*)^p$. We also identify $\mathcal{Z}$ with the coordinate ring of $(\mathbb{T} \times (\mathbb{C}^\times)^p)/W$ (see

Theorem 3.2.23(3)) and consider the W-orbit of (t_0, u_0) as a point in $(\mathbb{T}\times(\mathbb{C}^\times)^p))/W$. Here, the W-action on $\mathbb{T}\times(\mathbb{C}^\times)^p$ is determined by $\theta_x(w(t)) = \theta_{w^{-1}(x)}(t)$ $(t\in\mathbb{T})$ and $w(u') = u'$ $(u'\in(\mathbb{C}^\times)^p))$. Then define the maximal ideal

$$\mathcal{J}_{W(t_0,q_0)} := \{f\in\mathcal{Z} : f(t_0,q_0) = 0\}.$$

Let $\widehat{\mathcal{Z}}_{W(t_0,q_0)}$ be the $\mathcal{J}_{W(t_0,q_0)}$-adic completion of $\mathcal{Z}$. Let

$$\widehat{\mathcal{A}} = \widehat{\mathcal{Z}}_{W(t_0,q_0)} \otimes_{\mathcal{Z}} \mathcal{A},$$

$$\widehat{\mathcal{H}} = \widehat{\mathcal{Z}}_{W(t_0,q_0)} \otimes_{\mathcal{Z}} \mathcal{H}.$$

We now further assume that t_0 is *positive real*, meaning that $t_0\in\mathrm{Hom}(\mathcal{X},\mathbb{R}_{>0})\subset\mathrm{Hom}(\mathcal{X},\mathbb{C}^\times)$. This assumption is for the consistency of our simplified construction in the last section and is not necessary for [OS1, Theorem 2.8] (only the assumption (4.1) is required for t_0 in [OS1, Theorem 2.8]).

Let ζ_0 be the unique element in $V_0 = \mathbb{R}\otimes_{\mathbb{Z}}\mathcal{X}$ such that $\alpha(t) = e^{\alpha(\zeta_0)}$. (The uniqueness is guaranteed by (4.1)). Define the function $r_0 : \widetilde{S}\to\mathbb{C}$ such that $r_0(s) = \log q_0(s)$. In fact, $r_0(s)\in\mathbb{R}$, but we want to consider $r_0(s)$ as a point in $\mathbb{C}$ and so r_0 can be considered as a point in $\mathbb{C}^p$.

The algebra $\overline{\mathcal{A}}$ is identified with the coordinate ring of $V^\vee\times\mathbb{C}^p$ and we consider (ζ, r_0) as a point $V^\vee\times\mathbb{C}^p$. We again identify $\overline{\mathcal{Z}}$ with the coordinate ring of $(V^\vee\times\mathbb{C}^p)/W$ (see (viii) in Section 3.3) and consider the W-orbit of (ζ_0, r_0) as a point in $(V^\vee\times\mathbb{C}^p)/W$. Here, the W-action on $V^\vee\times\mathbb{C}^p$ is determined by $\theta_x(w(t)) = \theta_{w(x)}(t)$ $(t\in\mathbb{T})$ and $w(r_0') = r_0'$ $(r_0'\in\mathbb{C}^p)$. Then define the maximal ideal

$$\overline{\mathcal{J}}_{W(\zeta_0,r_0)} := \left\{f\in\overline{\mathcal{Z}} : f(\zeta_0,r_0) = 0\right\}.$$

Let $\widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)}$ be the $\overline{\mathcal{J}}_{W(\zeta_0,r_0)}$-adic completion of $\overline{\mathcal{Z}}$. Let

$$\widehat{\overline{\mathcal{A}}} = \widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)} \otimes_{\overline{\mathcal{Z}}} \overline{\mathcal{A}},$$

$$\widehat{\overline{\mathcal{H}}} = \widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)} \otimes_{\overline{\mathcal{Z}}} \overline{\mathcal{H}}.$$

We now turn to graded affine Hecke algebras in order to make some suitable identifications later. Define the parameter function $k_0 : S\to\mathbb{C}$ as follows (c.f. (vii) in Section 3.3):

(1) If $\alpha\notin 2\mathcal{Y}$, then

$$k_0(s_\alpha) = 2\log(q_0(s_\alpha)). \tag{4.3}$$

(2) If $\alpha \in 2\mathcal{Y}$, then

$$k_0(s_\alpha) = \log(q_0(s_\alpha)) + \log(q_0(a^\alpha s_\alpha)). \tag{4.4}$$

From the $W(\mathcal{X})$-invariance of q_0, we also have that k_0 is also W-invariant. The k_0 will be the parameter function for the corresponding graded affine Hecke algebra from $\overline{\mathcal{H}}$.

The following theorem an its consequence are the main goal of this chapter:

Theorem 3.4.26. *([OS2, Theorem 2.8], [Lu1, Theorem 9.3]) We retain the setting above. In particular, we are assuming (4.1) and (4.2). Then we have the following:*

(1) There are natural compatible $\mathbb{C}$-algebra isomorphisms between $\widehat{\mathcal{Z}}$ and $\widehat{\overline{\mathcal{Z}}}$, between $\widehat{\mathcal{A}}$ and $\widehat{\overline{\mathcal{A}}}$, and between $\widehat{\mathcal{H}}$ and $\widehat{\overline{\mathcal{H}}}$.

(2) Let $\overline{\mathcal{L}}$ be the ideal generated by $r_s - r_0(s)$ $(s \in \widetilde{S})$ in $\overline{\mathcal{H}}$. Then there are natural compatible $\mathbb{C}$-algebra isomorphisms between the center of $\overline{\mathcal{H}}/\overline{\mathcal{L}}$ and the center of $\mathbb{H}(W, k_0)$, between $\overline{\mathcal{A}}/(\overline{\mathcal{A}} \cap \overline{\mathcal{L}})$ and $S(V)$, and between $\overline{\mathcal{H}}/\overline{\mathcal{L}}$ and $\mathbb{H}(W, k_0)$.

Corollary 3.4.27. *([Lu1, Section 10.9], [OS2, Corollary 2.9]) With the setting above, we have the following:*

(1) The category of finite-dimensional $\mathcal{H}$-modules whose irreducible subquotients have central character $W(t_0, q_0)$ is equivalent to the category of finite-dimensional $\overline{\mathcal{H}}$-modules whose irreducible subquotients have central character $W(\zeta_0, r_0)$.

(2) Let $\mathcal{L}$ be the ideal generated by $q_s - q_0(s)$ $(s \in \widetilde{S})$ in $\mathcal{H}$. Let $\overline{\mathcal{L}}$ be the ideal generated by $r_s - r_0(s)$ $(s \in \widetilde{S})$ in $\overline{\mathcal{H}}$. Then the category of finite-dimensional $\mathcal{H}/\mathcal{L}$-modules whose irreducible subquotients have central character Wt_0 is equivalent to the category of finite-dimensional $\overline{\mathcal{H}}/\overline{\mathcal{L}}$-modules whose irreducible subquotients have central character $W\zeta_0$.

(3) The category of finite-dimensional $\overline{\mathcal{H}}/\overline{\mathcal{L}}$-modules whose irreducible subquotients have central character $W\zeta_0$ is equivalent to the category of finite-dimensional $\mathbb{H}(W, k_0)$-modules whose irreducible subquotients have the corresponding central character.

Proof. We briefly explain how to obtain the statements. Let $\widehat{\mathcal{J}}$ (resp. $\widehat{\overline{\mathcal{J}}}$) be the maximal ideal in $\widehat{\mathcal{Z}}_{W(t_0,q_0)}$ (resp. $\widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)}$) and set $\mathcal{J} = \mathcal{J}_{W(t_0,q_0)}$. Since $\widehat{\mathcal{H}}/\widehat{\mathcal{J}}^i\widehat{\mathcal{H}}$ is isomorphic to $\mathcal{H}/\mathcal{J}^i\mathcal{H}$, there is a bijection between finite-dimensional $\widehat{\mathcal{H}}$-modules which are annihilated by $\widehat{\mathcal{J}}^i$ and finite dimensional $\mathcal{H}$-modules which are annihilated by $\mathcal{J}^i$. The latter modules are also bijective to the finite-dimensional $\mathcal{H}$-modules whose irreducible subquotients have central characters $W(t_0, q_0)$. We then have a similar statement for $\widehat{\overline{\mathcal{H}}}$ and $\overline{\mathcal{H}}$. Combining with Theorem 3.4.26 (1), we have (1). For (2), we trace the isomorphism of Theorem 3.4.26 in the proof and see the isomorphism sends $\widehat{\mathcal{Z}}_{W(t_0,q_0)} \otimes_{\mathcal{Z}} \mathcal{L}$ to $\widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)} \otimes_{\overline{\mathcal{Z}}} \overline{\mathcal{L}}$. Hence, $\widehat{\mathcal{H}}/(\widehat{\overline{\mathcal{Z}}}_{W(t_0,q_0)} \otimes_{\overline{\mathcal{Z}}} \mathcal{L}) \cong \widehat{\overline{\mathcal{H}}}/(\widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)} \otimes_{\overline{\mathcal{Z}}} \overline{\mathcal{L}})$. Then we obtain (2) by similar argument as (1). For (3), it follows from the construction of $\overline{\mathcal{H}}$.

□

CHAPTER 4

EXT-GROUPS IN VARIOUS CATEGORIES

Given modules X, Y of an algebra, a module Z is called an extension of X by Y if there exists a short exact sequence:

$$0 \to X \to Z \to Y \to 0.$$

Those equivalence classes of short exact sequences can be equipped with a natural abelian group structure and form a group isomorphic to $\mathrm{Ext}^1(X, Y)$, the first dervied functor of $\mathrm{Hom}(\cdot, Y)$. The higher Ext-groups also have interpretations in terms of long exact sequences.

In this chapter, we apply discussions in the previous two chapters to transfer information of Ext-groups among several categories. However, we will not make any attempt to understand the Ext-groups in a specific category until next chapter. Indeed, we shall only focus the Ext-groups in only one specific category, that is the graded affine Hecke algebra modules. Results in this chapter are explicitly or implicitly known in the literature (e.g., [AP, OS4, So]).

4.1 Ext-groups and projective objects

Since we are going to work through Ext-groups over various categories, it is convenient for us to discuss the notation of Ext^i once and for all (see for example [We, Chapter 2] for details).

Let $\mathfrak{R}$ be an abelian category. Recall that an object P in $\mathfrak{R}$ is projective if $\mathrm{Hom}_{\mathfrak{R}}(P, .)$ is an exact functor. Assume $\mathfrak{R}$ has enough projective objects. For objects X, Y in $\mathfrak{R}$, define $\mathrm{Ext}^i_{\mathfrak{R}}$ as follows: Let $P^\bullet$ be a projective resolution for X

$$P^\bullet \to X \to 0.$$

Then $\mathrm{Ext}^i_{\mathfrak{R}}(X, Y)$ is defined to be the i-th homology of the complex $\mathrm{Hom}_{\mathfrak{R}}(X, Y)$. It is a general fact in homological algebra that the $\mathrm{Ext}^i_{\mathfrak{R}}(X, Y)$ does not depend on the choice of the projective resolution of X. It is immediate from the definitions that if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are two equivalent abelian categories of enough projective objects, then

$$\mathrm{Ext}^i_{\mathfrak{R}_1}(X,Y) \cong \mathrm{Ext}^i_{\mathfrak{R}_2}(F(X),F(Y)),$$

where F is the functor from $\mathfrak{R}_1$ to $\mathfrak{R}_2$ for the equivalence of categories.

We now refine our consideration to the category of nondegenerate $\mathcal{A}$-modules for some algebra $\mathcal{A}$.

Proposition 4.1.28. *Let $\mathcal{A}$ be an idempotented algebra as defined in Definition 2.2.6. Then the category $\mathfrak{R}(\mathcal{A})$ of nondegenerate $\mathcal{A}$-modules has enough projective objects.*

Proof. Let e be an idempotent in $\mathcal{A}$ and let X be a nondegenerate $\mathcal{A}$-module. We consider the $\mathcal{A}$-module $\mathcal{A}e \otimes_{\mathbb{C}} V$ with $\mathcal{A}$ acting on the first factor. Note that $\mathcal{A}\mathcal{A} = \mathcal{A}$ and so $\mathcal{A}e \otimes_{\mathbb{C}} V$ is nondegenerate. On the other hand, we note that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}e \otimes_{\mathbb{C}} Y, X) = eX \otimes Y^*$, where Y^* is the dual space of Y. Then for nondegenerate $\mathcal{A}$-modules X, X' and for a surjective map $f : X \to X'$, f induces a surjective map from $eX \otimes Y^*$ to $eX' \otimes Y^*$ and then by naturality, f induces a surjection from $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}e \otimes_{\mathbb{C}} Y, X)$ to $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}e \otimes_{\mathbb{C}} Y, X')$. Hence, $\mathcal{A}e \otimes_{\mathbb{C}} Y$ is a projective object in $\mathfrak{R}(G)$. Then for any $\mathcal{A}$-module X, the module

$$\bigoplus_e \mathcal{A}e \otimes_{\mathbb{C}} eX$$

is projective, where $\mathcal{A}$ acts on the first factor, and naturally surjects onto $X = \bigcup_e X$. Here, e runs for all the idempotents of $\mathcal{A}$. This shows the category has enough projective objects.

□

Corollary 4.1.29. *Suppose $\mathcal{A}$ is one of the following algebras:*

(1) the Hecke algebra in Section 2.2,

(2) the spherical function algebra in Definition 2.5.14,

(3) the generic affine Hecke algebra in Definition 3.2.21,

(4) the graded affine Hecke algebra in Definition 3.3.25.

The category $\mathfrak{R}(\mathcal{A})$ of nondegenerate $\mathcal{A}$-modules has enough projective objects.

Proof. The Hecke algebra is an idempotented algebra and other algebras have an unit.

□

Corollary 4.1.30. *Let G be a p-adic group and let $\mathfrak{B}(G)$ be the set parametrizing the Bernstein components in Definition 2.3.11. Then we have the following:*

(1) The category $\mathfrak{R}(G)$ of smooth representations of G has enough projective objects.

(2) For each $\mathfrak{s} \in \mathfrak{B}(G)$, $\mathfrak{R}^{\mathfrak{s}}(G)$ has enough projective objects.

(3) Let X and Y be objects in $\mathfrak{R}(G)$. Use Bernstein decomposition to decompose X into $\bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} X^{\mathfrak{s}}$ and similarly decompose Y into $\bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} Y^{\mathfrak{s}}$. Then

$$\mathrm{Ext}^i_{\mathfrak{R}(G)}(X,Y) \cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Ext}^i_{\mathfrak{R}^{\mathfrak{s}}(G)}(X^{\mathfrak{s}}, Y^{\mathfrak{s}}).$$

Proof. (1) follows from Corollary 4.1.29 (1) and Proposition 2.2.4. For (2), let X be an object in $\mathfrak{R}^{\mathfrak{s}}(G)$ and let P be a projective object such that there exists a surjection from P to X. Let $P^{\mathfrak{s}}$ be the factor of P in the Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$ of the Bernstein decomposition. Then we also have a surjection $P^{\mathfrak{s}}$ to X. By the Bernstein decomposition, we can also check $P^{\mathfrak{s}}$ is projective in $\mathfrak{R}^{\mathfrak{s}}(G)$. This shows (2).

For (3), let $P^{\bullet}$ be a projective resolution for X. Then by the Bernstein decomposition, we can write the resolution as the form:

$$\bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} (P^{\mathfrak{s}})^{\bullet} \to \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} X^{\mathfrak{s}}.$$

Then each sequence $(P^{\mathfrak{s}})^{\bullet}$ is a projective resolution of $X^{\mathfrak{s}}$. Then by definitions, we obtain (3).

□

4.2 Central characters and Ext-groups

We deal with the issue of central characters in this section. Corollary 4.2.34 will allow us to focus on modules of the same central characters for computing Ext-groups later. We work over a general setting in this section, but the major examples are affine Hecke algebras and graded affine Hecke algebras, where we have explicitly known their centers.

Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with an unit and let $\mathcal{Z}$ be the center of $\mathcal{A}$. An $\mathcal{A}$ module X is said to have a *central character* if there is a function $\chi_X : \mathcal{Z} \to \mathbb{C}$ such that for any $x \in X$, $z.x = \chi_X(z)x$.

Lemma 4.2.31. *Let $\mathcal{A}$ be an $\mathbb{C}$-algebra with an unit. Let X_1, X_2 be $\mathcal{A}$-modules. Then the left multiplication of $z \in \mathcal{Z}$ on X_1 and X_2 determines two algebra maps $z_1 : X_1 \to X_1$ and $z_2 : X_2 \to X_2$, respectively. Then the induced maps z_1 and z_2 on $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2)$ coincide.*

Proof. We prove by induction similar to the proof in [BW, Ch.I Sec.4 Lemma 4.4]. For $i = 0$, $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) = \mathrm{Hom}_{\mathfrak{R}(\mathcal{A})}(X_1, X_2)$. Thus, for $f \in \mathrm{Hom}_{\mathfrak{R}(\mathcal{A})}(X_1, X_2)$, $(z_1 f)(x) = f(zx) = zf(x) = (z_2 f)(x)$. Hence, $z_1 = z_2$. Now assume $i \geq 1$. Let P be a projective $\mathcal{A}$-module such that

$$0 \to X' \to P \to X_1 \to 0.$$

Then the associated long exact sequence for the $\mathrm{Hom}_{\mathfrak{R}(\mathcal{A})}(\cdot, X_2)$ is as follows:

$$\begin{aligned}\ldots \leftarrow \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathcal{A})}(P, X_2) \leftarrow \mathrm{Ext}^{i}_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) \leftarrow & \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathcal{A})}(X', X_2) \\ & \leftarrow \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathcal{A})}(P, X_2) \leftarrow \ldots .\end{aligned}$$

By using P is projective, we have $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) = \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathcal{A})}(X', X_2)$. Then by induction hypothesis, the action of z_1 and z_2 agree on $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2)$ as desired.

□

Proposition 4.2.32. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with an unit. Let X_1, X_2 be $\mathcal{A}$-modules with the central characters χ_{X_1} and χ_{X_2}, respectively. If $\chi_{X_1} \neq \chi_{X_2}$, then $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) = 0$ for all i.*

Proof. We follow the proof in [BW, Ch. I Sec. 4]. Since $\chi_{X_1} \neq \chi_{X_2}$ and $\mathcal{A}$ has an unit, there exists an element $z \in \mathcal{Z}$ such that $\chi_{X_1}(z) = 1$ and $\chi_{X_2}(z) = 0$. Then z acts on X_1 as an identity and so induces an identity map on $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2)$. Similarly z acts on X_2 as zero on X_2 and so induces a zero map on $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2)$. By Lemma 4.2.31, $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) = 0$ as desired.

□

Lemma 4.2.33. *Let X_k $(k = 1, 2)$ be $\mathcal{A}$-modules of finite length. Fix an integer i. If $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) \neq 0$, then there exists an (irreducible) composition factor Y of X_1 and a composition factor Z of X_2 such that $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(Y, Z) \neq 0$.*

Proof. If X_1 and X_2 have length 1, then the lemma is clear. We now assume X_2 has length 1 and proceed induction on the length of X_1. Assume the length of X_1 is greater than or equal to 2. Let $Y^0 \subset Y^1 \subset \ldots \subset Y^r = X$ be a composition series of X_1.

If $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(Y^0, X_2) \neq 0$, then we are done. Otherwise, by considering the short exact sequence

$$0 \to Y^0 \to X_1 \to X_1/Y^0 \to 0,$$

we have the associated long exact sequence

$$\ldots \leftarrow \mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(Y^0, X_2) \leftarrow \mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) \leftarrow \mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1/Y^0, X_2) \leftarrow \ldots$$

and so we have $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1/Y^0, X_2) \neq 0$. By an induction on the length of the composition series of X_1, we obtain the statement.

We now consider X_2 have length greater than or equal to 2. Indeed, this can be proven by a similar inductive argument to the case of X_1.

□

When $\mathcal{A}$ is an affine Hecke algebra or a graded affine Hecke algebra, any irreducible $\mathcal{A}$-module is finite-dimensional and so any irreducible $\mathcal{A}$-module has a central character by Schur's lemma.

Corollary 4.2.34. *Let $\mathcal{A}$ be an affine Hecke algebra or a graded affine Hecke algebra. Let X_k $(k = 1, 2)$ be $\mathcal{A}$-modules of finite length such that all the irreducible subquotients of X_k have the same central character, say χ_k. If $\chi_1 \neq \chi_2$, then $\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{A})}(X_1, X_2) = 0$ for all i.*

Proof. This follows from Proposition 4.2.32 and Lemma 4.2.33.

□

Corollary 4.2.35. *Let $\mathcal{A}$ be an affine Hecke algebra or a graded affine Hecke algebra. Let $\mathfrak{R}_{\mathrm{fin}}(\mathcal{A})$ be the category of $\mathcal{A}$-modules of finite length. Let Υ be the set of functions $\chi : \mathcal{Z} \to \mathbb{C}$ which are the central characters of irreducible $\mathcal{A}$-modules. For $\chi \in \Upsilon$, let $\mathfrak{R}_{\mathrm{fin},\chi}$ be the full subcategory of $\mathfrak{R}_{\mathrm{fin},\chi}(\mathcal{A})$ whose irreducible subquotients have the central characters χ. Then $\mathfrak{R}_{\mathrm{fin}}(\mathcal{A})$ can be decomposed into full subcategories as follows:*

$$\mathfrak{R}_{\mathrm{fin}}(\mathcal{A}) = \bigoplus_{\chi \in \Upsilon} \mathfrak{R}_{\mathrm{fin},\chi}(\mathcal{A}).$$

Proof. We proceed by an induction of the length of a module. It is nothing to prove for an irreducible module (of length 1). We now consider an $\mathcal{A}$-module X of length n and $n \geq 2$. Let Y be a submodule of X with length $n-1$. By the induction hypothesis, Y can be decomposed into $\bigoplus_{\chi \in \Upsilon} Y^{\chi}$, where Y^{χ} is the maximal submodule of Y whose irreducible quotients have the central character χ. Note that Y^{χ} is zero except for finitely many χ and let Υ_Y be the collection of $\chi \in \Upsilon$ such that Y^χ is nonzero. We now consider the irreducible subquotient X/Y and let χ' be the central character of X/Y. Let $Y' = \bigoplus_{\chi \in \Upsilon_Y \setminus \{\chi'\}} Y^\chi$. X/Y' has the central character χ' and χ' is not the central character of any irreducible subquotients of Y', then $\mathrm{Ext}^i_{\mathcal{A}}(Y', X/Y') = 0$ for all i by Corollary 4.2.34. By the well-known interpretation of Ext^1_i in terms of short exact sequence (see for example [We, Theorem 3.4.3]), we have

$$0 \to Y' \to X \to X/Y' \to 0$$

splits and so $X \cong Y' \oplus X/Y'$ as desired. This completes the proof.

□

4.3 Ext-groups for affine Hecke algebras and graded ones

In this section, we discuss the correspondence of the Ext-groups of affine Hecke algebra modules and the graded affine Hecke algebra modules. From Corollary 3.4.27, we expect the Ext-groups for those two algebras agree in a suitable sense whenever the equivalence of categories holds. However, it may not be completely clear since the equivalence of categories holds for finite-dimensional modules and the category of finite-dimensional modules may not have enough projective objects. Hence, we still want to take the Ext-groups in the categories $\mathfrak{R}(\mathcal{H})$ and $\mathfrak{R}(\mathbb{H})$. We are going to work out some detail. Most of the statements are quite standard. In view of Theorem 3.4.26, it is also natural to consider the Ext-groups for the formal completion of $\mathcal{H}$ and $\mathbb{H}$.

We use similar notation as in Section 3.4. Let $\mathcal{H}$ be a generic affine Hecke algebra (Definition 3.2.21). Let $\overline{\mathcal{H}}$ be the algebra in Section 3.3. Set $\mathcal{J} = \mathcal{J}_{W(t_0,q_0)}$ and set $\overline{J} = \overline{\mathcal{J}}_{W(\zeta_0,r_0)}$.

Lemma 4.3.36. *Let Y be a finite-dimensional $\mathcal{H}$-module and let $\widehat{Y} = \widehat{\mathcal{Z}}_{W(t_0,q_0)} \otimes_{\mathcal{Z}} Y$. Then $\mathrm{Hom}_{\widehat{\mathcal{H}}}(\widehat{\mathcal{H}}, \widehat{Y}) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, Y)$ as vector spaces and the isomorphism is natural.*

Proof. Since $\widehat{Y}$ is finite-dimensional, there exist an integer i such that $\mathcal{J}^i Y = 0$. For any $z' \in \widehat{\mathcal{Z}}$, there exists $z \in \mathcal{Z}$ such that $z' - z \in \widehat{\mathcal{Z}}\mathcal{J}^i$. Hence, $\widehat{Y}$ and Y are isomorphic as $\mathcal{H}$-modules via a natural map, say $\phi : \widehat{Y} \to Y$. We now define a map $F : \mathrm{Hom}_{\widehat{\mathcal{H}}}(\widehat{\mathcal{H}}, \widehat{Y}) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, Y)$ such that $F(f)(h) = \phi^{-1} \circ f(1 \otimes h)$. Similarly, define a map $F' : \mathrm{Hom}_{\widehat{\mathcal{H}}}(\widehat{\mathcal{H}}, \widehat{Y}) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, Y)$ such that $F'(f)(z \otimes h) = \phi(zf(h))$. It is straightforward to check that F and F' are inverse of each other.

□

Proposition 4.3.37. *Let X and Y be finite-dimensional $\mathcal{H}$-modules whose irreducible subquotients have central characters $W(t_0, q_0)$. Let $\widehat{X} = \widehat{\mathcal{Z}}_{W(t_0,q_0)} \otimes_{\mathcal{Z}} X$ and let $\widehat{Y} = \widehat{\mathcal{Z}}_{W(t_0,q_0)} \otimes_{\mathcal{Z}} Y$.*

$$\mathrm{Ext}^i_{\mathfrak{R}(\widehat{\mathcal{H}})}(\widehat{X}, \widehat{Y}) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathcal{H})}(X, Y).$$

Proof. We first construct a projective resolution for X as follow:

$$\ldots \to \mathcal{H} \otimes_{\mathbb{C}} \ker d_{r-1} \stackrel{d_r}{\to} \mathcal{H} \otimes_{\mathbb{C}} \ker d_{r-2} \to \ldots \to \mathcal{H} \otimes_{\mathbb{C}} \ker d_0 \stackrel{d_1}{\to} \mathcal{H} \otimes_{\mathbb{C}} X \stackrel{d_0}{\to} X \to 0 \quad (3.1)$$

Here, d_r is defined as $d_r(h \otimes x) = hx$. We now take the exact functor $\widehat{\mathcal{Z}} \otimes_{\mathcal{Z}}$ ([AM, Proposition 10.14]). Then we have the resolution

$$\ldots \to \widehat{\mathcal{H}} \otimes_{\mathbb{C}} \ker d_{r-1} \stackrel{d_r}{\to} \widehat{\mathcal{H}} \otimes_{\mathbb{C}} \ker d_{r-2} \to \ldots \to \widehat{\mathcal{H}} \otimes_{\mathbb{C}} \ker d_0 \stackrel{d_1}{\to} \widehat{\mathcal{H}} \otimes_{\mathbb{C}} X \stackrel{d_0}{\to} \widehat{X}, \widehat{Y} \to 0$$

Each $\widehat{\mathcal{H}} \otimes_{\mathbb{C}} \ker d_r$ and $\widehat{\mathcal{H}} \otimes_{\mathbb{C}} X$ is still a projective object in $\mathfrak{R}(\widehat{\mathcal{H}})$. Now taking the $\mathrm{Hom}_{\widehat{\mathcal{H}}}(., \widehat{Y})$ functor ([AM, Proposition 10.14]), we have the complex $\mathrm{Hom}_{\widehat{\mathcal{H}}}(\widehat{\mathcal{H}} \otimes \ker d_r, \widehat{Y})$. Then we also obtain a complex $\mathrm{Hom}_{\widehat{\mathcal{H}}}(\mathcal{H} \otimes_{\mathbb{C}} \ker d_r, \widehat{Y})$ by Lemma 4.3.36. By using the explicit map F given in the proof of Lemma 4.3.36, one also checks that the complex also agrees with the complex obtained by taking the $\mathrm{Hom}_{\mathcal{H}}(\cdot, Y)$ functor on the resolution (3.1). This implies $\mathrm{Ext}^i_{\mathfrak{R}(\widehat{\mathcal{H}})}(\widehat{X}, \widehat{Y}) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathcal{H})}(X, Y)$.

□

A similar proof of Lemma 4.3.36 and Proposition 4.3.37 then yields the following result:

Proposition 4.3.38. *Let X and Y be finite-dimensional $\overline{\mathcal{H}}$-modules whose irreducible subquotients have central characters $W(t_0, q_0)$. Let $\widehat{X} = \widehat{\overline{\mathcal{Z}}}_{W(\zeta_0,r_0)} \otimes_{\overline{\mathcal{Z}}} X$ and let $\widehat{Y} = \widehat{\overline{\mathcal{Z}}} \otimes_{\overline{\mathcal{Z}}} Y$.*

$$\mathrm{Ext}^i_{\mathfrak{R}(\widehat{\overline{\mathcal{H}}})}(\widehat{X}, \widehat{Y}) \cong \mathrm{Ext}^i_{\mathfrak{R}(\overline{\mathcal{H}})}(X, Y).$$

Corollary 4.3.39. *Let $\mathcal{H}_{q_0}$ be an affine Hecke algebra associated to a parameter function $q_0 : R \to \mathbb{R}_{>0}$. Let $\mathbb{H}$ be the corresponding graded affine Hecke algebra associated to a parameter k_0 defined in (4.3) and (4.4). Let $X_{\mathcal{H}_{q_0}}$ and $Y_{\mathcal{H}_{q_0}}$ be finite-dimensional $\mathcal{H}_{q_0}$-modules whose irreducible subquotients have positive real central characters and let $X_{\mathbb{H}}$ and $Y_{\mathbb{H}}$ be the corresponding finite-dimensional $\mathbb{H}$-modules, respectively, under the equivalences of categories in Corollary 3.4.27 (2) and (3). Then*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathcal{H}_q)}(X_{\mathcal{H}_{q_0}}, Y_{\mathcal{H}_{q_0}}) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X_{\mathbb{H}}, Y_{\mathbb{H}}).$$

Proof. It suffices to consider $X_{\mathcal{H}_{q_0}}$ and $Y_{\mathcal{H}_{q_0}}$ are indecomposable. We may further assume the central characters of irreducible subquotients of both $X_{\mathcal{H}_{q_0}}$ and $Y_{\mathcal{H}_{q_0}}$ are the same (otherwise by Corollary 4.2.34, the Ext-groups vanish and we are done). Then Proposition 4.3.37, Proposition 4.3.38 and Corollary 3.4.27 imply the corollary.

□

CHAPTER 5

RESOLUTIONS FOR $\mathbb{H}$-MODULES

Starting from this chapter, we study extensions of graded affine Hecke algebra modules from elementary principles. Our goal is to develop an algebraic approach for the study and we hope to bring another perspective to the extensions of smooth G-representations at the end via the equivalence of categories. (However, we will not address the latter part in this thesis apart from what we discussed in Chapter 4.)

In this chapter, we construct an explicit projective resolution for graded affine Hecke algebra modules, which is the main tool for studying Ext-groups later.

We fix and recall some notations. Fix a root datum $\Pi = (R, \mathcal{X}, R^\vee, \mathcal{Y}, \Delta)$ (see Section 3.1). Let $V = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$ and equip V with a natural W-action. For $v \in V$ and $w \in W$, write $w(v)$ to be the resulting element of the action w on v. Let $k : S \to \mathbb{C}$ be a parameter function such that $k(s) = k(s')$ if s and s' are W-conjugate. Write $k_\alpha = k_{s_\alpha} = k(s_\alpha)$ for all $\alpha \in \Delta$. Let $\mathbb{H}$ be the graded affine Hecke algebra associated to Π and k (Definition 3.3.25).

For any two complex vector spaces X_1 and X_2, we shall simply write $X_1 \otimes X_2$ for $X_1 \otimes_{\mathbb{C}} X_2$.

5.1 Projective objects and injective objects

In this section, we construct some projective objects and injective objects, which will be used to construct explicit resolutions for $\mathbb{H}$-modules in the next sections.

Let X be an $\mathbb{H}$-module and let U be a finite-dimensional $\mathbb{C}[W]$-module. $\mathbb{H}$ acts on the space $\mathbb{H} \otimes_{\mathbb{C}[W]} U$ by the left multiplication on the first factor while $\mathbb{H}$ acts on the space $\mathrm{Hom}_W(\mathbb{H}, U)$ by the right translation, explicitly that is for $f \in \mathrm{Hom}_W(\mathbb{H}, U)$, the action of $h' \in \mathbb{H}$ is given by:

$$(h'.f)(h) = f(hh'), \quad \text{for all } h \in \mathbb{H} .$$

Denote by Res_W the restriction functor from $\mathbb{H}$-modules to $\mathbb{C}[W]$-modules.

Lemma 5.1.40. *(Frobenius reciprocity) Let X be an $\mathbb{H}$-module. Let U be a $\mathbb{C}[W]$-module. Then*

$$\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(X, \mathrm{Hom}_W(\mathbb{H}, U)) = \mathrm{Hom}_W(\mathrm{Res}_W X, U),$$

and

$$\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(\mathbb{H} \otimes_W U, X) = \mathrm{Hom}_W(U, \mathrm{Res}_W X).$$

Proof. Let $F : \mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(X, \mathrm{Hom}_W(\mathbb{H}, U)) \to \mathrm{Hom}_W(\mathrm{Res}_W X, U)$ given by

$$(F(f))(x) = (f(x))(1).$$

Let $G : \mathrm{Hom}_W(\mathrm{Res}_W X, U) \to \mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(X, \mathrm{Hom}_W(\mathbb{H}, U))$ given by

$$((G(f)(x))(h) = f(hx).$$

It is straightforward to verify F and G are linear isomorphisms. This proves the second equation. The proof for the second one is similar.

□

Lemma 5.1.41. *Let U be a $\mathbb{C}[W]$-module. Then $\mathbb{H} \otimes_{\mathbb{C}[W]} U$ is projective and $\mathrm{Hom}_W(\mathbb{H}, U)$ is injective.*

Proof. We consider $\mathbb{H} \otimes_{\mathbb{C}[W]} U$. Every $\mathbb{C}[W]$-module is projective and so $\mathrm{Hom}_W(U, .)$ is an exact functor. The functor Res_W is also exact. Thus, the space $\mathrm{Hom}_W(U, \mathrm{Res}_W .)$, which is the composition of the two functors $\mathrm{Hom}_W(U, .)$ and Res_W, is also exact. Hence, by the Frobenius reciprocity, the space $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}[W]} U, .)$ is also exact. Thus, $\mathbb{H} \otimes_{\mathbb{C}[W]} U$ is projective. The proof for $\mathrm{Hom}_W(\mathbb{H}, U)$ being injective is similar.

□

5.2 Koszul-type resolution on $\mathbb{H}$-modules

Let X be an $\mathbb{H}$-module. Define a sequence of $\mathbb{H}$-module maps d_i as follows:

$$0 \to \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^n V) \overset{d_{n-1}}{\to} \ldots \overset{d_i}{\to} \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^i V)$$
$$\overset{d_{i-1}}{\to} \ldots \overset{d_0}{\to} \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X) \overset{\epsilon}{\to} X \to 0 \qquad (2.1)$$

such that $\epsilon : \mathbb{H} \otimes_{\mathbb{C}[W]} \mathrm{Res}_W X \to X$ given by

$$\epsilon(h \otimes x) = h.x$$

and for $i \geq 1$, $d_i : \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^{i+1} V) \to \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^{i} V)$ given by

$$d_i(h \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1})) \tag{2.2}$$

$$= \sum_{j=1}^{i+1} (-1)^{j+1} (hv_j \otimes x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1} - h \otimes v_j.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1}). \tag{2.3}$$

In priori, we do not know d_i is a well-defined $\mathbb{H}$-map, but we prove in the following:

Lemma 5.2.42. *The above d_i are well-defined $\mathbb{H}$-maps and $d^2 = 0$ i.e., (2.1) is a well-defined complex.*

Proof. We proceed by induction on i. It is easy to see that ϵ is well-defined. For convenience, we set $d_{-1} = \epsilon$. We now assume $i \geq 0$. To show d_i is independent of the choice of a representative in $\mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^{i+1} V)$, the nontrivial one is to show

$$d_i(t_w \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1}) = d_i(1 \otimes (t_w.x \otimes w(v_1) \wedge \ldots \wedge w(v_{i+1}))). \tag{2.4}$$

For simplicity, set

$$\begin{aligned} P^w &= d_i(t_w \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1})) \\ &= t_w \sum_{j=1}^{i+1} (-1)^{j+1} v_j \otimes (x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1}) \\ &\quad -1 \otimes (v_j.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1}) \end{aligned}$$

and

$$\begin{aligned} P_w &= d_i(1 \otimes (t_w.x \otimes w(v_1) \wedge \ldots \wedge w(v_{i+1})) \\ &= \sum_{j=1}^{i+1} (-1)^{j+1} w(v_j) \otimes (t_w.x \otimes w(v_1) \wedge \ldots \wedge \widehat{w(v_j)} \wedge \ldots \wedge w(v_{i+1})) \\ &\quad - \sum_{j=1}^{i+1} (-1)^{j+1} \otimes (w(v_j).t_w.x \otimes w(v_1) \wedge \ldots \wedge \widehat{w(v_j)} \wedge \ldots \wedge w(v_{i+1}). \end{aligned}$$

To show the equation (2.4), it is equivalent to show $P^w = P_w$. Regard $\mathbb{C}[W]$ as a natural subalgebra of $\mathbb{H}$. By using the fact that $t_w v - w(v) t_w \in \mathbb{C}[W]$ for $w \in W$, $P^w - P_w$ is an element of the form $1 \otimes u$ for some $u \in \mathrm{Res}_W X \otimes \wedge^i V$. Thus, it suffices to show that $u = 0$. To this end, a direct computation (from the original expressions of P^w and

P_w) shows that $d_{i-1}(P^w - P_w) = 0$. By induction hypothesis, d_{i-1} is well-defined and so $d_{i-1}(1 \otimes u) = d_{i-1}(P^w - P_w) = 0$. Write $1 \otimes u$ of the form

$$1 \otimes u = \sum_{1 \leq r_1 < \ldots < r_i \leq n} 1 \otimes (x_{r_1,\ldots,r_i}) \otimes e_{r_1} \wedge \ldots \wedge e_{r_i}, \tag{2.5}$$

where $x_{r_1,\ldots,r_i} \in \mathrm{Res}_W X$ and $e_1, \ldots, e_n$ is a fixed basis of V. By a direct computation of $d_{i-1}(1 \otimes u)$ from the expression (2.5), we have

$$\begin{aligned} & d_{i-1}(1 \otimes u) \\ = & \sum_{1 \leq r_1 < \ldots < r_i \leq n} \sum_{j=1}^{i} (-1)^{j+1} e_{r_j} \otimes (x_{r_1,\ldots,r_i}) \otimes e_{r_1} \wedge \ldots \wedge \widehat{e}_{r_j} \wedge \ldots \wedge e_{r_i}) \\ & - \sum_{1 \leq r_1 < \ldots < r_i \leq n} \sum_{j=1}^{i} (-1)^{j+1} 1 \otimes e_{r_j}.(x_{r_1,\ldots,r_i}) \otimes e_{r_1} \wedge \ldots \wedge \widehat{e}_{r_j} \wedge \ldots \wedge e_{r_i}). \end{aligned}$$

We have seen that $d_{i-1}(1 \otimes u) = 0$ and so $u = 0$ by using linearly independence arguments.

Verifying $d^2 = 0$ is straightforward.

□

Theorem 5.2.43. *(1) For any $\mathbb{H}$-module X, the complex (2.1) forms a projective resolution for X.*

(2) The global dimension of $\mathbb{H}$ is $\dim V$.

Proof. From Lemma 5.2.42, it remains to show the exactness for (1). This can be proven by an argument which imposes a filtration on $\mathbb{H}$ and uses a long exact sequence (see for example [HP, Section 5.3.8] or [Kn2, Chapter IV Section 6]). We provide some detail. Let $\mathbb{H}^r$ be the (vector) subspace of $\mathbb{H}$ spanned by the elements of the form

$$t_w v_1^{n_1} \ldots v_l^{n_l} \quad \text{for } w \in W,\ v_1, \ldots, v_l \in V,$$

with $n_1 + n_2 + \ldots + n_l \leq r$. Note that $\mathbb{H}^r$ is still (left and right) invariant under the action of W. Let

$$\mathbb{E}^{r,s} = \mathbb{H}^r \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^s V).$$

Then the differential d_{s-1} defines a map from $\mathbb{E}^{r,s}$ to $\mathbb{E}^{r+1,s-1}$. For convenience, also set $\mathbb{E}^{r+s+1,-1} = X$ and there is a map from $\mathbb{E}^{r+s,0}$ to X and $d_{-1} = \epsilon$. Then for a positive integer p, we denote by $\mathcal{E}(p)$ the complex $\{\mathbb{E}^{r,s}, d_{s-1}\}_{r+s=p,s\geq -1}$. We now define a graded structure. Let

$$\mathbb{F}^{r,s} = \mathbb{E}^{r,s}/\mathbb{E}^{r-1,s},$$

and let $\overline{d}_{r,s} : \mathbb{F}^{r,s} \to \mathbb{F}^{r+1,s-1}$ be the induced map from d_{s-1}. Then $\left\{\mathbb{F}^{r,s}, \overline{d}_{r,s}\right\}_{r+s=p}$ forms a complex for each p. Denote by $\mathcal{F}(p)$ for such complex. In fact, $\mathcal{F}(p)$ forms a standard Koszul complex and hence the homology $H^i(\mathcal{F}(p)) = 0$ for all i.

Now consider the following short exact sequences of the chain of complexes for $p \geq 1$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & X & \longleftarrow & \mathbb{E}^{p-1,0} & \longleftarrow & \mathbb{E}^{p-2,1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & X & \longleftarrow & \mathbb{E}^{p,0} & \longleftarrow & \mathbb{E}^{p-1,1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & \mathbb{F}^{p,0} & \longleftarrow & \mathbb{F}^{p-1,0} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The vertical map from $\mathbb{E}^{p-s-1,s}$ to $\mathbb{E}^{p-s,s}$ is the natural inclusion map. Then we have the associated long exact sequence:

$$\ldots \to H^{k+1}(\mathcal{F}(p)) \to H^k(\mathcal{E}(p-1)) \to H^k(\mathcal{E}(p)) \to H^k(\mathcal{F}(p)) \to \ldots$$

Since $H^k(\mathcal{F}(p)) = H^{k+1}(\mathcal{F}(p)) = 0$, $H^k(\mathcal{E}(p-1)) \cong H^k(\mathcal{E}(p))$. It remains to see $H^k(\mathcal{E}(0)) = 0$ for all k, but it follows from definitions.

We now prove (2). By (1), the global dimension of $\mathbb{H}$ is less than or equal to $\dim V$. We now show the global dimension attains the upper bound. Let $\gamma \in V^\vee$ be a regular element and let v_γ be a vector with weight $\gamma \in V^\vee$. Define $X = \mathrm{Ind}_{S(V)}^{\mathbb{H}} \mathbb{C}v_\gamma$. By Frobenius reciprocity and using γ is regular, $\mathrm{Ext}^i_{\mathbb{H}}(X, X) = \mathrm{Ext}^i_{S(V)}(\mathbb{C}v_\gamma, \mathbb{C}v_\gamma) \neq 0$ for all $i \leq \dim V$. This shows the global dimension has to be $\dim V$.

$\square$

5.3 Alternate form of the Koszul-type resolution

In this section, we give another form of the differential map d_i, which involves the terms $\widetilde{v}$ defined in (3.6). There are some advantages for computations later on.

For $v \in V$, we define the following element in $\mathbb{H}$:

$$\widetilde{v} \quad = v - \tfrac{1}{2} \textstyle\sum_{\alpha \in R^+} k_\alpha \langle v, \alpha^\vee \rangle t_{s_\alpha}. \tag{3.6}$$

This element is used by Barbasch-Ciubotaru-Trapa [BCT] for the study of the Dirac cohomology for graded affine Hecke algebras and is sometimes called the Drinfield presentation [Dr]. It turns out it is quite useful in several aspects. An important property of the element is the following:

Lemma 5.3.44. *[BCT, Proposition 2.10] For any $w \in W$ and $v \in V$, $t_w \widetilde{v} = \widetilde{w(v)} t_w$.*

Proof. It suffices to show for the case that w is a simple reflection $s_\beta \in W$.

$$\begin{aligned} t_{s_\beta} \widetilde{v} &= t_{s_\beta} \left(v - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle v, \alpha^\vee \rangle t_{s_\alpha} \right) \\ &= s_\beta(v) t_{s_\beta} + k_\beta \langle v, \beta^\vee \rangle - \frac{1}{2} k_\beta \langle v, \beta^\vee \rangle - \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\beta\}} k_\alpha \langle v, \alpha^\vee \rangle t_{s_{\beta(\alpha)}} t_{s_\beta} \\ &= s_\beta(v) t_{s_\beta} - \frac{1}{2} k_\beta \langle v, s_\beta(\beta^\vee) \rangle - \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\beta\}} k_\alpha \langle v, s_\beta(\alpha^\vee) \rangle t_{s_\alpha} t_{s_\beta} \\ &= s_\beta(v) t_{s_\beta} - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle s_\beta(v), \alpha^\vee \rangle t_{s_\alpha} t_{s_\beta} \\ &= \widetilde{s_\beta(v)} t_{s_\beta}. \end{aligned}$$

□

We consider the maps $\widetilde{d_i} : \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^{i+1} V) \to \mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^i V)$ as follows:

$$\widetilde{d_i}(h \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1})) \tag{3.7}$$

$$= \sum_{j=1}^{i+1} (-1)^{j+1} \left(h \widetilde{v}_j \otimes x \otimes v_1 \wedge \ldots \widehat{v}_j \ldots \wedge v_i - h \otimes \widetilde{v}_j . x \otimes v_1 \wedge \ldots \widehat{v}_j \ldots \wedge v_{i+1} \right). \tag{3.8}$$

This definition indeed coincides with the one in the previous subsection:

Proposition 5.3.45. $\widetilde{d_i} = d_i$.

Proof. Recall that for $v \in V$,

$$\widetilde{v} = v - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle v, \alpha^\vee \rangle t_{s_\alpha}.$$

Then

$$
\begin{aligned}
& \widetilde{v}_r \otimes (x \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1}) - 1 \otimes (\widetilde{v}_r.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1}) \\
= \; & v_r \otimes (x \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1}) - 1 \otimes (v_r.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1}) \\
& \quad -\frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle v_r, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes s_\alpha(v_1) \wedge \ldots \wedge s_\alpha(\widehat{v}_r) \wedge \ldots \wedge s_\alpha(v_{i+1}) \\
& \quad +\frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle v_r, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1} \\
= \; & v_r \otimes (x \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1}) - 1 \otimes (v_r.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{i+1}) \\
& -\frac{1}{2} \sum_{\alpha \in R^+} \sum_{p<r} (-1)^p k_\alpha \langle v_r, \alpha^\vee \rangle \langle v_p, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes \\
& \quad \alpha \wedge s_\alpha(v_1) \wedge \ldots s_\alpha(\widehat{v}_p) \wedge \ldots s_\alpha(\widehat{v}_r) \wedge \ldots \wedge s_\alpha(v_{i+1}) \\
& -\frac{1}{2} \sum_{\alpha \in R^+} \sum_{r<p} (-1)^{p-1} k_\alpha \langle v_r, \alpha^\vee \rangle \langle v_p, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes \\
& \quad \alpha \wedge s_\alpha(v_1) \wedge \ldots s_\alpha(\widehat{v}_r) \wedge \ldots s_\alpha(\widehat{v}_p) \wedge \ldots \wedge s_\alpha(v_{i+1}).
\end{aligned}
$$

The second equality follows from the expression of $\widetilde{v}_r$. Taking the alternating sum of the above expression with some standard computations can verify $\widetilde{d}_i = d_i$.

$\square$

5.4 Complex for computing Ext-groups

We now use the resolution in Section 5.2 to construct a complex for computing Ext-groups. Let X and Y be $\mathbb{H}$-modules.

Then taking the $\mathrm{Hom}_{\mathbb{H}}(\cdot, Y)$ functor on the projective resolution of X as the one in (2.1), we have the induced maps for $i \geq 1$,

$$
d_i : \mathrm{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^i V), Y) \to \mathrm{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^{i+1} V), Y).
$$

Then by using the Frobenius reciprocity, we have induced complex

$$
\begin{aligned}
0 \leftarrow \mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^n V, \mathrm{Res}_W Y) & \stackrel{d_{n-1}^*}{\longleftarrow} \ldots \stackrel{d_i^*}{\leftarrow} \mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y) \\
& \stackrel{d_{i-1}^*}{\longleftarrow} \ldots \stackrel{d_0^*}{\leftarrow} \mathrm{Hom}_W(\mathrm{Res}_W X, \mathrm{Res}_W Y) \leftarrow 0,
\end{aligned} \tag{4.9}
$$

where the map d_i^* can be explicitly written as:

$$
\begin{aligned}
d_i^*(f)(x \otimes v_1 \wedge \cdots \wedge v_{i+1}) = & \sum_{j=1}^{i+1} (-1)^{j+1} \widetilde{v}_j.f(x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1}) \\
& - \sum_{j=1}^{i+1} (-1)^{j+1} f(\widetilde{v}_j.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1}),
\end{aligned}
$$

where the action of $\widetilde{v}_j$ on the term $f(x \otimes v_1 \wedge \ldots \wedge \widehat{v}_j \wedge \ldots \wedge v_{i+1})$ is via the action of $\widetilde{v}_j$ on Y and the action of $\widetilde{v}_j$ on the term x is via the action of $\widetilde{v}_j$ on X.

Thus, we obtain the following:

Proposition 5.4.46. *Let X and Y be $\mathbb{H}$-modules. Then $\mathrm{Ext}^i_{\mathbb{H}}(X, Y)$ is naturally isomorphic to the i-th homology of the complex in (4.9).*

An immediate consequence is the following:

Corollary 5.4.47. *Let X and Y be finite-dimensional $\mathbb{H}$-modules. Then*

$$\dim \mathrm{Ext}^i_{\mathbb{H}}(X, Y) < \infty.$$

Proof. Since X and Y are finite dimensional, $\mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$ is finite dimensional. Then the statement follows from Proposition 5.4.46.

□

It is not hard to see that d_i^* can be naturally extended to a map from $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$ to $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Res}_W X \otimes \wedge^{i+1} V, \mathrm{Res}_W Y)$. We denote the map by $\overline{d}_i^*$, which will be used in Section 6.4.

CHAPTER 6

DUALITY FOR EXT-GROUPS

In this chapter, we prove a duality result for the Ext-groups of graded affine Hecke algebra modules, which can be thought of as an analogue of some classical dualities such as Poincaré duality or Serre duality (also see Poincaré duality for real reductive groups in [Kn2, Theorem 6.10]). Along the way, we discuss several duals of graded affine Hecke algebra modules involved in the duality result. The duality result is our first main theorem.

We keep using the notation from Chapter 5.

6.1 θ-action and θ-dual

We define an involution θ on $\mathbb{H}$ in this section. This θ is not needed in the duality result (Theorem 6.6.63), but it closely relates to the $*$ and $\bullet$ operations defined in the next section.

Let w_0 be the longest element in W. Let θ be an involution on $\mathbb{H}$ characterized by

$$\theta(v) = -w_0(v) \text{ for any } v \in V, \text{ and } \quad \theta(t_w) = t_{w_0 w w_0^{-1}} \text{ for any } w \in W, \tag{1.1}$$

where w_0 acts on v as the reflection representation of W. Since $\theta(\Delta) = \Delta$, $\langle ., . \rangle$ is W-invariant and $k_\alpha = k_{\theta(\alpha)}$ for any $\alpha \in \Delta$, it is straightforward to verify θ defines an automorphism on $\mathbb{H}$.

Note that θ also induces an action on $V^\vee$, still denoted as θ. For $\alpha \in R$, since $w_0(\alpha^\vee) = w_0(\alpha)^\vee$, we also have $\theta(\alpha^\vee) = \theta(\alpha)^\vee$. The action θ on $V^\vee$ will be used in Chapter 7 when we need to consider weights of an $\mathbb{H}$-module.

Recall that for $v \in V$, $\widetilde{v}$ is defined in (3.6).

Lemma 6.1.48. *For any $v \in V$, $\theta(\widetilde{v}) = \widetilde{\theta(v)}$.*

Proof.

$$\begin{aligned}\theta(\widetilde{v}) &= \theta\left(v - \frac{1}{2}\sum_{\alpha\in R^+} c_\alpha\langle\alpha, v\rangle s_\alpha\right) \\ &= \theta(v) - \frac{1}{2}\sum_{\alpha\in R^+} c_\alpha\langle\alpha, v\rangle s_{\theta(\alpha)} \\ &= \theta(v) - \frac{1}{2}\sum_{\alpha\in R^+} c_{\theta(\alpha)}\langle\theta(\alpha), v\rangle s_\alpha \\ &= \theta(v) - \frac{1}{2}\sum_{\alpha\in R^+} c_\alpha\langle\alpha, \theta(v)\rangle s_\alpha \\ &= \widetilde{\theta(v)}.\end{aligned}$$

□

Definition 6.1.49. For an $\mathbb{H}$-module X, define $\theta(X)$ to be the $\mathbb{H}$-module such that $\theta(X)$ is isomorphic to X as vector spaces and the $\mathbb{H}$-action is determined by:

$$\pi_{\theta(X)}(h)x = \pi_X(\theta(h))x,$$

where π_X and $\pi_{\theta(X)}$ are the maps defining the action of $\mathbb{H}$ on X and $\theta(X)$, respectively.

6.2 *-dual and •-dual

In this section, we study two anit-involutions on $\mathbb{H}$. These two anti-involutions are studied in [BC2], but we make a variation for our need. More precisely, those anti-involutions are linear rather than Hermitian, and we will discuss how to recover the results for the original anti-automorphisms at the end of this chapter. The linearity will make some construction easier. For instance, it is easier to make the identification of spaces in Section 6.4.

Define $* : \mathbb{H} \to \mathbb{H}$ to be the linear anit-involution determined by

$$v^* = t_{w_0}\theta(v)t_{w_0}^{-1} \quad \text{for } v \in V, \quad t_w^* = t_w^{-1} \quad \text{for } w \in W.$$

Define $\bullet : \mathbb{H} \to \mathbb{H}$ to be another linear anti-involution determined by

$$v^\bullet = v \quad \text{for } v \in V\ , \quad t_w^\bullet = t_w^{-1} \quad \text{for } w \in W.$$

It is straightforward to verify $*$ and $\bullet$ are well-defined maps. Indeed, for $\bullet$-operation, we have

$$\begin{aligned}(t_{s_\alpha}v - s_\alpha(v)t_{s_\alpha})^\bullet &= vt_{s_\alpha} - t_{s_\alpha}s_\alpha(v) \quad (\alpha \in \Delta, v \in V)\\ &= -k_\alpha\langle s_\alpha(v), \alpha^\vee\rangle\\ &= -k_\alpha\langle s_\alpha(v), \alpha^\vee\rangle\\ &= -k_\alpha\langle v, s_\alpha(\alpha^\vee)\rangle\\ &= (k_\alpha\langle v, \alpha^\vee\rangle)^\bullet.\end{aligned}$$

For $*$-operation, we need the following equation (see e.g., [BCT, Lemma 2.6]):

$$v^* = -v + \sum_{\beta>0} k_\beta\langle v, \beta^\vee\rangle t_{s_\beta}.$$

Then

$$\begin{aligned}(t_{s_\alpha}v - s(v)t_{s_\alpha})^* &= v^*t_{s_\alpha} - t_{s_\alpha}s_\alpha(v)^* \quad (\alpha \in \Delta, v \in V)\\ &= \left(-v + \sum_{\beta\in R^+} k_\beta\langle v, \beta^\vee\rangle t_{s_\beta}\right)t_{s_\alpha} - t_{s_\alpha}\left(-s_\alpha(v) + \sum_{\beta\in R^+} k_\beta\langle s_\alpha(v), \beta^\vee\rangle t_{s_\beta}\right)\\ &= t_{s_\alpha}s_\alpha(v) - vt_{s_\alpha} + 2k_\alpha\langle v, \alpha^\vee\rangle\\ &= k_\alpha\langle s(v), \alpha^\vee\rangle + 2k_\alpha\langle v, \alpha^\vee\rangle\\ &= k_\alpha\langle v, \alpha^\vee\rangle\\ &= (k_\alpha\langle v, \alpha^\vee\rangle)^*.\end{aligned}$$

Definition 6.2.50. Let X be an $\mathbb{H}$-module. A map $f : X \to \mathbb{C}$ is said to be a linear functional if $f(\lambda x_1 + x_2) = \lambda f(x_1) + f(x_2)$ for any $x_1, x_2 \in X$ and $\lambda \in \mathbb{C}$. The $*$-*dual of* X, denoted by X^*, is the space of linear functionals of X with the action of $\mathbb{H}$ determined

$$(h.f)(x) = f(h^*.x) \qquad \text{for any } x \in X. \tag{2.2}$$

We similarly define $\bullet$-*dual* of X, denoted by $X^\bullet$, by replacing h^* with $h^\bullet$ in equation (2.2).

Lemma 6.2.51. *Let X be an $\mathbb{H}$-module. Define a bilinear pairing $\langle , \rangle_X^* : X^* \times X \to \mathbb{C}$ (resp. ${}^\bullet\langle , \rangle_X : X^\bullet \times X \to \mathbb{C}$) such that $\langle f, x\rangle_X^* = f(x)$ (resp. ${}^\bullet\langle f, x\rangle_X = f(x)$). (We reserve $\langle , \rangle_X^\bullet$ for the use of another pairing later.) Then*

(1) for $v \in V$, $\langle \widetilde{v}.f, x\rangle_X^* = \langle f, -\widetilde{v}.x\rangle_X^*$ (resp. ${}^\bullet\langle \widetilde{v}.f, x\rangle_X = {}^\bullet\langle f, \widetilde{v}.x\rangle_X$),

(2) for $w \in W$, $\langle t_w.f, x\rangle_X^* = \langle f, t_w^{-1}.x\rangle_X^*$ (resp. ${}^\bullet\langle t_w.f, x\rangle_X = {}^\bullet\langle f, t_w^{-1}.x\rangle_X$),

(3) $\langle , \rangle_X^*$ (resp. ${}^\bullet\langle , \rangle_X$) is nondegenerate.

Proof. We first consider $*$-operation. Note that $(\widetilde{v})^* = t_{w_0}\theta(\widetilde{v})t_{w_0}^{-1} = \widetilde{w_0\theta(v)} = -\widetilde{v}$, where the second equality follows from Lemma 5.3.44 and Lemma 6.1.48. This implies (1). Other assertions follow immediately from the definitions.

For $\bullet$-operation, we have $(\widetilde{v})^\bullet = \widetilde{v}$ from the definitions. This then implies (1). Other assertions again follow from the definitions.

□

Lemma 6.2.52. *Let X be an $\mathbb{H}$-module. Then $(X^*)^\bullet \cong \theta(X)$.*

Proof. Let θ' be an automorphism on $\mathbb{H}$ such that $\theta'(h) = (h^*)^\bullet = t_{w_0}\theta(h)t_{w_0}^{-1}$ for any $h \in \mathbb{H}$. Then we have θ' sends $\mathbb{H}$-modules to $\mathbb{H}$-modules and we denote $\theta'(X)$ to be the image of the map of an $\mathbb{H}$-module X. Note that $\theta'(X) \cong (X^*)^\bullet$ by definitions. Then it suffices to show $\theta'(X) \cong \theta(X)$. We define a map $F : X \to X$, $x \mapsto t_{w_0}^{-1}.x$. Then by definitions $\theta(h).F(x) = F(\theta'(h).x)$. This implies $\theta'(X) \cong \theta(X)$ as desired.

□

6.3 Pairing for $\wedge^i V$ and $\wedge^{n-i} V$

Fix an ordered basis $e_1, \ldots, e_n$ for V. We define a nondegenerate bilinear pairing $\langle , \rangle_{\wedge^i V}$ as

$$\wedge^i V \times \wedge^{n-i} V \to \mathbb{C}$$

determined by

$$\langle v_1 \wedge \ldots \wedge v_i, v_{i+1} \wedge \ldots \wedge v_n \rangle_{\wedge^i V} = \det(v_1 \wedge \ldots \wedge v_n),$$

where det is the determinant function for the fixed ordered basis $e_1, \ldots, e_n$.

Define $(\wedge^i V)^\vee$ to be the dual space of $\wedge^i V$. For $\omega \in \wedge^{n-i} V$, define $\phi_\omega \in (\wedge^i V)^\vee$ by

$$\phi_\omega(\omega') = \langle \omega, \omega' \rangle_{\wedge^{n-i} V}. \tag{3.3}$$

By using $\det(w(v_1) \wedge \ldots \wedge w(v_n)) = \mathrm{sgn}(w)\det(v_1 \wedge \ldots \wedge v_n)$ for any $w \in W$, we see the map $\omega \mapsto \phi_\omega$ from $\wedge^{n-i} V$ to $(\wedge^i V)^\vee$ defines a W-representation isomorphism from $\wedge^{n-i} V$ to $\mathrm{sgn} \otimes (\wedge^i V)^\vee$.

We also define a pairing $\langle , \rangle_{(\wedge^{n-i} V)^\vee} : (\wedge^{n-i} V)^\vee \times (\wedge^i V)^\vee \to \mathbb{C}$ such that

$$\langle \phi_\omega, \phi_{\omega'} \rangle_{(\wedge^{n-i} V)^\vee} = \langle \omega, \omega' \rangle_{\wedge^i V},$$

where $\omega \in \wedge^i V$ and $\omega' \in \wedge^{n-i} V$.

By definitions, we have the following:

Lemma 6.3.53. *For $\omega \in \wedge^i V$ and $\omega' \in \wedge^{n-i} V$,*

$$\langle \phi_\omega, \phi_{\omega'} \rangle_{(\wedge^{n-i}V)^\vee} = \langle \omega, \omega' \rangle_{\wedge^i V} = \phi_\omega(\omega').$$

Lemma 6.3.54. *For any $w \in W$,*

$$\langle w.\phi_{v_1 \wedge \ldots \wedge v_i}, w.\phi_{v_{i+1} \wedge \ldots \wedge v_n} \rangle_{(\wedge^{n-i}V)^\vee} = \mathrm{sgn}(w) \langle \phi_{v_1 \wedge \ldots \wedge v_i}, \phi_{v_{i+1} \wedge \ldots \wedge v_n} \rangle_{(\wedge^{n-i}V)^\vee}.$$

Proof. As noted above,

$$w.\phi_{v_1 \wedge \ldots \wedge v_i} = \mathrm{sgn}(w) \phi_{w.(v_1 \wedge \ldots \wedge v_i)},$$
$$w.\phi_{v_{i+1} \wedge \ldots \wedge v_n} = \mathrm{sgn}(w) \phi_{w.(v_{i+1} \wedge \ldots \wedge v_n)}.$$

Then

$$\begin{aligned} \langle w.\phi_{v_1 \wedge \ldots \wedge v_i}, w.\phi_{v_{i+1} \wedge \ldots \wedge v_n} \rangle_{(\wedge^{n-i}V)^\vee} &= \langle w.(v_1 \wedge \ldots \wedge v_i), w.(v_{i+1} \wedge \ldots \wedge v_n) \rangle_{(\wedge^{n-i}V)^\vee} \\ &= \mathrm{sgn}(w) \langle v_1 \wedge \ldots \wedge v_i, v_{i+1} \wedge \ldots \wedge v_n \rangle_{(\wedge^{n-i}V)^\vee} \\ &= \mathrm{sgn}(w) \langle \phi_{v_1 \wedge \ldots \wedge v_i}, \phi_{v_{i+1} \wedge \ldots \wedge v_n} \rangle_{(\wedge^{n-i}V)^\vee}. \end{aligned}$$

□

The following technical lemma is a simple linear algebra consequence, but we shall use it a number of times.

Lemma 6.3.55. *Recall that $e_1, \ldots, e_n$ is a fixed basis of V. Consider $\phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \in (\wedge^i V)^\vee$ and $\phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \in (\wedge^{n-i+1} V)^\vee$. Suppose all $k_1, \ldots, k_{n-i}$ are mutually distinct (otherwise $\phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \in (\wedge^i V)^\vee = 0$) and also suppose all $k'_1, \ldots, k'_{i+1}$ are mutually distinct. If $|\{k_1, \ldots, k_{n-i}\} \cap \{k'_1, \ldots, k'_{i+1}\}| \geq 2$, then for any $p = 1, \ldots, n-i$ and $q = 1, \ldots, i+1$,*

$$\langle \phi_{e_{k_1} \wedge \ldots \wedge \widehat{e}_{k_p} \wedge \ldots \wedge e_{k_{n-i}}}, \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle_{\wedge^i V} = \langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}}, \phi_{e_{k'_1} \wedge \ldots \wedge \widehat{e}_{k'_q} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle_{\wedge^i V} = 0.$$

If $|\{k_1, \ldots, k_{n-i}\} \cap \{k'_1, \ldots, k'_{i+1}\}| = 1$, then there exists a unique pair of indices p and q such that

$$\begin{aligned} &\langle \phi_{e_{k_1} \wedge \ldots \wedge \widehat{e}_{k_p} \wedge \ldots \wedge e_{k_{n-i}}}, \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle_{(\wedge^{i-1}V)^\vee} \\ =&(-1)^{n-i+p+q+1} \langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}}, \phi_{e_{k'_1} \wedge \ldots \wedge \widehat{e}_{k'_q} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle_{(\wedge^i V)^\vee} \end{aligned}$$

and the terms do not vanish, and for $r \neq p$ or $s \neq q$,

$$\begin{aligned} &\langle \phi_{e_{k_1} \wedge \ldots \wedge \widehat{e}_{k_r} \wedge \ldots \wedge e_{k_{n-i}}}, \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle_{(\wedge^{i-1}V)^\vee} \\ =&\langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}}, \phi_{e_{k'_1} \wedge \ldots \wedge \widehat{e}_{k'_s} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle_{(\wedge^i V)^\vee} = 0. \end{aligned}$$

Proof. It is straightforward to verify this lemma with the fixed basis. We only prove the middle equality. Suppose $|\{k_1,\dots,k_{n-i}\}\cap\{k'_1,\dots,k'_{i+1}\}|=1$. Let e be the unique element in the ordered basis $\{e_1,\dots e_n\}$ such that $e=e_{k_p}=e_{k'_q}$ for the unique indexes k_p and k'_q. Then

$$\begin{aligned}&\langle \phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{(\wedge^i V)^\vee}\\ =&\det(e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}\wedge e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\\ =&(-1)^{n-i+p+q+1}\det(e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}\wedge e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k'_{i+1}})\\ =&(-1)^{n-i+p+q+1}\langle\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{(\wedge^i V)^\vee}.\end{aligned}$$

This proves the lemma.

□

6.4 Complexes involving duals

It is well-known that there is a natural identification between the spaces $\mathrm{Hom}_W(\mathrm{Res}_W X\otimes\wedge^i V,\mathrm{Res}_W Y)$ and $(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W$ (or $(X^\bullet\otimes Y\otimes(\wedge^i V)^\vee)^W$). Here, we consider a natural W-action on $X^*\otimes Y\otimes(\wedge^i V)^\vee$ and $(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W$ is the W invariant space.

In order to prove Theorem 6.6.63 later, we need to construct some pairing, which will be more convenient to be done for the spaces $(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W$ (or $(X^\bullet\otimes Y\otimes(\wedge^i V)^\vee)^W$). The goal of this section is to translate the differential maps d_i^* in Section 5.4 into the corresponding maps for $(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W$.

We define a (linear) map $\overline{D}_i : X^*\otimes Y\otimes(\wedge^i V)^\vee\to X^*\otimes Y\otimes(\wedge^{i+1}V)^\vee$ on the complex, which is determined by

$$\begin{aligned}&\overline{D}_i(f\otimes y\otimes\phi_{v_1\wedge\ldots\wedge v_{n-i}})\\ =&\sum_{j=1}^{n-i}(-1)^{j+1}(f\otimes\widetilde{v}_j.y\otimes\phi_{v_1\wedge\ldots\widehat{v}_j\ldots\wedge v_{n-i}}+\widetilde{v}_j.f\otimes y\otimes\phi_{v_1\wedge\ldots\widehat{v}_j\ldots\wedge v_{n-i}}),\end{aligned}\tag{4.4}$$

for $f\otimes y\otimes\phi_{v_1\wedge\ldots\wedge v_{n-i}}\in X^*\otimes Y\otimes(\wedge^i V)^\vee$, where $\widetilde{v}_j$ acts on f by the action on X^* and $\widetilde{v}_j$ acts on y by the action on Y.

Note that there is a natural W-action on $X^*\otimes Y\otimes(\wedge^i V)^\vee$ (from the W-action of X^*, Y and V). Such W action commutes with $\overline{D}_i$ and so $\overline{D}_i$ sends $(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W$ to $(X^*\otimes Y\otimes(\wedge^{i+1}V)^\vee)^W$. We denote the map $\overline{D}_i$ restricted to $(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W$ by D_i. If we want to emphasis the complexes that $\overline{D}_i$ or D_i refer to, we shall write $\overline{D}_{X^*\otimes Y\otimes(\wedge^i V)^\vee}$ for $\overline{D}_i$ and $D_{(X^*\otimes Y\otimes(\wedge^i V)^\vee)^W}$ for D_i.

In the priori, we do not have $D^2=0$, but we will soon prove it in Lemma 6.4.56.

We define another map $\overline{D}_i^{\bullet} : X^{\bullet} \otimes Y \otimes (\wedge^i V)^{\vee} \to X^{\bullet} \otimes Y \otimes (\wedge^{i+1} V)^{\vee}$ determined by:

$$\begin{aligned}&\overline{D}_i^{\bullet}(f \otimes y \otimes \phi_{v_1 \wedge \ldots \wedge v_{n-i}}) \\ =&\sum_{j=1}^{n-i}(-1)^{j+1}(f \otimes \widetilde{v}_j.y \otimes \phi_{v_1 \wedge \ldots \widehat{v}_j \ldots \wedge v_{n-i}} - \widetilde{v}_j.f \otimes y \otimes \phi_{v_1 \wedge \ldots \widehat{v}_j \ldots \wedge v_{n-i}}).\end{aligned}$$

Similar to $\overline{D}_i$, the restriction of $\overline{D}_i^{\bullet}$ to $(X^{\bullet} \otimes Y \otimes (\wedge^i V)^{\vee})^W$ has image in $(X^{\bullet} \otimes Y \otimes (\wedge^{i+1} V)^{\vee})^W$. Denote by $D_i^{\bullet}$ the restriction of $\overline{D}_i^{\bullet}$ to $(X^{\bullet} \otimes Y \otimes (\wedge^i V)^{\vee})^W$.

Define a linear isomorphism $\overline{\Psi} : X^* \otimes Y \otimes (\wedge^i V)^{\vee} \to \mathrm{Hom}_{\mathbb{C}}(X \otimes \wedge^i V, Y)$ as follows, where we also regard X and Y as vector spaces: for $f \otimes y \otimes \phi_{v_1 \wedge \ldots \wedge v_{n-i}}) \in X^* \otimes Y \otimes (\wedge^i V)^{\vee}$ has the action given by: for $f \otimes y \otimes \phi_{v_{k,1} \wedge \ldots \wedge v_{k,n-i}} \in X^* \otimes Y \otimes (\wedge^i V)^{\vee}$,

$$\overline{\Psi}(f \otimes y \otimes \phi_{v_{k,1} \wedge \ldots \wedge v_{k,n-i}})(x \otimes u_1 \wedge \ldots \wedge u_i) = f(x)\phi_{v_1 \wedge \ldots \wedge v_{n-l}}(u_1 \wedge \ldots \wedge u_i)y \in Y$$

for $x \otimes u_1 \wedge \ldots \wedge u_i \in X \otimes \wedge^i V$. The map $\overline{\Psi}$ indeed depends on i, but we shall suppress the index i. By taking restriction on the space $(X^* \otimes Y \otimes (\wedge^i V)^{\vee}))^W$, we obtain the linear isomorphism:

$$\Psi : (X^* \otimes Y \otimes (\wedge^i V)^{\vee})^W \to \mathrm{Hom}_W(X \otimes \wedge^i V, Y).$$

Since X^* and $X^{\bullet}$ can be naturally identified as vector spaces, the maps $\overline{\Psi}$ and Ψ are also defined for the corresponding spaces involving $\bullet$ instead of $*$.

Recall that d_i^* and $\overline{d}_i^*$ are defined in Section 5.4 and we remark that the $*$ on d_i^* has nothing to do with the $*$-involution on $\mathbb{H}$.

Lemma 6.4.56. *Let X and Y be $\mathbb{H}$-modules. Then*

(1) For any $\omega \in X^* \otimes Y \otimes (\wedge^i V)^{\vee}$, $\overline{\Psi}(\overline{D}_i(\omega)) = (-1)^{n-i+1}\overline{d}_i^*(\overline{\Psi}(\omega))$.

(2) For any $\omega \in X^{\bullet} \otimes Y \otimes (\wedge^i V)^{\vee}$, $\overline{\Psi}^{\bullet}(\overline{D}_i^{\bullet}(\omega)) = (-1)^{n-i+1}\overline{d}_i^*(\overline{\Psi}^{\bullet}(\omega))$.

Proof. Recall that $e_1, \ldots, e_n$ be the fixed basis for V. Let $\omega = f \otimes y \otimes \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \in X^* \otimes Y \otimes (\wedge^i V)^{\vee}$. By linearity, it suffices to check that

$$\overline{\Psi}(\overline{D}_i(\omega))(x \otimes e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) = (-1)^{n-i+1}\overline{d}_i^*(\overline{\Psi}(\omega))(x \otimes e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}})$$

for any $x \in X$ and any indices $k'_1, \ldots, k'_{i+1} \in \{1, \ldots, n\}$.

Suppose $|\{k_1,\ldots,k_{n-i}\}\cap\left\{k'_1,\ldots,k'_{i+1}\right\}|\geq 2$. By Lemma 6.3.53 and Lemma 6.3.55,

$$\begin{aligned}&\overline{d}_i^*\Psi(\omega)(x\otimes e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\\=&0\\=&\Psi(\overline{D}_i\omega)(x\otimes e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}).\end{aligned}$$

Suppose $|\{k_1,\ldots,k_{n-i}\}\cap\left\{k'_1,\ldots,k'_{i+1}\right\}|=1$. Let k_p and k'_q be the unique pair of indices such that $e_{k_p}=e_{k'_q}$. Then

$$\begin{aligned}&\overline{d}_i^*\Psi(\omega)(x\otimes e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\\=&\Psi(\omega)(d_i(x\otimes e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}))\\=&(-1)^{q+1}f(x)\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k'_{i+1}})\widetilde{e}_{k'_q}.y\\&\quad-(-1)^{q+1}f(\widetilde{e}_{k'_q}.x)\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k'_{i+1}})y\quad\text{(by Lemma 6.3.55)}\\=&(-1)^{n-i-p}f(x)\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\widetilde{e}_{k'_q}.y\\&\quad-(-1)^{n-i-p}f(\widetilde{e}_{k'_q}.x)\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})y\\&\qquad\qquad\text{(by Lemma 6.3.53 and Lemma 6.3.55)}\\=&(-1)^{n-i-p}f(x)\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\widetilde{e}_{k_p}.y\\&\quad-(-1)^{n-i-p}f(\widetilde{e}_{k_p}.x)\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})y\quad\text{(by }e_{k_p}=e_{k'_q})\\=&(-1)^{n-i-p}f(x)\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\widetilde{e}_{k_p}.y\\&\quad+(-1)^{n-i-p-1}(\widetilde{e}_{k_p}.f)(x)\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}}(e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})y\ \text{(by Lemma 6.2.51)}\\=&(-1)^{n-i+1}\Psi(D_i(f\otimes y\otimes\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}))(x\otimes e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}})\quad\text{(by Lemma 6.3.55)}\\=&(-1)^{n-i+1}\Psi(\overline{D}_i(\omega))(x\otimes e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}).\end{aligned}$$

This completes the proof for (1).

The proof for (2) follows the same style of computations. (One of the differences is in the fifth equality of the computation in the second case and that explains why the definition of $\overline{D}_i^\bullet$ and $\overline{D}_i$ differs by a sign in a term.)

□

Lemma 6.4.57. *We have the following:*

(1) $D^2=0$ *and* $(D^\bullet)^2=0$,

(2) The complex $\mathrm{Hom}_W(\mathrm{Res}_W X\otimes\wedge^i V,Y)$ *with differentials* d_i^* *is naturally isomorphic to the complex* $(X^*\otimes Y\otimes(\wedge^i V))^W$ *with differentials* D_i.

(3) The complex $\mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^i V, Y)$ *with differentials* d_i^* *is naturally isomorphic to the complex* $(X^\bullet \otimes Y \otimes (\wedge^i V))^W$ *with differentials* $D_i^\bullet$*.*

Proof. By Lemma 6.4.56 (1) and the fact that Ψ is an isomorphism, $D_i = \Psi^{-1} \circ d_i^* \circ \Psi$. Then (1) follows from Lemma 5.2.42. (2) follows from Lemma 6.4.56 (1). The proof for (3) and another assertion about $D^\bullet$ in (1) is similar.

□

Proposition 6.4.58. *Let* X, Y *be* $\mathbb{H}$*-modules.*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y^*, X^*) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y^\bullet, X^\bullet) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\theta(X), \theta(Y)),$$

and the isomorphisms as vector spaces between them are natural.

Proof. We first prove that $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y^*, X^*)$. By Lemma 6.4.57 (2),

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \cong \ker D_{(X^*\otimes Y \otimes (\wedge^i V)^\vee)^W} / \operatorname{im} D_{(X^*\otimes Y \otimes (\wedge^{i-1} V)^\vee)^W},$$

and

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y^*,X^*) \cong \ker D_{((Y^*)^*\otimes X^* \otimes (\wedge^i V)^\vee)^W} / \operatorname{im} D_{((Y^*)^*\otimes X^* \otimes (\wedge^{i-1} V)^\vee)^W}.$$

With $(Y^*)^* \cong Y$, there is a natural isomorphism between the spaces $(X^* \otimes Y \otimes (\wedge^{n-i} V)^\vee)^W$ and $((Y^*)^* \otimes X^* \otimes (\wedge^{n-i} V)^\vee)^W$. It is straightforward to verify the isomorphism induces an isomorphism between the corresponding complexes by using (4.4). It can be poven similarly for $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X^\bullet, Y^\bullet)$. For the equality of $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\theta(X), \theta(Y))$, it follows from the two equalities we have just proven. Indeed,

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y^*,X^*) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}((X^*)^\bullet,(Y^*)^\bullet) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\theta(X), \theta(Y)),$$

where the last equality follows from Lemma 6.2.52.

□

6.5 Iwahori-Matsumoto dual

Definition 6.5.59. The Iwahori-Matsumoto involution ι is an automorphism on $\mathbb{H}$ determined by

$$\iota(v) = -v \quad \text{for } v \in V, \iota(w) = \mathrm{sgn}(w)w \quad \text{for } w \in W.$$

This defines a map, still denoted ι, from the set of $\mathbb{H}$-modules to the set of $\mathbb{H}$-modules.

Lemma 6.5.60. *For any* $v \in V$*,* $\iota(\widetilde{v}) = -\widetilde{v}$*.*

Proof. This follows from $\iota(s_\alpha) = -s_\alpha$ and definitions.

□

Lemma 6.5.61. *Let κ be the natural (vector space) inclusion from Y to $\iota(Y)$ so that $h.\kappa(y) = \kappa(\iota(h).y)$. Let Y be an $\mathbb{H}$-module. Define a bilinear pairing $\langle , \rangle_Y^\bullet : Y \times \iota(Y)^\bullet \to \mathbb{C}$ by $\langle y, g\rangle_Y^\bullet = g(\kappa(y))$. Then*

(1) for $v \in V$, $\langle \widetilde{v}.y, g\rangle_Y^\bullet = \langle y, -\widetilde{v}.g\rangle_Y^\bullet$,

(2) for $w \in W$, $\langle t_w.y, g\rangle_Y^\bullet = \mathrm{sgn}(w)\langle y, t_w^{-1}.g\rangle_Y^\bullet$,

(3) $\langle , \rangle_Y^\bullet$ is nondegenerate.

Proof. By a direct computation, $\widetilde{v}^\bullet = \widetilde{v}$ and then (1) follows from Lemma 6.5.60 and the definitions. (2) and (3) follow from the definitions.

□

Proposition 6.5.62. *For $\mathbb{H}$-modules X and Y, $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, \iota(Y)) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\iota(X), Y)$.*

Proof. We sketch the proof. Let $P^i \to X$ be a projective resolution of X. (To avoid confusion, we do not use $P^\bullet$ for projective resolutions as before.) Then $\iota(P^i)$ is still a projective object and $\iota(P^i) \to \iota(X)$ is a projective resolution of $\iota(X)$. There is a natural isomorphism $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(\iota(P^i), Y) \cong \mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(P^i, \iota(Y))$. Hence, $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\iota(X), Y) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, \iota(Y))$.

□

6.6 Duality theorem

In this section, we state and prove our first main result.

Theorem 6.6.63. *Let $\mathbb{H}$ be the graded affine Hecke algebra associated to a based root datum $\Pi = (\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ and a parameter function $k : \Delta \to \mathbb{C}$ (Definition 3.3.25). Let $V = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$ and let $n = \dim V$. Let X and Y be finite dimensional $\mathbb{H}$-modules. Let X^* be the $*$-dual of X in Definition 6.2.50. Let $\iota(Y)$ be the Iwahori-Matsumoto dual in Definition 6.5.59 and let $\iota(Y)^\bullet$ be the $\bullet$-dual of $\iota(Y)$ in Definition 6.2.50. Then there exists a natural nondegenerate pairing*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y) \times \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*, \iota(Y)^\bullet) \to \mathbb{C}.$$

Proof. We divide the proof into a few steps.

Step 1: Construct nondegenerate bilinear pairings.

The space

$$\mathrm{Hom}_W(X \otimes \wedge^i V, Y) \times \mathrm{Hom}_W(X^* \otimes \wedge^{n-i} V, \iota(Y)^\bullet)$$

is identified with

$$(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \times (X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i} V)^\vee)^W$$

as in Section 6.4. Let $\langle , \rangle_X^*$ be the bilinear pairing on $X^* \times X$ such that $\langle f, x\rangle_X^* = f(x)$ for $f \in X^*$ and $x \in X$. Let $\langle , \rangle_Y^\bullet$ be the bilinear pairing on $Y \times \iota(Y)^\bullet$ such that $\langle y, g\rangle_Y^\bullet = g(\kappa(y))$ for $g \in \iota(Y)^\bullet$ and $y \in Y$. Here, κ is defined as in Lemma 6.5.61.

For each i, we first define the pairing $\langle , \rangle_{X,Y,\wedge^i V}$ on a larger space $(X^* \otimes Y \otimes (\wedge^i V)^\vee) \times (X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i} V)^\vee)$ via the product of the pairings $\langle , \rangle_X^*$, $\langle , \rangle_Y^\bullet$ and $\langle , \rangle_{(\wedge^i V)^\vee}$ i.e.,

$$\begin{aligned}&\langle f \otimes y \otimes \phi_{v_1 \wedge \ldots \wedge v_{n-i}}, x \otimes g \otimes \phi_{v_{n-i+1} \wedge \ldots \wedge v_n}\rangle_{X,Y.\wedge^i V}\\ =&\langle f, x\rangle_X^* \langle y, g\rangle_Y^\bullet \langle \phi_{v_1 \wedge \ldots \wedge v_{n-i}}, \phi_{v_{n-i+1} \wedge \ldots \wedge v_n}\rangle_{(\wedge^i V)^\vee}.\end{aligned}$$

Since all the pairings $\langle , \rangle_X^*$, $\langle , \rangle_Y^\bullet$ and $\langle , \rangle_{(\wedge^i V)^\vee}$ are bilinear, $\langle , \rangle_{X,Y,\wedge^i V}$ is bilinear and well-defined.

This pairing $\langle , \rangle_{X,Y,\wedge^i V}$ is nondegenerate because $\langle , \rangle_X^*$, $\langle , \rangle_Y^\bullet$ and $\langle , \rangle_{(\wedge^i V)^\vee}$ are nondegenerate. Note that $\langle , \rangle_{X,Y,\wedge^i V}$ is W-invariant, which follows from Lemma 6.2.51(2), Lemma 6.5.61(2) and Lemma 6.3.54.

In order to see $\langle , \rangle_{X,Y,\wedge^i V}$ restricted on $(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \times (X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i} V)^\vee)^W$ is still nondegenerate, we pick $\omega \in (X^* \otimes Y \otimes (\wedge^i V)^\vee)^W$. There exists $\omega' \in X \otimes (\wedge^{n-i} V)^\vee \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i} V)^\vee$ such that $\langle \omega, \omega'\rangle_{X,Y,\wedge^i V} \neq 0$. Then by the W-invariance of $\langle , \rangle_{X,Y,\wedge^i V}$, we have $\langle \omega, \sum_{w \in W} w(\omega')\rangle_{X,Y,\wedge^i V} \neq 0$, as desired. Hence, $\langle , \rangle_{X,Y,\wedge^i V}$ restricted on $(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \times (X \otimes \iota(Y^\bullet) \otimes (\wedge^{n-i} V)^\vee)^W$ is still nondegenerate.

Step 2: Compute the adjoint operator of $\overline{D}$ for $\langle , \rangle_{X,Y,\wedge^i V}$

Recall that $\overline{D}$ is defined in (4.4). For notational simplicity, set $\overline{D}_i^1 = \overline{D}_{(X^* \otimes Y \otimes (\wedge^i V)^\vee)}$ and $\overline{D}_{n-i-1}^2 = \overline{D}_{(X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i-1} V)^\vee)}$, where we regard $X = (X^*)^*$. Recall that we fixed a basis $e_1, \ldots, e_n$ for V. We first show that

$$\begin{aligned}&\langle \overline{D}_i(f \otimes y \otimes \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}}), x \otimes g \otimes \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}}\rangle_{X,Y,\wedge^i V}\\ =& \pm \langle f \otimes y \otimes \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}}, \overline{D}_{n-i-1}^2(x \otimes g \otimes \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}})\rangle_{X,Y,\wedge^{i-1} V}\end{aligned} \tag{6.5}$$

($f \in X^*$, $x \in X$, $g \in Y^*$, $y \in Y$, $\phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \in (\wedge^i V)^\vee$ and $\phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \in \wedge^{n-i-1} V$).

We divide into two cases. Suppose $|\{k_1,\ldots,k_{n-i}\}\cap\{k'_1,\ldots,k'_{i+1}\}|\geq 2$. Then by Lemma 6.3.55,

$$\begin{aligned}&\langle \overline{D}_i^1(f\otimes y\otimes\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}), x\otimes g\otimes\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{X,Y,\wedge^i V}\\ =&0\\ =&(-1)^{n-i}\langle f\otimes y\otimes\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}, \overline{D}^2_{n-i-1}(x\otimes g\otimes\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}}))\rangle_{X,Y,\wedge^{i-1}V}.\end{aligned}$$

For the second case, suppose $|\{k_1,\ldots,k_{n-i}\}\cap\{k'_1,\ldots,k'_{i+1}\}|=1$. Let k_p and k'_q be the unique pair of indices such that $e_{k_p}=e_{k'_q}$. Then

$$\begin{aligned}&\langle \overline{D}_i^1(f\otimes y\otimes\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}), x\otimes g\otimes\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{X,Y,\wedge^i V}\\
=&(-1)^{p+1}\langle \widetilde{e}_{k_p}.f,x\rangle_X^*\ \langle y,g\rangle_Y^\bullet\ \langle\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{(\wedge^{n-i-1}V)^\vee}\\
&+(-1)^{p+1}\langle f,x\rangle_X^*\ \langle \widetilde{e}_{k_p}.y,g\rangle_Y^\bullet\ \langle\phi_{e_{k_1}\wedge\ldots\wedge\widehat{e}_{k_p}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{(\wedge^{n-i-1}V)^\vee}\\
=&(-1)^{p+1}\langle f,-\widetilde{e}_{k_p}.x\rangle_X^*\ \langle y,g\rangle_Y^\bullet\ \langle\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{(\wedge^{n-i}V)^\vee}\\
&+(-1)^{p+1}\langle f,x\rangle_X^*\langle y,-\widetilde{e}_{k_p}.g\rangle_Y^\bullet\ \langle\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k'_{i+1}}}\rangle_{(\wedge^{n-i}V)^\vee}\\
=&(-1)^{n-i+q}\langle f,-\widetilde{e}_{k'_q}.x\rangle_X^*\ \langle y,g\rangle_Y^\bullet\ \langle\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k_{i+1}}}\rangle_{(\wedge^{n-i}V)^\vee}\\
&+(-1)^{n-i+q}\langle f,x\rangle_X^*\ \langle y,-\widetilde{e}_{k'_q}.g\rangle_Y^\bullet\ \langle\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}},\phi_{e_{k'_1}\wedge\ldots\wedge\widehat{e}_{k'_q}\wedge\ldots\wedge e_{k_{i+1}}}\rangle_{(\wedge^{n-i}V)^\vee}\\
=&(-1)^{n-i}\langle f\otimes y\otimes\phi_{e_{k_1}\wedge\ldots\wedge e_{k_{n-i}}}, \overline{D}^2_{n-i-1}(x\otimes g\otimes\phi_{e_{k'_1}\wedge\ldots\wedge e_{k'_{i+1}}})\rangle_{X,Y,\wedge^{i-1}V}.\end{aligned}$$

The first and last equalities follow from Lemma 6.3.55. The second equality follows from Lemma 6.2.51 (1) and Lemma 6.5.61 (1). The third equality follows from $e_{k_p}=e_{k'_q}$. Hence, we have shown the equation (6.5). By linearity, we have for $\omega_1\in(X^*\otimes Y\otimes(\wedge^iV)^\vee)^W$ and $\omega_2\in(X\otimes Y\otimes(\wedge^{n-i-1}V)^\vee)^W$,

$$\langle D_i^1\omega_1,\omega_2\rangle_{X,Y,\wedge^{i+1}V}=(-1)^{n-i}\langle\omega_1,D^2_{n-i-1}\omega_2\rangle_{X,Y,\wedge^iV},\tag{6.6}$$

where $D_i^1=D_{(X^*\otimes Y\otimes(\wedge^iV)^\vee)^W}$ and $D^2_{n-i-1}=D_{(X\otimes\iota(Y)^\bullet\otimes(\wedge^{n-i-1}V)^\vee)^W}$.

Step 3: Descend the pairing to $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y)\times\mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*,\iota(Y)^\bullet)\to\mathbb{C}$.

We use implicitly the fact that X and Y are finite-dimensional for linear algebra results below.

For $U\subset(X\otimes\iota(Y)^\bullet\otimes(\wedge^{n-i}V)^\vee)^W$, let $U^\perp$ to be a subspace of $(X^*\otimes Y\otimes(\wedge^iV)^\vee)^W$

$$U^\perp=\left\{\omega\in(X^*\otimes Y\otimes(\wedge^iV)^\vee)^W:\langle\omega,\omega'\rangle_{X,Y,\wedge^iV}=0\text{ for all }\omega'\in U\right\}.$$

For a subspace $U\subset(X^*\otimes Y\otimes(\wedge^iV)^\vee)^W$, define $U^\perp$ to be a subspace of $(X\otimes\iota(Y)^\bullet\otimes(\wedge^{n-i-1}V)^\vee)^W$ as

$$U^\perp=\left\{\omega'\in(X\otimes\iota(Y)^\bullet\otimes(\wedge^{n-i}V)^\vee)^W:\langle\omega,\omega'\rangle_{X,Y,\wedge^iV}=0\text{ for all }\omega\in U\right\}.$$

We first show the following two equations:

$$(\ker D^1_i)^\perp = \operatorname{im} {}^\iota D^2_{n-i-1} \tag{6.7}$$

and

$$(\operatorname{im} D^1_{i-1})^\perp = \ker D^2_{n-i}. \tag{6.8}$$

For the inclusion $(\ker D^1_i)^\perp \subseteq \operatorname{im} D^2_{n-i}$, we instead show the equivalent equation $\ker D^1_i \supseteq (\operatorname{im} D^2_{n-i-1})^\perp$. For $\omega \in (\operatorname{im} D^2_{n-i-1})^\perp$, $\langle \omega, D^2_{n-i-1}\omega' \rangle_{X,Y,\wedge^i V} = 0$ for all $\omega' \in (X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i-1}V)^\vee)^W$. Then with (6.6), we have $\langle D^1_i \omega, \omega' \rangle_{X,Y,\wedge^i V} = 0$ for all $\omega' \in (X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i-1}V)^\vee)^W$. By the nondegeneracy of $\langle , \rangle_{X,Y,\wedge^i V}$ (shown in step 1), we have $D^1_i \omega = 0$ and so $\omega \in \ker D^1_i$, as desired.

For another inclusion $(\ker D^1_i)^\perp \supseteq \operatorname{im} D^2_{n-i-1}$, let $\omega'' = D^2_{n-i-1}\omega' \in \operatorname{im} D^2_{n-i-1}$. Then for any $\omega \in \ker D^1_i$, $\langle \omega, D^2_{n-i-1}\omega' \rangle = \langle D^1_i \omega, \omega' \rangle = 0$. Hence, $\omega'' \in (\ker D^1_i)^\perp$, as desired. This completes the proof for the equation (6.7). The proof for the equation (6.8) is similar.

By (6.7), the pairing $\langle .,. \rangle_{X,Y.\wedge^i V}$ first descends to

$$\ker D^1_i \times ((X \otimes \iota(Y)^\bullet \otimes (\wedge^{n-i}V)^\vee)^W / \operatorname{im} D^2_{n-i-1}).$$

Then by (6.8), the pairing $\langle .,. \rangle_{X,Y.\wedge^i V}$ further descends to

$$\ker D^1_i / \operatorname{im} D^1_{i-1} \times \ker D^2_{n-i} / \operatorname{im} D^2_{n-i-1}.$$

By Proposition 5.4.46 and Lemma 6.4.57(2), we have a natural nondegenerate pairing on

$$\operatorname{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \times \operatorname{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*, \iota(Y)^\bullet) \to \mathbb{C}.$$

□

Remark 6.6.64. We give few comments concerning the statement and the proof of Theorem 6.6.63.

(1) If X and Y have the same central character, then X^* and $\iota(Y^\bullet)$ also have the same central character (see Example 6.7.68 below).

(2) The use of the element $\widetilde{v}$ makes the computation in step 2 of the proof easier.

(3) The choice of the duals is necessary to compute the adjoint operator for the pairing $\langle .,. \rangle_{X,Y,\wedge^i V}$ in step 2. By Proposition 6.4.58, one also obtains a nondegenerate pairing

$$\operatorname{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \times \operatorname{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^\bullet, \iota(Y)^*) \to \mathbb{C}.$$

(4) The Iwahori-Matsumoto involution is necessary to show the pairing $\langle .,. \rangle_{X,Y,\wedge^i V}$ is W-invariant and so nondegenerate in step 1.

6.7 Examples

In this section, we give few examples to illustrate how Theorem 6.6.63 is compatible with some known results.

Example 6.7.65. Let St be the Steinberg module of $\mathbb{H}$, which is an one-dimensional module $\mathbb{C}x$ with the $\mathbb{H}$-action defined as:

$$v.x = \langle v, \rho^\vee \rangle x \quad \text{for } v \in V, \quad w.x = \text{sgn}(w)x \quad \text{for } w \in W.$$

Here, $\rho^\vee$ is the half sum of all the positive coroots. Then Res_WSt is a sign representation and $\text{Res}_W \iota(\text{St})$ is a trivial representation. By Proposition 5.4.46,

$$\text{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\text{St}, \text{St}) = \frac{\ker d^* : \text{Hom}_W(\text{sgn} \otimes \wedge^i V, \text{sgn}) \to \text{Hom}_W(\text{sgn} \otimes \wedge^{i+1} V, \text{sgn})}{\text{im}\, d^* : \text{Hom}_W(\text{sgn} \otimes \wedge^{i-1} V, \text{sgn}) \to \text{Hom}_W(\text{sgn} \otimes \wedge^i V, \text{sgn})}.$$

It is well-known that $\left\{\wedge^i V\right\}_{i=0}^{\dim V}$ are irreducible and mutually nonisomorphic W-representations. Hence,

$$\text{Hom}_W(\text{sgn} \otimes \wedge^i V, \text{sgn}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have $\text{Ext}^i_{\mathbb{H}}(\text{St}, \text{St}) = \mathbb{C}$ for $i = 0$ and $\text{Ext}^i_{\mathbb{H}}(\text{St}, \text{St}) = 0$ for $i \neq 0$. By a similar consideration, we have $\text{Ext}^i_{\mathbb{H}}(\text{St}, \iota(\text{St})) = \mathbb{C}$ for $i = n$ and $\text{Ext}^i_{\mathbb{H}}(\text{St}, \text{St}) = 0$ for $i \neq n$. This agrees with the conclusion from Theorem 6.6.63.

Example 6.7.66. Let R be the root system of type A_2. Let α_1, α_2 be a fixed choice of simple roots of R. Assume R spans V. Let $\gamma = \alpha_1^\vee + \alpha_2^\vee$, which is the central character of the Steinberg module. There are four irreducible modules of $\mathbb{H}(A_2)$. We parametrize the four irreducible modules by their weights and denote the corresponding modules as $X(\gamma), X(\alpha_1^\vee, -\alpha_1^\vee), X(\alpha_2^\vee, -\alpha_2^\vee), X(-\gamma)$. For example, the weights of $X(\alpha_1^\vee, -\alpha_1^\vee)$ are $\alpha_1^\vee$ and $-\alpha_1^\vee$. For $\gamma' = \gamma$ or $-\gamma$,

$$X(\gamma')^* = X(\gamma'), \quad X(\gamma')^\bullet = X(\gamma'), \quad \iota(X(\gamma')) = X(-\gamma'),$$

For other irreducible modules,

$$X(\alpha_1^\vee, -\alpha_1^\vee)^* = X(\alpha_2^\vee, -\alpha_2^\vee), \quad X(\alpha_k^\vee, -\alpha_k^\vee)^\bullet = X(\alpha_k^\vee, -\alpha_k^\vee) \quad \text{for } k = 1, 2$$

$$\iota(X(\alpha_k^\vee, -\alpha_k^\vee)) = X(\alpha_k^\vee, -\alpha_k^\vee) \quad \text{for } k = 1, 2.$$

We have $X^* \not\cong \iota(X^\bullet)$ for any irreducible module X. Results in [Or], which in particular compute Ext-groups of all pairs of irreducible $\mathbb{H}$-modules of center character $W\gamma$, agree with Theorem 6.6.63.

Example 6.7.67. Let $k_\alpha \neq 0$ for all $\alpha \in \Delta$. Let X be the minimal parabolically induced with the central character 0. Then X is irreducible. Then by [OS3, Theorem 5.2], $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, X) \cong \wedge^i V$. We also have $X^* \cong X^\bullet \cong \iota(X) \cong X$. Then we have

$$\dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, X) = \begin{pmatrix} n \\ i \end{pmatrix} = \begin{pmatrix} n \\ n-i \end{pmatrix} = \dim \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*, \iota(X^\bullet)).$$

Example 6.7.68. Let $k_\alpha \neq 0$ for all $\alpha \in \Delta$. Let X be an irreducible principle series with a regular central character $W\gamma$ ($\gamma \in V^\vee$). Then X^* has the central character $W\theta(\gamma) = -W\gamma$ (since $\theta(\gamma) = -w_0(\gamma)$) and $\iota(X^\bullet)$ has the central character $-W\gamma$. By the irreducibility, we have $X^* \cong \iota(X^\bullet)$.

On another hand, by the Frobenius reciprocity and γ being regular,

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, X) = \mathrm{Ext}^i_{\mathfrak{R}(S(V))}(\mathbb{C}v_\gamma, \mathbb{C}v_\gamma) \cong \wedge^i V,$$

where v_γ is a vector with the $S(V)$-weight γ. We again have

$$\dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, X) = \begin{pmatrix} n \\ i \end{pmatrix} = \begin{pmatrix} n \\ n-i \end{pmatrix} = \dim \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*, \iota(X^\bullet)).$$

Similar consideration can extend to examples of (not necessarily irreducible) minimal principle series with a regular central character.

Example 6.7.69. Let $k \equiv 0$. Let X and Y be irreducible $\mathbb{H}$-modules with the central character 0. Then $\mathrm{Res}_W X$ and $\mathrm{Res}_W Y$ are also irreducible W-representations. By Proposition 5.4.46, $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y) = \mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$. Since all W-representations are self-dual, we have $X^* \cong X$ and $\iota(Y^\bullet) \cong \mathrm{sgn} \otimes Y$, where the sgn means the action of t_w ($w \in W$) is twisted by $\mathrm{sgn}(w)$. Then we have

$$\begin{aligned} \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^*, \iota(Y)^\bullet) &\cong \mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^{n-i} V, \mathrm{sgn} \otimes \mathrm{Res}_W Y) \\ &\cong \mathrm{Hom}_W(\mathrm{Res}_W X \otimes \mathrm{sgn} \otimes \wedge^{n-i} V, \mathrm{Res}_W Y) \\ &\cong \mathrm{Hom}_W(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y) \\ &\cong \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y), \end{aligned}$$

where the third isomorphism uses the fact that $\mathrm{sgn} \otimes \wedge^i V \cong \wedge^{n-i} V$ as W-representations.

6.8 Hermitian duals

As promised in Section 6.2, we will discuss the situation of the Hermitian anti-involutions usually encountered in the literature. Those anti-involutions are more natural because of their close relation to the unitary representations of p-adic groups [BC2]. An important assumption we have to make in this section is $k_\alpha \in \mathbb{R}$ for all $\alpha \in \Delta$. This is necessary for those anti-involutions to be well-defined.

Since $V = \mathbb{C} \otimes_{\mathbb{R}} V_0$, there is a natural complex conjugation on $\mathbb{H}$. Denote by $\overline{h}$ the complex conjugation of $h \in \mathbb{H}$. Define $\dagger : \mathbb{H} \to \mathbb{H}$ to be an Hermitian anit-involution determined by

$$v^\dagger = \overline{t_{w_0}\theta(v)t_{w_0}^{-1}} \quad \text{for } v \in V, \quad t_w^\dagger = t_w^{-1} \quad \text{for } w \in W.$$

Define $\circ : \mathbb{H} \to \mathbb{H}$ to be another Hermitian anti-involution determined by

$$v^\circ = \overline{v} \quad \text{for } v \in V \ , \quad t_w^\circ = t_w^{-1} \quad \text{for } w \in W.$$

It follows from similar calculations in Section 6.2 that both $\dagger$ and $\circ$ are well-defined. The analogue of Definition 6.2.50 is the following:

Definition 6.8.70. Let X be an $\mathbb{H}$-module. A map $f : X \to \mathbb{C}$ is said to be a Hermitian functional if $f(\lambda x_1 + x_2) = \overline{\lambda} f(x_1) + f(x_2)$ for any $x_1, x_2 \in X$ and $\lambda \in \mathbb{C}$. The $\dagger$-*dual of* X, denoted by $X^\dagger$, is the space of Hermitian functionals of X with the action of $\mathbb{H}$ determined

$$(h.f)(x) = f(h^\dagger.x) \quad \text{for any } x \in X. \tag{8.9}$$

We similarly define $\circ$-*dual* of X, denoted by X°, by replacing $h^\dagger$ with h° in equation (8.9).

In fact, we can define a conjugation involution ϵ as follows:

$$\epsilon(v) = \overline{v} \quad \text{for } v \in V \ , \quad \epsilon(t_w) = t_w \quad \text{for } w \in W.$$

Then $h^\dagger = \epsilon(h)^* = \epsilon(h^*)$ and $h^\circ = \epsilon(h)^\bullet = \epsilon(h^\bullet)$ for any $h \in \mathbb{H}$. The map ϵ induced a bijection, still denoted ϵ, from $\mathbb{H}$-modules to $\mathbb{H}$-modules.

Then using argument similar to Proposition 6.5.62, we have

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y) \cong \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(\epsilon(X), \epsilon(Y))$$

for any $\mathbb{H}$-modules X and Y. Then combining with Theorem 6.6.63, we have:

Theorem 6.8.71. *Let $\mathbb{H}$ be a graded affine Hecke algebra associated to a based root datum $\Pi = (\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ and a real parameter function $k : \Delta \to \mathbb{R}$ (Definition 3.3.25). Let $V = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$ and let $n = \dim V$. Then for any finite-dimensional $\mathbb{H}$-modules X and Y, there exists a natural nondegenerate pairing*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y) \times \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X^\dagger, \iota(Y)^\circ) \to \mathbb{C}.$$

CHAPTER 7

EXTENSIONS AND THE LANGLANDS CLASSIFICATION

To study the extensions of graded affine Hecke algebra modules, one may want to look for some natural construction of the extensions. A natural one is from the parabolic induction. In this chapter, we study a few classes of parabolically induced modules with the goal to understand some extensions. The most important class is the induced modules in the Langlands classification, from which we compare central characters to obtain information about Ext-groups. We also construct induced modules for discrete series and tempered modules and use them to study extensions. One may also compare the methods in [Hu2, Chapter 6] for the study of extensions in the BGG category $\mathcal{O}$. As an application of the study, we compute the Ext-groups among discrete series in the next chapter.

In this and next chapters, we make the following assumption: R spans V. In other words, $\mathbb{C} \otimes_{\mathbb{Z}} R = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$. This assumption will make the discussion more convenient and it is not hard to formulate corresponding results without the assumption.

7.1 Langlands classification

In this section, we review the Langlands classification for graded affine Hecke algebras in [Ev] (also see [KR]). We shall not reproduce a proof here, but we point out that the proof for the Langlands classification is algebraic (see [Ev], [KR]) and does not rely on results of affine Hecke algebras or p-adic groups.

We first need a notation of parabolic subalgebra of $\mathbb{H}$.

Notation 7.1.72. For any subset J of Δ, define V_J to be the complex subspace of V spanned by vectors in J and define $V_J^\vee$ to be the dual space of V_J lying in $V^\vee$. Let $R_J = V_J \cap R$ and let $R_J^\vee = V_J^\vee \cap R^\vee$. Let W_J be the subgroup of W generated by the elements s_α for $\alpha \in J$. Define

$$V_J^{\vee,\perp} = \left\{ v \in V : \langle v, u^\vee \rangle = 0 \quad \text{for all } u \in V_J^\vee \right\},$$

and

$$V_J^{\perp} = \left\{ v^\vee \in V^\vee : \langle u, v^\vee \rangle = 0 \quad \text{for all } u \in V_J \right\}.$$

For $J \subset \Delta$, let W_J be the subgroup of W generated by all s_α with $\alpha \in J$. Let $w_{0,J}$ be the longest element in W_J. Let W^J be the set of minimal representatives in the cosets in W/W_J.

Let $J \subset \Delta$. Define $\mathbb{H}_J$ to be the subalgebra of $\mathbb{H}$ generated by all $v \in V$ and t_w ($w \in W_J$). We also define $\overline{\mathbb{H}}_J$ to be the subalgebra of $\mathbb{H}$ generated by all $v \in V_J$ and t_w ($w \in W_J$). Note that $\mathbb{H}_J$ decomposes as

$$\mathbb{H}_J = \overline{\mathbb{H}}_J \otimes S(V_J^{\vee,\perp}).$$

Note $\mathbb{H}_J$ is the graded affine Hecke algebra associated to the root datum $(\mathcal{X}, R_J, \mathcal{Y}, R_J^\vee, J)$ and $\overline{\mathbb{H}}_J$ is the graded affine Hecke algebra associated to the root data $(\mathcal{Q}_J, R_J, \mathcal{P}_J^\vee, R_J^\vee, J)$, where $\mathcal{Q}_J$ is the root lattice of R_J and $\mathcal{P}_J^\vee$ is the corresponding coweight lattice.

We first describe the notion of parabolically induced modules. Denote by Ξ the set of pairs of (J, U) with $J \subset \Delta$ and irreducible $\mathbb{H}_J$-modules U. For $(J, U) \in \Xi$, $I(J, U)$ the induced module $\mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}} U$ from the $\mathbb{H}_J$-module U. Denote by $\mathrm{Res}_{\mathbb{H}_J}$ the right adjoint functor of $\mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}}$. We also denote by $\mathrm{Res}_{\overline{\mathbb{H}}_J}$ the restricton functor from $\mathbb{H}$-modules to $\overline{\mathbb{H}}_J$-modules.

Let $\nu \in V_J^{\perp} \subset V_J^\vee$ and let $\mathbb{C}_\nu$ be the corresponding one-dimensional $S(V_J^{\vee,\perp})$-module. For any $\alpha \in \Delta$, denote by $\omega_\alpha^\vee \in V_0^\vee$ the fundamental coweight corresponding to α i.e., for $\beta \in \Delta$,

$$\langle \beta, \omega_\alpha^\vee \rangle = \omega_\alpha^\vee(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

We define similarly for $\omega_\alpha \in V_0$ for $\alpha \in \Delta$.

Definition 7.1.73. An $\mathbb{H}$-module X is said to be *tempered* if any weight γ of X is of the form:

$$\mathrm{Re}\gamma = \sum_{\alpha \in \Delta} a_\alpha \alpha^\vee, \quad a_\alpha \leq 0.$$

Equivalently, an $\mathbb{H}$-module X is tempered if and only if $\langle \omega_\alpha, \gamma \rangle \leq 0$ for all $\alpha \in \Delta$ and for all weight γ of X. An $\mathbb{H}$-module X is said to be a *discrete series* if any weight γ of X is of the form:

$$\mathrm{Re}\gamma = \sum_{\alpha \in \Delta} a_\alpha \alpha^\vee, \quad a_\alpha < 0.$$

Equivalently, X is a discrete series if and only if $\langle \omega_\alpha, \mathrm{Re}\gamma \rangle < 0$ for all $\alpha \in \Delta$ and for all weight γ of X. In particular, an $\mathbb{H}$-discrete series is tempered.

For any $\gamma \in V^\vee$, we can uniquely write Reγ as the form (see [Kn1, Ch VIII Lemma 8.56]):

$$\text{Re}\gamma = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \sum_{\beta \in \Delta \setminus J} b_\beta \omega_\beta^\vee \quad \text{with } a_\alpha \leq 0,\, b_\beta > 0 \ .$$

Denote

$$\gamma_0 = \sum_{\beta \in \Delta \setminus J} b_\beta \omega_\alpha^\vee \in V_0^\vee.$$

Definition 7.1.74. We say a pair $(J, U) \in \Xi$ is a *Langlands classification parameter* if $U = \overline{U} \otimes \mathbb{C}_\nu$ as $\mathbb{H}_J \cong \overline{\mathbb{H}}_J \otimes S(V_J^{\vee,\perp})$-modules for some $\overline{\mathbb{H}}_J$-tempered module $\overline{U}$ and $\nu \in V_J^\perp$ with $\langle \text{Re}\alpha, \nu \rangle > 0$ for all $\alpha \in \Delta \setminus J$. Recall that ω_α is the fundamental weight associated to α. Set $\lambda(J, U) = \nu$. Denote by Ξ_L the set of all Langlands classification parameters. Hence, if $(J, U) \in \Xi_L$, any weight γ of U has the form

$$\text{Re}\gamma = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \sum_{\beta \in \Delta \setminus J} b_\beta \omega_\beta^\vee,$$

with $a_\alpha \leq 0$ and $b_\beta > 0$. Note that $\lambda(J, U) = \gamma_0$, where $\lambda(J, U)$ is independent of the choice of the weights γ of U.

We need a similar but stronger notation later. We say a pair (J, U) is a *strong Langlands classification parameter* if $U = \overline{U} \otimes \mathbb{C}_\nu$ as $\mathbb{H}_J \cong \overline{\mathbb{H}}_J \otimes S(V_J^{\vee,\perp})$-modules for some $\overline{\mathbb{H}}_J$-discrete series $\overline{U}$ and $\nu \in V_J^\perp$ with $\langle \alpha, \text{Re}\nu \rangle > 0$ for all $\alpha \in \Delta \setminus J$. Denote by $\Xi_{L,ds}$ the set of all strong Langlands classification parameters.

Theorem 7.1.75. *(Langlands classification) [Ev]*

(1) For any irreducible $\mathbb{H}$-module X, there exists $(J, U) \in \Xi_L$ such that X is isomorphic to the unique irreducible quotient of $I(J, U)$.

(2) Let $(J, U), (J_1, U_1) \in \Xi_L$. If the unique simple quotients of $I(J, U)$ and $I(J_1, U_1)$ are isomorphic, then $J = J_1$ and $U \cong U_1$ as $\mathbb{H}_J$-modules.

From Theorem 7.1.75, each irreducible $\mathbb{H}$-module can be associated to a pair in Ξ_L. We have the following terminology:

Definition 7.1.76. Let X be an irreducible $\mathbb{H}$-module. Let $(J, U) \in \Xi_L$ be such that X is the unique quotient of $I(J, U)$. We call (J, U) is the *Langlands classification parameter* for X.

There is an useful information about the weights in the Langlands classification. We need the following notation for comparing weights:

Definition 7.1.77. Let $\gamma, \gamma' \in V_0^\vee$. We write $\gamma \leq \gamma'$ if $\langle \omega_\alpha, \gamma \rangle \leq \langle \omega_\alpha, \gamma' \rangle$ for all $\alpha \in \Delta$. We write $\gamma < \gamma'$ if $\gamma \leq \gamma'$ and $\langle \omega_\alpha, \gamma \rangle < \langle \omega_\alpha, \gamma' \rangle$ for some $\alpha \in \Delta$.

The following lemma related the dual and the Langlands classification parameter will be useful later.

Lemma 7.1.78. *Let X be an irreducible $\mathbb{H}$-module. Let $(J, U) \in \Xi_L$ be the Langlands parameter for X and let $(J_*, U_*) \in \Xi_L$ be the Langlands classification parameter of X^*. Then $\lambda(J_*, U_*) = \theta(\lambda(J, U))$.*

Proof. Following from the construction in the Langlands classification (see [KR, (2.12)], also see Proposition 7.1.81 below), $\lambda(J, U)$ (resp. $\lambda(J_*, U_*)$) is the maximal element in the set

$$\{\gamma_0 : \gamma \text{ is a weight of } X\} \quad (\text{resp. } \{\gamma_0 : \gamma \text{ is a weight of } X^*\}).$$

On the other hand, by Lemma 6.2.52, γ is a weight of X if and only if $\theta(\gamma)$ is a weight of X^*. Hence, $\theta(\lambda(J_*, U_*)) = \lambda(J, U)$.

□

Proposition 7.1.81 below is in the proof of the Langlands classification [KR]. We reproduce the proof since the statement is crucial for our argument later.

Lemma 7.1.79. *(Lemma of Langlands) [Kn1, Ch VIII Lemma 8.59] Let $\gamma, \gamma' \in V_0^\vee$. If $\gamma \leq \gamma'$, then $\gamma_0 \leq \gamma_0'$.*

Lemma 7.1.80. *Let $J \subset \Delta$ and let $w \in W^J$. Then $w(\omega_\alpha^\vee) \leq \omega_\alpha^\vee$ for all $\alpha \in \Delta$ and $w(\omega_\alpha^\vee) < \omega_\alpha^\vee$ for some $\alpha \notin J$.*

Proof. Let $w = s_{\alpha_1} \dots s_{\alpha_l}$ be a reduced expression of w. Let $\beta_i^\vee = s_{\alpha_1} \dots s_{\alpha_{i-1}}(\alpha_i^\vee)$. Then

$$w(\omega_\alpha^\vee) = \omega_\alpha^\vee - \sum_{i=1}^{l} \langle \alpha_i, \omega_\alpha^\vee \rangle \beta_i^\vee \leq \omega_\alpha^\vee.$$

The last equality follows from $\langle \alpha_i, \omega_\alpha^\vee \rangle = 0, 1$ and $-\beta_i^\vee < 0$. Since $w \notin W^J$, $\alpha_k \notin J$ and so $\langle \alpha_k, \omega_{\alpha_k}^\vee \rangle = 1$. Hence, $w(\omega_{\alpha_k}^\vee) < \omega_{\alpha_k}^\vee$ as desired.

□

Proposition 7.1.81. *Let $(J, U) \in \Xi_L$. Let M be a composition factor of $I(J, U)$. Let $(J_1, U_1) \in \Xi_L$ be the Langlands classification parameter for M. Suppose M is not isomorphic to the unique simple quotient of $I(J, U)$ (i.e., $(J, U) \neq (J_1, U_1)$). Then $\lambda(J_1, U_1) < \lambda(J, U)$.*

Proof. Any weight of M is of the form $w(\mu)$ for some $w \in W^J \setminus \{1\}$ and some weight μ of U ([BM2, Theorem 6.4]). Write $\mathrm{Re}\mu$ in the form

$$\mathrm{Re}\mu = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \lambda(J, U) \quad \text{ with } a_\alpha \leq 0.$$

Then

$$\mathrm{Re}(w(\mu)) = \sum_{\alpha \in J} a_\alpha w(\alpha^\vee) + w(\lambda(J, U)).$$

Since $w(\alpha^\vee) > 0$ for all $\alpha^\vee$, we have $\mathrm{Re}(w(\mu)) \leq w(\lambda(J, U))$. For $w \neq 1$, by Lemma 7.1.80, we also have $w(\omega_\alpha) \leq \omega_\alpha$ for all $\alpha \notin J$ and $w(\omega_\alpha) < \omega_\alpha$ for some $\alpha \notin J$. Hence, we also have $\mathrm{Re}(w(\mu)) \leq w(\lambda(U)) < \lambda(J, U)$ and so $\mathrm{Re}(w(\mu))_0 < \lambda(J, U)$. Thus, for any weight γ of M, $\gamma_0 < \lambda(J, U)$ by Lemma 7.1.79.

On another hand, there is a surjective map from $I(J_1, U_1)$ to M by definition. Then by the Frobenius reciprocity, $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H}_J)}(U_1, \mathrm{Res}_{\mathbb{H}_J} M) \neq 0$. Hence, $\lambda(J_1, U_1) = \gamma_0$ for some weight γ of M. With the discussion in the previous paragraph, we have $\lambda(J_1, U_1) < \lambda(U)$.

□

Example 7.1.82. Let R be of type G_2 and let $k \equiv 1$. Let α, β be the simple roots of R with $\langle \alpha, \beta^\vee \rangle = -1$ and $\langle \beta, \alpha^\vee \rangle = -3$. We consider modules of the central character $\alpha^\vee + \beta^\vee$ and the possible weights are

$$\pm(\alpha^\vee + \beta^\vee), \pm(\alpha^\vee + 2\beta^\vee), \pm\beta^\vee.$$

Note that $\alpha^\vee + \beta^\vee = -\frac{1}{2}\beta^\vee + \frac{1}{2}(2\alpha^\vee + 3\beta^\vee)$. Let $J = \{\beta\}$ and let St be the Steinberg module of $\overline{\mathbb{H}}_J$ and let $\nu = \frac{1}{2}(2\alpha^\vee + 3\beta^\vee)$. The $\mathbb{H}_J$-module $\mathrm{St} \otimes \mathbb{C}_\nu$ has the weight $\alpha^\vee + \beta^\vee$ and $(J, \mathrm{St} \otimes \mathbb{C}_\nu)$ is a Langlands classification parameter. Let $I = \mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}}(\mathrm{St}, \mathbb{C}_\nu)$. It is known that Y has three composition factors. Denote by Y for the simple quotient of I, denote by DS for the composition factor of I being a discrete series and denote by Z for the remaining composition factor of I. We have

(1) The weight of Y is $\gamma^1 := \alpha^\vee + \beta^\vee$.

(2) The weights of DS are $\gamma^2 := -\alpha^\vee - 2\beta^\vee$ with multiplicity 2 and $\gamma^3 = -\alpha^\vee - \beta^\vee$.

(3) The weights of Z are $\gamma^4 := \beta^\vee$ and $\gamma^5 := -\beta^\vee$.

Then $\gamma^1_0 = \frac{1}{2}(2\alpha^\vee + 3\beta^\vee)$, $\gamma^2_0 = 0$, $\gamma^3_0 = 0$, $\gamma^4_0 = \frac{1}{2}(\alpha^\vee + 2\beta^\vee)$ and $\gamma^5_0 = 0$.

7.2 Inner product on V_0

Since V_0 is a real representation of W, there exists a W-isomorphism, denoted η from V_0 to $V_0^\vee$. Define a W-invariant bilinear form on V_0 by $(v_1, v_2) = \eta(v_2)(v_1)$. Since $V_0^\vee$ is irreducible, there exists a unique, up to a scalar, W-invariant bilinear form on V_0. Hence, $(.,.)$ is also symmetric. By the uniqueness, we also have the W-invariant bilinear form $(.,.)$ to be positive-definite. For $\gamma \in V_0$, denote by $||\gamma|| := \sqrt{(\gamma,\gamma)}$ the length of γ.

Furthermore, for each $\alpha \in \Delta$, $\mathbb{R}\alpha$ and $\mathbb{R}\alpha^\vee$ are the (-1)-eigenspaces of s_α on V_0 and $V_0^\vee$, respectively. Hence, $\eta(\alpha^\vee) \in \mathbb{R}\alpha$ for each $\alpha \in \Delta$. For $\alpha, \beta \in \Delta$ with $\beta \neq \alpha$, we also have

$$(\eta(\omega_\alpha^\vee), \beta) = (\beta, \eta(\omega_\alpha^\vee)) = \langle \beta, \omega_\alpha^\vee \rangle = 0.$$

Hence, $\eta(\omega_\alpha^\vee) \in \mathbb{R}\omega_\alpha$. Thus, the setting of Langlands classification in Section 7.1 can be naturally reformulated by the notations for V_0 (e.g., α, ω_α) if we identify V_0 with $V_0^\vee$ via the isomorphism η.

We also extend $(.,.)$ linearly to a symmetric bilinear form on V. We remark that in the Langlands classification, we mainly consider the real part of weights of a module. It will be important for $(.,.)$ to be an inner product on V_0 for comparing the length of weights later (see Definition 7.3.83 below).

7.3 Induced module for discrete series

We continue to assume R spans V. Let $n = \dim V = |\Delta|$. We keep using the notation in Section 7.2 (e.g., the bilinear form $(.,.)$ on V). Our goal is to construct a maximal parabolically induced module containing certain discrete series. We introduce the following notations to keep track useful information.

Definition 7.3.83. Let X be an irreducible $\mathbb{H}$-discrete series. Let Δ_{n-1} be the set containing all the subset of Δ of cardinality $n-1$. Let $\mathcal{W}(X)$ be the set of weights of X. Define a function $\Phi : \Delta_{n-1} \times \mathcal{W}(X) \to \mathbb{C}$ by

$$\Phi(J, \gamma) = -\frac{(\mathrm{Re}\gamma, \omega_\beta)}{||\omega_\beta||} = \frac{|(\mathrm{Re}\gamma, \omega_\beta)|}{||\omega_\beta||},$$

where β is the unique element in $\Delta \setminus J$. Denote by $(J_X, \gamma_X) \in \Delta_{n-1} \times \mathcal{W}(X)$ to be the pair such that $\Psi(J_X, \gamma_X)$ attains the minimum value among all pairs in $\mathcal{W}(X)$. Denote by β_X the unique element in $\Delta \setminus J_X$. Denote

$$L_{ds}(X) = \Phi(J_X, \gamma_X),$$

$$\lambda_{ds}(X) = L_{ds}(X) \frac{\omega_{\beta_X}}{||\omega_{\beta_X}||},$$

where ds stands for discrete series.

Example 7.3.84. We keep using the notation in Example 7.1.82. By fixing a choice of $(.,.)$, we have the following equations:

$$(\alpha, \alpha) = 2, \quad (\alpha, \beta) = -3, \quad (\beta, \beta) = 6.$$

We have $\omega_\alpha = 2\alpha + \beta$ and $\omega_\beta = 3\alpha + 2\beta$. The $||\omega_\alpha||^2 = 2$ and $||\omega_\beta||^2 = 6$. We consider the discrete series DS which has weights γ^2, γ^3.

$$\Phi(\{\alpha\}, \gamma^2) = \frac{2}{\sqrt{6}}, \quad \Phi(\{\beta\}, \gamma^2) = \frac{1}{\sqrt{2}},$$

$$\Phi(\{\alpha\}, \gamma^3) = \frac{1}{\sqrt{6}}, \quad \Phi(\{\beta\}, \gamma^3) = \frac{1}{\sqrt{2}}.$$

Thus, $J_{DS} = \{\alpha\}$. $L_{ds}(DS) = \frac{1}{\sqrt{6}}$ and $\lambda_{ds}(DS) = \frac{1}{6}\omega_\beta$. We also see $\eta(\lambda_{ds}(DS)) = \frac{1}{6}(3\alpha^\vee + 6\beta^\vee)$.

Before constructing an induced module for a discrete series, we mention useful results about duals (Proposition 7.3.85 and Lemma 7.3.87). Recall that $*$ and $\bullet$ are defined in Section 6.2.

Proposition 7.3.85. *[BM3, Corollary 1.4] For $(J, U) \in \Xi$, $I(J, U)^*$ is isomorphic to $I(J, U^{*_J})$, where $*_J$ is the corresponding $*$-operation for $\mathbb{H}_J$-modules.*

Proof. The proof is essentially the same as the one in [BM3, Corollary 1.4]. For any $h \in \mathbb{H}$, h can be uniquely written as the form

$$\sum_{w \in W^J} t_w h_w,$$

where $h_w \in \mathbb{H}_J$. Define the map $\sigma : \mathbb{H} \to \mathbb{H}_J$ by $\sigma(h) = h_e$ (e is the identity element in W^J). Then we define a bilinear form $\langle , \rangle^* : I(J, U) \times I(J, U^{*_M}) \to \mathbb{C}$ as

$$\langle h \otimes u, k \otimes f \rangle^* = f(\sigma(k^* h)u).$$

The hardest part is to check $\langle ., . \rangle$ is well-defined. By a slight consideration, we see it suffices to show $\sigma(h)^{*_M} = \sigma(h^*)$. We divide into two cases. For the first case, we consider $t_w p$ for $w \in W_J$ and $p \in S(V)$.

$$\begin{aligned}\sigma((t_w p)^*) &= \sigma(p^* t_{w^{-1}}) \\ &= \sigma\left(\left(-p + \sum_{\alpha>0} k_\alpha \frac{s_\alpha(p)-p}{\alpha} t_{s_\alpha}\right) t_{w^{-1}}\right) \\ &= \left(-p + \sum_{\alpha\in W_J, \alpha>0} k_\alpha \frac{s_\alpha(p)-p}{\alpha} t_{s_\alpha}\right) t_{w^{-1}} \\ &= (t_w p)^{*_J} \\ &= \sigma(t_w p)^{*_J}.\end{aligned}$$

We now consider $t_w p$ for $w \notin W_J$ and $p \in S(V)$.

$$\begin{aligned}\sigma((t_w p)^*) &= \sigma(p^* t_w^{-1}) \\ &= \sigma(t_{w_0}\theta(p) t_{w_0}^{-1} t_{w^{-1}}) \\ &= \sigma\left(t_{w_0} t_{w_0 w^{-1}} \sum_{\alpha>0, w_0 w^{-1}(\alpha)<0} k_\alpha t_{s_\alpha} p_\alpha\right) \qquad \text{where } p_\alpha \in S(V) \\ &= \sigma\left(\sum_{\alpha>0, w_0 w^{-1}(\alpha)<0} k_\alpha t_{w^{-1}s_\alpha} p_\alpha\right).\end{aligned}$$

We now claim that $w^{-1}s_\alpha \notin W_J$ for any $w_0 w^{-1}(\alpha) < 0$. In fact, since $w_0 w^{-1}(\alpha) < 0$, we have $w^{-1}(\alpha) > 0$. If $s_\alpha \in W_J$, $w^{-1}s_\alpha \notin W_J$ by $w \notin W_J$. If $\alpha \notin W_J$, then $w^{-1}s_\alpha(\alpha) < 0$ and so $w^{-1}s_\alpha \notin W_J$ by the fact that any element in W_J sends positive roots not in W_J to positive roots. This proves the claim. Now the claim implies $\sigma((t_w p)^*) = 0$. On the other hand, we also have $\sigma(t_w p) = 0$.

Then by linearity of σ, we proved $\sigma(h^*) = \epsilon(h)^{*_J}$, as desired.

□

Example 7.3.86. Let $J \subset \Delta$ be a singleton. Let $I = \mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}}(\mathrm{St} \otimes \mathbb{C}_\nu)$ for the Steinberg $\overline{\mathbb{H}}_J$-module St and $\nu \in V_J^\perp$. Then $I^* = \mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}}(\mathrm{St} \otimes \mathbb{C}_{-\nu})$. We now specify the example to type A_2 with the notation in Example 6.7.66. Take $J = \{\alpha_1\}$ and $\nu = -\frac{1}{2}\alpha_1^\vee - \alpha_2^\vee$. Then the composition factors of I are $X(-\gamma)$ and $X(\alpha_1^\vee, -\alpha_1^\vee)$ while the composition factors of I^* are $X(-\gamma)$ and $X(\alpha_2^\vee, -\alpha_2^\vee)$.

Lemma 7.3.87. *Let X be an irreducible $\mathbb{H}$-discrete series. Then X^*, $\theta(X)$ and $X^\bullet$ are also discrete series.*

Proof. By definitions, γ is a weight of X if and only if $\theta(\gamma)$ is a weight of $\theta(X)$. Since $\theta(\omega_\alpha) = \omega_{\theta(\alpha)}$, we have $\theta(X)$ is also a discrete series. For $X^\bullet$, X and $X^\bullet$ have the same weights and so $X^{\bullet}$ is also a discrete series. By Lemma 6.2.52, $X^* \cong \theta(X^{\bullet})$ is also a discrete series.

□

We now construct an induced module with respect to a discrete series. We also relate those induced modules to the Langlands classification parameter and so the composition factors of the induced module can also be better understood.

Proposition 7.3.88. *Let X be an irreducible $\mathbb{H}$-discrete series. Recall that J_X is defined in Definition 7.3.83. Then there exists a pair $(J_X, U') \in \Xi$ with the following properties:*

(1) X is a (not necessarily unique) irreducible quotient of $I(J_X, U')$, and

(2) $I(J_X, U') = I(J_X, U)^*$ for some $(J_X, U) \in \Xi_{L,ds}$ with $\lambda(J_X, U) = \lambda_{ds}(X)$.

Proof. Let (J_X, γ_X) be as in Definition 7.3.83. We shall construct an $\mathbb{H}_{J_X}$-module with some desired properties. We consider the $\mathbb{H}_{J_X}$-modules $\mathrm{Res}_{\mathbb{H}_{J_X}} X$, which can be written as the direct sum of $\mathbb{H}_{J_X}$-modules with distinct characters. Let Y be an indecomposable $\mathbb{H}_{J_X}$ submodule of X with the central character $W_J\gamma_X$. We want to show the composition factors of the $\overline{\mathbb{H}}_{J_X}$-module $\mathrm{Res}_{\overline{\mathbb{H}}_{J_X}} Y$ are discrete series. Let γ' be a weight of Y. Since Y has the W_J-central character $W_J\gamma_X$,

$$\mathrm{Re}\gamma' = \sum_{\alpha \in J_X} a_\alpha \alpha - L_{ds}(X)\frac{\omega_{\beta_X}}{||\omega_{\beta_X}||} = \sum_{\alpha \in J_X} a_\alpha \alpha - \lambda_{ds}(X), \quad \text{for some } a_\alpha \in \mathbb{R}. \tag{3.1}$$

Proving (1) is equivalent to proving that all $a_\alpha < 0$. Let $\alpha' \in J_X$. By using $\langle \gamma', \omega_{\alpha'} \rangle = a_{\alpha'} - L_{ds}(X)\frac{(\omega_{\beta_X}, \omega_{\alpha'})}{||\omega_{\beta_X}||}$, we have

$$\begin{aligned} \Phi(\Delta \setminus \{\alpha'\}, \gamma') &= -\frac{a_{\alpha'}}{||\omega_{\alpha'}||} + L_{ds}(X)\frac{(\omega_{\beta_X}, \omega_{\alpha'})}{||\omega_{\beta_X}||||\omega_{\alpha'}||} &(3.2)\\ &< -\frac{a_{\alpha'}}{||\omega_{\alpha'}||} + L_{ds}(X). &(3.3) \end{aligned}$$

Here, the second line also uses the fact that $(\omega_{\beta^*}, \omega_{\alpha'}) \geq 0$ ([Kn1, Lemma 8.57]) and $(.,.)$ is an inner product on V_0. By our choice of (J_X, γ_X), $a_\alpha < 0$ for all $\alpha \in J_X$. We now let Y' be an irreducible $\mathbb{H}_{J_X}$-submodule of Y and then we have $Y' = U_X \otimes \mathbb{C}_{-\lambda(X)}$ as $\mathbb{H}_{J_X} \cong \overline{\mathbb{H}}_{J_X} \otimes S(V_{J_X}^{\vee,\perp})$-module, where U_X is an irreducible $\overline{\mathbb{H}}_{J_X}$-discrete series. Then by construction and Frobenius reciprocity, one of the quotients of the parabolically induced module $I(J_X, U_X)$ is X. This shows (1).

For (2), by Proposition 7.3.85 and Lemma 7.3.87, $I(J_X, U_X)^* = I(J_X, U_X^{*J_X})$. Note that $U_X^{*J_X} = \overline{U} \otimes \mathbb{C}_{\lambda_{ds}(X)}$ for some $\overline{\mathbb{H}}_J$-discrete series $\overline{U}$ (by Lemma 7.3.87). Hence, $(J_X, U_X^{*J_X}) \in \Xi_{L,ds}$ with $\lambda(J_X, U_X^{*J_X}) = \lambda_{ds}(X)$ as desired.

□

Example 7.3.89. We continue to use the notations in Example 7.1.82 and Example 7.3.84. We consider the discrete series DS. From Example 7.3.84, we see that $\gamma^3 = -\frac{1}{2}\alpha^\vee - \eta(\lambda_{ds}(DS))$. Then there is a surjection from $I(\{\alpha\}, \mathrm{St} \otimes \mathbb{C}_{-\lambda_{ds}(DS)})$ to DS (by Frobenius reciprocity), where St is the Steinberg module of $\overline{\mathbb{H}}_\alpha$. This constructs the parabolically induced module for DS as the one in Proposition 7.3.88.

Recall that the Iwahori-Matsumoto involution ι is defined in Section 6.5. We have the following result describing the structure of parabolically induced modules which is proven by considering central characters and using Corollary 4.2.34.

Proposition 7.3.90. *Let X be an irreducible $\mathbb{H}$-discrete series. Let $(J, U) \in \Xi_L$ with $J \neq \Delta$. We have the following properties:*

(1) $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U)^*, \iota(X)) = 0$ *for all i.*

(2) Suppose $0 \neq \lambda(J,U) < \lambda_{ds}(X)$ or $0 \neq \theta(\lambda(J,U)) < \lambda_{ds}(X)$ (see Definition 7.1.74 for $\lambda(J,U)$). Then for all i

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U), \iota(X)) = 0.$$

Proof. We consider (1). By Proposition 7.3.85, $I(J,U)^* = I(J, U^{*_J})$ (where $*_J$ is the corresponding $*$-operation for $\mathbb{H}_J$) and the real part of the weights of U^{*_J} are of the form $\sum_{\alpha \in J} a_\alpha \alpha - \lambda(J, U^{*_J})$ for some $a_\alpha \leq 0$. Suppose $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U)^*, \iota(X)) \neq 0$ for some i. Then by Frobenius reciprocity, we have

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U^{*_J}, \mathrm{Res}_{\mathbb{H}_J} \iota(X)) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U)^*, \iota(X)) \neq 0$$

for some i. Then some composition factors of $\iota(X)$ and U^{*_J} have the same $\mathbb{H}_J$-central characters by Corollary 4.2.35. That implies $\iota(X)$ has a weight γ such that $\mathrm{Re}\gamma = \sum_{\alpha \in J} a'_\alpha \alpha - \lambda(J,U)$ for some $a_{\alpha'} \in \mathbb{R}$. Then for $\beta \notin J$, $(\mathrm{Re}\gamma, \omega_\beta) = -(\lambda(J,U), \omega_\beta) < 0$. However, this contradicts to the definition of ι and X being a discrete series.

(2) is a special case of Lemma 7.3.91 below (whose proof does not depend on this proposition). (In more detail, in the notation of Lemma 7.3.91, we choose $X_1 = X_2 = X$.)

□

We need an improved version of (2) of the above proposition for a better control in comparing the Ext-groups of two discrete series.

Lemma 7.3.91. *Let X_1 and X_2 be irreducible $\mathbb{H}$-discrete series. Assume $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J,U) \in \Xi_L$ with $J \neq \Delta$. Suppose $\lambda(J,U) \leq \lambda_{ds}(X_1)$. (resp. $\theta(\lambda(J,U)) \leq \lambda_{ds}(X_1)$) Then at least one of the following holds:*

(1) For all i

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U), \iota(X_2)) = 0.$$

(2) $\lambda(J,U) = \lambda_{ds}(X_1)$ and $J = J_{X_1}$ (resp. $\theta(\lambda(J,U)) = \lambda_{ds}(X_1)$ and $\theta(J) = J_{X_1}$). For any indecomposable $\mathbb{H}_J$-submodule Z of $\mathrm{Res}_{\mathbb{H}_J} \iota(X_2)$, if $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U,Z) \neq 0$ for some i, then all the composition factors of $\iota(\mathrm{Res}_{\overline{\mathbb{H}}_J} Z)$ are $\overline{\mathbb{H}}_J$-discrete series.

Proof. Suppose (1) is false to obtain (2). Then by the Frobenius reciprocity,

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U, \mathrm{Res}_{\mathbb{H}_J} \iota(X_2)) \neq 0,$$

for some i. This implies $\iota(X_2)$ contains a weight whose real part is of the form $\sum_{\alpha \in J} a_\alpha \alpha + \lambda(J,U)$ for some $a_\alpha \in \mathbb{R}$. Then by the definition of ι, X_2 contains a weight γ such that $\mathrm{Re}\gamma = -\sum_{\alpha\in J} a_\alpha \alpha - \lambda(J,U)$. Then for any $\alpha \notin J$,

$$\begin{aligned}
(-\mathrm{Re}\gamma, \omega_\alpha) &= (\lambda(J,U), \omega_\alpha) && (3.4)\\
&\leq (\lambda_{ds}(X_1), \omega_\alpha) \quad (\text{by } \lambda(J_{X_1}, U_1) \leq \lambda_{ds}(X_1)) && (3.5)\\
&\leq L_{ds}(X_1) \frac{(\omega_{\beta_{X_1}}, \omega_\alpha)}{||\omega_{\beta_{X_1}}||} \quad (\text{by Definition 7.3.83}) && (3.6)\\
&\leq L_{ds}(X_1) ||\omega_\alpha|| && (3.7)\\
&\leq L_{ds}(X_2) ||\omega_\alpha||. && (3.8)
\end{aligned}$$

By the definition of the function L_{ds}, all the inequalities become equalities. Then by the fourth line of the above computation, $\omega_\alpha = \omega_{\beta_X}$ and so $\Delta \setminus J = \{\beta_{X_1}\}$. Hence, $J = J_{X_1}$. By the second equality, we then have $\lambda(J,U) = L_{ds}(X_1) \frac{\omega_{\beta_{X_1}}}{||\omega_{\beta_{X_1}}||} = \lambda_{ds}(X_1)$. This proves the first assertion of (2).

For the second assertion of (2), let Z be an indecomposable $\mathbb{H}_J$-submodule of $\mathrm{Res}_{\mathbb{H}_J} \iota(X_2)$ with $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U,Z) \neq 0$ for some i. Then U and Z have the same $\mathbb{H}_J$-central character. With the definition of ι, for any weight γ' of $\iota(Z)$, γ satisfies

$$\mathrm{Re}\gamma' = \sum_{\alpha \in J} a'_\alpha \alpha - \lambda(J,U) = \sum_{\alpha\in J} a'_\alpha \alpha - \lambda_{ds}(X_1),$$

for some $a'_\alpha \in \mathbb{R}$. To show that any composition factors of $\iota(\mathrm{Res}_{\overline{\mathbb{H}}_J} Z)$ are discrete series, it suffices to show $a'_\alpha < 0$. To this end, let $\alpha \in J$ and we consider,

$$\begin{aligned}(-\mathrm{Re}\gamma', \omega_\alpha) &= -a'_\alpha + (\lambda_{ds}(X), \omega_\alpha) \\ &= -a'_\alpha + L_{ds}(X_1)\frac{(\omega_{\beta_{X_1}}, \omega_\alpha)}{||\omega_{\beta_{X_1}}||} \quad \text{(by Definition 7.3.83)} \\ &< -a'_\alpha + L_{ds}(X_1)||\omega_\alpha|| \quad \text{(because } \omega_\alpha \neq \omega_{\beta_X}).\end{aligned}$$

On the other hand, note that γ' is a weight of X_2 (since ι commutes with $\mathrm{Res}_{\mathbb{H}_J}$). By the definition of L_{ds} in Definition 7.3.83,

$$L_{ds}(X_2)||\omega_\alpha|| \leq -(\mathrm{Re}\gamma', \omega_\alpha).$$

Combining the equations and using $L_{ds}(X_1) \leq L_{ds}(X_2)$, we have $a'_\alpha < 0$ as desired. This proves (2).

We now comment on the θ-case. Again suppose (2) is false. Th equation (3.5) above will become $(\lambda(J,U), \omega_\alpha) \leq (\theta(\lambda_{ds}(X_1)), \omega_\alpha)$. Hence, we will obtain $J = \theta(J_{X_1})$. Then the similar line of argument in the non-θ case gives the remaining assertion.

□

7.4 Tempered modules

It is known that tempered modules can be parabolically induced from a discrete series twisted by an unitary character. We study those induced modules in this section.

Lemma 7.4.92. *Let X be an $\mathbb{H}$-tempered module, but not a discrete series. Then there exists $(J,U) \in \Xi$ with the following properties:*

(1) [BC2, Lemma 5.1.1] X is an irreducible (not necessarily unique) subquotient of $I(J,U)$ and $U = \overline{U} \otimes \mathbb{C}_\nu$ for an $\overline{\mathbb{H}}_J$-discrete series $\overline{U}$ and $\nu \in V_J^\perp$ with $\mathrm{Re}\nu = 0$, and

(2) $J \neq \Delta$, and

(3) any composition factor of $I(J,U)$ is tempered, and

(4) for any discrete series X',

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U), \iota(X')) = 0.$$

Proof. For each weight γ of X, let $J(\gamma)$ be a subset of Δ such that

$$\mathrm{Re}\gamma = \sum_{\alpha \in J} a_\alpha \alpha, \quad \text{with } a_\alpha < 0\ .$$

Set γ_X to be the weight of X such that $J(\gamma_X)$ has the minimal cardinality. Set $J = J(\gamma_X)$. Since X is not a discrete series, $J \neq \Delta$. Let Y be an indecomposable $\mathbb{H}_J$-module of $\mathrm{Res}_{\mathbb{H}_J} X$ with the $\mathbb{H}_J$-central character $W_J\gamma'$. We claim that any composition factor of $\mathrm{Res}_{\overline{\mathbb{H}}_J} Y$ is a discrete series. Let γ' be a weight of Y and write

$$\mathrm{Re}\gamma' = \sum_{\alpha \in J} a'_\alpha \alpha \quad \text{for some } a'_\alpha \in \mathbb{R}.$$

Since Y has a central character $W_J\gamma'$, $a'_\alpha = 0$ for all $\alpha \notin J$. It remains to show that $a'_\alpha < 0$ for all $\alpha \in J$. However, by the definition of a tempered module, $a'_\alpha = \langle \mathrm{Re}\gamma', \omega_\alpha \rangle \leq 0$ for all $\alpha \in J$. If $a'_\alpha = 0$ for some $\alpha \in J$, this will contradict our choice of γ'. This concludes that $a'_\alpha < 0$ for all $\alpha \in J$. Then we choose an irreducible $\mathbb{H}_J$-submodule U of Y and then $U = \overline{U} \otimes \mathbb{C}_\nu$ for an $\overline{\mathbb{H}}_J$-discrete series $\overline{U}$ and $\nu \in V_J^\perp$ with $\mathrm{Re}(\nu) = 0$. By Frobenius reciprocity, X is an irreducible quotient of $I(J, U)$ as desired. This proves (1) and (2).

We now prove (3). Any weight of $I(J, U)$ can be written of the form $w(\gamma_U)$ for $w \in W^J$ and a weight γ_U of U. Then

$$\mathrm{Re}(w(\gamma_U)) = \sum_{\alpha \in J} a_{\alpha,U} w(\alpha) \quad \text{for } a_{\alpha,U} \leq 0\ .$$

Since $w(\alpha) > 0$ for any $\alpha \in J$, $(\mathrm{Re}(w(\gamma_u)), \omega_\alpha) \leq 0$ for all $\alpha \in \Delta$. This proves (3).

We now prove (4) and continue to use the notations for (1). Let X' be a discrete series. Suppose the assertion is false to obtain a contradiction. Then by Frobenius reciprocity,

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U, \mathrm{Res}_{\mathbb{H}_J} \iota(X')) \neq 0.$$

Then by considering the $\mathbb{H}_J$-central character of U and Corollary 4.2.35, $\iota(X')$ has a weight $\gamma_{X'}$ such that

$$\mathrm{Re}\gamma_{X'} = \sum_{\alpha \in J} a_{\alpha,X'} \alpha.$$

Since we assume X is not a discrete series, $J \neq \Delta$. Then for $\alpha \notin J$, $(\mathrm{Re}\gamma_{X'}, \omega_\alpha) = 0$, which contradicts X' being a discrete series.

□

CHAPTER 8

EXTENSIONS OF DISCRETE SERIES

In this section, we compute the Ext-groups of discrete series. More precisely, we show that all the higher Ext-groups among discrete series vanish. The result for affine Hecke algebra is proven by Opdam-Solleveld in [OS1, Theorem 3.5] using the Schwartz algebra completion of an affine Hecke algebra. With the belief of the result from affine Hecke algebras, we take an algebraic approach for computing Ext-groups of discrete series for graded affine Hecke algebras. The results also cover complex parameter cases and non-crystallographic cases. We also hope some techniques can be extended for computing Ext-groups of more modules in the future (see an example in Section 8.2).

We briefly outline the strategy of our proof (also see the paragraph before Lemma 8.1.96). Instead of computing Ext-groups of discrete series directly, we first compute the Ext-groups for a tempered module and the Iwahori-Matsumoto dual of a discrete series, which has the advantage that the $\mathrm{Ext}^0_{\mathfrak{R}(\mathbb{H})}$-group vanishes. The next step is to construct parabolically induced modules for tempered modules (from Chapter 7). Then the parabolic induction allows us to make use of the knowledge from lower ranks (via Frobenius reciprocity and induction hypothesis), but in return, we have to deal with the Ext-groups for the composition factors in the related parabolically induced modules. Fortunately, those composition factors are well controlled by the Langlands classification and are manageable from the study in Chapter 7. Then the standard modules associated to those composition factors and the parabolic induction again gives some new information via the Frobenius reciprocity. This eventually leads to the computation of the Ext-groups of a tempered module and the Iwahori-Matsumoto dual of a discrete series. We finally apply the duality result (Theorem 6.6.63) to recover the Ext-groups between a tempered module and a discrete series.

We also want to point out that some argument can be simpler if we use some known results such as discrete series being unitary or self $*$-dual. However, the known proofs for those results rely on the setting in affine Hecke algebras or p-adic groups and we try to avoid using them.

8.1 Extensions of discrete series

In this section, we use the notation in Section 7.2 (i.e., identify $V^\vee$ with V and use the bilinear form $(.,.)$). We also continue to assume R spans V.

Theorem 8.1.93. *Let $\mathbb{H}$ be the graded affine Hecke algebra associated to a based root system $(\mathcal{X}, R, \mathcal{Y}, R^\vee, \Delta)$ and an arbitrary parameter function (Definition 3.3.25). Assume R spans V. Let X_1 be an irreducible tempered module and let X_2 be an irreducible $\mathbb{H}$-discrete series. Then*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X_1, X_2) = \begin{cases} \mathbb{C} & \text{if } X_1 \cong X_2 \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that each graded affine Hecke algebra is associated to a root system R. We shall call the rank of $\mathbb{H}$ to be the rank of R (i.e., the cardinality of Δ). We shall use induction on the rank of $\mathbb{H}$ to prove Theorem 8.1.93. The proof will be given at the end of this section. We first give few important lemmas.

Lemma 8.1.94. *Let $(J, U) \in \Xi$. Let $U = \overline{U} \otimes \mathbb{C}_\nu$ (as $\mathbb{H}_J = \overline{\mathbb{H}}_J \otimes S(V_J^{\vee,\perp})$-modules) for some $\overline{\mathbb{H}}_J$-module $\overline{U}$ and some $\nu \in V_J^\perp$. For an irreducible finite-dimensional $\mathbb{H}_J$-module Y, $Y = \overline{Y} \otimes \mathbb{C}\nu'$ for some irreducible $\overline{\mathbb{H}}_J$-module $\overline{Y}$ and for some $\nu' \in V_J^\perp$. Then*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U, Y) = \bigoplus_{k+l=i} \mathrm{Ext}^k_{\mathfrak{R}(S(V_J^\perp))}(\mathbb{C}_\nu, \mathbb{C}_{\nu'}) \otimes \mathrm{Ext}^l_{\mathfrak{R}(\overline{\mathbb{H}}_J)}(\overline{U}, \overline{Y}).$$

Proof. Let Y be an irreducible finite dimensional $\mathbb{H}$-module. Since $S(V_J^{\vee,\perp})$ is in the center of $\mathbb{H}_J$ and Y is finite-dimensional and irreducible, elements in $S(V_J^{\vee,\perp})$ act by a scalar on Y (by Schur's Lemma). This implies that $\mathrm{Res}_{\overline{\mathbb{H}}_J} Y$ is irreducible. Hence, $Y \cong \mathrm{Res}_{\overline{\mathbb{H}}_J} Y \otimes \mathbb{C}_\nu$ for some $\nu \in V_J^\perp$. This proves the first claim. The second assertion follows from the Künneth formula for complexes (see for example [We, Theorem 3.6.3]).

□

Lemma 8.1.95. *Let $\mathbb{H}$ be a graded affine Hecke algebra of rank n. Suppose Theorem 8.1.93 is true for all the graded affine Hecke algebra with rank $n-1$. Let X_1 and X_2 be irreducible $\mathbb{H}$-discrete series. Assume $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J, U) \in \Xi_L$ with $J \neq \Delta$. If $\lambda(J, U) \leq \lambda_{ds}(X_1)$ or $\theta(\lambda(J, U)) \leq \lambda_{ds}(X_1)$, then*

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J, U), \iota(X_2)) = 0$$

for all $i \leq n-2$.

Proof. By Frobenius reciprocity, it reduces to show

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U, \mathrm{Res}_{\mathbb{H}_J}\iota(X)) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I(J,U), \iota(X)) = 0$$

for all $i \leq n-2$. By Lemma 4.2.33, it suffices to show that $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}_J)}(U, Y) = 0$ for each $i \leq n-2$ and each composition factor Y of $\mathrm{Res}_{\mathbb{H}_J}\iota(X)$.

By Lemma 7.3.91, we only have to consider the composition factors Y of $\mathrm{Res}_{\mathbb{H}_J}\iota(X)$ for which the composition factors of $\iota(\mathrm{Res}_{\overline{\mathbb{H}}_J} Y)$ are discrete series. Let Y be such a composition factor of $\mathrm{Res}_{\mathbb{H}_J}\iota(X)$. By the irreducibility, we can then write $\iota(Y) = \overline{Y} \otimes \mathbb{C}_{\nu'}$ (as $\mathbb{H}_J \cong \overline{\mathbb{H}}_J \otimes S(V_J^{\vee,\perp})$-modules) for an irreducible $\overline{\mathbb{H}}_J$-discrete series $\overline{Y}$ and some $\nu' \in V_J^{\perp}$.

Similarly, we also write $U = \overline{U} \otimes \mathbb{C}_\nu$ for an $\overline{\mathbb{H}}_J$-discrete series $\overline{U}$ and $\nu \in V_J^{\perp}$. Then by Lemma 7.3.87, Theorem 6.6.63 and Theorem 8.1.93 for rank less than n, we have

$$\mathrm{Ext}^i_{\mathfrak{R}(\overline{\mathbb{H}}_J)}(\overline{U}, \overline{Y}) = \mathrm{Ext}^{n-i}_{\mathfrak{R}(\overline{\mathbb{H}}_J)}(\overline{U}^{*_J}, \iota(\overline{Y}^{\bullet_J})) = 0$$

for $i \leq n-2$. (Here, $*_J$ and $\bullet_J$ are the corresponding $*$ and $\bullet$ for $\mathbb{H}_J$, respectively.) With Lemma 8.1.94, this completes the proof.

□

The following lemma is the main technicality for Theorem 8.1.93. As mentioned in the beginning of this chapter, we have to deal with the composition factor in some parabolically induced modules. The assumptions in (2) of the following lemma pick out those composition factors. The assumption $L_{ds}(X_1) \leq L_{ds}(X_2)$ in (2) gives a much better control of what kind of composition factors are picked out.

The main idea of the proof of Lemma 8.1.96 is to use parabolically induced modules to construct short exact sequences. From those short exact sequences, we obtain associated long exact sequences by applying appropriate Hom-functors. Then Proposition 7.3.90 (1), Lemma 7.4.92 (4) and Lemma 8.1.95 makes the technique of dimension shifting work.

Lemma 8.1.96. *Let $\mathbb{H}$ be a graded affine Hecke algebra of rank n. Suppose Theorem 8.1.93 is true for all the graded affine Hecke algebra with rank less than or equal to $n-1$. Then we have the following:*

(1) Let X_1 be an $\mathbb{H}$-tempered module and let X_2 be an $\mathbb{H}$-discrete series. Then for all $i \leq n-1$

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) = 0.$$

(2) Let X_1 and X_2 be $\mathbb{H}$-discrete series. Assume $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J, U) \in \Xi_L$ with $\lambda(J, U) \leq \lambda_{ds}(X_1)$ or $\theta(\lambda(J, U)) \leq \lambda_{ds}(X_1)$. Let Y be the irreducible $\mathbb{H}$-module with the Langlands classification parameter $I(J, U)$. Suppose Y is not tempered (i.e., $J \neq \Delta$). Then for all $i \leq n-2$,

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) = 0,$$

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Y^*, \iota(X_2)) = 0.$$

((1) and (2) can be combined into one statement, but the separation into two statements will reflect how we prove in separate cases.)

Proof. (We shall use induction on i, which indexes $\mathrm{Ext}^i_{\mathbb{H}}$, but we do not fix any of X_1 or X_2.) When $i = 0$, any weight γ of $\iota(X_2)$ satisfy $(\mathrm{Re}\gamma, \omega_\alpha) > 0$ for all fundamental weights ω_α. Then by the definition of tempered modules, $\mathrm{Ext}^0_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) = \mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) = 0$. This proves (1) for $i = 0$.

For (2), we may assume $n \geq 2$ (otherwise there is nothing to prove). We consider the following exact sequence:

$$0 \to N \to I(J, U) \to Y \to 0,$$

where N is a proper (possibly zero) submodule of $I(J, U)$. Then we have the associated long exact sequence

$$0 \to \mathrm{Ext}^0_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) \to \mathrm{Ext}^0_{\mathfrak{R}(\mathbb{H})}(I(J, U), \iota(X_2)) \to \ldots$$

Then by Lemma 8.1.95, we have $\mathrm{Ext}^0_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) = 0$.

We now assume $1 \leq i \leq n-1$. Suppose X_2 is a discrete series. We first consider X_1 is a discrete series. By using $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X_1, X_2) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}({X_2}^*, X_1^*)$ (Proposition 6.4.58) and ${X_1}^*$, X_2^* being discrete series (Lemma 7.3.87), we can just consider $L_{ds}(X_1) \leq L_{ds}(X_2)$. Then by Proposition 7.3.88, we have the following short exact sequence:

$$0 \to N \to I(J_{X_1}, U_1)^* \to X_1 \to 0,$$

for $(J_{X_1}, U_1) \in \Xi_L$ with $\lambda(J_{X_1}, U_1) = \lambda_{ds}(X_1)$ (and $J_{X_1} \neq \Delta$). Then by applying the $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(., \iota(X_2))$ functor to obtain a long exact sequence and using Proposition 7.3.90(1),

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) \cong \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(N, \iota(X_2)).$$

Then it remains to show $\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(N, X_2) \neq 0$. By Lemma 4.2.33, it suffices to show that $\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(Y, X_2) = 0$ for any composition factor Y of N. Let $(J, U) \in \Xi_L$ be the Langlands parameter of Y. We first consider $J \neq \Delta$. Note that Y^* is a composition factor of $I(J_{X_1}, U_1)$ and so with Lemma 7.1.78 and Proposition 7.1.81, $\theta(\lambda(J,U)) \leq \lambda(J_{X_1}, U_1) = \lambda_{ds}(X_1)$ satisfies the assumption in (2). By the induction hypothesis, $\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) = 0$. It remains to consider $J = \Delta$. Then we have Y is tempered and so by the induction hypothesis for (1), we also have $\mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) = 0$. This proves $\mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) = 0$ for the case that X_1 is a discrete series.

We now consider X_1 is tempered but not a discrete series. By Lemma 7.4.92, there exists $(\widetilde{J}, \widetilde{U}) \in \Xi$ such that X_1 is an irreducible quotient of $I(\widetilde{J}, \widetilde{U})$ with the properties in Lemma 7.4.92. By Lemma 7.4.92(4), using the long exact sequence associated to $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(., \iota(X_2))$, we again obtain $\mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) \cong \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(\widetilde{N}, \iota(X_2))$ for some irreducible submodule $\widetilde{N}$ of $I(\widetilde{J}, \widetilde{U})$. Since any composition factor of $I(\widetilde{J}, \widetilde{U})$ is tempered (Proposition 7.4.92(3)), the induction hypothesis with Lemma 4.2.33 again yields $\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(\widetilde{N}, \iota(X_2) = 0$. Then $\mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(X_1, \iota(X_2)) = 0$ as desired.

We now prove (2) and assume $1 \leq i \leq n-2$. Let X_1 and X_2 be irreducible discrete series with $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J', U') \in \Xi_L$ with $\lambda(J', U') \leq \lambda_{ds}(X_1)$ and $J' \neq \Delta$. Let Y' be the irreducible $\mathbb{H}$-module with the Langlands classification parameter (J', U'). We consider the short exact sequence

$$0 \to N' \to I(J', U') \to Y' \to 0,$$

where N' is some submodule of $I(J', U')$. Then apply the functor $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(\cdot, \iota(X_2))$ to obtain a long exact sequence

$$\begin{aligned} \ldots \to &\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(I(J', U'), \iota(X_2)) \to \mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(N', \iota(X_2)) \\ &\to \mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) \to \mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(I(J', U'), \iota(X_2)) \to \ldots \end{aligned}$$

By Lemma 8.1.95,

$$\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(N', \iota(X_2)) \cong \mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)). \tag{1.1}$$

Then again consider the composition factors of N' and using a similar argument as the proof for (1) in the previous paragraph. Let M be a composition factor of N' and let (J^M, U^M) be the Langlands classification parameter of M. Then $\lambda(J^M, U^M) \leq \lambda(J', U') \leq \lambda_{ds}(X_1)$ by Proposition 7.1.81. Then by the induction hypothesis of (1) and (2), we obtain $\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(M, \iota(X_2)) = 0$. Then by Lemma 4.2.33, we have $\mathrm{Ext}^{i-1}_{\mathfrak{R}(\mathbb{H})}(N', \iota(X_2)) = 0$. Then $\mathrm{Ext}^{i}_{\mathfrak{R}(\mathbb{H})}(Y, \iota(X_2)) = 0$ by (1.1).

We continue to prove the remaining assertion in (2). We now consider the dual Y'^* of Y'. Let $(J_*, U_*) \in \Xi_L$ be the Langlands classification parameter of Y'^*. Then by Lemma 7.1.78,

$\lambda(J_*, U_*) = \theta(\lambda(J', U'))$ and so $\theta(\lambda(J_*, U_*)) \le \lambda_{ds}(X_1)$. For any composition factor M_* with the Langlands classification parameter $(J^{M_*}, U^{M_*}) \in \Xi_L$, we also have a similar inequality $\theta(\lambda(J^{M_*}, U^{M_*})) \le \theta(\lambda(J_*, U_*)) \le \lambda_{ds}(X_1)$ by Proposition 7.1.81. Thus, the induction step applies with a similar argument.

□

Proof of Theorem 8.1.93. For the case of $|\Delta| = 1$, it is easy to verify. In fact, in that case, when the parameter function $k \ne 0$, there is only one discrete series. When the parameter function $k = 0$, there is no discrete series. Assume $|\Delta| \ge 2$. Let X_1 be an irreducible tempered module and X_2 be an irreducible discrete series. By the induction hypothesis, Lemma 8.1.96, Lemma 7.3.87 and Theorem 6.6.63,

$$\dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X_1, X_2) = \dim \mathrm{Ext}^{n-i}_{\mathfrak{R}(\mathbb{H})}(X_1^*, \iota(X_2)^\bullet) = 0$$

for all $i \ge 1$. The case for $i = 0$ follows from $\mathrm{Hom}_{\mathbb{H}} = \mathrm{Ext}^0_{\mathbb{H}}$ and the Schur's lemma. This completes the proof.

8.2 Beyond discrete series: an example

In this section, we continue Example 7.1.82 and use the notation in the example. We go back to use $V^\vee$ for weights (to be consistent with the notation in Example 7.1.82). In particular, we use the notation Y, Z, DS to denote the irreducible modules as in Example 7.1.82. There are two more irreducible modules with the central character $W(\alpha^\vee + \beta^\vee)$ not in the list. One is a discrete series, denoted by DS'. Another one is the spherical module, denoted S.

Let $I' = \mathrm{Ind}^{\mathbb{H}}_{\mathbb{H}_{\{\alpha\}}}(\mathrm{St}_\alpha \otimes \mathbb{C}_{\frac{1}{2}(\alpha^\vee+\beta^\vee)})$, where $\mathrm{St}_{\{\alpha\}}$ is the Steinberg module of $\overline{\mathbb{H}}_{\{\alpha\}}$ (c.f. Proposition 7.3.88 and Example 7.3.89). Then by considering the weights, we have the following short exact sequence:

$$0 \to DS \oplus DS' \to I' \to Z \to 0. \tag{2.2}$$

(We use Theorem 8.1.93 for the maximal submodule of I' to be a direct sum of DS and DS'.) Then taking the $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(\cdot, X)$ $(X = DS, DS')$ and computing $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(I, DS)$, we have for $X = DS, DS'$,

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Z, X) = \begin{cases} \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By taking $\bullet$-operation, we have

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Z) = \begin{cases} \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 6.6.63 and taking $\mathrm{Hom}_{\mathfrak{R}(\mathbb{H})}(\cdot, Z)$ functor on (2.2), we have

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Z,Z) = \begin{cases} \mathbb{C} & \text{if } i = 0,2 \\ 0 & \text{otherwise.} \end{cases}$$

For $X' = Y, S$, Theorem 6.6.63 implies

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X',Z) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(Z,X') = \begin{cases} \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $X = DS, DS'$ and $X' = S, Y$, we also have

$$\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X',X) = \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,X') = \begin{cases} \mathbb{C} & \text{if } i = 2 \text{ and} \\ & \text{either } (X = DS \text{ and } X' = S) \\ & \text{or } (X = DS' \text{ and } X = Y) \\ 0 & \text{otherwise.} \end{cases}$$

The Ext-groups among Y and S are similar to the Ext-groups of DS and DS'. We skip the detail.

CHAPTER 9

EULER-POINCARÉ PAIRING

Euler-Poincaré pairing is defined as the alternating sum of the dimension of the $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}$-groups and is usually easier to be understood than the $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}$-groups. For instance, the pairing only depends on the composition factors of modules. An important aspect is that the pairing defines an inner product on an appropriate subspace of the virtual representations of a graded affine Hecke algebra, and is an analogue of the elliptic pairing for the p-adic groups [SS]. The representation theory related to the Euler-Poincaré pairing has been studied in the literature (e.g., [Ch2, CH, Re, OS1, OS3, SS]) and is closely related to the spin representations of Weyl groups and Dirac cohomology (e.g., [Ch1, Ci2, CT, COT]).

In this chapter, we will show that the Euler Poincaré pairing only depends on the Weyl group structure of graded affine Hecke algebra modules by using the projective resolution constructed in Chapter 5. The statement is first proven by Reeder [Re] (for equal parameters) and also independently proven by Opdam-Solleveld [OS1] (for arbitrary parameters) with different proofs. We also prove a similar statement for a twisted Euler-Poincaré pairing in connection with the twisted elliptic pairing of Weyl groups studied by Ciubotaru-He [CH]. Applications are given at the end of the chapter.

9.1 Euler-Poincaré pairing

We keep using the notation of a graded affine Hecke algebra in Definition 3.3.25. (In this chapter, R does not necessarily span V.) Define the Euler-Poincaré pairing for $\mathbb{H}$-modules X and Y as:

$$\mathrm{EP}_{\mathbb{H}}(X,Y) = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y).$$

This pairing can be realized as an inner product on a certain elliptic space for $\mathbb{H}$-modules analogous to the one in p-adic reductive groups in the sense of Schneider-Stuhler [SS].

The elliptic pairing $\langle , \rangle_W^{\mathrm{ellip},V}$ for W-representations U and U' is defined as

$$\langle U, U' \rangle_W^{\mathrm{ellip},V} = \frac{1}{|W|} \sum_{w \in W} \mathrm{tr}_U(w)\overline{\mathrm{tr}_{U'}(w)}\mathrm{det}_V(1-w).$$

Proposition 9.1.97. *For any finite-dimensional $\mathbb{H}$-modules X and Y,*

$$\mathrm{EP}_{\mathbb{H}}(X,Y) = \langle \mathrm{Res}_W(X), \mathrm{Res}_W(Y)\rangle_W^{\mathrm{ellip},V}.$$

In particular, the Euler-Poincare pairing depends only on the W-module structure of X and Y.

Proof.

$$\begin{aligned}
& \mathrm{EP}_{\mathbb{H}}(X,Y) \\
= & \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \\
= & \sum_i (-1)^i (\ker d_i^* - \mathrm{im}\, d_{i-1}^*) \quad \text{(by Proposition 5.4.46)} \\
= & \sum_i (-1)^i \dim \mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W(X) \otimes \wedge^i V, \mathrm{Res}_W(Y)) \quad \text{(by Euler-Poincaré principle)} \\
= & \sum_{w\in W} \mathrm{tr}_{\mathrm{Res}_W X}(w) \overline{\mathrm{tr}_{\mathrm{Res}_W Y}(w)} \mathrm{tr}_{\wedge^{\pm} V}(w) \\
= & \langle \mathrm{Res}_W(X), \mathrm{Res}_W(Y)\rangle_W^{\mathrm{ellip},V}.
\end{aligned}$$

Here, $\wedge^{\pm} V = \bigoplus_{i\in\mathbb{Z}} (-1)^i \wedge^i V$ as a virtual representation. The last equality follows from $\mathrm{tr}_{\wedge^i V}(w) = \det(1-w)$ and definitions.

□

9.2 Twisted Euler-Poincaré pairing

Recall that θ is defined in Section 6.1. For any $\mathbb{H} \rtimes \langle\theta\rangle$-module X, denote $\mathrm{Res}_W X$ to be the restriction of X to a $\mathbb{C}[W]$-algebra module (Definition 3.3.25 (1)). The notion $\mathrm{Res}_{W\rtimes\langle\theta\rangle}$ is similarly defined.

Let X and Y be $\mathbb{H} \rtimes \langle\theta\rangle$-modules. Recall that the map d^* from $\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$ to $\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W X \otimes \wedge^{i+1} V, \mathrm{Res}_W Y)$ is defined in Section 5.4.

Define θ^* to be the linear automorphism on $\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$ given by

$$\theta^*(\psi)(x\otimes v_1 \wedge \ldots \wedge v_i) \;=\; \theta\circ\psi(\theta(x)\otimes\theta(v_1)\wedge\ldots\wedge\theta(v_i)). \tag{2.1}$$

Here, θ-actions on $\mathrm{Res}_W X$ and $\mathrm{Res}_W Y$ are the natural actions from the θ-actions on X and Y (as $\mathbb{H} \rtimes \langle\theta\rangle$-modules), and furthermore the θ-action on v_i comes from the action of θ on V.

Lemma 9.2.98. $\theta^* \circ d^* = d^* \circ \theta^*$

Proof.

$$
\begin{aligned}
& (\theta^* \circ d^*)(\psi)(x \otimes v_1 \wedge \ldots \wedge v_k) \\
= \ & \theta \circ d^*(\psi)(\theta(x) \otimes \theta(v_1) \wedge \ldots \wedge \theta(v_k)) \\
= \ & \theta \circ \psi(d(\theta(x) \otimes \theta(v_1) \wedge \ldots \wedge \theta(v_k))) \\
= \ & \sum_i (-1)^i v_r.\theta \circ \psi(\theta(x) \otimes \theta(v_1) \wedge \ldots \wedge \theta(\widehat{v}_i) \wedge \ldots \wedge \theta(v_k)) \\
& \quad - \sum_i (-1)^i \theta \circ \psi(\theta(v_r).\theta(x) \otimes \theta(v_1) \wedge \ldots \wedge \theta(\widehat{v}_i) \wedge \ldots \wedge \theta(v_k)) \\
= \ & \sum_i (-1)^i v_r.\theta^*(\psi)(x \otimes v_1 \wedge \ldots \wedge \widehat{v}_i \wedge \ldots \wedge v_k) \\
& \quad - \sum_i (-1)^i \theta^*(\psi)(v_r.x \otimes v_1 \wedge \ldots \wedge \widehat{v}_i \wedge \ldots \wedge v_k) \\
= \ & (d^* \circ \theta^*)(\psi)(x \otimes v_1 \wedge \ldots \wedge v_k).
\end{aligned}
$$

□

By Lemma 9.2.98, θ^* induces an action, still denoted θ^* on $\mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, X)$. We can then define the θ-twisted Euler-Poincaré pairing $\mathrm{EP}^\theta_{\mathbb{H}}$ as follows:

Definition 9.2.99. For $\mathbb{H} \rtimes \langle\theta\rangle$-modules X and Y, define

$$\mathrm{EP}^\theta_{\mathbb{H}}(X, Y) = \sum_i (-1)^i \mathrm{trace}(\theta^* : \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y) \to \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X, Y)).$$

Here, we also regard X and Y to be $\mathbb{H}$-modules equipped with the θ-action.

We remark that this definition also makes sense for θ to be any automorphism of $\mathbb{H}$. However, when we prove Theorem 9.3.104 later, we use θ to arise from w_0 in (1.1).

9.3 Relation between two twisted elliptic pairings

In this section, we relate the twisted Euler-Poincaré pairing to the twisted elliptic pairing of Weyl groups defined in [CH]. We first recall the definition of the twisted elliptic pairing of Weyl groups.

Definition 9.3.100. [CH] For any $W \rtimes \langle\theta\rangle$-representation U and U', the θ-twisted elliptic pairing on U and U' is defined as:

$$\langle U, U' \rangle_W^{\theta-\mathrm{ellip},V} = \frac{1}{|W|} \sum_{w \in W} \mathrm{tr}_U(w\theta)\overline{\mathrm{tr}_{U'}(w\theta)}\det_V(1 - w\theta).$$

Since $w_0\theta = -\mathrm{Id}_V$ on V, it is equivalent that

$$\langle U, U'\rangle_W^{\theta-\mathrm{ellip},V} = \frac{1}{|W|}\sum_{w\in W} \mathrm{tr}_{U^+-U^-}(ww_0)\overline{\mathrm{tr}_{U'^+-U'^-}(ww_0)}\mathrm{det}_V(1+ww_0),$$

where U^+ and U^- (resp. U'^+ and U'^-) are the $+1$ and -1-eigenspaces of $w_0\theta$ of U (resp. U'), and $U^+ - U^-$ and $U'^+ - U'^-$ are regarded as virtual representations of W.

For some basic properties of $\langle ., .\rangle_W^{\theta-\mathrm{ellip}}$, one may refer to [CH, Section 5] (also see [Ch2, Section 4]).

Definition 9.3.101. Let X be an $\mathbb{H} \rtimes \langle\theta\rangle$-module. Define $X^{\pm}$ to be the ± 1 eigenspaces of the action of θt_{w_0} on X, respectively. It is easy to see $X^{\pm}$ are invariant under the action of t_w for $w \in W$ (see Lemma 9.3.102 below). We shall regard $X^{\pm}$ as W-representations or $W \rtimes \langle\theta\rangle$-representations. Moreover, since θt_{w_0} is diagonalizable, we also have $X = X^+ \oplus X^-$.

Lemma 9.3.102. *Let X be an $\mathbb{H} \rtimes \langle\theta\rangle$-module. Then*

(1) X^+ and X^- are $W \rtimes \langle\theta\rangle$-invariant.

(2) Let X be an $\mathbb{H} \rtimes \langle\theta\rangle$-module. For any $v \in V$, $\widetilde{v}.X^{\pm} \subset X^{\mp}$.

Proof. (1) follows from $\theta t_{w_0} t_w = t_w t_{w_0}\theta$. (2) follows from $w_0\theta(v) = -v$ and Lemma 5.3.44.

□

Lemma 9.3.103. *For $\mathbb{H} \rtimes \langle\theta\rangle$-modules X and Y, define*

$$\mathrm{Hom}_i^+ = \mathrm{Hom}_{\mathbb{C}[W]}(X^+ \otimes \wedge^i V, Y^+) \oplus \mathrm{Hom}_{\mathbb{C}[W]}(X^- \otimes \wedge^i V, Y^-)$$

and

$$\mathrm{Hom}_i^- = \mathrm{Hom}_{\mathbb{C}[W]}(X^+ \otimes \wedge^i V, Y^-) \oplus \mathrm{Hom}_{\mathbb{C}[W]}(X^- \otimes \wedge^i V, Y^+).$$

The map d_i^* sends $\mathrm{Hom}_i^{\pm} \to \mathrm{Hom}_{i+1}^{\mp}$. Moreover, θ^* acts identically as $(-1)^i$ on Hom_i^+ and acts identically as $-(-1)^i$ on Hom_i^-.

Proof. The first assertion follows from Lemma 9.3.102 and Proposition 5.3.45. For the second assertion, we pick $\psi \in \mathrm{Hom}_i^+$. Suppose $x \in X^+$ and $v_1, \ldots, v_i \in V$. Then

$$\begin{aligned}
&\theta^*(\psi)(x \otimes v_1 \wedge \ldots \wedge v_i)\\
=&\theta.\psi(\theta(x) \otimes \theta(v_1) \wedge \ldots \wedge \theta(v_i))\\
=&t_{w_0}\theta.\psi((t_{w_0}\theta.x) \otimes w_0\theta(v_1) \wedge \ldots \wedge w_0\theta(v_i))\\
=&(-1)^i t_{w_0}\theta.\psi(x \otimes v_1 \wedge \ldots \wedge v_i)\\
=&(-1)^i \psi(x \otimes v_1 \wedge \ldots \wedge v_i).
\end{aligned}$$

The forth equality follows from $w_0\theta(v) = -v$, $t_{w_0}\theta.x = x$, and the last equality follows from $\operatorname{im}\psi \in Y^+$. Other cases are similar.

□

With $\mathrm{Hom}_i^{\pm}$ defined in Lemma 9.3.103, we also define that

$$\mathrm{Ext}^i(X,Y)^+ = \frac{\ker(d_i^* : \mathrm{Hom}_i^+ \to \mathrm{Hom}_i^-)}{\operatorname{im}(d_i^* : \mathrm{Hom}_i^- \to \mathrm{Hom}_i^+)},$$

and similarly,

$$\mathrm{Ext}^i(X,Y)^- = \frac{\ker(d_i^* : \mathrm{Hom}_i^- \to \mathrm{Hom}_i^+)}{\operatorname{im}(d_i^* : \mathrm{Hom}_i^+ \to \mathrm{Hom}_i^-)}.$$

Note that by the projective resolution in (2.1),

$$\mathrm{Ext}^i_{\mathbb{H}}(X,Y) = \mathrm{Ext}^i(X,Y)^+ \oplus \mathrm{Ext}^i(X,Y)^-. \tag{3.2}$$

Theorem 9.3.104. *For any finite-dimensional $\mathbb{H} \rtimes \langle\theta\rangle$-modules X and Y with θ defined as in (1.1),*

$$\mathrm{EP}^\theta_{\mathbb{H}}(X,Y) = \langle \mathrm{Res}_{W\rtimes\langle\theta\rangle} X, \mathrm{Res}_{W\rtimes\langle\theta\rangle} Y\rangle_W^{\theta-\mathrm{ellip},V}.$$

In particular, the θ-twisted elliptic pairing $\mathrm{EP}^\theta_{\mathbb{H}}$ depends on the W-module structures of X and Y only.

Proof. Set $d_i^{*,+} = d_i^*|_{\mathrm{Hom}_i^+}$ and $d_i^{*,-} = d_i^*|_{\mathrm{Hom}_i^-}$.

$$\begin{aligned}
& \mathrm{EP}^\theta_{\mathbb{H}}(X,Y) \\
= & \sum_i (-1)^i \mathrm{trace}(\theta^* : \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \to \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y)) \\
= & \sum_i (-1)^i [(-1)^i \dim \mathrm{Ext}^i(X,Y)^+ - (-1)^i \dim \mathrm{Ext}^i(X,Y)^-] \quad \text{(by Lemma 9.3.103)} \\
= & \sum_i (\dim \mathrm{Ext}^i(X,Y)^+ - \dim \mathrm{Ext}^i(X,Y)^-) \\
= & \sum_i [(\dim\ker d_i^{*,+} - \dim\operatorname{im} d_{i-1}^{*,-}) - (\dim\ker d_i^{*,-} - \dim\operatorname{im} d_{i-1}^{*,+})] \\
= & \sum_i (\dim\ker d_i^{*,+} + \dim\operatorname{im} d_{i-1}^{*,+})) - (\dim\ker d_i^{*,-} + \dim\operatorname{im} d_{i-1}^{*,-}) \\
= & \sum_i (\dim \mathrm{Hom}_i^+ - \dim \mathrm{Hom}_i^-) \quad \text{(definition of } \mathrm{Hom}^{\pm} \text{ in Lemma 9.3.103)} \\
= & \frac{1}{|W|} \sum_{w\in W} \mathrm{tr}_{X^+-X^-}(w)\overline{\mathrm{tr}_{Y^+-Y^-}(w)}\mathrm{det}_V(1+w) \quad \text{(as virtual representations)} \\
= & \frac{1}{|W|} \sum_{w\in W} \mathrm{tr}_X(ww_0\theta)\overline{\mathrm{tr}_Y(ww_0\theta)}\mathrm{det}_V(1-ww_0\theta) \\
= & \langle \mathrm{Res}_{W\rtimes\langle\theta\rangle}(X), \mathrm{Res}_{W\rtimes\langle\theta\rangle}(Y)\rangle_W^{\theta-\mathrm{ellip},V}.
\end{aligned}$$

The third last equality follows from the fact that $\sum_i \mathrm{tr}_{\wedge^i V}(w) = \det_V(1+w)$ and $w_0\theta = -\mathrm{Id}_V$.

$\square$

Remark 9.3.105. We give an example to show that Theorem 9.3.104 is not true in general if θ is replaced by an outer automorphism on W. Let R be of type $A_1 \times A_1$. Let θ' be the Dynkin diagram automorphism interchanging two factors of A_1. Let $\mathbb{H}$ be the graded Hecke algebra of type $A_1 \times A_1$. Note that $\langle , \rangle_W^{\theta'-\mathrm{ellip},V} \equiv 0$ as $\mathrm{tr}(w\theta') = 0$ for all $w \in W$. Here, $W = S_2 \times S_2$ and $V = \mathbb{C} \oplus \mathbb{C}$. However, we may choose an $\mathbb{H}$-module X (e.g., the exterior tensor product of Steinberg modules) such that $\mathrm{EP}_{\mathbb{H}}^{\theta'}(X,X) \neq 0$.

We give an interpretation of θ-twisted Euler-Poincaré pairing with the Euler-Poincaré pairing of $\mathbb{H} \rtimes \langle\theta\rangle$-modules. Define

$$\mathrm{EP}_{\mathbb{H}\rtimes\langle\theta\rangle}(X,Y) = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}\rtimes\langle\theta\rangle)}(X,Y).$$

Corollary 9.3.106. *For any finite-dimensional $\mathbb{H} \rtimes \langle\theta\rangle$-modules X and Y,*

$$\begin{aligned}&\dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H}\rtimes\langle\theta\rangle)}(X,Y)\\ =&\frac{1}{2}\dim \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) + \frac{1}{2}\mathrm{trace}(\theta^* : \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y) \to \mathrm{Ext}^i_{\mathfrak{R}(\mathbb{H})}(X,Y)),\end{aligned}$$

and

$$\mathrm{EP}_{\mathbb{H}\rtimes\langle\theta\rangle}(X,Y) = \frac{1}{2}\mathrm{EP}_{\mathbb{H}}(X,Y) + r\frac{1}{2}\mathrm{EP}^{\theta}_{\mathbb{H}}(X,Y).$$

Proof. Note that

$$\mathrm{Hom}_{\mathbb{C}[W]\rtimes\langle\theta\rangle}(\mathrm{Res}_{W\rtimes\langle\theta\rangle}X \otimes \wedge^i V, \mathrm{Res}_{W\rtimes\langle\theta\rangle}Y) \cong \begin{cases} \mathrm{Hom}_i^+ & \text{if } i \text{ is even} \\ \mathrm{Hom}_i^- & \text{if } i \text{ is odd} \end{cases}.$$

Then by using a Koszul type resolution as in (2.1), one could see that

$$\mathrm{Ext}^i_{\mathbb{H}\rtimes\langle\theta\rangle}(X,Y) = \begin{cases} \mathrm{Ext}_i^+ & \text{if } i \text{ is even} \\ \mathrm{Ext}_i^- & \text{if } i \text{ is odd} \end{cases}.$$

By Lemma 9.3.103, the latter expression above is equal to

$$\frac{1}{2}\dim \mathrm{Ext}^i_{\mathbb{H}}(X,Y) + \frac{1}{2}\mathrm{trace}(\theta^* : \mathrm{Ext}^i_{\mathbb{H}}(X,Y) \to \mathrm{Ext}^i_{\mathbb{H}}(X,Y)).$$

It follows from the proof of Proposition 9.1.97 that

$$\begin{aligned}&\mathrm{Ext}_{\mathbb{H}\rtimes\langle\theta\rangle}(X,Y)\\ =&\frac{1}{2|W|}\sum_{w\in W}\mathrm{tr}_X(w)\overline{\mathrm{tr}_Y(w)}\det{}_V(1-w) + \frac{1}{2|W|}\sum_{w\in W}\mathrm{tr}_X(w\theta)\overline{\mathrm{tr}_Y(w\theta)}\det{}_V(1-w\theta)\\ =&\frac{1}{2}\langle \mathrm{Res}_W(X), \mathrm{Res}_W(Y)\rangle_W^{\mathrm{ellip},V} + \frac{1}{2}\langle \mathrm{Res}_{W\rtimes\langle\theta\rangle}(X), \mathrm{Res}_{W\rtimes\langle\theta\rangle}(Y)\rangle_W^{\theta-\mathrm{ellip},V}.\end{aligned}$$

Now the statement follows from Theorem 9.3.104 and Proposition 9.1.97.

□

9.4 Applications

We give two applications of the Euler-Poincaré pairing in this section.

An element $w \in W$ is said to be *elliptic* if $\det_V(1-w) \neq 0$. A conjugacy class of W is said to be elliptic if any element in the conjugacy class is elliptic. The first application is to give an upper bound of the number irreducible discrete series. The statement for affine Hecke algebra is proven by Opdam-Solleveld in [OS1, Proposition 3.9] and is applied to classify discrete series of affine Hecke algebras for arbitrary parameters.

Corollary 9.4.107. *Let $\mathbb{H}$ be the graded affine Hecke algebra associated to a root datum and an arbitrary parameter function. The number of irreducible $\mathbb{H}$-discrete series are less than or equal to the number of elliptic conjugacy classes of W. In particular, there are only a finite number of nonisomorphic irreducible $\mathbb{H}$-discrete series.*

Proof. Let $R(W)$ be the virtual representation ring of W. Let $\overline{R}(W) = R(W)/\mathrm{rad}\langle,\rangle_W^{\mathrm{ellip}}$. Then by the definition of $\mathrm{rad}\langle,\rangle_W^{\mathrm{ellip}}$, the dimension of $\overline{R}(W)$ is the number of elliptic conjugacy classes (see [Re, Section 2] for the detail). Let $R(\mathbb{H})$ be the Grothendieck group of the category of finite-dimensional $\mathbb{H}$-modules.

On the other hand, the restriction map Res_W defines an isometry from $R(\mathbb{H})$ to $\overline{R}(W)$ with respect to the paring $\mathrm{EP}_{\mathbb{H}}$ and $\langle.,.\rangle_W^{\mathrm{ellip}}$, respectively. By Theorem 8.1.93, discrete series form an orthonormal set for the pairing $\mathrm{EP}_{\mathbb{H}}$. Hence, the number of discrete series is less or equal to the number of elliptic conjugacy classes.

□

Example 9.4.108. For R of type A_n, there is only one elliptic conjugacy classes for $W = S_{n+1}$. Hence, the Steinberg module is the only irreducible discrete series for $\mathbb{H}$ of type A_n.

The second application concerns the duals of discrete series. For real parameter functions, it is even known that those discrete series are even $*$-unitary (by some analytic results in affine Hecke algebras, see [So, Theorem 7.2]).

Corollary 9.4.109. *Let $\mathbb{H}$ be the graded affine Hecke algebra associated to a root datum and an arbitrary parameter function. Let X be an $\mathbb{H}$-discrete series. Then*

(1) X, $X^{\bullet}$, X^* and $\theta(X)$ are isomorphic,

(2) Let $W\gamma$ be the central character of X. Then $W\theta(\gamma) = W\gamma$.

Proof. Since X and X^* have the same W-module structure, by Proposition 9.1.97 we have

$$\mathrm{EP}_{\mathbb{H}}(X, X^*) = \mathrm{EP}_{\mathbb{H}}(X, X) = \mathbb{C}.$$

The second equality follows from Theorem 8.1.93. Since X^* is also a discrete series (Lemma 6.4.58), Theorem 8.1.93 forces $X^* \cong X$. The assertion for $X^\bullet$ and $\theta(X)$ in (1) can be proven similarly. For (2), the central character of $\theta(X)$ is $W\theta(\gamma)$. Then (2) follows from $X \cong \theta(X)$ in (1).

$\square$

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