



Hilbert Synthesis on Foundation of Geometry

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Abstract— *The relations between intuition, axiomatic method and formalism in Hilbert's foundational studies has been discussed several times, but geometrical ones still have unclear sides and there is not a commonly held opinion.*

In this article we try to frame Hilbert's geometrical works within a historical context. The aim is to show that intuition and nature of the axioms in Grundlagen der Geometrie do not derive from a mature philosophical awareness of the author, but from the development of a historical path of the idea of geometry and of its foundations. The path begins with the discovery of non-Euclidean geometry and finds in Hilbert's work its final and definitive synthesis for Euclidean geometry.

Keywords— *Foundation of Geometry, History of geometry, Hilbert, Philosophy of mathematics.*

I. INTRODUCTION

The studies on non-Euclidean geometries by Lobačevskij and Bolyai dates to the 1820-30s [49, 6], but initially, except from Gauss and his student Riemann, very few mathematicians recognized the meaning of the new geometries. Only after over thirty years some of them, like Helmholtz, Klein, Lie, Poincaré and Russell began to study the new geometries. It is known that Gauss had warned about the rising of “the cry of the Boetians”, because the subject was destined to clash with Kant's conception of geometry. The debate took place inevitably by the hands both of Kantians and the different mathematical schools, which intended to propose their foundational visions.

The debate went on for almost fifty years until 1899, when Hilbert published the *Grundlagen der Geometrie*. In what follows a brief description of the milestones on foundation of geometry from Riemann to Russell and of the consequent debate, in particular on the use of analytical method in geometry; finally the synthesis made by Hilbert in the *Grundlagen*.

II. BRIEF SKETCH ON FOUNDATIONS OF GEOMETRY

2.1 Riemann's manifold

Riemann's inaugural lecture for academic teaching (*Habilitationvortrag*) was published posthumous in 1868, two years after his death, with the title *On the Hypotheses which lie at the Base of Geometry* [66]. Following Torretti's notation [72, p. 156], Riemann's hypotheses are:

Ri1 Space is a continuous manifold of n dimensions, i.e. a multiplicity of points, each of them identified by n coordinates that vary continuously with the displacement of the point;

Ri2 For any n -dimensional manifold the algebraic expression of the distance between two infinitely close points is “the square root of an always positive integral homogeneous function of the second order of the quantities dx_i [with i from 1 to n], in which the coefficients are continuous functions of the quantities x_i that is

$$ds = \sqrt{\sum_{ij} g_{ij} dx_i dx_j}$$

where g_{ij} is the matrix of coefficients.¹

¹ The simplest examples are Euclidean space and spherical surface.

For three-dimensional Euclidean space g_{ij} is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, if

$x_1 = x$, $x_2 = y$, $x_3 = z$, the distance of a point infinitely close to the origin of the reference system is given by

$$ds^2 = \sum_{ij} g_{ij} dx_i dx_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \cdot [dx \quad dy \quad dz] =,$$

$$= dx^2 + dy^2 + dz^2$$

that is the usual Pythagorean distance. For the spherical bidimensional surface of radius R , g_{ij} is $\begin{bmatrix} 1 & 0 \\ R^2 & 0 \\ 0 & \frac{\sin^2 \theta}{R^2} \end{bmatrix}$ if $x_1 = \theta$ and

$x_2 = \phi$, the distance of a point infinitely close to the origin of the reference system is given by

Direct consequences of these two hypotheses are the possibility of conceiving spaces having more than three dimensions and the possibility of conceiving a virtually infinite amount of new geometries “without having the slightest spatial intuition”,² by changing the elements of the matrix g_{ij} .³

The virtually conceivable geometries are limited by a third hypothesis:

Ri3 The length of lines is independent of their position, and consequently every line is measurable by means of every other.

Therefore, of all the possible geometries, only those in which the length unit of measurement is constant are useful: this means Euclidean, hyperbolic and elliptical geometries.⁴

Another further limitation derives from the application of previous hypotheses to physical space, so Riemann introduces two other hypotheses:

Ri4 Space is an unbounded three-fold manifoldness, [this] is an assumption which is developed by every conception of the outer world;

Ri5 From astronomical measurements [the curvature] cannot be different from zero.

Riemann, in other words, while showing the logical possibility of many independent from spatial intuition geometries, admits it will be “extremely unfruitful”⁵ to

study those geometries that have no actual physical evidence.

2.2 Helmholtz’s geometry of rigid bodies

Helmholtz replied to the Riemann’s *Habilitationvortrag* publication with an article with a significantly similar title *On the Facts Underlying Geometry*.⁶

In Helmholtz opinion the weakness of Riemann’s hypotheses was having ignored the right starting point, that is “the primary measurement of space is entirely based upon the observation of congruence” [25, p. 41]; observation that presupposes the existence of rigid bodies free to move in space, unchanged in shape and size, during displacement or rotation.

According to Helmholtz, the existence of mobile, but rigid, bodies is a preliminary condition for the foundation not only of any kind of metric, but also of any geometry [26, p. 24].

Helmholtz opposes to Riemann’s “hypotheses” the foundation of geometry on “the observational fact, that in our space the motion of fixed spatial structures is possible with that degree of freedom with which we are acquainted, and from this fact the necessity of the algebraic expression which Riemann set down as an axiom” [26, p. 15].

Then he advances a system of axioms to describe the spatial relations and the invariant motion of rigid bodies:

He1 Space of n dimensions is an n -fold extended manifold. In other words, the individual specified in it, the point, is specifiable by measuring any continuously and independently vary-

$$ds^2 = \sum_{ij} g_{ij} dx_i dx_j = \begin{bmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{\sin^2 \theta}{R^2} \end{bmatrix} \cdot \begin{bmatrix} d\theta \\ d\phi \end{bmatrix} \cdot \begin{bmatrix} d\theta & d\phi \end{bmatrix} = \frac{1}{R^2} d\theta^2 + \frac{\sin^2 \theta}{R^2} d\phi^2$$

² Fragment XVI, 40^r of Riemann’s *Nachlass*, quoted in [71].

³ The elements of g_{ij} , and therefore the coefficients of ds , can take any values with any dependence on the coordinates, hence the different geometries are virtually infinite.

⁴ These geometries are the only ones with constant curvature; Euclidean geometry has a zero curvature, hyperbolic a negative curvature and elliptical a positive one. The concept of surface curvature was initially developed by Gauss and generalized for n -dimensional manifolds by Riemann. The curvature k of a surface

at a point P is $k_P = \frac{1}{r_{\min}} \cdot \frac{1}{r_{\max}}$, where r_{\min} and r_{\max} are the radii

of the minimum and maximum tangent circle to the surface in P . If c remains constant as P moves on the surface, then the surface has a constant curvature [18, pp. 17-18].

⁵ Fragment XVI, 40^r of Riemann’s *Nachlass*, quoted in [71].

⁶ Helmholtz’s articles on this topic are actually two [24, 25]. On the dating of the first one there was a misunderstanding due to a typo in the original publication in *Verhandlungen des naturhistorisch-medizinischen Vereins zu Heidelberg*; “22 may 1866” instead of “22 may 1868” [76], which was then repeated in the Helmholtz’s scientific work collection [28, pp. 610-617] and in other subsequent editions. This mistake led some historians to anticipate Helmholtz’s first article on geometry by two years, when Riemann was still alive [e.g. 57, 72], generating a misinterpretation of Helmholtz’s works. In January 1867 Schering wrote an obituary for Riemann [70], from which Helmholtz knew that the topic of the Riemann’s *Habilitationvortrag* was the hypotheses at the basis of geometry, which he had also been working on for about two years, without having yet published anything. Thus on 21 April 1867, Helmholtz sent a letter to Schering to ask him for a copy of Riemann’s *Habilitationvortrag* and a month later, on 18 May 1868, he wrote again to Schering to thank him for receiving the requested work and to send him for publication the manuscript of [24], correctly received on “22 may 1868” [43, pp. 254-255]. So Helmholtz was definitely aware of Riemann’s work before writing his own.

ing magnitudes (coordinates), whose total number is n ;

He2 Between the $2n$ coordinates of any point pair belonging to a body fixed in itself, there exists an equation which is independent of the motion of the latter, and which for all congruent point pairs is the same;

He3 Completely free mobility of fixed bodies is presupposed;

He4 If a fixed body rotates about $n-1$ of its points, and these are so chosen that its position then depends only upon one independent variable, then rotation without reversal finally returns it to the initial situation from which it started;

He5 Space has three dimensions;

He6 Space is infinitely extended.

Helmholtz, like Riemann did, attributes a fundamental role to the coordinates since the very first axiom, but he adds another condition, not mentioned by Riemann, besides continuity: derivability; “by continuity of change during motion we do not merely mean that all values of the changing magnitudes intermediate between the terminal values are run through, but also that derivatives exist” [25, p. 42].⁷

Even if improperly, as Lie will later show [47], Helmholtz uses axioms “only for points having infinitely small coordinate differences”, and calculates the algebraic expression of a quantity ds “which remains unchanged during any motion of two points which are fixedly connected to each other and whose separation is vanishingly small” [25, p. 56]. Then he concludes: “With this we have got to the starting point of Riemann's investigations” that is a homogeneous second degree expression of the differentials of the coordinate x_1, x_2, x_3 , ultimately equal to Riemann's expression for metric.

So despite the apparent initial differences and the contrasting titles of the two papers, Helmholtz himself states that he “had essentially taken the same path as that followed by Riemann”.

⁷ In spite of Riemann was one of the few mathematicians of the time who disagreed with the common opinion that a continuous function at every point of its domain was always derivable, it is not known why in his hypotheses he did not specify derivability condition. He even proposed an example of a continuous function that could not be derived at any point, which turned out to be wrong [9 p. 231]. Probably Riemann did not refer to the derivability condition either because in 1854 he had not yet come to that conviction or he omitted it for the composition of the audience during his inaugural lecture, mostly philosophers and not mathematicians.

2.3 Klein's and Lie's transformation groups

The same path has been also taken by Klein and Lie. Starting from Helmholtz's empirical observations on the motion of rigid bodies and preservation of shapes and size, Klein and Lie review Helmholtz's approach on the basis of their new theory of transformation groups.

In 1872 Klein, with the famous *Erlangen Program* [42], unifies under the more general projective geometry the different geometries known at that time: Euclidean, hyperbolic and elliptic. He lays at the basis of each geometry a particular group of transformations. Each geometry is defined by the invariant properties with respect to a specific group, i.e. those properties that are preserved during the transformation.

Euclidean geometry is characterized by a group of transformations called isometries that preserve distances and therefore the shape and size of rigid bodies.

The most general geometry in the *Erlangen Program* is projective geometry, where the properties of incidence, belonging and alignment are preserved. Within projective geometry there are also hyperbolic and elliptical geometries, which are obtained from the more general one through transformations that keep unchanged other characteristics, such as the distance between two points, a property common to Euclidean geometry.

Projective geometry is in turn part of topological geometry, where only some very general properties, such as orientation and connection, are preserved, it is characterized by those transformations called continuous deformations.

Based on this program, Lie develops the theory of continuous transformations [47], in the language of whose he reformulates Helmholtz's axioms and corrects a couple of errors.

2.4 Poincaré's conventionalism

In the 1880s Poincaré had already arrived at a group interpretation of geometry, independently by Klein's *Program*, of which he was not aware [23]. At the inauguration of the 1886/87 academic year, he states that “geometry is nothing more than the study of a group”, but unlike Helmholtz he doesn't think that experience can help settle the real nature of space. He argues “if these hypotheses were experimental facts, Geometry would be subjected to an incessant revision, it would not be an exact science” [60].

The common property of not changing distances implies that, assuming true the hypotheses of Euclidean geometry, every experiment could also be carried out in hyperbolic or elliptical geometry. Geometry is therefore a convention, such as the choice of a coordinate system or the adoption of a metric system, “one geometry cannot be more true than another; it can only be more convenient” [62].

In 1898 he substantially reaffirms his agreement with group theory, while confirming his conventionalism about the real nature of space: “We owe the theory which I have just sketched to Helmholtz and Lie. I differ from them in one point only, but probably the difference is in the mode of expression only and at bottom we are completely in accord” [63].

The point on which Poincaré does not agree is the need to consider matter pre-existing to the group concept that describes geometry of the space in which the matter itself is in there. The mathematical form of a group, i.e. its properties and its dimension, is independent of the existence of matter; in other words the different ways in which a cube can superposed upon itself do not depend on the matter of which it is made. Thus we return to the impossibility of defining the nature of space through experience [5].

By reversing the priority between matter and form, Poincaré also tries to solve another problem that, as it will be seen later, is a key point in the debate on the foundation of geometry, namely the *a priori* use of analytic mathematic, which already presupposes a three-dimensional, continuous and differentiable space.

2.5 Pasch’s and Peano’s axiomatics

The Euclidean model of rigorous science based on the hypothetical-deductive method was taken up by Pasch to give a new foundation to projective geometry [54].

Pasch’s novelty is not only the utmost rigour, so much so as to receive the epithet “father of rigour in geometry” [16], but the conviction that

for a truly deductive geometry, the process of inference must be completely independent of the meaning of geometric concepts as it must be independent of the figures. The relations we should consider are only those one established between the geometrical concepts in the theorems and in the definitions that have been used” [54, p 98].

Therefore, in order to make deduction in a truly independent way from the meaning, no relationship must be omitted, implied or taken for granted and axioms and definitions must be well formulated.

Today it seems an obvious statement but in the tradition of geometry, which considered the axioms as evident and unprovable truths, often these were not even explained and it was not uncommon the use of an axiom, never presented before, to fill a gap in a proof [16].

Pasch is the first to overcome Euclid in setting up a rigorous axiomatic system without neglecting any obvious relation. In Pasch the absolute abstraction of logical deduction process, which should be totally unrelated to intuition, is contrasted by a radical empiricism in the axiom formula-

tion [17]. Indeed, primitive notions of his system derive from direct empirical observations: points are defined as bodies that cannot be further divided, segments as the shortest path between two points and flat surfaces as the external limits of a physical object. Instead, straight lines and planes are excluded from primitive notions as entities that cannot be directly observed [54, pp. 4 and 20].

Another characteristic of Pasch’s extreme empiricism is the rejection of continuity, as it is not sufficiently supported by empirical evidence. He argues “in empirical observation you can never consider an infinite number of things” and also “a segment cannot contain an infinite number of points unless you extend the definition of point away from its intuitive meaning” [54, pp. 125-127].

With his radical thought Pasch refuses to admit not only that a segment is a continuous set of points, isomorphic to \mathbb{R} , but also a dense set, isomorphic to \mathbb{Q} , this means denying that between any two points there is always another point.

In Germany, in the last quarter of the nineteenth century, the idea that continuity was not a property of space was also shared by two other leading geometers of the time: Wiener and Schur [11]. The latter, in 1899, proves, following an intuition of Wiener, the fundamental theorem of projective geometry without using continuity, thus made analytic geometry free from the axiom of continuity [10].

The path opened by Pasch was taken up mainly by Italian geometers, in particular by Peano and his school.

In the two works *I Principii di geometria logicamente esposti* [56] and *Sui fondamenti della geometria* [57], Peano proposes an alternative system of axioms with attention to two fundamental questions: the independence of the axioms and the use of implicit definitions for primitive objects.⁸

Independence of the axioms had already been addressed by Peano for natural numbers [55] and, considering himself “morally certain” also of the independence of his geometric axioms [56, p. 5], Peano tries to proceed in the same way. Nonetheless the task turns out to be more difficult than expected and a few years later, with enviable intellectual honesty, he declares to be still “far from having completed this proof” [57].

Introduction of implicit definitions completes Pasch’s project of a geometry developed only with deductive processes without any direct link with intuition. In *Principii*, prim-

⁸ The concept of implicit definition dates back to the French mathematician Gergonne [19], but its first use in an axiomatic system is found in Peano’s work.

itive entities are not defined; instead of points and segments, Peano simply uses signs, such as 1 and a, b, c, \dots

With the help of other notations and abbreviations, explained in the beginning of the book, Peano lists axioms, theorems and definitions of non-primitive objects in terms of primitive ones:

the reader can understand with the sign 1 any category of entities, and with $c \in ab$ any relationship between any three entities of that category; all the following definitions will always be valid, and all the propositions will subsist. Depending on the meaning given to the undefined signs 1 and may be satisfied, or not, the axioms. If a certain group of axioms is verified, all the propositions that are deduced will also be true, since these propositions are only transformations of those axioms and definitions [56, p. 5].

In *Sui fondamenti*, actually, it clearly states the idea of implicit definition “it will be necessary to determine the properties of the undefined entity p , and of the relationship, by means of axioms, or postulates” [57, p. 55].

2.6 Kantism of early Russell

In the last years of the nineteenth century, a very young Russell entered in the debate basically with a Kantian philosophical point of view. In the conclusions of his *Essay on Foundation of Geometry* [69], he states that the only geometry “wholly” a priori is projective geometry, because axioms of projective geometry “appear as a priori, as essential to the existence of any Geometry and experience of an external world as such” [69, p. 146].

Axioms of projective geometry are common to Euclidean and non-Euclidean geometry, and what distinguishes Euclidean from other geometries are empirical characteristics. The axioms of projective geometry are three [69, p. 132]:

- Ru1 We can distinguish different parts of space, but all parts are qualitatively similar, and are distinguished only by the immediate fact that they lie outside one another;
- Ru2 Space is continuous and infinitely divisible; the result of infinite division, the zero of extension, is called a point;
- Ru3 Any two points determine a unique figure, called a straight line; any three in general determine a unique figure, the plane. Any four determine a corresponding figure of three dimensions, and for aught that appears to the contrary, the same may be true of any number of points.

These axioms, according to Russell, are authentically transcendental because they are inferable from the fundamental principle of homogeneity of space or, more generally, from the possibility of experiencing externality.

Russell raises numerous criticisms to Riemann, Helmholtz, Klein and Lie. In the *Essay* he notes that the notion of manifold, even if exhaustive for the purposes of analytic geometry, is not sufficient to define space in its broadest generality, because spatial relations precede the possibility of expressing them quantitatively. Projective geometry, for example, has no metric relationships.

But for Russell, Riemann and Helmholtz's quantitative method contains an even more difficult assumption to justify: the superiority of algebra over geometry. Through algebraic calculations on coordinate numbers, it is possible to obtain theorems and information initially not known, but “perception of space being wholly absent, Algebra rules supreme, and no inconsistency can arise”.

Russell is therefore aware of the power of analytical method, but warns: “Finally [...] only a knowledge of space, not a knowledge of Algebra, can assure us that any given set of quantities will have a spatial correlate, and in the absence of such a correlate, operations with these quantities have no geometrical import” [69, p. 46].

Another section of his essay is dedicated to philosophical insights into the foundations of geometry, Russell identifies some contradictions that are present both in the concept of space and in all geometric theories

The contradictions in space are an ancient theme as ancient, in fact, as Zeno's refutation of motion. They are, roughly, of two kinds, though the two kinds cannot be sharply divided. There are the contradictions inherent in the notion of the *continuum*, and the contradictions which spring from the fact that space, while it must, to be knowable, be pure relativity, must also, it would seem, since it is immediately experienced, be something more than mere relations [69, p. 188].

These “inevitable” contradictions give rise to three other recurrent antinomies in all geometric theories:

- I Though the parts of space are intuitively distinguished, no conception is adequate to differentiate them. Hence arises a vain search for elements, by which the differentiation could be accomplished, and for a whole, of which the parts of space are to be components. Thus we get the point, or zero extension, as the spatial element, and an infinite regress or a vicious circle in the search for a whole;
- II All positions being relative, positions can only be defined by their relations, i.e. by the

straight lines or planes through them; but straight lines and planes, being all qualitatively similar, can only be defined by the positions they relate. Hence, again, we get a vicious circle;

- III Spatial figures must be regarded as relations. But a relation is necessarily indivisible, while spatial figures are necessarily divisible ad infinitum.

The antinomy I emerges wherever there is a *continuum* because by its nature “a point must be spatial, otherwise it would not fulfil the function of a spatial element; but again it must contain no space, for any finite extension is capable of further analysis”, therefore “points can never be given in intuition, which has no concern with the infinitesimal; they are a purely conceptual construction”.

On the antinomy II we can observe that the point and the line can be defined through their mutual relations “two points lie on one straight line which they completely determine; and two straight lines meet in one point, which they completely determine” [69, p. 127]. In two dimensions the perfect duality does not make trouble because two elements of one type (points or lines) define an element of another type (a line or a point respectively). It becomes more cumbersome when we switch to three dimensions because a plane can be defined by three points (not aligned) or by a straight line and a point (not lying on it). The perfect duality is therefore not preserved.

Antinomy III means that the infinite divisibility of space is at odd with the fact that reciprocal relations between the figures ordered in space cannot be divided indefinitely.

The solution proposed by Russell is considering the space only as an ordered space, a set of “relations between unextended material atoms”. Absolute space for Russell “arises, by an inevitable illusion, out of the spatial element in sense-perception, may be regarded, if we wish to retain it, as the bare principle of relativity, the bare logical possibility of relations between diverse things” [69, p. 198].

To sum up, in Russell opinion there are at least four problems in the path on the foundations of geometry by Riemann and Helmholtz, the last three of which are unavoidable also in the following developments by Klein and Lie: the superiority of algebra and analytical geometry over synthetic geometry, the problem of definition of the point, the problem of circular definitions and finally the contradiction between the infinite divisibility of space and the need to have finite spatial relations.

III. DEBATE ON FOUNDATIONS OF GEOMETRY AND ANALYTIC METHOD

The path covered by Riemann, Helmholtz, Klein, Lie and Poincaré is essentially the same, and it is very distant from the traditional Kantian way that identified Euclidean space as an *a priori* form of our intuition.

For Kant axioms are “synthetic *a priori* principles, insofar as they are immediately certain” [40, B760], position that philosophers will still keep for a long time. Mathematicians, instead, at the end of the 1800s tend to derive axioms from sensitive intuitions, while having different views about their nature: for Riemann axioms are “hypotheses”, for Helmholtz “observational facts” [26], for Klein “idealization of empirical data” [8] and for Poincaré “conventions” [62].

By the way Helmholtz, Poincaré, Klein and Lie formulate geometric axioms from factual statements or sensitive intuition and Riemann uses observations on physical space to identify Euclidean geometry, after having assumed a metric for any geometry, even more general than the projective one.

Pasch formulates axioms of projective geometry after having made a laborious abstraction and conceptualization of empirical material. He defines terms and logical relations “so that there is no need, after their definition, to return to sensitive perception” [54, p. 17]. A similar process is also carried out by Peano.

Since the discovery of non-Euclidean geometry, mathematicians, in particular in Germany, have tried to found Euclidean geometry on sensitive intuition, this is the background in which Hilbert forms his conception of intuition, he does not know if “the origin of the intuition is *a priori* or empiric” [32, p. 303], but this is the intuition he has in mind.

On the contrary, early Russell starts from *a priori* intuition and, following Kant's “transcendental aesthetics”, elaborates new axioms for projective and metric geometry.

Riemann studied Herbart's philosophy and he was deeply influenced by him, so much that he declared himself “Herbartian in psychology and epistemology (methodology and eidology)” [53].

Herbart succeeded at the chair of Kant at Königsberg, in 1809 after 7 years in Göttingen. He recognizes the problem of the thing-in-itself inherited by his predecessor and tries to solve it by not completely excluding the empirical datum. For Herbart knowledge “is constituted by a deliberate conceptual creation and serves as a theoretical system of reference for empirical investigations and thus plays a formative role for the cognition of the empirical world” [72].

Herbart's philosophy was welcomed by the scientists of the period because, contrary to the idealistic development of Kantian thought, it acknowledged full autonomy to the sciences in particular mathematics and logic. This view saves "realism" of the sciences and in the meantime frees science from an immediate correspondence with the empirical reality [59, 70].

Riemann affirmed to adopt Herbart's methodology and wrote on his notebook these steps

1. Formation of concepts from perceptions, through abstraction
2. Change or integration of already formed concepts, to overcome contradictions or implausibility.⁹

Until the 1980s, it was thought that the key to Herbart's influence on Riemann's work was the philosophy of space [72, pp. 107-109] but thanks to a detailed analysis of Riemann's notes, made available by Göttingen University, Scholz (1982) and Pettoello (1986) have revealed that the most influence is due to epistemology and the method for knowledge, even if in the *Habilitationvortrag* Riemann does not respect this method, because he forms immediately his hypotheses neglecting the phase of abstraction and conceptualization. This issue about Riemann is still controversial [4, pp. 55-56], but Herbart's process toward knowledge seems very closer to the Pasch one.

Riemann geometry breaks with Kant tradition; Riemann conceives a general n -dimensional space and the three-dimensional physical space is just a particular case. The only way to distinguish physical space from other spaces is through empirical properties, such as the preservation of lengths and the possibility for a rigid body to undergo transformations that keep unchanged its shape. Euclidean space has no longer a transcendental nature and its characterization is *a posteriori*.

Even more openly in contrast with Kantian doctrine seems to be Helmholtz's approach because the assumptions that had led him to Euclidean geometry were purely empirical. The reaction to writings of Riemann and Helmholtz, in defence of Kantian orthodoxy, is not to be expected [45, 46], as well as the Helmholtz's rejoinder [27] and the contribution of other eminent philosophers more or less related to Kant [13, 49].

Krause's criticism of Helmholtz's idea of geometry is radical and moved by a pride of belonging to the German Kantian School:

If Helmholtz is right, and if Kantian groundwork is wrong, the content and method, which necessarily

derive from it, also fall; then the secular direction of German philosophy is wrong, and it remains only for us to send the young Germans to study philosophy in the English schools [45, introduction].

Whereas the criticism of Land will be taken up even outside the German Empire. Land accuses Helmholtz that "from geometry proper, there is an easy transit into metaphysics, by the road of analytical geometry" [46]. This kind of objection is generalized and strengthened some years later by Veronese:

The analysis applied to geometry serves to give us directions also in the study of the principles, but *a priori* it is not known if these directions can be used from a purely geometric point of view [74, p. XXIII]

and then by others; for example by Russell:

His [of Riemann] definition of space as a species of manifold, therefore, though for analytical purposes it defines, most satisfactorily, the nature of spatial magnitudes, leaves obscure the true ground for this nature, which lies in the nature of space as a system of relations, and is anterior to the possibility of regarding it as a system of magnitudes at all [69, p.16].

By Poincaré

Your group [i.e. Helmholtz and Lie's group] presupposes space; to construct it you are obliged to assume a continuum of three dimensions. You proceed as if you already knew analytical geometry [63, p. 40]

and finally by Hilbert:

In consequence of his method of development, Lie has also necessitated the express statement of the axiom that the group of displacements is produced by infinitesimal transformations. These requirements, as well as essential parts of Lie's fundamental axioms concerning the nature of the equation defining points of equal distance, can be expressed geometrically in only a very unnatural and complicated manner. Moreover, they appear only through the analytical method used by Lie and not as a necessity of the problem itself [35].

In response to the criticisms received, Helmholtz expressly reiterates its opposition to Kant:

I am not the one "who is acquainted with a transcendental space having laws proper to itself" but I am instead here seeking to draw the consequences of what I consider to be Kant's unproved and incorrect hypothesis, according to which the axioms are taken to be propositions given by transcendental intuition, and I do this in order to demonstrate that a

⁹ Fragment XVIII, 9 of Riemann's *Nachlass*, quoted in [60].

geometry based upon such intuition would be wholly useless as regards objective knowledge [27]. Erdmann, in the 1870s a “disciple of Helmholtz”¹⁰ and later editor of Kant's works on behalf of the Berlin Academy, tries instead to reconcile the uniqueness of *a priori* spatial intuition with the different geometries determined by Riemann and Helmholtz:

The fact that our spatial intuition is single is not contradicted: we can only conceptualize the general intuition of a pseudo-spherical or spherical space of a certain measure of curvature. Such uniqueness, however, is not absolute anymore because we can fix homogeneous parts of those spaces intuitively and compare them with the metrical relations between partial representations of space [13, p. 135]¹¹.

On the contrary Lotze, which took over Herbart's chair of philosophy at Göttingen, thought that the new geometric speculations were “just one big connected mistake” [50, p. 234]. In his opinion Euclidean geometry provides the only description of the world, because of the self-evidence of axioms and he discussed Helmholtz arguments against the creation of different geometries [72, p. 286].

IV. HILBERT'S SYNTHESIS

In 1895 Hilbert moved from Königsberg to Göttingen, at the request of Klein who wanted to bring Göttingen University back to the centre of German and European mathematics, as it was at Gauss time [68]. The professional relationship between Hilbert and Klein dates back to the mid-1880s, when Hilbert attended some lectures of Klein in Leipzig [65, p. 19].

Klein had understood fecundity of formalism and algebraic invariants in the study of geometric structures, as shown in the *Erlangen's Program*, even if he continued to rely on intuition and mental visualization in geometric reasoning. In a note to the *Program* he wrote “it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aid of analysis is only a first, though a very important step” [43]. For this reason, in more recent times, he has been accused of superficiality and lack of rigour [15].

On the contrary, Hilbert is considered the champion of rigour, founder of the formalism and axiomatic school. But the differences between Klein and Hilbert are much less marked than the appearance, because finally they “shared

understanding of mathematics as a multi-faceted but fundamentally unified body of knowledge” [67].

Hilbert's geometric researches are in a logical continuity with Klein's one. Klein had proved the coherence of non-Euclidean geometries by having assumed the coherence of Euclidean geometry [41]; Hilbert, in *Grundlagen*, proves the coherence of Euclidean geometry by assuming the coherence of arithmetic.¹² Klein moreover approved the work of his colleague, giving it immediate publication for the inauguration of the monument in memory of Gauss and Weber in Göttingen in 1899.

Hilbert opens the *Grundlagen* with a quotation from the *Critics to the Pure Reason* of his fellow citizen Kant:

Thus all human cognition begins with intuitions, goes from there to concepts, and ends with ideas [40, B730].

Hilbert's starting point is intuition. The same intuition that excelled in Klein's geometric studies and that Hilbert clearly distinguishes from the abstraction used by Pasch and Peano and suggested by Herbart:

In mathematics, as in all scientific research, we encounter two tendencies: the tendency toward abstraction – which seeks to extract the logical elements from diverse material and bring this together systematically – and the other tendency toward intuition [*Anschaulichkeit*], which begins instead with the lively comprehension of objects and their substantial [*inhaltliche*] interrelations.... The many-sidedness of geometry and its connections with the most diverse branches of mathematics enable us in this way [namely, through the *anschauliche* approach] to gain an overview of mathematics in its entirety and an impression of the abundance of its problems and the rich thought they contain [32, p. V].

Hilbert finds the origin of geometry in spatial intuition, with two fundamental differences from Klein and Helmholtz.

The first is the continuity of geometric space, a feature considered essential also by Riemann and Lie [78]. The second concerns a methodological aspect, already present in many criticisms: the priority of algebra and analytical geometry on synthetic geometry.

Hilbert does not think that space is a Riemannian, continuous and differentiable, manifold. Indeed in the *Grundlagen* he proves that continuity is not necessary to found Euclidean geometry, because the one-to-one correspondence between the points of a straight line and the set of real num-

¹⁰ As Russell defined him [69, p. 71].

¹¹ Translation in English by Biagioli in [4, p. 89].

¹² Chapter 2 of the *Grundlagen* contains the coherence proof of Euclidean geometry

bers \mathbb{R} is not strictly necessary. It is sufficient the density of the rational numbers \mathbb{Q} ,¹³ together with a numerable amount of square roots.

On this point Hilbert is in the middle between Riemann, Helmholtz and Lie, who saw in continuity a fundamental element of their geometry, and the extreme empiricism of Pasch, who did not even accept the match between rational numbers and points of a straight line.

On *continuum* problem, most probably, Hilbert also underwent the influence of Kronecker. The two mathematicians met at least a couple of times [65, p. 31], during which Kronecker transmitted to the young Hilbert the nodes of his mathematical thought, summarized in the famous sentence “God created the integers, all else is the work of man”.¹⁴

Mathematics for Kronecker, as well as geometry for Pasch who was a student of his [17], has its foundation in experience. Mathematics is a natural science whose objects exist in relation to our experience of counting. Consequently, the only existing numbers are the positive integers and all other mathematical objects must be constructed from these ones [7]. Kronecker, as well as Wiener and Schur, is therefore contrary to a mathematics based on the continuity of real numbers, and he claims the non-existence of those numbers that cannot be built through a finite number of operations on integers, such as transcendent numbers.

From Kronecker's ideas a particular constructivist conception of mathematics, called finitism, has developed. Finitists admit the existence of mathematical objects only if they can be constructed through a finite number of operations, a conception that Hilbert will declare in the 1930s substantially coinciding with his way of thinking [51, p. 267].

Axiom of completeness, that gives continuity to the system of the *Grundlagen*, is not mentioned in the first edition [33], it is added for the first time in the French edition a year later [34]. Some different hypotheses have been advanced on the reason for its adjunction [e.g. 12, 74]. We can here suppose Hilbert adds it to reconcile his axiomatic treatment with the one of Riemann, Helmholtz and Lie to account for the motions of rigid bodies, defined by them as continuous spatial transformations. Hilbert dedicates to this topic an article [35], then attached in a much larger

¹³ See axiom II.2 “If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D” [33, p. 6].

¹⁴ Quoted in [77].

form as appendix in the second German edition of the *Grundlagen* [36]. Here we can read in the incipit:

I have attempted in what follows to formulate a set of axioms for plane geometry which while resting on the concept of a group contain only simple and geometrically clear requirements, and in particular assume in no way the differentiability of functions included by the motions. The axioms of the set formulated by me are contained as particular parts in the Lie axioms or, as I believe, can be derived immediately from them.

The axioms are:

Hi1 The motions form a group;

Hi2 Every true circle consists of an infinite number of points;

Hi3 The motions form a closed set.

and in the conclusion:

the arrangement of the axioms [in the *Grundlagen*] is such that continuity is required **last**¹⁵ among the axioms so that than the questions as to what extent the well-known theorems and arguments of elementary geometry which are independent of continuity arises in the foreground in natural way. In the present investigation, however, continuity is required **first** among the axioms by the definition of the plane and a motion so that here the most important task has been rather to determine the least number of conditions from which to obtain by the most extensive use of continuity the elementary figures of geometry (circle and line) and their properties necessary for the construction of geometry. Indeed, the present investigation has shown that to this end the conditions stated in Axioms I-III are sufficient.¹⁶

Thus for Euclidean geometry, as already proved in *Grundlagen*, continuity is not necessary, it becomes necessary to account for the rigid motions of bodies. It is not so for the differentiability, which remains a strong and avoidable condition, if we proceed synthetically rather than analytically.

Blumenthal reports that Hilbert began working on projective geometry presumably in 1891, after following a Wiener lecture on the theorems of Pappus and Desargues. Previously he had dealt with algebraic invariants and with the theory of numerical fields [78].

The theory of numerical fields has played an important role in drafting the *Grundlagen*. Very few researches have highlighted this point, but recent studies, after publication

¹⁵ Both here and below the bold typeface is by Hilbert.

¹⁶ English translation by L. Unger in [38].

of Hilbert's geometry lectures [22], have shown that one of *Grundlagen* main aims was the introduction of number in geometry from inside and not from outside through an artificial Cartesian reference system [1, 3, 20].

The analytical method, thanks to the certainty of its results, had gradually subordinated synthetic geometry to algebra, imposing very demanding hypotheses on the idea of space, in particular continuity and derivability.

In the last quarter of nineteenth century, at the height of the debate on the foundations of geometry, the need to restore the autonomy of geometry was felt not only by "pure" geometers but also by supporters of Land's criticism to Helmholtz, including illustrious mathematicians and philosophers such as Poincaré and Russell.

Since his first works on projective geometry, Hilbert has tried to introduce numbers and arithmetic operations exclusively through internal relations to geometry, using only lines and points [30]; at the beginning following von Staudt's *würfe* method [52] and then using the theorems of Pappus-Pascal and Desargues [31, 32, 33].

His intent was to reaffirm the independence of geometry from algebra, and perhaps even the priority of the first over the second, but without giving up the algebraic tools of metric geometry [21].

The antinomies II and III rose up by Russell on foundation of geometry, i.e. the definition of point and the circular definitions, are solved by Hilbert with implicit definitions, as Peano did. Hilbert opens the first chapter of the *Grundlagen* by considering three "systems of objects" (*Systeme von Dingen*): points, lines and planes. These objects are not explicitly defined, but their mutual relations are made explicit through the axioms. What matters about these objects is not the shape, the dimension, or whether they are further divisible, but only the relations described through the axioms.

Virtually, all types of objects that possess the characteristics given by the axioms can equally be taken into account as geometric entities. To describe Hilbert's approach, it is always appropriate the famous anecdote according to which at the end of a Wiener lesson, on foundations of geometry, he said "One must be able to say at all times - instead of points, straight lines, and planes - tables, chairs, and beer mugs" [65, p. 39].

In *Grundlagen* implicit definitions express "certain related facts basic to our intuition (*Anschauung*)", but do not define objects as such. They attribute to points, lines and planes precise relations that Hilbert calls concepts (*Be-*

griffen): "to be between", "congruent", "parallel", etc. [38, p. 39].¹⁷

Here we find a similarity with Russell solution of antinomy III. For early Russell the empty space is wholly conceptual, because it "is the logical possibility of relation between things" [69, p. 198], as well as for Hilbert, space is ultimately determined by concepts, which are relations among fundamental objects: points, straight lines and planes.

If axioms that describe relations among these objects are all satisfied, then space is Euclidean; if all the axioms are satisfied except the parallel one, then the space is non-Euclidean. If, on the other hand, there are other relations among points, straight lines and planes, space can be Desarguesian or non-Desarguesian, Pascalian or non-Pascalian, Archimedean or non-Archimedean and so on. It is the set of fundamental concepts that determines the geometry of space.

The difference between early Russell's and Hilbert's vision is that for Russell, as well as for Kant, the truthfulness of geometry was based on the *a priori* truth of concepts, which were based on the more general transcendental principle of homogeneity of space. For Hilbert, instead, the truthfulness of geometry is given by the logical coherence of concepts and therefore of axioms that define them. With this statement Hilbert replied to a Frege's observation:

if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist [14].

For Hilbert a geometric space exists simply just if axioms that describe it are not contradictory.

Nevertheless the quotation from Kant at the beginning of the *Grundlagen* suggests that Hilbert does not stray as far away from Kantian thought as it seems and has been argued elsewhere [e.g. 2]. Better yet, it provides the philosophical key to the reading of the whole work. Hilbert hints that in writing the *Grundlagen*, and in general in his study on geometry, he followed the path indicated by his fellow citizen.

Another passage of the *Critique of Pure Reason*, very similar to the opening quotation of the *Grundlagen*, better specifies the nature of intuitions from which the knowledge starts, and which Hilbert makes his own also for geometry:

¹⁷ It is noteworthy that in the first edition sometime Hilbert uses the term *Begriffen* referred to points, lines and planes [33, p. 5], but in the seventh edition he corrects the text, referring to them with the term *Dingen* [37, p. 3].

All our cognition starts from the senses, goes from there to the understanding, and ends with reason, beyond which there is nothing higher to be found in us to work on the matter of intuition and bring it under the highest unity of thinking [40, B355].

A little further on, Kant explains what he means by intuition, concept and idea:

A perception that refers to the subject as a modification of its state is a sensation (*sensatio*); an objective perception is cognition (*cognitio*). The latter is either an intuition or a concept (*intuitus vel conceptus*). The former is immediately related to the object and is singular; the latter is mediate, by means of a mark, which can be common to several things. A concept is either an empirical or a pure concept, and the pure concept, insofar as it has its origin solely in the understanding (not in a pure image of sensibility), is called *notio*. A concept made up of notions, which goes beyond the possibility of experience, is an idea or a concept of reason [40, B377].

Hilbert starts his researches from the intuition of the fundamental objects; points, straight lines and planes. He then relates these objects through common characteristics as described in the axioms.

These characteristics define pure concepts, “notions” in Kantian words, because they originate in the understanding, since we cannot have sensitive images for example of all the points lying between two other points or of two parallel lines that never meet. The cognitive process ends with the idea of geometry determined by the concepts made up of these notions.

In the *Grundlagen* we are dealing with Euclidean geometry, but Hilbert has also shown how changing one or more axioms, and thus modifying the concepts, other perfectly consistent geometries are obtained. Hilbert's path differs therefore from the Kantian one only due to the not clarified nature of the intuitions at the base of the geometry.

V. CONCLUSIONS

Although the *Grundlagen* do not contain any real discoveries or original methodological innovations [15], they have been, and still are, the book par excellence of modern Euclidean geometry. One of the reasons for this immediate success was Hilbert's ability to make a synthesis of the main schools features that had given rise to the debate on the foundations of geometry in the second half of the nineteenth century. He took from each one the key elements and solved problems and contradictions that emerged in the debate.

Hilbert took up Pasch's axiomatic method, in order to be unassailable, as Euclid's system was; it had to be absolute-

ly complete and not to omit any concept. With Pasch Hilbert also shared the perplexity about the *continuum* with a slight difference: if for Pasch it was impossible because it was not found in nature, for Hilbert it was not necessary because for the construction of Euclidean geometry a countable quantity of points, with the property of density, common to the set of rational numbers, was sufficient.

Another important methodological aspect in the *Grundlagen* is the use of implicit definitions, whose primary birth was often wrongly attributed to Hilbert. The choice, borrowed from Peano, was essential to overcome some issues in the Euclidean's *Elements* that Russell considered inherent in the treatment of any geometry.

In common with Riemann, Helmholtz and Klein, Hilbert had the starting point of his path towards the foundation of Euclidean geometry, which was intuition. The difference was that for Hilbert real space was not a three-dimensional *continuum* to be investigated with the means of analytical geometry. Hilbert's approach was all within the geometry itself. He introduced numbers and arithmetic operations thanks to the relations between points and straight lines defined through the axioms, so as to keep independent geometry from algebra.

The effectiveness of the achievement was also recognized by Poincaré [64], at the time one of the most profound mathematician to have criticized the not fully justified use of analytic geometry by Riemann, Helmholtz and Lie.

In his research work on geometry Hilbert followed a philosophical reference that did not contradict the Kantian text, as it might seem at first glance. Rather he moved himself in the path traced by Kant to achieve knowledge. Hilbert expressed the “fundamental facts relating to our intuition” through the axioms, which define the fundamental concepts. Then he concluded with the definition of the idea of Euclidean geometry, opening the door to the study of other new geometries as the axioms vary.

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