Extending two fixpoint theorems of Langley and Zheng

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Extending Two Fixpoint Theorems of Langley and Zheng

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Abstract

We extend two theorems on fixpoints of \( f(z) \) by Langley and Zheng [1] to the consideration of points where \( f(z) = Q(z) \) for some rational function \( Q \) such that \( Q(\infty) = \infty \). In addition, we extend the class of functions \( f \) from transcendental entire functions to meromorphic functions with relatively few poles.

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1 Introduction

Let \( \mathcal{B} \) denote the class of functions \( f \) meromorphic in the plane, for which the set of finite singular values of the inverse function \( f^{-1} \) is bounded, that is, the class of all functions \( f \) whose set of finite asymptotic and critical values is bounded. This class \( \mathcal{B} \) has been considered extensively in iteration theory, see [2], [3], [1].

In [1], Langley and Zheng prove the following two fixpoint theorems, for certain types of functions in the class \( \mathcal{B} \).

**Theorem 1.1 (Langley and Zheng, [1])** Let \( 0 < \alpha < 1 \). There is a positive constant \( c \), depending only on \( \alpha \), such that if \( f \) is a transcendental entire function in the class \( \mathcal{B} \), then there are infinitely many fixpoints \( z \) satisfying

\[
f(z) = z, \quad |f'(z)| > c \log M(\alpha|z|, f).
\]

**Theorem 1.2 (Langley and Zheng, [1])** Let \( f \) be a meromorphic function in the class \( \mathcal{B} \), with order \( \infty \geq \rho(f) > \mu > 0 \). Then \( f \) has infinitely many fixpoints \( z \) with

\[
f(z) = z, \quad |f'(z)| > |z|^\mu/2.
\]

Given a transcendental function \( f \) and a rational function \( Q \), we define a \( Q \)-point of \( f \) to be any solution \( z_0 \) of the equation \( f(z) = Q(z) \). We then extend these two fixpoint theorems to \( Q \)-point theorems. In particular, we extend them to the case where \( f(z) \) is equal to a rational function \( Q(z) \) such that \( Q(\infty) = \infty \), that is, where \( Q \) is a rational function with a pole of multiplicity \( p \geq 1 \) at \( \infty \).

In addition, we further extend Theorem 1.1 to transcendental meromorphic functions \( f \) such that \( \delta(\infty, f) > 0 \), that is, meromorphic functions with relatively few poles. We state the extended theorems as follows.
Theorem 1.3 Let $0 < \alpha < 1$. There is a positive constant $c$, depending only on $\alpha$, such that if $f$ is a transcendental meromorphic function in the class $\mathcal{B}$ with $\delta(\infty, f) > 0$, and $Q$ is a rational function with a pole of multiplicity $p \geq 1$ at $\infty$, then there are infinitely many points $z$ satisfying

$$f(z) = Q(z), \quad |f'(z)| > c\delta(\infty, f)\frac{|Q(z)|}{|z|}T(\alpha|z|, f).$$

(2)

Theorem 1.4 Let $f$ be a meromorphic function in the class $\mathcal{B}$, with order $\infty \geq \rho(f) > \mu > 0$. Let $Q$ be a rational function with a pole of multiplicity $p \geq 1$ at $\infty$. Then $f$ has infinitely many points $z$ with

$$f(z) = Q(z), \quad |f'(z)| > |z|^{\mu/2+\mu-1}.$$

In § 2 we prove a lemma that is used in the proof of Theorem 1.3 and Theorem 1.4. In § 3 and § 4 we prove Theorem 1.3 and Theorem 1.4 respectively.

2 A useful lemma

The following lemma is used in the proof of Theorem 1.3 and Theorem 1.4. We note that it is an extension of [1, Lemma 1], and we will provide the proof for completeness.

Lemma 2.1 Suppose that $f$ is a transcendental meromorphic function in the class $\mathcal{B}$. Let $Q$ be a rational function such that $Q(\infty) = \infty$. Define a function $G$ by

$$G(z) = f(z)/Q(z), \quad G'(z)/G(z) = f'(z)/f(z) - Q'(z)/Q(z).$$

(3)

Suppose that $\delta$ is a positive constant. Then there exists a positive constant $\epsilon$ such that the following is true. If $|z_1|$ is large and $|G(z_1) - 1| < \frac{1}{2}\epsilon$, then $z_1$ lies in a component $C$ of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$, such that $C$ is contained in $B(z_1, \delta|z_1|)$. Define a function $H$ by

$$H(z) = \frac{2}{\epsilon}(G(z) - G(z_1)).$$

(4)

Then $C$ is mapped conformally onto $B(0,1)$ by $H$. Furthermore, $|Q(z)G'(z)|$ is large on $C$, and given any $z_0 \in C$ such that $|G(z_0) - G(z_1)| < \frac{1}{4}\epsilon$, we have that

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_0)| \leq \frac{27}{4}|G'(z_1)|.$$

(5)

The following two results are used in the proof of Lemma 2.1.

Lemma 2.2 (Eremenko, Lyubich and Bergweiler, [2], [4], [3], [5]) Suppose that $f$ is a transcendental meromorphic function in the class $\mathcal{B}$. Then there are positive constants $R, S$ and $c$ such that

$$|zf'(z)/f(z)| \geq c\log^+ |f(z)/R|$$

for $|z| > S$. Here, $R$ and $S$ depend on $f$, and $c$ does not.

Theorem 2.3 (Koebe distortion theorem) Let $f$ be an analytic univalent function on $B(0,1)$. Then for $0 < r < 1$,

$$\frac{1 - r}{(1 + r)^3}|f'(0)| \leq \max_{|z| \leq r} |f'(z)| \leq \frac{1 + r}{(1 - r)^3}|f'(0)|.$$
Proof of Lemma 2.1  First we choose a large positive $R_1$ and a small positive constant $\epsilon$ such that $|G(z) - 1| > \epsilon$ on $|z| = R_1$. Suppose that $|z| > R_1$ and that $|G(z) - 1| < \epsilon$. Then we have that $|G(z)| > \frac{1}{2}$, and so by (3), $|f(z)| > \frac{1}{2}|Q(z)|$. Then by Lemma 2.2, there is a positive constant $R$ such that

$$|zf'(z)/f(z)| \geq c \log^3 |f(z)/R| > c \log |Q(z)|,$$

where $c$ denotes a positive constant which does not depend on $R_1$ or $\epsilon$, but not necessarily the same constant at each occurrence.

For such $z$, we have by (3) that

$$|zG'(z)/G(z)| = |zf'(z)/f(z) - zQ'(z)/Q(z)| \geq |zf'(z)/f(z)| - |zQ'(z)/Q(z)|. \tag{7}$$

We recall that $|G(z) - 1| < \epsilon$ and so $|G(z)|$ is close to 1, and since $Q(\infty) = \infty$ and $z$ is large, we have that $|zQ'(z)/Q(z)| = O(1)$ as $z \to \infty$. Then by (6) and (7) we have that

$$|zG'(z)| > R_2 = c \log |Q(R_1)|. \tag{8}$$

Then since $Q(\infty) = \infty$ and $z$ is large, we have that $|Q(z)| \geq c|z|$ and so,

$$|Q(z)G'(z)| > R_3 = cR_2. \tag{9}$$

Suppose now that $\delta$ is a positive constant and that $z_1$ is as in the statement of the lemma, with $|z_1| > 2R_1$. That is, $|G(z_1) - 1| < \frac{1}{4} \epsilon$, and $z_1$ lies in a component $C$ of the set $\{z : |G(z) - G(z_1)| < \frac{1}{4} \epsilon\}$. Let $H$ be defined as in the statement of the lemma, that is, $H(z) = \frac{2}{\epsilon} (G(z) - G(z_1))$. Then $H(z_1) = 0$ and $|H'(z_1)| = \frac{2}{\epsilon} |G'(z_1)| > 2R_2/c|z_1|$ by (8), and so $H'(z_1) \neq 0$. Then $H$ is a conformal mapping in a neighbourhood of $z_1$.

Next define

$$h(w) = \sum_{k=0}^{\infty} c_k w^k, \quad c_0 = z_1, \tag{10}$$

to be that branch of the inverse function $H^{-1}$ which maps 0 to $z_1$, and let $r_1$ be the radius of convergence of this series. Then, by a standard compactness argument, there is some $w^* = r_1 e^{i\theta^*}$, for some real $\theta^*$, such that $h$ has no analytic continuation to a neighbourhood of $w^*$. Thus the image of the path $w = te^{i\theta^*}$, $0 \leq t < r_1$, under $h$ must, as $t \to r_1$, either tend to $\infty$ or to a multiple point $z^*$ of $H$, with $H(z^*) = w^*$.

Set $r_2 = \min\{r_1, 1\}$. Let $\gamma$ be the path $\gamma(t) = te^{i\theta^*}$, $0 \leq t < r_2$. For $|z| = R_1$, we have that

$$|H(z)| = \frac{2}{\epsilon} |G(z) - G(z_1)| \geq \frac{2}{\epsilon} (|G(z) - 1| - |G(z_1) - 1|) > \frac{2}{\epsilon} (\epsilon - \frac{1}{4} \epsilon) = \frac{3}{2},$$

and since $h(0) = z_1$ we must have that the image path $h(\gamma)$ lies in $|z| > R_1$. Now $H(h(w)) = w$, which by differentiation gives $(H(h(w)))' = H'(h(z))h'(w) = 1$ where $z = h(w)$. Then,

$$h(w)/h'(w) = zH'(z) = \frac{2}{\epsilon} zG'(z) \tag{11}$$

and this is large on $\gamma$ by (8). Then for $w$ on $\gamma$ we have by (8) and (11) that

$$\log |h(w)/z_1| \leq \int_0^{|w|} |h'(se^{i\theta^*})/h(se^{i\theta^*})| ds \leq \frac{\epsilon}{2} |w| \leq \frac{\epsilon}{2R_2}.$$
Then if \( \epsilon \) was chosen small enough, the path \( h(\gamma) \) lies in \( B(z_1, \delta|z_1|) \), and replacing \( \theta^* \) by any \( \theta \in [0, 2\pi] \) we see that \( h(B(0, r_2)) \subseteq B(z_1, \delta|z_1|) \).

On the path \( h(\gamma) \) we have

\[
|G(z) - 1| \leq |G(z) - G(z_1)| + |G(z_1) - 1| < \frac{1}{2}\epsilon + \frac{1}{4}\epsilon = \frac{3}{4}\epsilon,
\]

and, using (8), we have that \( |H'(z)| = \frac{3}{2}|G'(z)| \geq 2R_2/\epsilon|z| \). In particular, we have that \( h(\gamma) \) is bounded and does not tend to a critical point of \( H \). Since \( r_2 = \min\{r_1, 1\} \), we must then have that \( r_1 \geq 1 \). Thus, \( C \) is contained in \( B(z_1, \delta|z_1|) \) and is mapped conformally onto \( B(0, 1) \) by \( H \). Furthermore, by (9), \( |Q(z)G'(z)| \) is large on \( C \).

Next, let \( z_0 \in C \) such that \( |G(z_0) - G(z_1)| < \frac{1}{4}\epsilon \). Then \( |H(z_0)| = \frac{2}{3}|G(z_0) - G(z_1)| < \frac{1}{2} \). Then since \( h \) is univalent on \( B(0, 1) \), there exists some \( w_0 \in B(0, 1) \), such that \( h(w_0) = z_0 \). Further, \( |w_0| < \frac{1}{2} \), and so by the Koebe distortion theorem we have,

\[
\frac{4}{27}|h'(0)| \leq |h'(w_0)| \leq 12|h'(0)|.
\]

Then since \( H'(z)h'(w) = 1 \) we have \( h'(w) = 1/H'(z) \) for \( w \in B(0, 1) \), and in particular, \( h'(0) = 1/H'(z_1) \) and \( h'(w_0) = 1/H'(w_0) \). Also, by (4), \( H'(z) = \frac{2}{3}G'(z) \), and so by (12) we have

\[
\frac{1}{12}|G'(z_1)| \leq |G'(z_0)| \leq \frac{27}{4}|G'(z_1)|.
\]

### 3 Proof of Theorem 1.3

We need the following theorem.

**Theorem 3.1 (Toppila and Winkler, [6])** Let \( f \) be a transcendental meromorphic function of order \( \lambda \) such that \( \delta(\infty, f) > 0 \). Then

\[
\limsup_{z \to \infty, z \in E(f)} \frac{|z| f'(z)}{T(|z|, f)} \geq A_0 \delta(\infty, f)(1 + \lambda)
\]

where \( E(f) = \{ z : |f(z)| = 1 \} \), \( A_0 \) is a positive absolute constant and \( f^\sharp(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \) is the spherical derivative.

We now prove Theorem 1.3.

**Proof of Theorem 1.3** Let \( 0 < \alpha < 1 \) and let \( R \) be a large positive constant. Let \( \delta \) and \( \epsilon \) be as in Lemma 2.1, with \( \frac{1}{1 + \theta} > \alpha \). Define functions \( G \) and \( f_1 \) by

\[
G(z) = \frac{f(z)}{Q(z)} = 1 + \frac{\epsilon f_1(z)}{16}.
\]

Then \( f_1(z) = \frac{16}{\epsilon} (\frac{f(z)}{Q(z)} - 1) = c_1 \frac{f(z)}{Q(z)} - c_2 \) where \( c_j \) denotes a positive constant not depending on \( R \). Then since \( f \) is a transcendental meromorphic function and since \( Q \) is a rational function, we have that \( f_1 \) is a transcendental meromorphic function.
Further, we have that $\delta(\infty, f_1) > 0$ since $f(z) = (c_3 f_1(z) + c_4)Q(z)$ and,

$$0 < \delta(\infty, f) = \delta(\infty, (c_3 f_1 + c_4)Q) = \liminf_{r \to \infty} \frac{m(r, (c_3 f_1 + c_4)Q)}{T(r, (c_3 f_1 + c_4)Q)} \leq \liminf_{r \to \infty} \frac{m(r, c_3 f_1 + c_4) + m(r, Q) + O(1)}{T(r, c_3 f_1 + c_4) - T(r, Q) - O(1)} = \liminf_{r \to \infty} \frac{m(r, f_1) + O(\log r) + O(1)}{T(r, f_1) - O(\log r) - O(1)} = \delta(\infty, f_1),$$

since $Q$ is a rational function and $f_1$ is a transcendental function function.

Let $\lambda$ be the order of $f_1$, and let $E(f_1) = \{ z : |f_1(z)| = 1 \}$. Then by Theorem 3.1, we have that

$$\limsup_{z \to \infty, z \in E(f_1)} \frac{|z| f_1'(z)}{T(|z|, f_1)} \geq A_0 \delta(\infty, f_1)(1 + \lambda),$$

where $A_0$ is a positive absolute constant. We note that for $z \in E(f_1)$ we have that $f_1'(z) = \frac{|f_1(z)|}{1 + |f_1(z)|^2} = \frac{|f_1(z)|}{2}$.

Then we have

$$\limsup_{z \to \infty, z \in E(f_1)} \frac{|z| f_1'(z)}{T(|z|, f_1)} > d_1 = A_0 \delta(\infty, f_1),$$

and so there exists a sequence $(\zeta_n)$ in $E(f_1)$, $\zeta_n \to \infty$ as $n \to \infty$, such that

$$\frac{|\zeta_n| f_1'(\zeta_n)}{T(|\zeta_n|, f_1)} > d_1.$$

Thus we can choose $z_1$ in $E(f_1)$ arbitrarily large, and in particular such that $|z_1| > R$, with

$$|z_1| |f_1'(z_1)| > d_1 T(|z_1|, f_1). \quad (14)$$

Next, since $f(z) = (c_3 f_1(z) + c_4)Q(z)$ we have that

$$T(|z_1|, f) = T(|z_1|, (c_3 f_1 + c_4)Q) \leq T(|z_1| c_3 f_1 + c_4) + T(|z_1|, Q) \leq T(|z_1|, f_1) + O(\log |z_1|),$$

and so, if $R$ is large enough,

$$T(|z_1|, f_1) \geq T(|z_1|, f) - O(\log |z_1|) > \frac{1}{2} T(|z_1|, f), \quad (15)$$

since $f$ is a transcendental function. Then since $|G'(z)| = \frac{e}{16} |f_1'(z)|$, we have by (14) and (15) that

$$|G'(z_1)| > \frac{e}{16} \frac{d_1}{|z_1|} T(|z_1|, f_1) > \frac{d_2}{|z_1|} T(|z_1|, f), \quad (16)$$

where $d_2 = \frac{3}{2} d_1$.

We recall that $G(z) = 1 + \frac{f(z)}{16}$, and thus since $z_1 \in E(f_1)$, we have that $|G(z_1) - 1| = \frac{1}{16} < \frac{\epsilon}{2}$. Then by Lemma 2.1, $z_1$ lies in a component $C$ of the set $\{ z : |G(z) - G(z_1)| < \frac{1}{2} \epsilon \}$ such that $C$ is contained in $B(z_1, \delta|z_1|)$. Also, by Lemma 2.1, $C$ is mapped conformally onto $B(0, 1)$ by the function $H(z) = \frac{2}{\epsilon}(G(z) -$
If $Q$ is a rational function with a pole of multiplicity $p \geq 1$ at $\infty$, then we have that $\left|2\frac{Q(z)}{Q(z)}\right| = O(1)$ as $z \to \infty$. Since $z_2 \in B(z_1, \delta|z_1|)$, we have that $|z_1| \leq |z_1 - z_2| + |z_2| \leq \delta|z_1| + |z_2|$, which gives that $\left|\frac{z_2}{z_1}\right| \geq 1 - \delta$. Then by (17) and (18), we have that

$$\left|2\frac{f'(z_2)}{Q(z_2)}\right| \geq |z_2G'(z_2)| - \left|2\frac{Q'(z_2)}{Q(z_2)}\right|.$$  

(18)

Then since $d_3 = \frac{d_3}{484}A_0\delta(\infty, f)$ and $A_0$ is an absolute constant, we have that

$$|f'(z_2)| > c\delta(\infty, f)\left|\frac{Q(z_2)}{z_2}\right|T(\alpha|z_2|, f).$$

where $c$ is a positive constant depending only on $\alpha$.

4 Proof of Theorem 1.4

We need the following theorem.

Theorem 4.1 (Langley and Zheng, [1]) Let $f$ be a transcendental meromorphic function in the class $B$. If $Q$ is a rational function with a pole of multiplicity $p \geq 1$ at $\infty$, then

$$m(r, \frac{1}{f - Q}) = O(\log rT(r, f))$$  

(19)

as $r \to \infty$ outside a set of finite measure.

We note that the Nevanlinna counting function $N(r, \frac{1}{f - z})$ counts the poles of $\frac{1}{f - z}$, that is, the fixpoints of $f$. In [1] the following implication of Theorem 4.1 is noted, namely, that $N(r, \frac{1}{f - z})$ cannot satisfy $N(r, \frac{1}{f - z}) = o(T(r, f))$ as $r \to \infty$. We extend this to where $Q$ is a rational function such that $Q(\infty) = \infty$, and state the result as a corollary. We provide a proof for completeness.
Corollary 4.2 Let $f$ and $Q$ be as in the statement of Theorem 1.4. Then the Nevanlinna counting function $N(r, \frac{1}{T-f})$, of points where $f(z) = Q(z)$, cannot satisfy $N(r, \frac{1}{T-f}) = o(T(r, f))$ as $r \to \infty$. Also, for $\sigma > 0$, we may choose arbitrarily large $r$ such that there are at least $2r^\sigma$ Q-points $z_j$, such that $f(z_j) = Q(z_j)$, in the annulus $\frac{1}{2}r \leq |z| \leq r$.

**Proof** Since $Q$ is a rational function we have that

$$T(r, f - Q) \leq T(r, f) + T(r, Q) + O(1) = T(r, f) + O(\log r) + O(1),$$

and that,

$$T(r, f) = T(r, f - Q + Q) \leq T(r, f - Q) + T(r, Q) + O(1) = T(r, f - Q) + O(\log r) + O(1).$$

Thus

$$|T(r, f) - T(r, f - Q)| \leq O(\log r) + O(1), \quad (20)$$

and since, $T(r, \frac{1}{T-f}) = T(r, f - Q) + O(1)$ we have that

$$|T(r, f) - T(r, \frac{1}{T-Q}) - O(1)| \leq O(\log r) + O(1). \quad (21)$$

Then since $T(r, \frac{1}{T-f}) = N(r, \frac{1}{T-f}) + m(r, \frac{1}{T-f})$ we have that

$$|T(r, f) - T(r, \frac{1}{T-f}) - O(1)| \geq |T(r, f) - N(r, \frac{1}{T-f})| - |m(r, \frac{1}{T-f}) + O(1)|.$$ 

We may assume by (19) that $m(r, \frac{1}{T-f}) = O(\log r T(r, f))$ as $r \to \infty$, with $r \notin E$, for some set $E$ of finite measure. Then by (21) and Theorem 4.1 we have

$$|T(r, f) - N(r, \frac{1}{T-f})| \leq O(\log r) + O(1) + |m(r, \frac{1}{T-f}) + O(1)| = O(\log r) + O(1) + O(\log r T(r, f)), \quad r \notin E.$$ 

Then since $f$ is a transcendental function, we have that $N(r, \frac{1}{T-f})$ cannot satisfy $N(r, \frac{1}{T-f}) = o(T(r, f))$ as $r \to \infty$.

Next we show that all large $z_j$ such that $f(z_j) = Q(z_j)$ are simple zeros of $f - Q$. Let $z_0$ be large and suppose $f(z_0) = Q(z_0)$. Then since $f$ and $Q$ satisfy the hypotheses of Lemma 2.1 and Lemma 2.2, we have by (6) that

$$|z_0 f'(z_j)/f(z_j)| > R_1 = c \log |Q(z_0)|, \quad (22)$$

where $c$ is a positive constant. Also, since $Q$ is a rational function with a pole of multiplicity $p \geq 1$ at $\infty$, we have that $|Q'(z_0)/Q(z_0)| \leq \frac{p}{|z_0|}(1 + o(1))$. Then, by (22) and since $f(z_0) = Q(z_0)$, we have that

$$|f'(z_0)| \geq \frac{R_1}{|z_0|}|Q(z_0)| \geq \frac{p}{|z_0|}(1 + o(1))|Q(z_0)| \geq |Q'(z_0)|,$$

since we can choose $z_0$ so large that $p/R_1$ is very small. Then this gives that $f'(z_0) - Q'(z_0) \neq 0$ and so $z_0$ is a simple zero of $f - Q$.

Suppose now that $r_0 \geq 0$ is such that for $r \geq r_0$, there are less than $2r^\sigma$ points $z_j$, such that $f(z_j) = Q(z_j)$, in $\frac{1}{2} |z| \leq r$. Then for $m \in \mathbb{N}$ we have that

$$n \left( \frac{2^m r_0}{1 - Q} \right) < n \left( \frac{r_0}{1 - f} \right) + 2 \left( (2^m r_0)^\sigma + (2^{m-1} r_0)^\sigma + \ldots + (2r_0)^\sigma \right). \quad (23)$$
Let \( r \) be large, with, in particular, \( r > r_0 \). Then there exists \( m \in \mathbb{N} \) such that \( 2^{m-1}r_0 \leq r < 2^mr_0 \). Then by (23) we have

\[
\begin{align*}
\frac{n\left( r, \frac{1}{f - Q} \right)}{n\left( 2^m r_0, \frac{1}{f - Q} \right)} &\leq \frac{n\left( r_0, \frac{1}{f - Q} \right) + 2 \left[ (2^m r_0)^\sigma + (2^{m-1} r_0)^\sigma + \ldots + (2r_0)^\sigma \right]}{c_1 + 2r_0^\sigma \left[ \frac{2^\sigma (2^m - 1)}{2^\sigma - 1} \right]} \\
&\leq c_1 + c_2 r^\sigma,
\end{align*}
\]

where \( c_1 = n(r_0, \frac{1}{f - Q}) \) and \( c_2 \) is a positive constant. Then we have \( N(r, \frac{1}{f - Q}) = O(r^\sigma) \).

Next, by (19), we may choose \( r \leq r_1 \leq 2r \) such that \( m(r_1, \frac{1}{f - Q}) = O(\log r_1 T(r_1, f)) \). Then

\[
T(r_1, \frac{1}{f - Q}) = N(r_1, \frac{1}{f - Q}) + m(r_1, \frac{1}{f - Q}) = O(r_1^\sigma) + O(\log r_1 T(r_1, f)),
\]

and so,

\[
T(r, \frac{1}{f - Q}) \leq T(r_1, \frac{1}{f - Q}) = O(r^\sigma).
\]

Therefore we have that \( T(r, f) = O(r^\sigma) \), and thus,

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r} \leq \sigma.
\]

This is a contradiction since \( \rho(f) > \sigma \). Therefore, we may choose arbitrarily large \( r \) such that there are at least \( 2r^\sigma \) points \( z_j \) such that \( f(z_j) = Q(z_j) \) in the annulus \( \frac{1}{2}r \leq |z| \leq r \).

We now prove Theorem 1.4.

**Proof of Theorem 1.4** Let \( f \) and \( Q \) be as in the statement of Theorem 1.4. Choose \( \sigma \) with \( \rho(f) > \sigma > \mu \). Since \( \rho(f) > 0 \), we have that \( f \) is a transcendental function, and so by Corollary 4.2, we can choose arbitrarily large \( r \) such that \( f \) has at least \( 2r^\sigma \) points \( z_j \) such that \( f(z_j) = Q(z_j) \) in the annulus \( \frac{1}{2}r \leq |z| \leq r \).

Suppose \( \delta \) is a small positive constant. Let \( G(z) = f(z)/Q(z) \) be as defined in Lemma 2.1. Then for each \( z_j \) we have that \( G(z_j) = 1 \). Then by Lemma 2.1, we have that to each \( z_j \) there corresponds a component \( C_j \) of the set \( \{ z : |G(z) - 1| < \frac{1}{2} \epsilon \} \), and that each \( C_j \) is contained in \( B(z_j, \delta |z_j|) \). Then choosing \( \delta \) small enough gives that each \( C_j \) lies in the annulus \( \frac{1}{2}r \leq |z| \leq 2r \). These \( C_j \) are disjoint simple islands since, by Lemma 2.1, they are mapped conformally onto \( B(0,1) \) by the function \( H(z) = \frac{2}{\epsilon} (G(z) - G(z)) \).

Let \( h_j : B(0,1) \to C_j \) be the inverse function of \( H \). Then \( h_j \) is a univalent function on \( B(0,1) \) and by the Koebe distortion theorem we have that for \( 0 < r_0 < 1 \),

\[
c_1 |h_j'(0)| \leq \max_{|w| \leq r_0} |h_j'(w)| \leq c_2 |h_j'(0)|,
\]

where \( c_1 \) and \( c_2 \) are positive constants, depending only on \( r_0 \).

In particular, choose \( r_0 = \frac{1}{2} \) and let \( \tilde{C}_j = h_j(B(0, \frac{1}{2})) \). Then \( |h_j'(w)| \geq c_1 |h_j'(0)| \) for \( |w| \leq \frac{1}{2} \). Then since \( h_j'(w) = 1/H'(z) \) where \( z = h_j(w) \), and in particular since \( h_j'(0) = 1/H'(z_j) = \epsilon/2G'(z_j) \) we have that

\[
|h_j'(w)| \geq c/|G'(z_j)|, \quad \text{for} \ |w| \leq \frac{1}{2},
\] (24)
where, from here on, $c$ denotes a positive constant, not necessarily the same at each occurrence, but not depending on $r$ or $\sigma$.

Next, since the area of the annulus $\frac{1}{4}r \leq |z| \leq 2r$ is $cr^2$, and since there are at least $2r^\sigma$ disjoint components $\hat{C}_j$ in the annulus, we have that at least $r^\sigma$ of these $\hat{C}_j$ have area at most $cr^{2-\sigma}$. Then, for these $z_j$, we have

$$z_j \frac{f'(z_j)}{f(z_j)} = z_j \frac{G'(z_j)}{G(z_j)} + z_j \frac{Q'(z_j)}{Q(z_j)},$$

and since $f(z_j) = Q(z_j)$ and $G(z_j) = 1$, we have that

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq \left| z_j G'(z_j) \right| - \left| z_j \frac{Q'(z_j)}{Q(z_j)} \right|. \quad (25)$$

Now since $Q$ is a rational function with a pole of multiplicity $p \geq 1$ at $\infty$, we have that $\left| z_j \frac{Q'(z_j)}{Q(z_j)} \right| = O(1)$ as $z \to \infty$. Therefore, (25) gives that

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq \left| z_j G'(z_j) \right| - O(1). \quad (26)$$

Now by [7, p.4] and (24) we have that

$$\text{area of } \hat{C}_j = \iint_{B(0, \frac{1}{2})} |h'(w)|^2 du \, dv \geq c/|G'(z_j)|^2.$$

And so, since the area of $\hat{C}_j$ is at most $cr^{2-\sigma}$, we have that $1/|G'(z_j)|^2 \leq cr^{2-\sigma}$, which gives $|G'(z_j)| \geq cr^{\sigma/2-1}$. Then by (26) and since $r \geq |z_j|$, and $z_j$ is large, and $p \geq 1$, we have

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq cr^{\sigma/2} - O(1) \geq c|z_j|^\sigma/2.$$

Then since $|Q(z_j)| > c|z_j|^p$, we have that

$$|f'(z_j)| > c|z_j|^\sigma/2 + p - 1 \geq |z_j|^\mu/2 + p - 1.$$

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