A Note on Belyi's Theorem for Klein Surfaces

by

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Abstract. Singerman and the first named author have recently developed a real Belyi theory, leaving open a particular case in the proof of Belyi's theorem for Klein surfaces. We answer their question affirmatively by a descent argument which turns out to extend to a much more general context.

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Introduction

Compact Klein surfaces correspond to smooth projective algebraic curves over $\mathbb{R}$ in the same way as compact Riemann surfaces correspond to smooth projective algebraic curves over $\mathbb{C}$. This well-known fact was the starting point for David Singerman and the first named author to generalize the famous Belyi theory for Riemann surfaces (see e.g. [JS] or [Wo]) to Klein surfaces (see [KS]).

Let S be a compact connected Klein surface. A Belyi map on S is a meromorphic function β from S to the compactified closed upper half plane Δ such that the complex double cover $\beta^c : S^c \to \Delta^c = \hat{\mathbb{C}}$ has at most three critical values on each component of S^c. By [KS] there is a Belyi map if and only if S allows uniformizations of a particular type (see conditions (iii) – (v) in the introduction of *loc. cit.*) or if and only if S carries an embedded graph, called map, of a certain type (see condition (vi) in *loc. cit.*).

Furthermore, if the curve over $\mathbb{R}$ corresponding to S can be defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$ then S admits a Belyi map. The converse is proved in [KS] as well except in the case where S is non-orientable with empty boundary and the genus of S^c is at least 2. The object of this note is to show the converse in general, answering Question 2.7 in *loc. cit.* affirmatively. The proof, given in section 1, relies on a descent from $\mathbb{C}$ to $\bar{\mathbb{Q}}$ of Galois descent data. This type of argument generalises naturally to varieties with finite automorphism groups in a more general context (see section 2).

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1. Descending the field of definition

Let S be a compact connected Klein surface. We recall that the field $K(S)$ of meromorphic functions from S to Δ is a real function field in one variable, which corresponds to a smooth projective curve X over $\mathbb{R}$. The ring of meromorphic functions on the Riemann surface S^c can be identified with $K(S) \otimes_{\mathbb{R}} \mathbb{C}$, and $X_{\mathbb{C}}$ is the smooth projective curve over $\mathbb{C}$ corresponding to this ring. The resulting isomorphism of Riemann surfaces $S^c \cong X(\mathbb{C})$ is compatible with complex conjugation on both sides. In particular the boundary of S can be identified with the set $X(\mathbb{R})$ of real points on X. We note that $X_{\mathbb{C}}$ is reducible if and only if S is the Klein surface associated with a Riemann surface, i.e. if and only if S is orientable without boundary. Hence S is non-orientable without boundary if and only if X is geometrically irreducible without real points.

We now assume that S admits a Belyi map $S \to \Delta$. The "converse" of the classical Belyi theorem (see e.g. [JS]) implies that $X_{\mathbb{C}}$ can be defined over $\bar{\mathbb{Q}}$. In order to show that X can be defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$ it therefore suffices to prove Proposition 1.

Proposition 1. Let X be a smooth projective curve over $\mathbb{R}$. If $X_{\mathbb{C}}$ can be defined over $\bar{\mathbb{Q}}$ then X can be defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$.

Proof. Clearly X may be assumed connected. If $X_{\mathbb{C}}$ is reducible the assertion is easy because then X carries the structure of a curve over $\mathbb{C}$ so that $X \times_{\mathbb{R}} \mathbb{C}$ is the disjoint union of X and its complex conjugate X^σ. Hence we may also assume that X is

geometrically irreducible.
If the genus of $X_{\mathbb{C}}$ is at most 1 the assertion is proved in Example 2.8 of [KS], but let us briefly recall the argument. In the case $g(X_{\mathbb{C}}) = 0$ the function field $K(X)$ of X is isomorphic to $\mathbb{R}(t)$ or to the field of fractions of the integral domain $\mathbb{R}[s,t]/(s^2+t^2+1)$; thus, X can in fact be defined over $\mathbb{Q}$. If $g(X_{\mathbb{C}}) = 1$ the theory of real elliptic curves implies that $K(X)$ is isomorphic to the field of fractions of an integral domain of the form

$$\mathbb{R}[s,t]/(t^2 \pm (1 \pm s^2)(1 \pm \lambda s^2))$$

where $\lambda \in \mathbb{R}$ denotes the Legendre modulus of X. Here λ is algebraic over $\mathbb{Q}(j)$ where j denotes the j-invariant of the elliptic curve $X_{\mathbb{C}}$. But j is an algebraic number because $X_{\mathbb{C}}$ can be defined over $\bar{\mathbb{Q}}$. Thus X can be defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$.
We now assume that $g(X_{\mathbb{C}}) \geq 2$. By assumption there exists a curve Y over $\bar{\mathbb{Q}}$ and an isomorphism $X \times_{\mathbb{R}} \mathbb{C} \cong Y \times_{\bar{\mathbb{Q}}} \mathbb{C}$ over $\mathbb{C}$. Via this isomorphism, complex conjugation acting on the second factor of the left-hand side induces an $\mathbb{R}$-automorphism τ of the right-hand side. Since $Y \times_{\bar{\mathbb{Q}}} \mathbb{C} = Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \mathbb{R}$ we may view $\mathrm{Aut}_{\bar{\mathbb{Q}} \cap \mathbb{R}}(Y)$ as a subgroup of $\mathrm{Aut}_{\mathbb{R}}(Y \times_{\bar{\mathbb{Q}}} \mathbb{C})$. If we can show that τ lies in this subgroup then the subfield of $K(Y)$ fixed by the automorphism induced by τ is a function field in one variable over $\bar{\mathbb{Q}} \cap \mathbb{R}$ which corresponds to a smooth projective curve Z over $\bar{\mathbb{Q}} \cap \mathbb{R}$ such that $Z \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \mathbb{R} \cong X$. Hence it suffices to prove Lemma 1. □

Lemma 1. Let Y be a connected smooth projective curve over $\bar{\mathbb{Q}}$ of genus at least 2 and let $Y_{\mathbb{C}} = Y \times_{\bar{\mathbb{Q}}} \mathbb{C}$. Then the canonical map

$$\mathrm{Aut}_{\bar{\mathbb{Q}} \cap \mathbb{R}}(Y) \to \mathrm{Aut}_{\mathbb{R}}(Y_{\mathbb{C}})$$

is bijective.

Proof. Let σ denote the complex conjugation acting on the second factor of $Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}$ and of $Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C}$. Then $\mathrm{Aut}_{\mathbb{R}}(Y_{\mathbb{C}})$ can be identified with the subgroup of those elements of $\mathrm{Aut}_{\mathbb{C}}(Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C})$ which commute with σ. As we have a similar description for $\mathrm{Aut}_{\bar{\mathbb{Q}} \cap \mathbb{R}}(Y)$ it suffices to show that the canonical map

$$\mathrm{Aut}_{\bar{\mathbb{Q}}}(Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}) \to \mathrm{Aut}_{\mathbb{C}}(Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C})$$

is bijective. We may identify the $\mathbb{C}$-scheme $Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C}$ with the disjoint union of $Y \times_{\bar{\mathbb{Q}}} \mathbb{C}$ and of its complex conjugate. Similarly we may identify the $\bar{\mathbb{Q}}$-scheme $Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}$ with the disjoint union of Y and of its complex conjugate Y^{σ}. Hence we get a decomposition

$$\mathrm{Aut}_{\bar{\mathbb{Q}}}(Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}) = \mathrm{Aut}_{\bar{\mathbb{Q}}}(Y) \times \mathrm{Aut}_{\bar{\mathbb{Q}}}(Y^{\sigma}) \ \sqcup \ \mathrm{Isom}_{\bar{\mathbb{Q}}}(Y, Y^{\sigma}) \times \mathrm{Isom}_{\bar{\mathbb{Q}}}(Y^{\sigma}, Y)$$

and a similar decomposition for $\mathrm{Aut}_{\mathbb{C}}(Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C})$. Since Y^{σ} is a connected smooth projective curve of genus at least 2 as well, Lemma 1 follows from Lemma 2. $\square$

Lemma 2. Let X, Y be connected smooth projective curves over $\bar{\mathbb{Q}}$ of genus at least 2. Then the canonical map

$$\mathrm{Isom}_{\bar{\mathbb{Q}}}(X, Y) \to \mathrm{Isom}_{\mathbb{C}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$$

is bijective.

See [Kö], Lemma 1.12 for an elementary proof of this lemma using the language of function fields.

2. A broader context

In this section we give an axiomatic generalization of (the main case of) Proposition 1 built on the observation that the key to proving Lemma 2 is the finiteness of the automorphism group, hoping this will more clearly reveal the conceptual nature of the argument. We begin by setting up the context.

Let K/k and l/k be extensions of fields. We assume that k is algebraically closed in K and that l/k is a finite Galois extension with Galois group G. Then $L := K \otimes_k l$ is a field as well, l is algebraically closed in L (cf. Lemma 1.1 in [KS]) and L/K is a finite Galois extension with Galois group G again. The following diagram visualizes the situation:

$$\begin{array}{ccc} K & \overset{G}{\text{———}} & L \\ \text{closed}\,\Big| & & \Big|\,\text{closed} \\ k & \overset{G}{\text{———}} & l \end{array}$$

We recover the situation considered in Section 1 when we put $k = \bar{\mathbb{Q}} \cap \mathbb{R}, l = \bar{\mathbb{Q}}$ and $K = \mathbb{R}$. The following proposition generalizes and refines Proposition 1 (if the genus of $X_{\mathbb{C}}$ is at least 2).

Proposition 2. Let X and Y be projective schemes over the fields K and l, respectively, and let $\alpha : X_L \xrightarrow{\sim} Y_L$ be an isomorphism of L-schemes. We assume that $\mathrm{Aut}_{\bar{l}}(Y \times_l \bar{l})$ is finite. Then there exists a projective scheme Z over k, a K-isomorphism $\beta : Z_K \xrightarrow{\sim} X$, and an l-isomorphism $\gamma : Z_l \xrightarrow{\sim} Y$ such that the following diagram commutes:

$$\begin{array}{ccc} (Z_K)_L & \xrightarrow{\text{can}} & (Z_l)_L \\ \beta_L \downarrow & & \downarrow \gamma_L \\ X_L & \xrightarrow{\alpha} & Y_L \end{array}$$

Proof. Via the isomorphism α we obtain an action

$$\tau : G \to \mathrm{Aut}_K(Y \times_l L)$$

of G on $Y \times_l L = Y \times_k K$ which is compatible with the action of G on L. By Lemma 3 below τ is induced by an action $\tau_\circ : G \to \mathrm{Aut}_k(Y)$ which is compatible with the given action of G on l. We call any l-scheme equipped with such an action a *G-scheme over l*. By Galois descent (see Lemma 4 below) it follows that $Y \cong Z \times_k l$ for some projective scheme Z over k such that $\tau_\circ$ corresponds to the G-action on the second factor. Then Z satisfies the required conditions. □

Lemma 3. Let Y be a projective scheme over l such that $\mathrm{Aut}_{\bar{l}}(Y \times_l \bar{l})$ is finite. Then the canonical monomorphism

$$\mathrm{Aut}_k(Y) \to \mathrm{Aut}_K(Y \times_k K)$$

is bijective.

Proof. Since l/k is finite Y is projective over k as well. Then, by Theorem (3.7) in [MO], the functor $T \mapsto \mathrm{Aut}_T(Y \times_k T)$ from the category of schemes over k to the category of groups is representable by a group scheme H which is locally of finite type over k. As in the proof of Lemma 1 we may identify the $\bar{l}$-scheme $Y \times_k \bar{l}$ with the disjoint union of $Y_{\bar{l}} = Y \times_l \bar{l}$ and its G-conjugates. Since these all have the same number of $\bar{l}$-automorphisms, the finiteness of $\mathrm{Aut}_{\bar{l}}(Y_{\bar{l}})$ implies the finiteness of $\mathrm{Aut}_l(Y \times_k \bar{l}) = H(\bar{l})$. Thus the group scheme H is in fact finite over k. Now a K-automorphism σ of $Y \times_k K$ is by definition a K-valued point of H. Since the residue fields at all points of H are finite extensions of k and since k is algebraically closed in K every K-valued point of H is already k-valued. In particular σ is defined over k, as was to be shown. □

Lemma 4 (Galois descent). The functor $Z \mapsto Z \times_k l$ induces an equivalence between the category of (quasi-)projective schemes over k and the category of (quasi-)projective G-schemes over l.

This is well-known, see for example [Mi], Proposition 1.8. For a given (quasi-)projective G-scheme Y over l the associated k-scheme is the quotient $Z = Y/G$, which exists as Y is (quasi-)projective over k. Since the projection $Y \to Z$ is finite etale and surjective Z is (quasi-)projective over k.

Remark. Proposition 2 also holds if the field extension l/k is only assumed algebraic and separable instead of finite Galois.
Indeed, since the isomorphism α involves only finitely many elements of L we may assume that l/k is finite (and separable). Let n/k be the normal closure of l/k and let $N := K \otimes_k n$. We put $H := \mathrm{Gal}(n/l)$ and $G := \mathrm{Gal}(n/k)$. Then as in the proof of Proposition 2 we obtain an action of G on $Y \times_l N = (Y \times_l n) \times_k K$ which is compatible with the given action of G on n and which extends the obvious action of H on $Y \times_l N$. Replacing l with n and Y with $Y \times_l n$ in Lemma 3, we conclude that this action is induced by an action of G on $Y \times_l n$ which is compatible with the given action of G on n and which extends the obvious action of H on $Y \times_l n$. Now an obvious generalization of Lemma 4 finishes the proof.

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