

# A Note on Belyi's Theorem for Klein Surfaces

by

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**Abstract.** Singerman and the first named author have recently developed a real Belyi theory, leaving open a particular case in the proof of Belyi's theorem for Klein surfaces. We answer their question affirmatively by a descent argument which turns out to extend to a much more general context.

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## Introduction

Compact Klein surfaces correspond to smooth projective algebraic curves over  $\mathbb{R}$  in the same way as compact Riemann surfaces correspond to smooth projective algebraic curves over  $\mathbb{C}$ . This well-known fact was the starting point for David Singerman and the first named author to generalize the famous Belyi theory for Riemann surfaces (see e.g. [JS] or [Wo]) to Klein surfaces (see [KS]).

Let  $S$  be a compact connected Klein surface. A Belyi map on  $S$  is a meromorphic function  $\beta$  from  $S$  to the compactified closed upper half plane  $\Delta$  such that the complex double cover  $\beta^c : S^c \rightarrow \Delta^c = \hat{\mathbb{C}}$  has at most three critical values on each component of  $S^c$ . By [KS] there is a Belyi map if and only if  $S$  allows uniformizations of a particular type (see conditions (iii) – (v) in the introduction of *loc. cit.*) or if and only if  $S$  carries an embedded graph, called map, of a certain type (see condition (vi) in *loc. cit.*).

Furthermore, if the curve over  $\mathbb{R}$  corresponding to  $S$  can be defined over  $\bar{\mathbb{Q}} \cap \mathbb{R}$  then  $S$  admits a Belyi map. The converse is proved in [KS] as well except in the case where  $S$  is non-orientable with empty boundary and the genus of  $S^c$  is at least 2. The object of this note is to show the converse in general, answering Question 2.7 in *loc. cit.* affirmatively. The proof, given in section 1, relies on a descent from  $\mathbb{C}$  to  $\bar{\mathbb{Q}}$  of Galois descent data. This type of argument generalises naturally to varieties with finite automorphism groups in a more general context (see section 2).

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## 1. Descending the field of definition

Let  $S$  be a compact connected Klein surface. We recall that the field  $K(S)$  of meromorphic functions from  $S$  to  $\Delta$  is a real function field in one variable, which corresponds to a smooth projective curve  $X$  over  $\mathbb{R}$ . The ring of meromorphic functions on the Riemann surface  $S^c$  can be identified with  $K(S) \otimes_{\mathbb{R}} \mathbb{C}$ , and  $X_{\mathbb{C}}$  is the smooth projective curve over  $\mathbb{C}$  corresponding to this ring. The resulting isomorphism of Riemann surfaces  $S^c \cong X(\mathbb{C})$  is compatible with complex conjugation on both sides. In particular the boundary of  $S$  can be identified with the set  $X(\mathbb{R})$  of real points on  $X$ . We note that  $X_{\mathbb{C}}$  is reducible if and only if  $S$  is the Klein surface associated with a Riemann surface, i.e. if and only if  $S$  is orientable without boundary. Hence  $S$  is non-orientable without boundary if and only if  $X$  is geometrically irreducible without real points.

We now assume that  $S$  admits a Belyi map  $S \rightarrow \Delta$ . The “converse” of the classical Belyi theorem (see e.g. [JS]) implies that  $X_{\mathbb{C}}$  can be defined over  $\bar{\mathbb{Q}}$ . In order to show that  $X$  can be defined over  $\bar{\mathbb{Q}} \cap \mathbb{R}$  it therefore suffices to prove Proposition 1.

**Proposition 1.** Let  $X$  be a smooth projective curve over  $\mathbb{R}$ . If  $X_{\mathbb{C}}$  can be defined over  $\bar{\mathbb{Q}}$  then  $X$  can be defined over  $\bar{\mathbb{Q}} \cap \mathbb{R}$ .

*Proof.* Clearly  $X$  may be assumed connected. If  $X_{\mathbb{C}}$  is reducible the assertion is easy because then  $X$  carries the structure of a curve over  $\mathbb{C}$  so that  $X \times_{\mathbb{R}} \mathbb{C}$  is the disjoint union of  $X$  and its complex conjugate  $X^{\sigma}$ . Hence we may also assume that  $X$  is

geometrically irreducible.

If the genus of  $X_{\mathbb{C}}$  is at most 1 the assertion is proved in Example 2.8 of [KS], but let us briefly recall the argument. In the case  $g(X_{\mathbb{C}}) = 0$  the function field  $K(X)$  of  $X$  is isomorphic to  $\mathbb{R}(t)$  or to the field of fractions of the integral domain  $\mathbb{R}[s, t]/(s^2 + t^2 + 1)$ ; thus,  $X$  can in fact be defined over  $\mathbb{Q}$ . If  $g(X_{\mathbb{C}}) = 1$  the theory of real elliptic curves implies that  $K(X)$  is isomorphic to the field of fractions of an integral domain of the form

$$\mathbb{R}[s, t]/(t^2 \pm (1 \pm s^2)(1 \pm \lambda s^2))$$

where  $\lambda \in \mathbb{R}$  denotes the Legendre modulus of  $X$ . Here  $\lambda$  is algebraic over  $\mathbb{Q}(j)$  where  $j$  denotes the  $j$ -invariant of the elliptic curve  $X_{\mathbb{C}}$ . But  $j$  is an algebraic number because  $X_{\mathbb{C}}$  can be defined over  $\bar{\mathbb{Q}}$ . Thus  $X$  can be defined over  $\bar{\mathbb{Q}} \cap \mathbb{R}$ .

We now assume that  $g(X_{\mathbb{C}}) \geq 2$ . By assumption there exists a curve  $Y$  over  $\bar{\mathbb{Q}}$  and an isomorphism  $X \times_{\mathbb{R}} \mathbb{C} \cong Y \times_{\bar{\mathbb{Q}}} \mathbb{C}$  over  $\mathbb{C}$ . Via this isomorphism, complex conjugation acting on the second factor of the left-hand side induces an  $\mathbb{R}$ -automorphism  $\tau$  of the right-hand side. Since  $Y \times_{\bar{\mathbb{Q}}} \mathbb{C} = Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \mathbb{R}$  we may view  $\text{Aut}_{\bar{\mathbb{Q}} \cap \mathbb{R}}(Y)$  as a subgroup of  $\text{Aut}_{\mathbb{R}}(Y \times_{\bar{\mathbb{Q}}} \mathbb{C})$ . If we can show that  $\tau$  lies in this subgroup then the subfield of  $K(Y)$  fixed by the automorphism induced by  $\tau$  is a function field in one variable over  $\bar{\mathbb{Q}} \cap \mathbb{R}$  which corresponds to a smooth projective curve  $Z$  over  $\bar{\mathbb{Q}} \cap \mathbb{R}$  such that  $Z \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \mathbb{R} \cong X$ . Hence it suffices to prove Lemma 1.  $\square$

**Lemma 1.** Let  $Y$  be a connected smooth projective curve over  $\bar{\mathbb{Q}}$  of genus at least 2 and let  $Y_{\mathbb{C}} = Y \times_{\bar{\mathbb{Q}}} \mathbb{C}$ . Then the canonical map

$$\text{Aut}_{\bar{\mathbb{Q}} \cap \mathbb{R}}(Y) \rightarrow \text{Aut}_{\mathbb{R}}(Y_{\mathbb{C}})$$

is bijective.

*Proof.* Let  $\sigma$  denote the complex conjugation acting on the second factor of  $Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}$  and of  $Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C}$ . Then  $\text{Aut}_{\mathbb{R}}(Y_{\mathbb{C}})$  can be identified with the subgroup of those elements of  $\text{Aut}_{\mathbb{C}}(Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C})$  which commute with  $\sigma$ . As we have a similar description for  $\text{Aut}_{\bar{\mathbb{Q}} \cap \mathbb{R}}(Y)$  it suffices to show that the canonical map

$$\text{Aut}_{\bar{\mathbb{Q}}}(Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}) \rightarrow \text{Aut}_{\mathbb{C}}(Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C})$$

is bijective. We may identify the  $\mathbb{C}$ -scheme  $Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C}$  with the disjoint union of  $Y \times_{\bar{\mathbb{Q}}} \mathbb{C}$  and of its complex conjugate. Similarly we may identify the  $\bar{\mathbb{Q}}$ -scheme  $Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}$  with the disjoint union of  $Y$  and of its complex conjugate  $Y^{\sigma}$ . Hence we get a decomposition

$$\text{Aut}_{\bar{\mathbb{Q}}}(Y \times_{\bar{\mathbb{Q}} \cap \mathbb{R}} \bar{\mathbb{Q}}) = \text{Aut}_{\bar{\mathbb{Q}}}(Y) \times \text{Aut}_{\bar{\mathbb{Q}}}(Y^{\sigma}) \sqcup \text{Isom}_{\bar{\mathbb{Q}}}(Y, Y^{\sigma}) \times \text{Isom}_{\bar{\mathbb{Q}}}(Y^{\sigma}, Y)$$

and a similar decomposition for  $\text{Aut}_{\mathbb{C}}(Y_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C})$ . Since  $Y^{\sigma}$  is a connected smooth projective curve of genus at least 2 as well, Lemma 1 follows from Lemma 2.  $\square$

**Lemma 2.** Let  $X, Y$  be connected smooth projective curves over  $\bar{\mathbb{Q}}$  of genus at least 2. Then the canonical map

$$\text{Isom}_{\bar{\mathbb{Q}}}(X, Y) \rightarrow \text{Isom}_{\mathbb{C}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$$

is bijective.

See [Kö], Lemma 1.12 for an elementary proof of this lemma using the language of function fields.

## 2. A broader context

In this section we give an axiomatic generalization of (the main case of) Proposition 1 built on the observation that the key to proving Lemma 2 is the finiteness of the automorphism group, hoping this will more clearly reveal the conceptual nature of the argument. We begin by setting up the context.

Let  $K/k$  and  $l/k$  be extensions of fields. We assume that  $k$  is algebraically closed in  $K$  and that  $l/k$  is a finite Galois extension with Galois group  $G$ . Then  $L := K \otimes_k l$  is a field as well,  $l$  is algebraically closed in  $L$  (cf. Lemma 1.1 in [KS]) and  $L/K$  is a finite Galois extension with Galois group  $G$  again. The following diagram visualizes the situation:

$$\begin{array}{ccc} K & \xrightarrow{G} & L \\ \text{closed} \downarrow & & \downarrow \text{closed} \\ k & \xrightarrow{G} & l \end{array}$$

We recover the situation considered in Section 1 when we put  $k = \bar{\mathbb{Q}} \cap \mathbb{R}, l = \bar{\mathbb{Q}}$  and  $K = \mathbb{R}$ . The following proposition generalizes and refines Proposition 1 (if the genus of  $X_{\mathbb{C}}$  is at least 2).

**Proposition 2.** Let  $X$  and  $Y$  be projective schemes over the fields  $K$  and  $l$ , respectively, and let  $\alpha : X_L \xrightarrow{\sim} Y_L$  be an isomorphism of  $L$ -schemes. We assume that  $\text{Aut}_{\bar{l}}(Y \times_l \bar{l})$  is finite. Then there exists a projective scheme  $Z$  over  $k$ , a  $K$ -isomorphism  $\beta : Z_K \xrightarrow{\sim} X$ , and an  $l$ -isomorphism  $\gamma : Z_l \xrightarrow{\sim} Y$  such that the

following diagram commutes:

$$\begin{array}{ccc} (Z_K)_L & \xrightarrow{\text{can}} & (Z_l)_L \\ \beta_L \downarrow & & \downarrow \gamma_L \\ X_L & \xrightarrow{\alpha} & Y_L \end{array}$$

*Proof.* Via the isomorphism  $\alpha$  we obtain an action

$$\tau : G \rightarrow \text{Aut}_K(Y \times_l L)$$

of  $G$  on  $Y \times_l L = Y \times_k K$  which is compatible with the action of  $G$  on  $L$ . By Lemma 3 below  $\tau$  is induced by an action  $\tau_\circ : G \rightarrow \text{Aut}_k(Y)$  which is compatible with the given action of  $G$  on  $l$ . We call any  $l$ -scheme equipped with such an action a  $G$ -scheme over  $l$ . By Galois descent (see Lemma 4 below) it follows that  $Y \cong Z \times_k l$  for some projective scheme  $Z$  over  $k$  such that  $\tau_\circ$  corresponds to the  $G$ -action on the second factor. Then  $Z$  satisfies the required conditions.  $\square$

**Lemma 3.** Let  $Y$  be a projective scheme over  $l$  such that  $\text{Aut}_{\bar{l}}(Y \times_l \bar{l})$  is finite. Then the canonical monomorphism

$$\text{Aut}_k(Y) \rightarrow \text{Aut}_K(Y \times_k K)$$

is bijective.

*Proof.* Since  $l/k$  is finite  $Y$  is projective over  $k$  as well. Then, by Theorem (3.7) in [MO], the functor  $T \mapsto \text{Aut}_T(Y \times_k T)$  from the category of schemes over  $k$  to the category of groups is representable by a group scheme  $H$  which is locally of finite type over  $k$ . As in the proof of Lemma 1 we may identify the  $\bar{l}$ -scheme  $Y \times_k \bar{l}$  with the disjoint union of  $Y_{\bar{l}} = Y \times_l \bar{l}$  and its  $G$ -conjugates. Since these all have the same number of  $\bar{l}$ -automorphisms, the finiteness of  $\text{Aut}_{\bar{l}}(Y_{\bar{l}})$  implies the finiteness of  $\text{Aut}_l(Y \times_k \bar{l}) = H(\bar{l})$ . Thus the group scheme  $H$  is in fact finite over  $k$ . Now a  $K$ -automorphism  $\sigma$  of  $Y \times_k K$  is by definition a  $K$ -valued point of  $H$ . Since the residue fields at all points of  $H$  are finite extensions of  $k$  and since  $k$  is algebraically closed in  $K$  every  $K$ -valued point of  $H$  is already  $k$ -valued. In particular  $\sigma$  is defined over  $k$ , as was to be shown.  $\square$

**Lemma 4 (Galois descent).** The functor  $Z \mapsto Z \times_k l$  induces an equivalence between the category of (quasi-)projective schemes over  $k$  and the category of (quasi-)projective  $G$ -schemes over  $l$ .

This is well-known, see for example [Mi], Proposition 1.8. For a given (quasi-)projective  $G$ -scheme  $Y$  over  $l$  the associated  $k$ -scheme is the quotient  $Z = Y/G$ , which exists as  $Y$  is (quasi-)projective over  $k$ . Since the projection  $Y \rightarrow Z$  is finite étale and surjective  $Z$  is (quasi-)projective over  $k$ .

**Remark.** Proposition 2 also holds if the field extension  $l/k$  is only assumed algebraic and separable instead of finite Galois.

Indeed, since the isomorphism  $\alpha$  involves only finitely many elements of  $L$  we may assume that  $l/k$  is finite (and separable). Let  $n/k$  be the normal closure of  $l/k$  and let  $N := K \otimes_k n$ . We put  $H := \text{Gal}(n/l)$  and  $G := \text{Gal}(n/k)$ . Then as in the proof of Proposition 2 we obtain an action of  $G$  on  $Y \times_l N = (Y \times_l n) \times_k K$  which is compatible with the given action of  $G$  on  $n$  and which extends the obvious action of  $H$  on  $Y \times_l N$ . Replacing  $l$  with  $n$  and  $Y$  with  $Y \times_l n$  in Lemma 3, we conclude that this action is induced by an action of  $G$  on  $Y \times_l n$  which is compatible with the given action of  $G$  on  $n$  and which extends the obvious action of  $H$  on  $Y \times_l n$ . Now an obvious generalization of Lemma 4 finishes the proof.

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