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EXISTENCE OF SOLUTIONS FOR NONLINEAR MIXED TYPE INTEGRODIFFERENTIAL EQUATION OF SECOND ORDER

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Abstract. In this paper, we investigate the existence of solutions for nonlinear mixed Volterra-Fredholm integrodifferential equation of second order with nonlocal conditions in Banach spaces. Our analysis is based on Leray-Schauder alternative, rely on a priori bounds of solutions and the inequality established by B. G. Pachpatte.

1 Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $B = C([t_0, t_0 + \beta], X)$ be the Banach space of all continuous functions from $[t_0, t_0 + \beta]$ into X endowed with supremum norm

$$\|x\|_B = \sup\{\|x(t)\| : t \in [t_0, t_0 + \beta]\}.$$

Motivated by the work of [6], in this paper we consider the following nonlinear mixed Volterra-Fredholm integrodifferential equation of the form:

$$\begin{aligned} x''(t) + Ax(t) = f(t, x(t), &\int_{t_0}^{t} a(t,s)k(s,x(s))ds, \\ &\int_{t_0}^{t_0+\beta} b(t,s)h(s,x(s))ds), \quad t \in [t_0, t_0+\beta] \end{aligned} \tag{1.1}$$

$$x(t_0) + g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0, \quad x'(0) = \eta, \tag{1.2}$$

where $-A$ is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in Banach space X, $0 \le t_0 < t_1 < t_2 < \cdots < t_p \le \beta$, $f : [t_0, t_0 + \beta] \times X \times X \times X \to X$, $k,\ h : [t_0, t_0 + \beta] \times X \to X$, $g(t_1, t_2, \cdots, t_p, \cdot) : C([t_0, t_0 + \beta], X) \to X$, $a,\ \ b : [t_0, t_0 + \beta] \times [t_0, t_0 + \beta] \to R$ are functions and x_0 is a given element of X.

The nonlocal condition, which is a generalization of the classical initial condition, was motivated by physical problems. The problem of existence of solutions of

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evolution equation with nonlocal conditions in Banach space was first studied by [2] and he investigated the existence and uniqueness of mild, strong and classical solutions of the nonlocal Cauchy problem. As indicated in [2, 3] and references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be applied in physics with better effect than the classical condition $y(0) = y_0$. For example, in [3], the author used

$$g(y) = \sum_{i=1^p} c_i y(t_i), \tag{1.3}$$

where $c_i, i = 1, 2, \cdots, p$ and $0 < t_1 < t_2 < \cdots \leq b$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, the equation (1.3) allows the additional measurements at $t_i, i = 1, 2, \cdots, p$. The study of differential and integrodifferential equations in abstract spaces with nonlocal condition has received much attention in recent years. We refer to the papers [4, 6, 7, 8, 9, 12, 13, 14] and the references cited therein.

The objective of the present paper is to study the global existence of solutions of the equations (1.1)–(1.2).

The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions and the inequality established by B. G. Pachpatte. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

The paper is organized as follows. In section 2, we present the preliminaries and the statement of our main result. Section 3 deals with proof of Theorem.

2 Preliminaries and Main Result

We give the following preliminaries and hypotheses used in our subsequent discussion.

In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert into first order systems. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family. We only mention a few results and notations needed to establish our results. A one parameter family $\{C(t) : t \in R\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

(a) $C(0) = I$ (I is the identity operator);

(b) $C(t)x$ is strongly continuous in t on R for each fixed $x \in X$;

(c) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t,\ s \in R$.

If $\{C(t) : t \in R\}$ is a strongly continuous cosine family in X, then $\{S(t) : t \in R\}$, associated to the given strongly continuous cosine family, is defined by

$$S(t)x = \int_0^t C(s)xds, \quad x \in X, \quad t \in R.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in R\}$ is defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(R, X)\}$.

Definition 1. *Let $f \in L^1(t_0, t_0 + \beta; X)$. The function $x \in B$ given by*

$$x(t) = C(t-t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))) + S(t-t_0)\eta + \int_{t_0}^t S(t-s)f(s, x(s),$$
$$\int_{t_0}^s a(s,\tau)k(\tau, x(\tau))d\tau, \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, x(\tau))d\tau)ds, \quad t \in [t_0, t_0+\beta], \tag{2.1}$$

is called mild solution of the initial value problem (1.1)–(1.2).

In the sequel we will use the following results:

Lemma 2. *([15]). Let $C(t)$, (resp.$S(t)$), $t \in R$ be a strongly continuous cosine (resp. sine) family on X. Then there exist constants $N^* \geq 1$and $w \geq 0$ such that*

$$\|C(t)\| \leq N^* e^{|t|}, \quad \textit{for all} \quad t \in R,$$
$$\|S(t_1) - S(t_2)\| \leq N^* |\int_{t_1}^{t_2} e^{w|s|}ds|, \quad \textit{for all} \quad t_1, \ t_2 \in R.$$

Theorem 3. *([11], p-47). Let $z(t)$, $u(t)$, $v(t)$, $w(t) \in C([\alpha, \beta], R_+)$ and $k \geq 0$ be a real constant and*

$$z(t) \leq k + \int_\alpha^t u(s)[z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma]ds, \quad \textit{for} \quad t \in [\alpha, \beta].$$

If

$$r = \int_\alpha^\beta w(\sigma)\exp(\int_\alpha^\sigma [u(\tau) + v(\tau)]d\tau)d\sigma < 1,$$

then

$$z(t) \leq \frac{k}{1-r}\exp(\int_\alpha^t [u(s) + v(s)]ds), \quad \textit{fort} \in [\alpha, \beta].$$

Theorem 4. *([5], p-61). Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let $\varepsilon(F) = \{x \in S : x = \lambda Fx \quad for\ some 0 < \lambda < 1\}$. Then either $\varepsilon(F)$ is unbounded or F has a fixed point.*

We list the following hypotheses for our convenience.

(H_1) $-A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ which is compact for $t > 0$, and there exists a constant K such that

$$K = \sup\{C(t) : t \in [t_0, t_0 + \beta]\}.$$

(H_2) There exists a constant G such that

$$\|g(t_1, t_2, \cdots, t_p, x(\cdot))\| \leq G,$$

for all $x \in C([t_0, t_0 + \beta], X)$.

(H_3) There exists a continuous function $p : [t_0, t_0 + \beta] \to R_+$ such that

$$\|k(t, , x(t))\| \leq p(t)\|x(t)\|,$$

for every $t \in [t_0, t_0 + \beta]$ and $x \in X$.

(H_4) There exists a continuous function $q : [t_0, t_0 + \beta] \to R_+$ such that

$$\|h(t, x(t))\| \leq q(t)\|x(t)\|,$$

for every $t \in [t_0, t_0 + \beta]$ and $x \in X$.

(H_5) There exists a continuous function $l : [t_0, t_0 + \beta] \to R_+$ such that

$$\|f(t, x, y, z)\| \leq l(t)(\|x\| + \|y\| + \|z\|),$$

for every $t \in [t_0, t_0 + \beta]$ and $x,\ y,\ z \in X$.

(H_6) There exists a constant M such that

$$|a(t, s)| \leq M, \quad \text{for} \quad t \geq s \geq t_0.$$

(H_7) There exists a constant N such that

$$|b(t, s)| \leq N, \quad \text{for} \quad t, s \in [t_0, t_0 + \beta].$$

(H_8) For each $t \in [t_0, t_0 + \beta]$ the function $f(t, \cdot, \cdot, \cdot) : [t_0, t_+\beta] \times X \times X \times X \to X$ is continuous and for each $x,\ y,\ z \in X$ the function $f(\cdot, x, y, z) : [t_0, t_0 + \beta] \times X \times X \times X \to X$ is strongly measurable.

(H_9) For each $t \in [t_0, t_0 + \beta]$ the functions $k(t, \cdot),\ h(t, \cdot) : [t_0, t_0 + \beta] \times X \to X$ are continuous and for each $x \in X$ the functions $k(\cdot, x),\ h(\cdot, x) : [t_0, t_0 + \beta] \times X \to X$ are strongly measurable.

(H_{10}) For every positive integer m there exists $\alpha_m \in L^1(t_0, t_0 + \beta)$ such that

$$\sup_{\|x\| \le m, \|y\| \le m, \|z\| \le m} \|f(t, x, y, z)\| \le \alpha_m(t), \quad \text{for} \quad t \in [t_0, t_0 + \beta] \quad \text{a. e.}$$

3 Existence Result

Theorem 5. *Suppose that the hypotheses $(H_1) - (H_{10})$ hold. If*

$$r^* = \int_{t_0}^{t_0+\beta} Nq(\sigma) \exp\left(\int_{t_0}^{\sigma} [K\beta l(\tau) + Mp(\tau)] d\tau\right) d\sigma < 1, \tag{3.1}$$

then the initial value problem (1.1)–(1.2) has a mild solution on $[t_0, t_0 + \beta]$.

Proof. To prove the existence of a solution of nonlinear mixed Volterra-Fredholm integrodifferential equations (1.1)–(1.2), we apply topological transversality theorem and Pachpatte's inequality. First we establish the priori bounds for the initial value problem

$$\begin{aligned} x''(t) + Ax(t) = \lambda f(t, x(t), &\int_{t_0}^{t} a(t, s) k(s, x(s)) ds, \\ &\int_{t_0}^{t_0+\beta} b(t, s) h(s, x(s)) ds), \quad t \in [t_0, t_0 + \beta]. \end{aligned} \tag{3.2}$$

with condition (1.2). Let $x(t)$ be a mild solution of the IVP (1.1)–(1.2). Then from

$$\begin{aligned} x(t) = [C(t - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))) + S(t - t_0)\eta] + \lambda \int_{t_0}^{t} S(t - s) f(s, x(s), \\ \int_{t_0}^{s} a(s, \tau) k(\tau, x(\tau)) d\tau, \int_{t_0}^{t_0+\beta} b(s, \tau) h(\tau, x(\tau)) d\tau) ds, \quad t \in [t_0, t_0 + \beta] \end{aligned} \tag{3.3}$$

and using hypotheses $(H_1) - (H_7)$ and the fact that $\lambda \in (0, 1)$, we have

$$\|x(t)\| \le \|C(t - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot)))\| + \|S(t - t_0)\eta\| + \int_{t_0}^{t} \|S(t - s)\|$$

$$
\begin{aligned}
&\times \|f(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau, \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, x(\tau))d\tau)\|ds \\
&\leq K(\|x_0\| + G) + K\beta\|\eta\| + \int_{t_0}^{t} K\beta l(s) \\
&\times [\|x(s)\| + \int_{t_0}^{s} \|a(s,\tau)k(\tau, x(\tau))\|d\tau + \int_{t_0}^{t_0+\beta} \|b(s,\tau)h(\tau, x(\tau))\|d\tau]ds \\
&\leq K[(\|x_0\| + G) + \beta\|\eta\|] + \int_{t_0}^{t} K\beta l(s) \\
&\times [\|x(s)\| + \int_{t_0}^{s} Mp(\tau)\|x(\tau)\|d\tau + \int_{t_0}^{t_0+\beta} Nq(\tau)\|x(\tau)\|d\tau]ds. \qquad (3.4)
\end{aligned}
$$

Using the condition (3.1) and Pachpatte's inequality given in Theorem 3 with $z(t) = \|x(t)\|$ in (3.4) , we obtain

$$
\begin{aligned}
\|x(t)\| &\leq \frac{K[(\|x_0\| + G) + \beta\|\eta\|]}{1 - r^*} \exp(\int_{t_0}^{t} [K\beta l(s) + Mp(s)]ds) \\
&\leq \frac{k^*}{1 - r^*} \exp(\beta[K\beta L + MP]) = \gamma, \qquad (3.5)
\end{aligned}
$$

where $k^* = K[(\|x_0\| + G) + \beta\|\eta\|], \qquad L = \sup_{t\in[t_0,t_0+\beta]} \{l(t)\},$

and $P = \sup_{t\in[t_0,t_0+\beta]} \{p(t)\}.$

Hence there exists a constant γ independent of $\lambda \in (0, 1)$ such that $\|x(t)\| \leq \gamma$ and consequently

$$\|x\|_B = \sup\{\|x(t)\| : t \in [t_0, t_0 + \beta]\} \leq \gamma.$$

Now, we rewrite the problem (1.1)–(1.2) as follows: If $y \in B$ and $x(t) = C(t - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))) + y(t), \quad t \in [t_0, t_0 + \beta]$, where $y(t)$ satisfies

$$
\begin{aligned}
y(t) = S(t - t_0)\eta + &\int_{t_0}^{t} S(t - s)f(s, y(s) + C(s - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))), \\
&\int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + C(\tau - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau, \\
&\int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, y(\tau) + C(\tau - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau)ds,
\end{aligned}
$$

$t \in [t_0, t_0 + \beta]$ if and only if $x(t)$ satisfies

$$x(t) = C(t - t_0)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))) + S(t - t_0)\eta + \int_{t_0}^{t} S(t - s)$$

$$\times f(s,x(s),\int_{t_0}^{s} a(s,\tau)k(\tau,x(\tau))d\tau,\int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau,x(\tau))d\tau)ds.$$

Define $F : B_0 \to B_0, \quad B_0 = \{y \in B : y(t_0) = 0\}$ by

$$\begin{aligned}(Fy)(t) = S(t-t_0)\eta + \int_{t_0}^{t} S(t-s)f(s,y(s)+C(s-t_0)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))),\\ \int_{t_0}^{s} a(s,\tau)k(\tau,y(\tau)+C(\tau-t_0)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau,\\ \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau,y(\tau)+C(\tau-t_0)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau)ds,\end{aligned} \tag{3.6}$$

$t \in [t_0, t_0+\beta]$.

First, we prove that $F : B_0 \to B_0$ is continuous. Let $\{u_n\}$ be a sequence of elements of B_0 converging to u in B_0. Then

$$\begin{aligned}(Fu_n)(t) = S(t)\eta + \int_{t_0}^{t} S(t-s)f(s,u_n(s)+C(s)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))),\\ \int_{t_0}^{s} a(s,\tau)k(\tau,u_n(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau,\\ \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau,u_n(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau)ds,\end{aligned} \tag{3.7}$$

$t \in [t_0, t_0+\beta]$. Now, $\|Fu_n - Fu\|_B = \sup_{t\in[t_0,t_0+\beta]} \|(Fu_n)(t) - (Fu)(t)\|$. Since $\{u_n\}$ be the sequence of elements of B_0 converging to u in B_0 and by hypotheses $(H_8)-(H_9)$, we have

$$\begin{aligned}&f(t,u_n(t)+C(t)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))),\\ &\quad\int_{t_0}^{t} a(t,s)k(s,u_n(s)+C(s)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))ds,\\ &\quad\int_{t_0}^{t_0+\beta} b(t,s)h(s,u_n(s)+C(s)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))ds)\\ &\to f(t,u(t)+C(t)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))),\\ &\quad\int_{t_0}^{t} a(t,s)k(s,u(s)+C(s)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))ds,\\ &\quad\int_{t_0}^{t_0+\beta} b(t,s)h(s,u(s)+C(s)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))ds)\end{aligned}$$

for each $t \in [t_0, t_0+\beta]$. Then by dominated convergence theorem, we have

$$\|(Fu_n)(t) - (Fu)(t)\|$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t} \|S(t-s)\| \|f(s, u_n(s) + C(s)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))),\\
&\quad \int_{t_0}^{s} a(s,\tau)k(\tau, u_n(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau,\\
&\quad \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, u_n(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau)\\
&\quad - f(s, u(s) + C(s)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))),\\
&\quad \int_{t_0}^{s} a(s,\tau)k(\tau, u(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau,\\
&\quad \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, u(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau)\|ds\\
&\to 0
\end{aligned}
$$

and consequently $\|Fu_n - Fu\|_B \to 0$ as $n \to \infty$ i.e. $Fu_n \to Fu$ in B_0 as $u_n \to u \in B_0$. Therefore, F is continuous.

Now, we prove that F maps a bounded set of B_0 into a precompact set of B_0. Let $B_m = \{y \in B_0 : \|y\|_B \leq m\}$ for some $m \geq 1$. We first show that F maps B_m into an equicontinuous family of functions with values in X. Let $y \in B_m$ and $t_0 \leq s < t \leq t_0 + \beta$. Then we have,

$$
\begin{aligned}
&\|(Fy)(s) - (Fy)(t)\|\\
&\leq \|S(s)\eta - S(t)\eta\|\\
&\quad + \int_{t_0}^{s} \|S(s-\tau) - S(t-\tau)\| l(\tau)[\|y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot)))\|\\
&\qquad + \|\int_{t_0}^{\tau} a(\tau,\sigma)k(\sigma, y(\sigma) + C(\sigma)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\sigma\|\\
&\qquad + \|\int_{t_0}^{t_0+\beta} b(\tau,\sigma)h(\sigma, y(\sigma) + C(\sigma)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\sigma\|]d\tau\\
&\quad + \int_{s}^{t} K\beta l(\tau)[\|y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot)))\|\\
&\qquad + \|\int_{t_0}^{\tau} a(\tau,\sigma)k(\sigma, y(\sigma) + C(\sigma)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\sigma\|\\
&\qquad + \|\int_{t_0}^{t_0+\beta} b(\tau,\sigma)h(\sigma, y(\sigma) + C(\sigma)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\sigma\|]d\tau\\
&\leq \|S(s)\eta - S(t)\eta\| + \int_{t_0}^{s} \|S(s-\tau) - S(t-\tau)\| L[m + K(\|x_0\| + G)\\
&\qquad + \int_{t_0}^{\tau} MP(m + K(\|x_0\| + G))d\sigma + \int_{t_0}^{t_0+\beta} NQ(m + K(\|x_0\| + G))d\sigma]d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_s^t K\beta L[m+K(\|x_0\|+G)\\
&\quad+\int_{t_0}^{\tau} MP(m+K(\|x_0\|+G))d\sigma\|+\int_0^{\beta} NQ(m+K(\|x_0\|+G))d\sigma\|]d\tau\\
&\leq \|S(s)\eta-S(t)\eta\|+\int_{t_0}^{s}\|S(s-\tau)-S(t-\tau)\|\\
&\times L[(m+k_1^*)+\int_{t_0}^{\tau} MP(m+k_1^*)d\sigma+\int_{t_0}^{t_0+\beta} NQ(m+k_1^*)d\sigma]d\tau\\
&+\int_s^t K\beta L[(m+k_1^*)+\int_{t_0}^{\tau} MP(m+k_1^*)d\sigma+\int_{t_0}^{t_0+\beta} NQ(m+k_1^*)d\sigma]d\tau\\
&\leq \|S(s)\eta-S(t)\eta\|+\int_{t_0}^{s}\|S(s-\tau)-S(t-\tau)\|\\
&\times L[(m+k_1^*)+\beta MP(m+k_1^*)+\beta NQ(m+k_1^*)]d\tau\\
&+\int_s^t K\beta L[(m+k_1^*)+\beta MP(m+k_1^*)+\beta NQ(m+k_1^*)d\sigma]d\tau, \qquad (3.8)
\end{aligned}
$$

where $k_1^*=K(\|x_0\|+G)$ and $Q=\sup_{t\in[0,\beta]}\{q(t)\}$. The right hand side of (3.8) is independent of $y\in B_m$ and tends to zero as $s-t\to 0$, since $C(t)$, $S(t)$ are uniformly continuous for $t\in[t_0,t_0+\beta]$ and the compactness of $C(t)$, $S(t)$ for $t>0$ imply the continuity in the uniform operator topology. Thus F_{B_m} is an equicontinuous family of functions with values in X.

We next show that F_{B_m} is uniformly bounded. From the equation (3.6) and using hypotheses $(H_1)-(H_7)$ and the fact that $\|y\|_B\leq m$, we obtain

$$
\begin{aligned}
\|(Fy)(t)\| &\leq K\beta\|\eta\|+\int_{t_0}^{t} K\beta l(s)[\|y(s)+C(s)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot)))\|\\
&\quad+\|\int_{t_0}^{s} a(s,\tau)k(\tau,y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau\|\\
&\quad+\|\int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau,y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau\|]ds\\
&\leq K\beta\|\eta\|+\int_{t_0}^{t} K\beta L[m+K(\|x_0\|+G)\\
&\quad+\int_{t_0}^{s} Mp(\tau)\|y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot)))\|d\tau\\
&\quad+\int_{t_0}^{t_0+\beta} Nq(\tau)\|y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot)))\|d\tau]ds\\
&\leq K\beta\|\eta\|+\int_{t_0}^{t} K\beta L[(m+k_1^*)+\int_{t_0}^{s} MP(m+k_1^*)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_0}^{t_0+\beta} NQ(m+k_1^*)d\tau]ds \\
&\leq K\beta\|\eta\| + \int_{t_0}^{t} K\beta L[(m+k_1^*) + \beta MP(m+k_1^*) + \beta NQ(m+k_1^*)]ds \\
&\leq K\beta\|\eta\| + K\beta^2 L(m+k_1^*)[1+\beta MP+\beta NQ].
\end{aligned}
$$

This implies that the set $\{(Fy)(t) : \|y\|_B \leq m, \quad t_0 \leq t \leq t_0+\beta\}$ is uniformly bounded in X and hence $\{F_{B_m}\}$ is uniformly bounded.

We have already shown that F_{B_m} is an equicontinuous and uniformly bounded collection. To prove the set F_{B_m} is precompact in B, it is sufficient, by Arzela-Ascoli's argument, to show that the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X for each $t \in [t_0, t_0+\beta]$. Since $(Fy)(t_0) = 0$ for $y \in B_m$, it suffices to show this for $t_0 < t \leq t_0+\beta$. Let $t_0 < t \leq t_0+\beta$ be fixed and ϵ a real number satisfying $t_0 < \epsilon < t$. For $y \in B_m$, we define

$$
\begin{aligned}
(F_\epsilon y)(t) = S(t)\eta + \int_{t_0}^{t-\epsilon} & S(t-s)f(s, y(s) + C(s)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))), \\
& \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau, \\
& \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau)ds. \qquad (3.9)
\end{aligned}
$$

Since $C(t), \; S(t)$ are compact operators , the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_m\}$ is precompact in X, for every $\epsilon, \; t_0 < \epsilon < t$. Moreover for every $y \in B_m$, we have

$$
\begin{aligned}
&(Fy)(t) - (F_\epsilon y)(t) \\
&= \int_{t-\epsilon}^{t} S(t-s)f(s, y(s) + C(s)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))), \\
&\qquad \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau, \\
&\qquad \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau)ds. \qquad (3.10)
\end{aligned}
$$

By making use of hypotheses $(H_1) - (H_7)$ and the fact that $\|y(s)\| \leq m$, we have

$$
\begin{aligned}
&\|(Fy)(t) - (F_\epsilon y)(t)\| \\
&\leq \int_{t-\epsilon}^{t} K\beta l(s)[\|y(s) + C(s)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot)))\| \\
&\quad + \|\int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + C(\tau)(x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))))d\tau\|
\end{aligned}
$$

$$
\begin{aligned}
&+\|\int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau,y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot))))d\tau\|]ds\\
&\leq \int_{t-\epsilon}^{t} K\beta L[m+K(\|x_0\|+G)\\
&\quad+\int_{t_0}^{s} Mp(\tau)\|y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot)))\|d\tau\\
&\quad+\int_{t_0}^{t_0+\beta} Nq(\tau)\|\|y(\tau)+C(\tau)(x_0-g(t_1,t_2,\cdots,t_p,x(\cdot)))\|d\tau]ds\\
&\leq \int_{t-\epsilon}^{t} K\beta L[(m+k_1^*)+\int_{t_0}^{s} MP(m+k_1^*)d\tau+\int_{t_0}^{t_0+\beta} NQ(m+k_1^*)d\tau]ds\\
&\leq K\beta L(m+k_1^*)[1+\beta MP+\beta NQ]\epsilon.
\end{aligned}
$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_m\}$. Hence the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X. Thus we have shown that F is completely continuous operator.

Moreover, the set

$$\varepsilon(F)=\{y\in B_0 : y=\lambda Fy \quad \text{for some} \quad 0<\lambda<1\},$$

is bounded in B, since for every y in $\varepsilon(F)$, the function $x(t) = y(t) + C(t)(x_0 - g(t_1,t_2,\cdots,t_p,x(\cdot)))$ is a mild solution of (3.2)–(1.2) for which we have proved $\|x\|_B \leq \gamma$ and hence $\|y\|_B \leq \gamma + k_1^*$. Now, by virtue of Theorem 4, the operator F has a fixed point in B_0. Therefore, the initial value problem (1.1)–(1.2) has a solution on $[t_0, t_0+\beta]$. □

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