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A NOTE ON ARBITRARILY VERTEX DECOMPOSABLE GRAPHS

Abstract. A graph G of order n is said to be arbitrarily vertex decomposable if for each sequence $(n_1, \ldots, n_k)$ of positive integers such that $n_1 + \ldots + n_k = n$ there exists a partition $(V_1, \ldots, V_k)$ of the vertex set of G such that for each $i \in \{1, \ldots, k\}$, V_i induces a connected subgraph of G on n_i vertices.

In this paper we show that if G is a two-connected graph on n vertices with the independence number at most $\lceil n/2 \rceil$ and such that the degree sum of any pair of non--adjacent vertices is at least $n-3$, then G is arbitrarily vertex decomposable. We present another result for connected graphs satisfying a similar condition, where the bound $n-3$ is replaced by $n-2$.

Keywords: arbitrarily vertex decomposable graphs, traceable graphs, independence number, perfect matching.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph of order n. Let $\tau = (n_1, \ldots, n_k)$ denote a sequence of positive integers such that $n_1 + \ldots + n_k = n$. Such a sequence will be called *admissible for* G. If $\tau = (n_1, \ldots, n_k)$ is an admissible sequence for G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set V such that for each $i \in \{1, \ldots, k\}$, $|V_i| = n_i$ and the subgraph induced by V_i is connected, then τ is called *realizable in* G and the sequence $(V_1, \ldots, V_k)$ is said to be *a* G*-realization of* τ or *a realization of* τ *in* G. A graph G is *arbitrarily vertex decomposable* (avd for short) if for each admissible sequence τ for G there exists a G-realization of τ.

It is clear that each avd graph admits a perfect matching or a matching that omits exactly one vertex. Note also that if G_1 is a spanning subgraph of a graph G_2 and G_2 is not avd, then neither is G_1.

Let r, $1 \leq r \leq n$ be a fixed integer. G is *r-vertex decomposable* if each admissible sequence $(n_1, \ldots, n_r)$ of r components is realizable in G.

There are several papers concerning avd trees. In [2] Barth and Fournier proved a conjecture of Horňák and Woźniak [8] stating that any tree T with maximum degree $\Delta(T)$ at least five is not avd. The first result characterizing avd caterpillars with three leaves was found by Barth et al. [1] and, independently, by Horňák and Woźniak [7] (see Section 3). In [1] and [2] Barth et al. and Barth and Fournier investigated trees homeomorphic to $K_{1,3}$ or $K_{1,4}$ and showed that determining if such a tree is avd can be done using a polynomial algorithm. Cichacz et al. [4] gave a complete characterization of arbitrarily vertex decomposable caterpillars with four leaves. They also described two infinite families of arbitrarily vertex decomposable trees with maximum degree three or four. The complete characterization of on-line avd trees has been recently found by Horňák et al. [6].

In [9] Kalinowski et al. studied a family of unicyclic avd graphs. It is worth recalling an old result of Győri [5] and Lovász [11] stating that every k-connected graph is k-vertex decomposable.

However, it is evident that each traceable graph is avd. Therefore, each sufficient condition for a graph to have a hamiltonian path also implies that the graph is avd. We can try to replace some known conditions for traceability by weaker ones implying that the graphs satisfying these conditions are avd.

Observe that any necessary condition for a graph to contain a perfect matching (or a matching that omits exactly one vertex) is a necessary condition for a graph to be arbitrarily vertex decomposable. Thus we will assume that the independence number of an n-vertex graph is at most $\lceil n/2 \rceil$.

It follows from Ore's theorem [12] that every graph G of order n such that the degree sum of any two nonadjacent vertices is at least $n-1$ (i.e. G satisfies an Ore-type condition with the bound $n-1$), is traceable. The aim of this paper is to show that every 2-connected graph satisfying a similar condition, where the bound $n-1$ is replaced by $n-3$, is avd, provided its independence number is at most $\lceil n/2 \rceil$. We also prove a similar theorem for connected graphs verifying the above condition with the bound $n-2$. These two results (Theorems 5 and 4) are presented in Section 4. In Section 5 we examine the structure of graphs that satisfy Ore-type conditions and are not avd; we also present the admissible sequences which are not realizable in graphs under consideration.

Notice that the problem of deciding whether a given graph is arbitrarily vertex decomposable is NP-complete [1] but we do not know if this problem is NP-complete when restricted to trees.

2. TERMINOLOGY AND NOTATION

Let $T = (V, E)$ be a tree. A vertex $x \in V$ is called *primary* if $d(x) \geq 3$. A *leaf* (or a *hanging vertex*) is a vertex of degree one. A path P of T is an *arm* if one of its endvertices is a leaf in T, the other one is primary and all internal vertices of

P have degree two in T. A graph T is a *star-like tree* if it is a tree homeomorphic to a star $K_{1,q}$ for some $q \geq 3$. Such a tree has one primary vertex and q arms $A_1, \ldots, A_q$. For each A_i, let $a_i \geq 2$ be the order of A_i. We will denote the above defined star-like tree by $S(a_1, \ldots, a_q)$. Notice that the order of this star-like tree is equal to $1 + \sum_{i=1}^{q}(a_i - 1)$.

Let G be a graph and let C be a cycle of G with a given orientation. Suppose a is a vertex of C. We shall denote by a^+ the successor of a on C and by a^- its predecessor. We write a^{+2} for $(a^+)^+$, a^{-2} for $(a^-)^-$, a^{+k} for $(a^{+(k-1)})^+$ and a^{-k} for $(a^{-(k-1)})^-$. If A is a subset of $V(C)$, then $A^+ = \{v \in V(C) |\ v^- \in A\}$ and $A^- = \{v \in V(C) |\ v^+ \in A\}$. Let a and b be two vertices of C. By $a\,C\,b$ we denote the set of consecutive vertices of C from a to b (a and b included) in the direction specified by the orientation of C. It will be called a *segment of C from a to b*. Throughout the paper the indices of a cycle $C = x_1, x_2, \ldots, x_p$ are to be taken modulo p. If $x \notin V(C)$ we write $N_C(x)$ for the set of neighbors of x on C and we denote by $d_C(x)$ the number $\mid N_C(x) \mid$.

A *sun with r rays* is a graph of order $n \geq 2r$ with r hanging vertices $u_1, \ldots, u_r$ whose deletion yields a cycle C_{n-r}, and each vertex v_i on C_{n-r} adjacent to u_i is of degree three. If the sequence of vertices v_i is situated on the cycle C_{n-r} in such a way that there are exactly $b_i \geq 0$ vertices, each of degree two, between v_i and v_{i+1}, $i = 1, \ldots, r$, (the indices taken modulo r), then this sun is denoted by $\mathrm{Sun}(b_1, \ldots, b_r)$, and it is unique up to an isomorphism. Clearly, every sun with one ray is avd since it is traceable.

Let G be a graph of order n. Define

$$\sigma_2(G) := \min\{d(x) + d(y) |\ x, y \text{ are nonadjacent vertices in } G\}$$

if G is not a complete graph, and $\sigma_2(G) = \infty$ otherwise. Ore's well known theorem [12] states that every graph G with $\sigma_2(G) \geq n \geq 3$ is hamiltonian. This immediately implies that if $\sigma_2(G) \geq n - 1$ then G is traceable, so also avd.

3. PREPARATORY RESULTS

The first result characterizing avd star-like trees (i.e. caterpillars with one single leg) was established by Barth, Baudon and Puech [1] and, independently, by Horňák and Woźniak [7].

Proposition 1. *A star-like tree $S(2, a, b)$ is avd if and only if the integers a and $n = a + b$ are coprime. Moreover, each admissible and non-realizable sequence in $S(2, a, b)$ is of the form $(d, d, \ldots, d)$, where $a \equiv n \equiv 0 \pmod{d}$ and $d > 1$.*

Proposition 2. *Let G be the graph of order $n \geq 4$ obtained by taking a path $P = x_1, \ldots, x_{n-1}$, a single vertex x and by adding the edges xx_{i_1}, $xx_{i_2}, \ldots, xx_{i_p}$, where $1 < i_1 < \ldots < i_p < n - 1$ and $p \geq 1$. Then G is not avd if and only if there are integers $d > 1$, λ, $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$.*

Proof. Suppose that the integers $d > 1$, λ, $\lambda_1, \lambda_2, \ldots, \lambda_p$ satisfy the condition of the theorem and consider the admissible sequence $\tau = (\underbrace{d, \ldots, d}_{\lambda})$ for G. Observe that if G' is a connected subgraph of G of order d which contains the vertex x, then the connected component of $G - V(G')$ containing the vertex x_1 is a path P' such that d does not divide the order of P'. Thus, τ is not realizable in G. Conversely, if $\tau = (n_1, n_2, \ldots, n_\lambda)$ is an admissible sequence for G that is not realizable in G, then τ is also not realizable in the caterpillar $S(2, i_1, n - i_1)$. By Proposition 1, there are two integers $d > 1$ and λ_1 such that $n_1 = n_2 = \ldots = n_\lambda = d$ and $i_1 = \lambda_1 d$. The sequence τ cannot be realizable in the caterpillar $S(2, i_2, n - i_2)$ therefore, again by Proposition 1, $i_2 = \lambda_2 d$ for some integer λ_2. Repeating the same argument we prove that the condition of the proposition holds. $\square$

The following result is due to Kalinowski *et al.* [9]. However, for the sake of completeness we give a short proof of this theorem here.

Theorem 1. $\mathrm{Sun}(a, b)$ *with two rays is arbitrarily vertex decomposable if and only if either its order* n *is odd or both* a *and* b *are even. Moreover, each sequence which is admissible and non realizable in* $\mathrm{Sun}(a, b)$ *is of the form* $(2, 2, \ldots, 2)$.

Proof. If $\mathrm{Sun}(a, b)$ is avd and n is even, then the sequence $(2, \ldots, 2)$ is realizable. It easily follows that both a and b have to be even.

Suppose now that there exists an admissible and non-realizable sequence $(n_1, \ldots, n_k)$ for $\mathrm{Sun}(a, b)$. If we choose a vertex of degree three and delete a non-hanging edge incident to it, then we obtain a star-like tree isomorphic either to $S(a+1, b+3)$ or to $S(a+3, b+1)$. Clearly, the sequence $(n_1, \ldots, n_k)$ cannot be realized in any of these two trees. Hence, by Proposition 1, this sequence is of the form $n_1 = \ldots = n_k = d$ with d being a common divisor of four numbers $a+1$, $a+3$, $b+1$, $b+3$. This implies that $d = 2$, and both a and b are odd, contrary to the assumption. $\square$

In the proofs of the main results of this paper we will need two generalizations of Ore's theorem [12]. The first one is due to Pósa [13].

Theorem 2. *Let* G *be a connected graph of order* $n \geq 3$ *such that*

$$\sigma_2(G) \geq d.$$

If $d < n$*, then* G *contains a path of length* d*, and if* $d \geq n$*, then* G *is hamiltonian.*

The second one was found by Bermond [3] and, independently, by Linial [10].

Theorem 3. *Let* G *be a 2-connected graph such that*

$$\sigma_2(G) \geq d.$$

Then G *contains either a cycle of length at least* d *or a hamiltonian cycle.*

4. MAIN RESULTS

Theorem 4. *Let G be a connected graph of order n such that $\sigma_2(G) \geq n-2$ and $\alpha(G)$ is at most $\lceil n/2 \rceil$. Then G is avd.*

Proof. Suppose G is not avd. Then G is not traceable, so $n \geq 4$, and by Theorem 2, there exists in G a path $P = x_1, x_2, \ldots, x_{n-1}$ of length $n-2$. Let x be the unique vertex outside P and let $N(x) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_p}\}$, $1 \leq i_1 < \ldots \leq i_p \leq n-1$, be the set of neighbors of x. Since G is connected and non-traceable, we have $p \geq 1$, $i_1 > 1$, $i_p < n-1$ and $x_1 x_{n-1} \notin E(G)$. By Proposition 2, there are integers $d > 1$, λ, $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$. Furthermore, because G is not traceable, there is at least one vertex between any two consecutive neighbors of x on P.

Since $x_1 x \notin E(G)$, it follows by assumption that $d(x_1) \geq n-2-p$. Observe now that $x_1 x_{i_j+1} \notin E(G)$ for each $j = 1, \ldots, p$, for otherwise $x, x_{i_j}, x_{i_j-1}, x_1, x_{i_j+1}, \ldots, x_{n-1}$ is a hamiltonian path in G, a contradiction. Therefore, $d(x_1) \leq n-2-p$, hence $d(x_1) = n-2-p$ and x_1 is adjacent to any x_i with $i \in \{2, \ldots, n-1\} \setminus \{i_1+1, \ldots, i_p+1\}$. Since $x_1 x_{n-1} \notin E(G)$, we have $x_{n-1} = x_{i_p+1}$, thus $2 = n - i_p = (\lambda - \lambda_p)d$, so $d = 2$ and n is even. Moreover, $x_1 x_{i_j-1} \notin E(G)$ for each $j = 2, \ldots, p$, $p \geq 2$, for otherwise we could easily find a hamiltonian path in G: $x_{i_{j-1}+1}, \ldots, x_{i_j-1}, x_1, \ldots, x_{i_{j-1}}, x, x_{i_j}, \ldots, x_{n-1}$. Thus, every set $x_{i_j} P x_{i_{j+1}}$ with $j \in \{1, \ldots, p-1\}$ contains exactly three vertices. Now, because x_{n-1} and x are not adjacent, we can in a similar way deduce that $x_2 \in N(x)$, i.e., $N(x) = \{x_2, x_4, \ldots, x_{n-2}\}$ and $d(x) = d(x_1) = (n-2)/2$. It is obvious that any edge of the form $x_{2i-1}x_{2j-1}$ would create a hamiltonian path in G, so the set $\{x, x_1, x_3, \ldots, x_{n-3}, x_{n-1}\}$ of $(n+2)/2$ vertices is independent and we obtain a contradiction. □

Theorem 5. *Let G be a 2-connected graph on n vertices such that $\alpha(G) \leq \lceil n/2 \rceil$ and $\sigma_2(G) \geq n-3$. Then G is avd.*

Proof. By Theorem 3, G contains a cycle of length at least $n-3$. If G has a hamiltonian cycle or a C_{n-1}, then G is traceable, so also avd. Moreover, as $n \leq 6$, it follows that $\sigma_2(G) \geq 4 \geq n-2$ and Theorem 4 can be applied. Therefore, we shall assume that it contains neither C_n nor C_{n-1} and $n \geq 7$. Suppose, contrary to our claim, that G is not avd.

Case 1. G has no cycle of length $n-2$, i.e., the circumference of G equals $n-3$. Denote by C a cycle of length $n-3$ with a given orientation and let $X := V(G) \setminus V(C) = \{x, y, z\}$.

Case 1.1. X is an independent set. Assume without loss of generality that $d(x) \geq d(y) \geq d(z) \geq 2$. Let $A = N(x) = N_C(x)$. Since x and y are not adjacent and $d(x) \geq d(y)$, we have $d(x) \geq (n-3)/2$. Note that no two neighbors of x are consecutive on C, for otherwise G would contain a cycle of length $n-2$. Hence $d(x) = (n-3)/2$ and n is odd. Furthermore, since X is independent and $\sigma_2(G) \geq n-3$, we have

$d(y) = d(z) = (n-3)/2$. If u and v belong to $N(x)$, then $u^+v^+ \notin E(G)$, because otherwise $u^+, v^+, v^{++}, \ldots, u^-, u, x, v, v^-, \ldots, u^+$ would be a cycle of length $n-2$, contradicting our assumption. In the same manner (replacing the path $u, u^+, \ldots, v, v^+$ with the path $u, x, v, v^-, \ldots, u^+, y, v^+$), we show that $u^+y \notin E(G)$ or $v^+y \notin E(G)$. Thus, the set $A^+ \cup \{x\}$ is independent and $|N(y) \cap A^+| \le 1$. Suppose $|N(y) \cap A^+| = 1$. Then, because $d(y) = (n-3)/2$ and there are no consecutive neighbors of y on C, we have $N(y) = A^+$, hence $|N(y) \cap A^+| = |A^+| = (n-3)/2 \ge 2$, a contradiction. Finally, we conclude that $N(x) = N(y) = N(z) = A$ and $A^+ \cup X$ is an independent set of cardinality $(n-3)/2 + 3 > \lceil n/2 \rceil$, a contradiction.

Case 1.2. The set X induces the disjoint union $K_2 \cup K_1$. We may assume without loss of generality that $xy \in E(G)$, $xz \notin E(G)$ and $yz \notin E(G)$. Note that if $d(z) = 2$, then $n - 3 \le d(x) + d(z) = (d_C(x) + 1) + 2$, therefore, $d_C(x) \ge n - 6 > \frac{n-3}{2}$ for $n > 9$, so G contains a C_{n-2}, a contradiction. For $n = 9$ we conclude as in Case 1.1 that $d_C(x) = d_C(y) = 3$, $N_C(x) = N_C(y)$, so G has a C_{n-2} containing x and y, a contradiction. It is easy to see that for $n = 7, 8$ the circumference of G is at least $n-2$, again a contradiction. Thus, we shall assume $d(z) \ge 3$.

Suppose first $d(z) \ge \frac{n-3}{2}$. We can show as in the previous case that $d(z) = \frac{n-3}{2} \ge 3$ and $n \ge 9$ is odd. Moreover, we may assume $d_C(x) = d_C(y) = (n-5)/2 \ge 2$, for otherwise $d_C(x) = d_C(y) = (n-3)/2$, $N_C(x) = N_C(y) = N(z)$ and G would contain a C_{n-2} passing through x and y. If u belongs to $N_C(x)$, then $\{u^+, u^{+2}, u^-, u^{-2}\} \cap N_C(y) = \emptyset$, because the circumference of G equals $n-3$. Then the number of neighbors of y belonging to the path $u^{-2}Cu^{+2}$ is at most one. Furthermore, $u^{+3}, u^{-3} \in N_C(y)$ (possibly $u^{+3} = u^{-3}$), for otherwise

$$d_C(y) \le \frac{n-3-5}{2} + 1 < \frac{n-5}{2},$$

a contradiction. Thus, $(u^{-2}Cu^{+5} \setminus \{u, u^{+3}\}) \cap N_C(x) = \emptyset$, so

$$d_C(x) \le \frac{n-3-7}{2} + 2 < \frac{n-5}{2},$$

again a contradiction. If $3 \le d(z) < \frac{n-3}{2}$, then both $d(x)$ and $d(y)$ are at least $(n-1)/2$, hence $d_C(x) = d_C(y) = \frac{n-3}{2}$ and G contains a C_{n-2}, a contradiction.

Case 1.3. The set X induces a connected subgraph H of G. Then H contains a path of length two, and, because at least two vertices of X are joined to $G - X$ (G is 2-connected), G is traceable, a contradiction.

Case 2. G contains a cycle C of length $n-2$ (i.e., the circumference of G is $n-2$). Let x and y be two vertices of G outside C. Since G is 2-connected, these two vertices together with C and two independent edges connecting $\{x, y\}$ with C form a spanning subgraph H of G isomorphic to the sun $\mathrm{Sun}(a, b)$ with two rays. By our assumption, G is not avd, so it follows from Theorem 1 that n is even and a is odd. Suppose first that x and y are adjacent in G. By Theorem 1, the only admissible sequence of G which is not realizable in $\mathrm{Sun}(a,b)$ is of the form $(2,\ldots,2)$; however, we can easily find a perfect matching in G (we cover the set $\{x,y\}$ with a path of length one and the cycle C with $(n-2)/2$ paths of length one), a contradiction. Assume now $xy \notin E$ and $d(x) \geq d(y)$. Thus $d(x) \geq \frac{n-3}{2}$, hence, since n is even and any two consecutive vertices of C do not belong both to $N(x)$, we have $d(x) = \frac{n-2}{2}$. Set $A = N(x)$. Clearly, the set A^+ is independent. Moreover, $N(y) \subseteq A$, for otherwise there is $u \in V(C)$ such that $ux \in E(G)$ and $u^+y \in E(G)$, therefore G is traceable, a contradiction. Finally, the set $A^+ \cup \{x,y\}$ is independent and has $\frac{n-2}{2} + 2 > \lceil \frac{n}{2} \rceil$ vertices, a contradiction. □

5. CONCLUSIONS

Corollary 1. *If G is a graph of order n with $\sigma_2(G) \geq n-2$, then G is avd or the union of two disjoint cliques, or n is even and G satisfies $K_{\frac{n+2}{2},\frac{n-2}{2}} \subseteq G \subseteq \overline{K}_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}}$.*

Proof. If G is not connected and $\sigma_2(G) \geq n-2$, then G is the union of two disjoint cliques, so G is not avd. Suppose then that G is a connected graph, $\sigma_2(G) \geq n-2$ and G is not avd. It follows from the proof of Theorem 4 that n is even and G contains an independent set S on $\frac{n}{2}+1 \geq 3$ vertices. We have $d(x) \leq \frac{n}{2}-1$ for each $x \in S$, and, since $\sigma_2(G) \geq n-2$, $d(x) = \frac{n}{2}-1$ for every $x \in S$. Moreover, for every $y \in V(G) \setminus S$ we have $d(y) \geq \frac{n}{2}+1$ (since y is joined to each vertex of S), therefore G is the join $\overline{K}_{(n+2)/2} \vee H$, where H is any graph on $(n-2)/2$ vertices. Thus G has the structure as claimed. □

The proof of Theorem 4 also implies the following.

Corollary 2. *If G is a connected graph of order n such that $\sigma_2(G) \geq n-2$, then G is k-vertex decomposable for any $k \neq n/2$. Moreover, $(2,\ldots,2)$ is the unique admissible sequence for G which is not realizable in G.*

Corollary 3. *If G is a 2-connected graph of order n with $\sigma_2(G) \geq n-3$, then G is avd, or $n \geq 7$ is odd and $K_{\frac{n+3}{2},\frac{n-3}{2}} \subseteq G \subseteq \overline{K}_{\frac{n+3}{2}} \vee K_{\frac{n-3}{2}}$, or $n \geq 6$ is even, $K_{\frac{n+2}{2},\frac{n-2}{2}} \subseteq G \subseteq \overline{K}_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}}$, or $K_{\frac{n+2}{2},\frac{n-2}{2}} - e \subseteq G \subseteq (\overline{K}_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}}) - e$, where e is an arbitrary edge of the last graph.*

Proof. If the circumference of G is $n-3$ and G is not avd we find the situation described in Case 1.1 in the proof of Theorem 5, so $n \geq 7$ is odd and G contains an independent set S on $\frac{n+3}{2} \geq 5$ vertices. Because $\sigma_2(G) \geq n-3$, every vertex of S is adjacent to every vertex of $G-S$, thus G is the join $\overline{K}_{(n+3)/2} \vee H$, where H is any graph on $\frac{n-3}{2}$ vertices and the first assertion of the corollary follows.

Suppose G is not avd with circumference $n-2$ and consider the Case 2 of Theorem 5. Now n is even and G contains an independent set of $\frac{n+2}{2} \geq 4$ vertices, hence all of them except at most one are of degree $\frac{n-2}{2}$ and the only exceptional vertex must have the degree at least $(n-4)/2$, so G is contained in the join $\overline{K}_{(n+2)/2} \vee H$, where H is an arbitrary graph on $\frac{n-2}{2}$ vertices and can miss only one edge between $\overline{K}_{(n+2)/2}$ and H. □

Corollary 4. *If G is a 2-connected graph of order n such that $\sigma_2(G) \geq n-3$, then for every integer $k \notin \{(n-1)/2, n/2, (n+1)/2\}$ G is k-vertex decomposable. Moreover, each admissible and non-avd sequence is of the form $(2, 2, \dots, 2, 2, 3)$ or $(2, 2, \dots, 2)$ or else $(1, 2, 2, \dots, 2)$.*

Proof. Graphs that are not avd appear in Cases 1.1 and 2 of the proof of Theorem 5. In the latter situation, n is even and the graph contains a graph $\mathrm{Sun}(a, b)$ on n vertices therefore, by Theorem 1, $(2, 2, \dots, 2)$ is the only sequence which is not realizable in G.

Suppose then n is odd, G is not avd and consider the admissible sequences $\tau_1 = (1, 2, \dots, 2)$ and $\tau_2 = (2, 2, \dots, 2, 3)$ for G. Assume τ_1 or τ_2 are realizable in G. Then, since the vertex set of a connected graph of order three can be partitioned so that the parts induce K_1 and K_2, there exists a partition $(V_1, \dots, V_{(n+1)/2})$ of $V(G)$ into $\frac{n+1}{2}$ parts inducing complete subgraphs. Now, if S is an independent set of G, then each set V_i contains at most one vertex of S. Therefore, by Corollary 3, $\frac{n+3}{2} \leq \alpha(G) \leq \frac{n+1}{2}$, so we get a contradiction. Thus τ_1 and τ_2 are not realizable in G.

Assume now $\tau = (n_1, n_2, \dots, n_k)$ is another admissible sequence for G. If $n_i \leq 2$ for all $i \in \{1, \dots, k\}$ and $\tau \neq \tau_1$, then, by Corollary 3, τ is realizable in G. Consider again the Case 1.1 of Theorem 5, where x, y and z are three vertices outside the cycle C of length $n-3$. Let $C = x_1, \dots, x_{n-3}, x_1$ and suppose without loss of generality that $x_1 \in N(x) = N(y) = N(z)$. Now the spanning subgraph of G consisting of the path $x_1, \dots, x_{n-3}$ and three vertices x, y and z together with the edges xx_1, yx_1 and zx_1 is isomorphic to the star-like tree $S(2, 2, 2, b)$, where $b = n-3$. Suppose for some i, say $i = 1$, $n_i = n_1 \geq 4$. Set $V_1 = \{x, y, z, x_1\}$. Clearly, V_1 induces a connected subgraph of G and the graph $G - V_1$ contains a hamiltonian path, so it is easy to find a realization of τ in G. Suppose then $n_j \leq 3$ for all j and there is i, say $i = 1$, such that $n_i = n_1 = 3$. Now the set $V_1 = \{x, y, x_1\}$ induces a connected subgraph of G and, because z is adjacent to x_3 in G, $G - V_1$ has a spanning subgraph G' which is isomorphic to the star-like tree $S(2, 2, n-5)$. By Proposition 1, every admissible sequence for G' which is different from $(2, 2, \dots, 2)$ is realizable in G', thus τ is realizable in G provided $\tau \neq \tau_2$. □

We can also formulate an immediate corollary of Theorem 5 involving a Dirac-type condition.

Corollary 5. *If G is a 2-connected graph on n vertices such that $\alpha(G) \leq \lceil n/2 \rceil$ and minimum degree $\delta(G) \geq \frac{n-3}{2}$, then G is avd.*

Let G_1 be the join $K_1 \vee (K_1 \cup 2K_2)$, where $2K_2$ denotes two disjoint copies of K_2. This graph is not avd, because the sequence $(3, 3)$ is not realizable in G_1. It is easy to check that $\sigma_2(G_1) = n - 3 = 3$ and $\alpha(G_1) = 3 = \lceil n/2 \rceil$. Consider now the graph $G_2 = K_1 \vee 3K_2$. It can be easily seen that the sequences $(3, 3, 1)$ and $(4, 3)$ are not realizable in G_2, but $\sigma_2(G_2) = n - 3 = 4$ and $\alpha(G_2) = 3 < \lceil n/2 \rceil$. We conjecture that every connected graph G of order n such that $\sigma_2(G) \geq n - 3$, $\alpha(G)$ is at most $\lceil n/2 \rceil$ and G is isomorphic neither to G_1 nor to G_2 is avd.

Consider now the join $G_3 = K_2 \vee 4K_2$. Clearly, G_3 is a 2-connected graph such that $\sigma_2(G_3) = n - 4 = 6$, $\alpha(G_3) = 4 < \lceil n/2 \rceil$, but the sequence $(3, 3, 3, 1)$ is not realizable in G_3. This example shows that if we lower the bound $n - 3$ in Theorem 5, then the structure of non-avd graphs verifying the corresponding Ore-type condition becomes more diversified.

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