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DIFFERENT STRUCTURES ON SUBSPACES OF $Osc^k M$

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Abstract *The geometry of $Osc^k M$ spaces was introduced by R. Miron and Gh. Atanasiu in [6] and [7]. The theory of these spaces was developed by R. Miron and his cooperators from Romania, Japan and other countries in several books and many papers. Only some of them are mentioned in references. Here we recall the construction*

of adapted bases in $T(Osc^k M)$ and $T^(Osc^k M)$, which are comprehensive with the J structure. The theory of two complementary family of subspaces is presented as it was done in [2] and [4].*

The operators $J, \bar{J}, \theta, \bar{\theta}, p, p^$ are introduced in the ambient space and subspaces. Some new relations between them are established. The action of these operators on Liouville vector fields are examined.*

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1. TANGENT AND COTANGENT BUNDLES ON $Osc^k M$

Let $E = Osc^k M$ be a C^∞ , $(k+1)n$ dimensional space. Some point $u \in E$ in some local chart has coordinates:

$$u = (y^{0a}, y^{1a}, \dots, y^{ka}) = (y^{Aa}), \quad A = \overline{0, k}, \quad a = \overline{0, n}.$$

The set of allowable coordinate transformations are given by

$$y^{0a'} = y^{0a'}(y^{0a}) \quad \text{or} \quad x^{a'} = x^{a'}(x^a) \quad (1.1)$$

$$y^{1a'} = (\partial_{0a} y^{0a'}) y^{1a}, \quad \partial_{Aa} = \frac{\partial}{\partial y^{Aa}}, \quad A = \overline{0, k},$$

$$y^{2a'} = (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a}, \dots,$$

$$y^{ka'} = (\partial_{0a} y^{(k-1)a'}) y^{1a} + (\partial_{1a} y^{(k-1)a'}) y^{2a} + \dots + (\partial_{(k-1)a} y^{(k-1)a'}) y^{ka}$$

Theorem 1.1. *The transformations of type (1.1) form a pseudo-group.*

The natural basis $\overline{B^*}$ of $T^*(E)$ is

$$\overline{B^*} = \{ dy^{0a}, dy^{1a}, \dots, dy^{ka} \}.$$

As elements of this basis are not transforming as tensors, we introduce the special adapted basis

$$\overline{B^*} = \{ \delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka} \}$$

where

$$\begin{bmatrix} \delta y^{0b} \\ \delta y^{1b} \\ \delta y^{2b} \\ \vdots \\ \delta y^{kb} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \delta_c^b & 0 & 0 & \dots & 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} M_{0c}^{1b} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta_c^b & 0 & \dots & 0 \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} M_{0c}^{2b} & \begin{pmatrix} 2 \\ 1 \end{pmatrix} M_{0c}^{1b} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} \delta_c^b & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \begin{pmatrix} k \\ 0 \end{pmatrix} M_{0c}^{kb} & \begin{pmatrix} k \\ 1 \end{pmatrix} M_{0c}^{(k-1)b} & \begin{pmatrix} k \\ 2 \end{pmatrix} M_{0c}^{(k-2)b} & \dots & \begin{pmatrix} k \\ k \end{pmatrix} \delta_c^b \end{bmatrix} \begin{bmatrix} dy^{0c} \\ dy^{1c} \\ dy^{2c} \\ \vdots \\ dy^{kc} \end{bmatrix} \quad (1.2)$$

or shorter

$$[\delta y^b] = [M_{(c)}^{(b)}] [dy^c]. \quad (1.3)$$

Theorem 1.2. *The necessary and sufficient conditions that δy^{Aa} are transformed as d -tensors, i.e.*

$$\delta y^{Aa'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{Aa}, \quad A = \overline{0, k} \quad (1.4)$$

are the following equations:

$$M_{0b}^{1a} \partial_{0a} y^{0b'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} M_{0c}^{1b'} \partial_{0b} y^{0c'} + \partial_{0b} y^{1b'}, \quad (1.5)$$

$$M_{0b}^{2a} \partial_{0a} y^{0b'} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} M_{0c}^{2b'} \partial_{0b} y^{0c'} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} M_{0c}^{1b'} \partial_{0b} y^{1c'} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \partial_{0b} y^{2b'}, \dots,$$

$$M_{0b}^{ka} \partial_{0a} y^{0b'} = \begin{pmatrix} k \\ 0 \end{pmatrix} M_{0c}^{kb'} \partial_{0b} y^{0c'} + \begin{pmatrix} k \\ 1 \end{pmatrix} M_{0c}^{(k-1)b'} \partial_{0b} y^{1c'} + \dots + \begin{pmatrix} k \\ k \end{pmatrix} \partial_{0b} y^{kb'}.$$

The natural basis \overline{B} of $T(E)$ is

$$\overline{B} = \{ \partial_{0a}, \partial_{1a}, \dots, \partial_{ka} \}.$$

The elements of this basis are not tensors, so we introduce the special adapted basis

$$B = \{ \delta_{0a}, \delta_{1a}, \dots, \delta_{ka} \},$$

where

$$[\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}] = [\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}] [N_{(a)}^{(b)}], \quad (1.6)$$

$$[N_{(a)}^{(b)}] = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \delta_a^b & 0 & 0 & \dots & 0 \\ -\begin{pmatrix} 1 \\ 0 \end{pmatrix} N_{0a}^{1b} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta_a^b & 0 & \dots & 0 \\ -\begin{pmatrix} 2 \\ 0 \end{pmatrix} N_{0a}^{2b} & -\begin{pmatrix} 2 \\ 1 \end{pmatrix} N_{0a}^{1b} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} \delta_a^b & \dots & 0 \\ \vdots & & & & \\ -\begin{pmatrix} k \\ 0 \end{pmatrix} N_{0a}^{kb} & -\begin{pmatrix} k \\ 1 \end{pmatrix} N_{0a}^{(k-1)b} & -\begin{pmatrix} k \\ 2 \end{pmatrix} N_{0a}^{(k-2)b} & \dots & \begin{pmatrix} k \\ k \end{pmatrix} \delta_a^b \end{bmatrix} \begin{bmatrix} dy^{0c} \\ dy^{1c} \\ dy^{2c} \\ \vdots \\ dy^{kc} \end{bmatrix} \quad (1.7)$$

or shorter

$$[\delta_a] = [\partial_b] [N_{(a)}^{(b)}]. \quad (1.8)$$

Theorem 1.3. *The necessary and sufficient conditions that δy_{Aa} , $A = \overline{0, k}$ are transformed as d -tensors, i.e.*

$$\delta_{Aa'} = \frac{\partial x^a}{\partial x^{a'}} \delta_{Aa} \quad (= (\partial_{0a'} y^{0a}) \delta_{Aa}) \quad (1.9)$$

are the following equations:

$$N_{0a'}^{Bb'} (\partial_a x^{a'}) = N_{0a}^{Bc} \partial_{Bc} y^{Bb'} + N_{0a}^{(B-1)c} \partial_{(B-1)c} y^{Bb'} + \dots + N_{0a}^{1c} \partial_{1c} y^{Bb'} - \partial_{0a} y^{Bb'}, \quad (1.10)$$

$$1 \leq B \leq k .$$

Theorems 1.2 and 1.3 first time was proved in [6] and [7] for the special adapted bases B and B^* which are comprehensive with the structure J . In [6] and [7] one solution of (1.5) and (1.10) using only the metric tensor in $M(x^a)$ was given. They are formally different from these theorems because in [5]-[12] instead of y^{Aa} from (1.1) it appears $\frac{y^{Aa}}{A!}$, $A = \overline{1, k}$.

Theorem 1.4. *If the bases $\overline{B^*}$ and \overline{B} are dual to each other, then B^* and B will be dual if $NM = I$ i.e. the matrices N and M are inverse to each other.*

Proof. By assumption is $[dy^c][\partial_b] = \delta_b^c I$. From (1.3) and (1.8) we have

$$[\delta y^d][\delta_a] = [M_{(c)}^{(d)}][dy^c][\partial_b][N_{(a)}^{(b)}] = [M_{(b)}^{(d)}]\delta_b^c I [N_{(a)}^{(b)}] = [M_{(b)}^{(d)}][N_{(a)}^{(b)}] = \delta_a^d I \quad (1.11)$$

In this form, Theorem 1.4 was proved in [2], [3], but in the explicit form it was given already in [6], [7].

2. THE PROJECTION OPERATORS

The projection operators are well known in linear algebra. Here they are presented in the tensor form in the special adapted bases of $Osc^k M$.

Let us denote by $H^*, V_1^*, V_2^*, \dots, V_k^*$ the subspaces of $T^*(E)$ generated by $\{\delta y^{0a}\}, \{\delta y^{1a}\}, \dots, \{\delta y^{ka}\}$ respectively. The following decomposition is true:

$$T^*(E) = H^* \oplus V_1^* \oplus V_2^* \oplus \dots \oplus V_k^* .$$

Let us denote by H, V_1, V_2, \dots, V_k the subspaces of $T(E)$ generated by $\{\delta_{0a}\}, \{\delta_{1a}\}, \{\delta_{2a}\}, \dots, \{\delta_{ka}\}$ respectively. Then

$$T(E) = H \oplus V_1 \oplus V_2 \oplus \dots \oplus V_k .$$

If $X \in T(E)$, then we can write

$$X = X^{0a} \delta_{0a} + X^{1a} \delta_{1a} + X^{2a} \delta_{2a} + \dots + X^{ka} \delta_{ka} = X^{Aa} \delta_{Aa} \quad (2.1)$$

Let us define p, p_1, p_2, \dots, p_k , the projector operators of $T(E)$ on H, V_1, V_2, \dots, V_k respectively, where:

$$p_0 = \delta_{0b} \otimes \delta y^{0b} = \begin{bmatrix} \delta_{0b} & \delta_{1b} & \dots & \delta_{kb} \end{bmatrix} \otimes \begin{bmatrix} \delta y^{0b} \\ \delta y^{1b} \\ \vdots \\ \delta y^{kb} \end{bmatrix}, \quad (2.2)$$

$$p_1 = \delta_{1b} \otimes \delta y^{1b},$$

$$\vdots$$

$$p_k = \delta_{kb} \otimes \delta y^{kb} = \begin{bmatrix} \delta_{0b} & \delta_{1b} & \dots & \delta_{kb} \end{bmatrix} \otimes \begin{bmatrix} \delta y^{0b} \\ \delta y^{1b} \\ \vdots \\ \delta y^{kb} \end{bmatrix}.$$

We have:

$$p_0 + p_1 + p_2 + \dots + p_k = I.$$

Theorem 2.1. *For the vector field X given by (2.1) and the projector operators defined by (2.2) the following decomposition is valid*

$$X = p_0 X + p_1 X + p_2 X + \dots + p_k X = (p_0 + p_1 + p_2 + \dots + p_k) X,$$

where

$$p_0 X = X^{0b} \delta_{0b}, \quad p_1 X = X^{1b} \delta_{1b}, \dots, \quad p_k X = X^{kb} \delta_{kb}.$$

The one form field $\omega \in T^*(E)$ can be written:

$$\omega = \omega_{0a} \delta y^{0a} + \omega_{1a} \delta y^{1a} + \dots + \omega_{ka} \delta y^{ka}. \quad (2.3)$$

Let us define $p_0^*, p_1^*, \dots, p_k^*$ the projector operators of $T^*(E)$ on $H^*, V_1^*, V_2^*, \dots, V_k^*$ respectively, with

$$p_0^* = \delta y^{0b} \otimes \delta_{0b} \quad (2.4)$$

$$p_1^* = \delta y^{1b} \otimes \delta_{1b}, \dots,$$

$$p_k^* = \delta y^{kb} \otimes \delta_{kb}.$$

If we write $p_0^*, p_1^*, \dots, p_k^*$ in the matrix form, it is easy to see that

$$p_0^* = \overline{p_0}, p_1^* = \overline{p_1}, \dots, p_k^* = \overline{p_k} \quad (2.5)$$

where "-" means: transposed of.

We have

$$p_0^* + p_1^* + \dots + p_k^* = I \quad (2.6)$$

Theorem 2.2. For the one-form field w given by (2.3) and the projector operators $p_0^*, p_1^*, \dots, p_k^*$ defined by (2.4) the following relation is valid:

$$w = p_0^* w + p_1^* w + \dots + p_k^* w = wp_0 + wp_1 + \dots + wp_k, \quad (2.7)$$

where

$$w_{0b} \delta y^{0b} = p_0^* w = wp_0,$$

$$w_{1b} \delta y^{1b} = p_1^* w = wp_1, \dots,$$

$$w_{kb} \delta y^{kb} = p_k^* w = wp_k.$$

3. THE J AND θ STRUCTURES

The structure J is a tensor field on space $T(E) \otimes T^*(E)$ defined by

$$J = [\delta_{0a} \delta_{1a} \delta_{2a} \dots \delta_{ka}] = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \dots & k & 0 \end{bmatrix} \otimes \begin{bmatrix} \delta y^{0a} \\ \delta y^{1a} \\ \delta y^{2a} \\ \vdots \\ \delta y^{ka} \end{bmatrix} =$$

$$\delta_{1a} \otimes \delta y^{0a} + 2\delta_{2a} \otimes \delta y^{1a} + 3\delta_{3a} \otimes \delta y^{2a} + \dots + k\delta_{ka} \otimes \delta y^{(k-1)a} =$$

$$\partial_{1a} \otimes dy^{0a} + 2\partial_{2a} \otimes dy^{1a} + 3\partial_{3a} \otimes dy^{2a} + \dots + k\partial_{ka} \otimes dy^{(k-1)a}. \quad (3.1)$$

From Definition 3.1, we get

Remark 3.1. We have

$$J\delta_{0b} = \delta_{1b}, J\delta_{1b} = 2\delta_{2b}, J\delta_{2b} = 3\delta_{3b}, \dots, J\delta_{(k-1)b} = k\delta_{kb}, J\delta_{kb} = 0 \quad (3.2)$$

From the above it follows

$$J : H \rightarrow V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_{k-1} \rightarrow V_k \rightarrow 0.$$

From (3.1) it follows

Remark 3.2.

$$\delta y^{0b} J = 0, \delta y^{1b} J = \delta y^{0b}, \delta y^{2b} J = 2\delta y^{1b}, \dots, \delta y^{kb} J = k\delta y^{(k-1)b} \quad (3.3)$$

i.e.

$$0 \leftarrow H^* \leftarrow V_1^* \leftarrow V_2^* \leftarrow \dots \leftarrow V_{k-1}^* \leftarrow V_k^* : J.$$

Definition 3.2. The structure θ is a tensor field on space $T(E) \otimes T^*(E)$ defined by

$$\theta = \left[\begin{array}{cccccc} \delta_{0a} & \delta_{1a} & \delta_{2a} & \dots & \delta_{ka} \\ \delta_{0a} & \delta_{1a} & \delta_{2a} & \dots & \delta_{ka} \\ \delta_{0a} & \delta_{1a} & \delta_{2a} & \dots & \delta_{ka} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{0a} & \delta_{1a} & \delta_{2a} & \dots & \delta_{ka} \\ \delta_{0a} & \delta_{1a} & \delta_{2a} & \dots & \delta_{ka} \end{array} \right] \otimes \left[\begin{array}{c} \delta y^{0a} \\ \delta y^{1a} \\ \delta y^{2a} \\ \vdots \\ \delta y^{(k-1)a} \\ \delta y^{ka} \end{array} \right] =$$

$$\delta_{0a} \otimes \delta y^{1a} + \frac{1}{2} \delta_{1a} \otimes \delta y^{2a} + \dots + \frac{1}{k} \delta_{(k-1)a} \otimes \delta y^{ka} = \quad (3.4)$$

$$\partial_{0a} \otimes dy^{1a} + \frac{1}{2} \partial_{1a} \otimes dy^{2a} + \dots + \frac{1}{k} \partial_{(k-1)a} \otimes dy^{ka}.$$

Remark 3.3. The structure θ satisfies the following relations:

$$\delta y^{0b} \theta = \delta y^{1b}, \delta y^{1b} \theta = \frac{1}{2} \delta y^{2b}, \dots, \delta y^{(k-1)b} \theta = \frac{1}{k} \delta y^{kb}, \delta y^{kb} \theta = 0 \quad (3.5)$$

i.e.

$$0 \leftarrow V_k^* \leftarrow V_{k-1}^* \leftarrow \dots \leftarrow V_2^* \leftarrow V_1^* \leftarrow H : \theta.$$

From (3.4) it follows

Remark 3.4. The following relations are valid

$$\theta \delta_{0b} = 0, \theta \delta_{1b} = \delta_{0b}, \theta \delta_{2b} = \frac{1}{2} \delta_{1b}, \dots, \theta \delta_{kb} = \frac{1}{k} \delta_{(k-1)b}, \quad (3.6)$$

i.e.

$$\theta : V_k \rightarrow V_{k-1} \rightarrow \dots \rightarrow V_2 \rightarrow V_1 \rightarrow H \rightarrow 0.$$

Definition 3.3. The transpose of the structure J denoted by \bar{J} is a tensor field on $T^*(E) \otimes T(E)$ defined by

$$\bar{J} = \left[\begin{array}{cccccc} \delta y^{0a} & \delta y^{1a} & \delta y^{2a} & \dots & \delta y^{ka} \\ \delta y^{0a} & \delta y^{1a} & \delta y^{2a} & \dots & \delta y^{ka} \\ \delta y^{0a} & \delta y^{1a} & \delta y^{2a} & \dots & \delta y^{ka} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta y^{0a} & \delta y^{1a} & \delta y^{2a} & \dots & \delta y^{ka} \\ \delta y^{0a} & \delta y^{1a} & \delta y^{2a} & \dots & \delta y^{ka} \end{array} \right] \otimes \left[\begin{array}{c} \delta_{0a} \\ \delta_{1a} \\ \vdots \\ \delta_{ka} \end{array} \right] =$$

$$\delta y^{0a} \otimes \delta_{1a} + 2\delta y^{1a} \otimes \delta_{2a} + 3\delta y^{2a} \otimes \delta_{3a} + \dots + k\delta y^{(k-1)a} \otimes \delta_{ka} \quad (3.7)$$

Definition 3.4. The transpose of the structure θ denoted by $\bar{\theta}$ is a tensor field on $T^*(E) \otimes T(E)$ defined by

$$\bar{\theta} = \begin{bmatrix} \delta y^{0a} & \delta y^{1a} & \delta y^{2a} & \dots & \delta y^{ka} \end{bmatrix} \otimes \begin{bmatrix} \delta_{0a} \\ \delta_{1a} \\ \delta_{2a} \\ \vdots \\ \delta_{ka} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \frac{1}{k} 0 \end{bmatrix} \quad (3.8)$$

$$\delta y^{1a} \otimes \delta_{0a} + \frac{1}{2} \delta y^{2a} \otimes \delta_{1a} + \dots + \frac{1}{k} \delta y^{ka} \otimes \delta_{(k-1)a}$$

Remark 3.5. For the structure \bar{J} the following relations are valid:

$$\bar{J} \delta y^{0b} = 0, \quad \bar{J} \delta y^{1b} = \delta y^{0b}, \quad \bar{J} \delta y^{2b} = 2\delta y^{1b}, \dots, \quad \bar{J} \delta y^{kb} = k\delta y^{(k-1)b}, \quad (3.9)$$

i.e.

$$\bar{J} : V_k^* \rightarrow V_{k-1}^* \rightarrow \dots \rightarrow V_2^* \rightarrow V_1^* \rightarrow H^* \rightarrow 0$$

$$\delta_{0b} \bar{J} = \delta_{1b}, \quad \delta_{1b} \bar{J} = 2\delta_{2b}, \quad \delta_{2b} \bar{J} = 3\delta_{3b}, \dots, \quad \delta_{(k-1)b} \bar{J} = k\delta_{kb}, \quad \delta_{kb} \bar{J} = 0 \quad (3.10)$$

i.e.

$$0 \leftarrow V_k \leftarrow V_{k-1} \leftarrow \dots \leftarrow V_2 \leftarrow V_1 \leftarrow H : \bar{J}.$$

Remark 3.6. For the structure $\bar{\theta}$ the following relations are valid:

$$\bar{\theta} \delta y^{0b} = \delta y^{1b}, \quad \bar{\theta} \delta y^{1b} = \frac{1}{2} \delta y^{2b}, \dots, \quad \bar{\theta} \delta y^{(k-1)b} = \frac{1}{k} \delta y^{kb}, \quad \bar{\theta} \delta y^{kb} = 0, \quad (3.11)$$

i.e.

$$\bar{\theta} : H^* \rightarrow V_1^* \rightarrow V_2^* \rightarrow \dots \rightarrow V_k^* \rightarrow 0$$

$$\delta_{0b} \bar{\theta} = 0, \quad \delta_{1b} \bar{\theta} = \delta_{ob}, \quad \delta_{2b} \bar{\theta} = \frac{1}{2} \delta_{1a}, \dots, \quad \delta_{kb} \bar{\theta} = \frac{1}{k} \delta_{(k-1)b}, \quad (3.12)$$

i.e.

$$0 \leftarrow H \leftarrow V_1 \leftarrow V_2 \leftarrow \dots \leftarrow V_{k-1} \leftarrow V_k : \bar{\theta}.$$

Theorem 3.1. *The structures J , \bar{J} , θ , $\bar{\theta}$, the projectors $p_0, p_1, \dots, p_k, p_0^*, p_1^*, \dots, p_k^*$ are connected by*

$$J\theta = I - p_0, \quad \bar{J}\bar{\theta} = I - p_k^* \quad (3.13)$$

$$\theta J = I - p_k, \quad \bar{\theta}\bar{J} = I - p_0^*. \quad (3.14)$$

Proof. It is easy to see that

$$\begin{aligned} J\theta &= \delta_{1a} \otimes \delta^{1a} + \delta_{2a} \otimes \delta^{2a} + \dots + \delta_{ka} \otimes \delta^{ka}, \\ \theta J &= \delta_{0a} \otimes \delta^{0a} + \delta_{1a} \otimes \delta^{1a} + \dots + \delta_{(k-1)a} \otimes \delta^{(k-1)a}, \\ \bar{J}\bar{\theta} &= \delta y^{0a} \otimes \delta_{0a} + \delta y^{1a} \otimes \delta_{1a} + \dots + \delta y^{(k-1)a} \otimes \delta_{(k-1)a}, \\ \bar{\theta}\bar{J} &= \delta y^{1a} \otimes \delta_{1a} + \delta y^{2a} \otimes \delta_{2a} + \dots + \delta y^{ka} \otimes \delta_{ka}. \end{aligned}$$

Proposition 3.1. *From the above it follows that*

$$\begin{aligned} J\theta &\text{ is the identity operator on } V_1 \oplus V_2 \oplus \dots \oplus V_k, \\ \theta J &\text{ is the identity operator on } H^* \oplus V_1^* \oplus \dots \oplus V_{k-1}^*, \\ J\theta &\text{ is the identity operator on } H \oplus V_1 \oplus \dots \oplus V_{k-1}, \\ \theta J &\text{ is the identity operator on } V_1^* \oplus V_2^* \oplus \dots \oplus V_k^*, \end{aligned}$$

Theorem 3.2. *The structures J , \bar{J} , θ , $\bar{\theta}$ are k -tangent structures, namely:*

$$J^{k+1} = 0, \quad \bar{J}^{k+1} = 0, \quad \theta^{k+1} = 0, \quad \bar{\theta}^{k+1} = 0$$

Proof. The proof is obtained by direct calculation.

Remark 3.7. In $Osc^1 M$ $p_k = p_1, p_k^* = p_1^*, p_0 + p_1 = I, p_0^* + p_1^* = I$, and from (3.13) we obtain $J\theta + \theta J = I, \bar{J}\bar{\theta} + \bar{\theta}\bar{J} = I$.

The first relation can be found in [5].

One kind of the Liouville vector fields in the natural basis of $T(Osc^k M)$ [1], have the form

$$\Gamma_{(i)} = \begin{pmatrix} k \\ 0 \end{pmatrix} y^{1a} \partial_{ka}, \quad (3.15)$$

$$\begin{aligned}\Gamma_{(2)} &= \binom{k}{1} y^{2a} \partial_{ka} + \binom{k-1}{0} y^{1a} \partial_{(k-1)a}, \\ \Gamma_{(3)} &= \binom{k}{3} y^{3a} \partial_{ka} + \binom{k-1}{1} y^{2a} \partial_{(k-1)a} + \binom{k-2}{0} y^{1a} \partial_{(k-2)a}, \dots, \\ \Gamma_{(k)} &= \binom{k}{k-1} y^{ka} \partial_{ka} + \binom{k-1}{k-2} y^{(k-1)a} \partial_{(k-1)a} + \dots + \binom{2}{1} y^{2a} \partial_{2a} + \binom{1}{0} y^{1a} \partial_{1a}.\end{aligned}$$

It can be proved that $(k-(i-1))!\Gamma_{(i)}$ given above are exactly the Liouville vector fields $\Gamma^{(i)}$ given by R. Miron and Gh. Atanasiu in [6], [7].

The action of J structure on Liouville vector fields was determined in [6], [7] and in some modified version in [1]-[3].

It is known that the k -structure J transform the Liouville vector fields in the following way [1]:

$$J\Gamma = 2\Gamma_{(k-1)}, J\Gamma_{(k-1)} = 3\Gamma_{(k-2)}, \dots, J\Gamma_{(3)} = (k-1)\Gamma_2, J\Gamma_{(2)} = k\Gamma_{(1)}, J\Gamma_{(1)} = 0 \quad (3.16)$$

The connection between Liouville vector fields and the structure θ are given by

Theorem 3.3. *The action of the structure θ on the vector fields Γ are given by*

$$\begin{aligned}\Gamma_{(2)} &= k\theta\Gamma_{(1)} + \binom{k}{1} y^{2a} \partial_{ka}, \\ \Gamma_{(3)} &= (k-1)\theta\Gamma_{(2)} + \binom{k}{2} y^{3a} \partial_{ka}, \\ \Gamma_{(4)} &= (k-2)\theta\Gamma_{(3)} + \binom{k}{3} y^{4a} \partial_{ka}, \\ \Gamma_{(k)} &= 2\theta\Gamma_{(k-1)} + \binom{k}{k-1} y^{ka} \partial_{ka}.\end{aligned} \quad (3.17)$$

Proof. As $J\theta = \partial_{1a} \otimes dy^{1a} + \partial_{2a} \otimes dy^{2a} + \dots + \partial_{ka} \otimes dy^{ka}$, the action of structure J on $\Gamma_{(2)}, \Gamma_{(3)}, \dots, \Gamma_{(k)}$ gives ($J\partial_{ka} = 0$)

$$j\Gamma_{(2)} = k\Gamma_{(1)}, j\Gamma_{(3)} = (k-1)\Gamma_{(2)}, j\Gamma_{(4)} = (k-2)\Gamma_{(3)}, \dots, j\Gamma_{(k)} = 2\Gamma_{(k)}.$$

The above equations are (3.16).

4. THE SUBSPACES OF $Osc^k M$

The theory of subspaces of $Osc^k M$ in the form used here, is given in [4]. For the understanding the action of operators J , θ and p on subspaces, we recall the notations, definitions and theorems, which give relations between different adapted bases.

Here some special case of the general transformation (1.1) of M will be considered, namely, when

$$y^{0a} = y^{0a}(u^{01}, \dots, u^{0m}, v^{0(m+1)}, \dots, v^{0n}) = y^{0a}(u^{0a}, v^{0\hat{a}}), \quad (4.1)$$

$$a, b, c, \dots = 1, 2, \dots, n, \quad \alpha, \beta, \gamma, \delta, \dots = 1, 2, \dots, m, \quad \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \dots = m+1, \dots, n$$

and the new coordinates of the point u in the base manifold M with respect to another chart (U', φ') are $(u^{01'}, \dots, u^{0m'}, v^{0(m+1)'}, \dots, v^{0n'})$, where

$$\begin{aligned} u^{0a'} &= u^{0a'}(u^{01}, \dots, u^{0m}), \quad v^{0\hat{a}'} = v^{0\hat{a}'}(v^{0(m+1)}, \dots, v^{0n}), \\ y^{0a'} &= y^{0a'}(u^{01'}, \dots, u^{0m'}, v^{0(m+1)'}, \dots, v^{0n'}) = y^{0a'}(u^{0a'}, v^{0\hat{a}'}). \end{aligned}$$

We shall use the notations

$$\partial_\alpha = \partial_{0\alpha} = \frac{\partial}{\partial u^{0\alpha}}, \quad \partial_{\hat{\alpha}} = \partial_{0\hat{\alpha}} = \frac{\partial}{\partial v^{0\hat{\alpha}}}.$$

$$B_\alpha^{a'} = \partial_{0\alpha} u^{0a'}, \quad B_{\hat{\alpha}}^{\hat{a}'} = \partial_{0\hat{\alpha}} v^{0\hat{a}'},$$

$$B_\alpha^a = \partial_{0\alpha} y^{0a} = \partial_\alpha x^a, \quad B_{\hat{\alpha}}^{\hat{a}} = \partial_{0\hat{\alpha}} y^{0a} = \partial_{\hat{\alpha}} x^a.$$

If the transformation (4.1) is regular, then there exists an inverse transformation:

$$u^{0\alpha} = u^{0\alpha}(y^{0a}), \quad v^{0\hat{\alpha}} = v^{0\hat{\alpha}}(y^{0a}).$$

Let us denote

$$B_a^\beta = \frac{\partial u^\beta}{\partial x^a} = \frac{\partial u^{0\beta}}{\partial x^{0a}}, \quad B_{\hat{a}}^{\hat{\beta}} = \frac{\partial v^{\hat{\beta}}}{\partial x^a} = \frac{\partial v^{0\hat{\beta}}}{\partial y^{0a}}$$

then the following equations are valid:

$$B_a^\beta B_\alpha^a = \delta_\alpha^\beta, \quad B_a^{\hat{\beta}} B_\alpha^a = 0, \quad B_a^\beta B_\alpha^a = 0, \quad B_a^{\hat{\beta}} B_{\hat{\alpha}}^a = \delta_{\hat{\alpha}}^{\hat{\beta}}, \quad B_\alpha^a B_b^a + B_{\hat{\alpha}}^a B_b^{\hat{\alpha}} = \delta_b^a \quad (4.2)$$

We shall use the notations:

$$y^{1a} = \frac{dy^{0a}}{dt}, \dots, y^{ka} = \frac{d^k y^{0a}}{dt^k},$$

$$u^{1\alpha} = \frac{du^{0\alpha}}{dt}, \dots, u^{k\alpha} = \frac{d^k u^{0\alpha}}{dt^k},$$

$$v^{1\hat{\alpha}} = \frac{dv^{0\hat{\alpha}}}{dt}, \dots, v^{k\hat{\alpha}} = \frac{d^k v^{0\hat{\alpha}}}{dt^k}$$

In the base manifold we can construct two families of subspaces M_1 and M_2 given by equations

$$M_1 : y^{0\alpha} = y^{0\alpha}(u^{0\alpha}, C^{0\hat{\alpha}}), \quad M_2 : y^{0\alpha} = y^{0\alpha}(C^{0\alpha}, v^{0\hat{\alpha}}),$$

where we suppose, that the functions appeared in (4.1) are C^∞ . The subspaces M_1 and M_2 of M induces subspaces $E_1 = Osc^k M_1$ and $E_2 = Osc^k M_2$ in $Osc^k M$. Some point $u \in E_1$ has coordinates $(u^{0\alpha}, u^{1\alpha}, \dots, u^{k\alpha})$ and some point $v \in E_2$ has coordinates $(v^{0\hat{\alpha}}, v^{1\hat{\alpha}}, \dots, v^{k\hat{\alpha}})$. We have

$$\dim(Osc^k M_1) = (k+1)m, \quad \dim(Osc^k M_2) = (k+1)(n-m)$$

We can construct the special adapted bases B_1 and B_1^* of $T(E_1)$ and $T^*(E_1)$, further B_2 and B_2^* of $T(E_2)$ and $T^*(E_2)$ respectively.

$$B_1 = \{\delta_{0\alpha}, \delta_{1\alpha}, \dots, \delta_{k\alpha}\}, \quad B_2 = \{\delta_{0\hat{\alpha}}, \delta_{1\hat{\alpha}}, \dots, \delta_{k\hat{\alpha}}\},$$

$$B_1^* = \{\delta u^{0\alpha}, \delta u^{1\alpha}, \dots, \delta u^{k\alpha}\}, \quad B_2^* = \{\delta v^{0\hat{\alpha}}, \delta v^{1\hat{\alpha}}, \dots, \delta v^{k\hat{\alpha}}\}.$$

Definition 3.4. *The special adapted bases B_1, B_1^*, B_2, B_2^* are defined by*

$$[\delta_{(\alpha)}] = [\partial_{(\beta)}] [N_{(\alpha)}^{(\beta)}], \quad \left[\delta_{\binom{\hat{\alpha}}{\alpha}} \right] = \left[\partial_{\binom{\hat{\beta}}{\beta}} \right] \left[N_{\binom{\hat{\alpha}}{\alpha}}^{\binom{\hat{\beta}}{\beta}} \right],$$

$$[\delta u^{(\alpha)}] = [M_{(\beta)}^{(\alpha)}] [du^{(\beta)}], \quad \left[\delta v^{\binom{\hat{\alpha}}{\alpha}} \right] = \left[M_{\binom{\hat{\beta}}{\beta}}^{\binom{\hat{\alpha}}{\alpha}} \right] \left[dv^{\binom{\hat{\beta}}{\beta}} \right],$$

where $[N_{\beta}^{\alpha}]$ and $\left[N_{\binom{\hat{\beta}}{\beta}}^{\binom{\hat{\alpha}}{\alpha}} \right]$ are matrices obtained from (1.7) by substitution $(a,b) \rightarrow (\alpha, \beta)$

and $(a,b) \rightarrow \left(\binom{\hat{\alpha}}{\alpha}, \binom{\hat{\beta}}{\beta} \right)$ respectively. The matrices $[M_{\beta}^{\alpha}]$ and $\left[M_{\binom{\hat{\beta}}{\beta}}^{\binom{\hat{\alpha}}{\alpha}} \right]$ are obtained from

(1.2) by substitution $(a,b) \rightarrow (\alpha,\beta)$ and $(a,b) \rightarrow \left(\overset{\wedge}{\alpha}, \overset{\wedge}{\beta}\right)$ respectively.

Theorem 4.1. *The necessary and sufficient condition that $\delta u^{a\alpha}(\delta v^{A\hat{\alpha}})$ are transformed as d -tensors, i.e.*

$$\delta u^{A\alpha'} = B_{\alpha}^{\alpha'} \delta u^{A\alpha} \left[\delta v^{a\alpha'} = B_{\alpha}^{\hat{\alpha}} \delta v^{A\hat{\alpha}} \right], A = \overline{0, k} \quad (4.3)$$

are given by (1.5) if (a,b,c,y) is substituted by (α,β,γ,u) [(a,b,c,y) is substituted by $(\overset{\wedge}{\alpha}, \overset{\wedge}{\beta}, \overset{\wedge}{\gamma}, \hat{v})$].

Theorem 4.2. *The necessary and sufficient condition that $\delta_{A\alpha}[\delta_{A\hat{\alpha}}]$ are transformed as d -tensors, i.e.*

$$\delta_{A\alpha'} = B_{\alpha}^{\alpha'} \delta u_{A\alpha'} \left[\delta_{A\hat{\alpha}} = B_{\alpha'}^{\hat{\alpha}} \delta_{A\hat{\alpha}} \right] A = \overline{0, k} \quad (4.4)$$

are given by (1.10) if (a,b,c,y) is substituted by (α,β,γ,u) [(a,b,c,y) is substituted by $(\overset{\wedge}{\alpha}, \overset{\wedge}{\beta}, \overset{\wedge}{\gamma}, \hat{v})$].

Theorem 4.3. *The necessary and sufficient conditions that B_1^* be dual B_1 , B_2^* be dual to B_2 are the following equations*

$$\begin{aligned} \left[M_{(\gamma)}^{(\beta)} \right] \left[N_{(\alpha)}^{(\gamma)} \right] &= \delta_{\alpha}^{\beta} I_{m \times m} \\ \left[M_{\left(\overset{\wedge}{\gamma}\right)}^{\left(\overset{\wedge}{\beta}\right)} \right] \left[N_{\left(\overset{\wedge}{\alpha}\right)}^{\left(\overset{\wedge}{\gamma}\right)} \right] &= \delta_{\overset{\wedge}{\alpha}}^{\overset{\wedge}{\beta}} I_{(n-m) \times (n-m)}. \end{aligned}$$

Now we want to obtain the relations between the adapted bases B and B' , where

$$\begin{aligned} B &= \{\delta_{0\alpha}, \delta_{1\alpha}, \dots, \delta_{k\alpha}\}, \\ B' &= B_1 \cup B_2 = \{\delta_{0\alpha}, \delta_{0\hat{\alpha}}, \delta_{1\alpha}, \delta_{1\hat{\alpha}}, \dots, \delta_{k\alpha}, \delta_{k\hat{\alpha}}\}, \end{aligned}$$

further between B^* and B'^* , where

$$\begin{aligned} B^* &= \{\delta y^{0\alpha}, \delta y^{1\alpha}, \dots, \delta y^{k\alpha}\}, \\ B'^* &= B_1^* \cup B_2^* = \{\delta u^{0\alpha}, \delta v^{0\hat{\alpha}}, \delta u^{1\alpha}, \delta v^{1\hat{\alpha}}, \dots, \delta u^{k\alpha}, \delta v^{k\hat{\alpha}}\}. \end{aligned}$$

The adapted basis B, B', B^*, B'^* are functions of

$$\left[M_{(b)}^{(a)} \right], \left[N_{(b)}^{(a)} \right], \left[M_{(\beta)}^{(\alpha)} \right], \left[N_{(\beta)}^{(\alpha)} \right], \left[M_{\hat{\beta}}^{\hat{\alpha}} \right], \left[N_{\hat{\beta}}^{\hat{\alpha}} \right],$$

which have to satisfy the conditions given in previous text. It is clear, that the adapted bases are not uniquely determined.

For the easier calculations we want to obtain such adapted bases, for which the following relations are valid.

$$\delta_{Aa} = B_a^\alpha \delta_{A\alpha} + B_a^{\hat{\alpha}} \delta_{A\hat{\alpha}}, \quad A = 0, 1, \dots, k \quad (4.5)$$

$$\delta y^{Aa} = B_\alpha^a \delta u^{A\alpha} + B_{\hat{\alpha}}^a \delta v^{A\hat{\alpha}}, \quad A = 0, 1, \dots, k \quad (4.6)$$

The adapted bases $B, B^{\wedge}, B', B'^{\wedge}$ satisfy (4.5) and (4.6)

Theorem 4.4. *The adapted bases B, B^*, B', B'^* satisfy (4.5) and (4.6) if different M and N are connected by*

$$\left[M_{(b)}^{(a)} \right] \left[B_{(\beta)}^{(b)} \right] = \left[B_{(\alpha)}^{(a)} \right] \left[M_{(\beta)}^{(\alpha)} \right], \quad \left[M_{(b)}^{(a)} \right] \left[B_{\hat{\beta}}^{(b)} \right] = \left[B_{\hat{\alpha}}^{(a)} \right] \left[M_{\hat{\beta}}^{\hat{\alpha}} \right] \quad (4.7)$$

and

$$\left[B_{(b)}^{(\beta)} \right] \left[N_{(a)}^{(b)} \right] = \left[N_{(\alpha)}^{(\beta)} \right] \left[B_{(a)}^{(\alpha)} \right], \quad \left[B_{\hat{\beta}}^{(\beta)} \right] \left[N_{(a)}^{(b)} \right] = \left[N_{\hat{\alpha}}^{\hat{\beta}} \right] \left[B_{(a)}^{\hat{\alpha}} \right]. \quad (4.8)$$

The proof of Theorems 4.1-4.4 are given in [4].

5. THE STRUCTURES P, J AND θ ON THE SUBSPACES

Let us denote by $H', V'_1, V'_2, \dots, V'_k$ the subspaces of $T(E_1)$ generated by $\{\delta_{0\alpha}\}, \{\delta_{1\alpha}\}, \dots, \{\delta_{k\alpha}\}$ respectively and by $H'', V''_1, V''_2, \dots, V''_k$ the subspaces of $T(E_2)$ generated by $\{\delta_{0\hat{\alpha}}\}, \{\delta_{1\hat{\alpha}}\}, \dots, \{\delta_{k\hat{\alpha}}\}$ respectively.

Now we have

$$T(E_1) = H' \oplus V'_1 \oplus \dots \oplus V'_k,$$

$$T(E_2) = H'' \oplus V''_1 \oplus \dots \oplus V''_k$$

Let us denote by H^*, V^*_1, \dots, V^*_k the subspaces of $T^*(E_1)$ generated by $\{\delta u^{0\alpha}\}, \{\delta u^{1\alpha}\}, \dots, \{\delta u^{k\alpha}\}$ respectively and by H^*, V^*_1, \dots, V^*_k the subspaces of $T^*(E_2)$ generated by $\{\delta v^{0\hat{\alpha}}\}, \{\delta v^{1\hat{\alpha}}\}, \dots, \{\delta v^{k\hat{\alpha}}\}$.

The following relation is valid

$$T^*(E_1) = H^* \oplus V^*_1 \oplus \dots \oplus V^*_k,$$

$$T^*(E_2) = H^* \oplus V^*_1 \oplus \dots \oplus V^*_k.$$

The basis vectors of B , B^* , B_1 , B_2 , B_1^* and B_2^* are connected by (4.5) and (4.6) i.e.

$$\delta_{A\alpha} = B_a^\alpha \delta_{A\alpha} + B_a^{\hat{\alpha}} \delta_{A\hat{\alpha}}, \quad \delta y^{A\alpha} = B_a^\alpha \delta u^{A\alpha} + B_a^{\hat{\alpha}} \delta v^{A\hat{\alpha}}, \quad A = \overline{0, k} \quad (5.1)$$

Let us examine the operators p , J and θ on the subspaces.

Proposition 5.1. *The projector operators p_0, p_1, \dots, p_k given by (2.2) can be decomposed in the following way:*

$$p_0 = p'_0 + p''_0 \quad (5.2)$$

$$p_1 = p'_1 + p''_1, \dots,$$

$$p_k = p'_k + p''_k$$

where

$$p_A' = \delta_{A\alpha} \otimes \delta u^{A\alpha}, \quad p_A'' = \delta_{A\hat{\alpha}} \otimes \delta v^{A\hat{\alpha}}, \quad A = \overline{0, k},$$

(no summation over A).

Proof. From (2.2) and (5.1) we get

$$\begin{aligned} p_A = \delta_{A\alpha} \otimes \delta y^{A\alpha} &= \left(B_a^\alpha \delta_{A\alpha} + B_a^{\hat{\alpha}} \delta_{A\hat{\alpha}} \right) \otimes \left(B_\beta^\alpha \delta u^{A\beta} + B_\beta^{\hat{\alpha}} \delta v^{A\hat{\beta}} \right) = \\ & B_a^\alpha B_\beta^\alpha \delta_{A\alpha} \otimes \delta u^{A\beta} + B_a^{\hat{\alpha}} B_\beta^{\hat{\alpha}} \delta_{A\hat{\alpha}} \otimes \delta u^{A\beta} + B_a^\alpha B_\beta^{\hat{\alpha}} \delta_{A\alpha} \otimes \delta v^{A\hat{\beta}} + B_a^{\hat{\alpha}} B_\beta^\alpha \delta_{A\hat{\alpha}} \otimes \delta v^{A\hat{\beta}} \end{aligned}$$

From (4.2) it follows

$$B_a^\alpha B_\beta^a = \delta_\beta^a, \quad B_a^{\hat{\alpha}} B_\beta^a = 0, \quad B_a^\alpha B_\beta^{\hat{\alpha}} = 0, \quad B_a^{\hat{\alpha}} B_\beta^{\hat{\alpha}} = \delta_\beta^{\hat{\alpha}}.$$

Now we get

$$p_A = \delta_{A\alpha} \otimes \delta u^{A\alpha} + \delta_{A\hat{\alpha}} \otimes \delta v^{A\hat{\alpha}} = p'_A + p''_A, \quad A = \overline{0, k}.$$

Arbitrary vector field $X \in T(E)$ given by (2.1) can be expressed in basis $B' = B_1 \cup B_2$ in the following way

$$\begin{aligned} X &= X^{Aa} \delta_{Aa} = X^{Aa} \left(B_a^\alpha \delta_{A\alpha} + B_a^{\hat{\alpha}} \delta_{A\hat{\alpha}} \right) = \\ &= X^{A\alpha} \delta_{A\alpha} + X^{A\hat{\alpha}} B_{A\hat{\alpha}}, \end{aligned} \quad (5.3)$$

where

$$X^{A\alpha} = B_a^\alpha X^{Aa}, \quad X^{A\hat{\alpha}} = B_a^{\hat{\alpha}} X^{Aa}$$

Proposition 5.2. *The vector field X can be expressed in the form*

$$X = X' + X'',$$

where

$$\begin{aligned} X' &= (p_0' + p_1' + \dots + p_k') X, \\ X'' &= (p_0'' + p_1'' + \dots + p_k'') X \end{aligned}$$

Proof. Using (5.2) and (5.3) we get

$$X' = \left(\delta_{0\beta} \otimes \delta u^{0\beta} + \delta_{1\beta} \otimes \delta u^{1\beta} + \dots + \delta_{k\beta} \otimes \delta u^{k\beta} \right) \left(X^{0\alpha} \delta_{0\alpha} + X^{1\alpha} \delta_{1\alpha} + \dots + X^{k\alpha} \delta_{k\alpha} + \right. \\ \left. X^{0\hat{\alpha}} \delta_{0\hat{\alpha}} + X^{1\hat{\alpha}} \delta_{1\hat{\alpha}} + \dots + X^{k\hat{\alpha}} \delta_{k\hat{\alpha}} \right)$$

when B_1^* is dual to B_1 , B_2^* is dual to B_2 , i.e. the conditions of Theorem 4.3 are satisfied, i.e.

$$\begin{aligned} \left(\delta u^{A\alpha}, \delta_{A\beta} \right) &= \delta_\beta^\alpha, \quad \left(\delta v^{A\hat{\alpha}}, \delta_{A\hat{\beta}} \right) = \delta_{\hat{\beta}}^{\hat{\alpha}}, \\ \left(\delta u^{A\alpha}, \delta_{A\hat{\beta}} \right) &= 0, \quad \left(\delta v^{A\hat{\alpha}}, \delta_{A\beta} \right) = 0, \quad A = \overline{0, k} \end{aligned}$$

we get

$$X' = X^{0\alpha} \delta_{0\alpha} + X^{1\alpha} \delta_{1\alpha} + \dots + X^{k\alpha} \delta_{k\alpha} = X^{A\alpha} \delta_{A\alpha}.$$

Similar for X'' .

Proposition 5.3. *The projection operator*

$$\begin{aligned} p'_0 + p'_1 + \dots + p'_k &= I_{m \times m} \text{ on } T(E_1) \\ p''_0 + p''_2 + \dots + p''_k &= I_{(n-m) \times (n-m)} \text{ on } T(E_2) \end{aligned}$$

i.e. p'_0 is the projection operator of $T(E)$ on H' , $P_A'(A = \overline{1, k})$ are the projections of $T(E)$ on V_A' . Similarly for the second part of theorem.

Proposition 5.4. The one-form field $w \in T^*(E)$ given by (2.3) can be written in the basis $B^{*'} = B_1^* \cup B_2^*$ in the form

$$w = w' + w'',$$

where

$$\begin{aligned} w' &= w_{0\alpha} \delta u^{0\alpha} + w_{1\alpha} \delta u^{1\alpha} + \dots + w_{k\alpha} \delta u^{k\alpha} \\ w'' &= w_{\hat{0}\alpha} \delta u^{\hat{0}\alpha} + w_{\hat{1}\alpha} \delta u^{\hat{1}\alpha} + \dots + w_{\hat{k}\alpha} \delta u^{\hat{k}\alpha}. \end{aligned}$$

Proposition 5.5. The projection operators $p_0^*, p_1^*, \dots, p_k^*$ defined by (2.4) can be decomposed in the form

$$p_0^* = p_0^{*'} + p_0^{*''}, \quad (5.4)$$

$$p_1^* = p_1^{*'} + p_1^{*''}, \dots, p_k^* = p_k^{*'} + p_k^{*''}$$

where

$$\begin{aligned} p_A^{*'} &= \delta u^{A\alpha} \otimes \delta_{A\alpha} \\ p_A^{*''} &= \delta v^{A\hat{\alpha}} \otimes \delta_{A\hat{\alpha}}, \quad A \text{ fixed, } A = \overline{0, k}. \end{aligned}$$

Proposition 5.6. For the 1-form field w given by (2.3) and the projector operators $p_A^{*'}$ and $p_A^{*''}$ the following relation is valid

$$w' = (p_0^{*'} + p_1^{*'} + \dots + p_k^{*'}) w'$$

$$w'' = (p_0^{*''} + p_1^{*''} + \dots + p_k^{*''}) w''$$

where

$$\begin{aligned} p_A^{*'} w' &= (\delta u^{A\alpha} \otimes \delta_{A\alpha}) w_{B\beta} \delta^{B\beta} = \\ w_{B\beta} \delta u^{A\alpha} \delta_A^B \delta_\alpha^\beta &= w_{A\alpha} \delta u^{A\alpha}, \quad A \text{ fixed.} \end{aligned}$$

From the above it is obvious that

$$p_0^{*'} + p_1^{*'} + \dots + p_k^{*'} = I_{m \times m} \quad \text{on } T^*(E_1)$$

$$p_0^{*''} + p_1^{*''} + \dots + p_k^{*''} = I_{(n-m) \times (n-m)} \quad \text{on } T^*(E_2)$$

and $p_0^{*'}$ is the projection of $T^*(E_1)$ on $H^{*'}$, $p_A^{*'}$ is the projection of $T^*(E_1)$ on $V_A^{*'}$; $p_0^{*''}$ is the projection of $T^*(E_2)$ on $H^{*''}$; $p_A^{*''}$ is the projection of $T^*(E_2)$ on $V_A^{*''}$, $A = \overline{1, k}$.

Proposition 5.7. *The structure J which in B and B^* can be expressed by (3.1) in the bases $B' = B_1 \cup B_2$ and $B^* = B_1^* \cup B_2^*$ can be written in the form*

$$J = J' + J'', \quad (5.5)$$

where

$$J' = \delta_{1\alpha} \otimes \delta u^{0\alpha} + 2\delta_{2\alpha} \otimes \delta u^{1\alpha} + \dots + k\delta_{k\alpha} \otimes \delta u^{(k-1)\alpha}, \quad (5.6)$$

$$J'' = \delta_{1\hat{\alpha}} \otimes \delta v^{0\hat{\alpha}} + 2\delta_{2\hat{\alpha}} \otimes \delta v^{1\hat{\alpha}} + \dots + k\delta_{k\hat{\alpha}} \otimes \delta v^{(k-1)\hat{\alpha}}.$$

Proof. The proof is similar to the proof of Proposition 5.1, where the relations (4.2) are used.

Remark 5.1. The structure \bar{J} defined by (3.7) in the basis B' and $B^{*'}$ can be expressed by

$$\bar{J} = \bar{J}' + \bar{J}'', \quad (5.7)$$

where

$$\bar{J}' = \delta u^{0\alpha} \otimes \delta_{1\alpha} + 2\delta u^{1\alpha} \otimes \delta_{2\alpha} + \dots + k\delta u^{(k-1)\alpha} \otimes \delta_{k\alpha} \quad (5.8)$$

$$\bar{J}'' = \delta u^{0\hat{\alpha}} \otimes \delta_{1\hat{\alpha}} + 2\delta u^{1\hat{\alpha}} \otimes \delta_{2\hat{\alpha}} + \dots + k\delta u^{(k-1)\hat{\alpha}} \otimes \delta_{k\hat{\alpha}}.$$

Proposition 5.8. The structure θ defined by (3.4) in the bases B' and $B^{*'}$ can be expressed by

$$\theta = \theta' + \theta'', \quad (5.9)$$

where

$$\theta' = \delta_{0\alpha} \otimes \delta u^{1\alpha} + \frac{1}{2}\delta_{1\alpha} \otimes \delta u^{2\alpha} + \dots + \frac{1}{k}\delta_{(k-1)\alpha} \otimes \delta u^{k\alpha}, \quad (5.10)$$

$$\theta'' = \delta_{0\hat{\alpha}} \otimes \delta v^{1\hat{\alpha}} + \frac{1}{2}\delta_{1\hat{\alpha}} \otimes \delta v^{2\hat{\alpha}} + \dots + \frac{1}{k}\delta_{(k-1)\hat{\alpha}} \otimes \delta v^{k\hat{\alpha}}.$$

Remark 5.2. The structure $\bar{\theta}$ defined by (3.8) in the bases B' and B' can be expressed by

$$\bar{\theta} = \bar{\theta}' + \bar{\theta}'', \quad (5.11)$$

where

$$\bar{\theta}' = \delta u^{1\alpha} \otimes \delta_{0\alpha} + \frac{1}{2}\delta u^{2\alpha} \otimes \delta_{1\alpha} + \dots + \frac{1}{k}\delta u^{k\alpha} \otimes \delta_{(k-1)\alpha} \quad (5.12)$$

$$\bar{\theta}'' = \delta u^{1\bar{\alpha}} \otimes \delta_{0\bar{\alpha}} + \frac{1}{2} \delta u^{2\bar{\alpha}} \otimes \delta_{1\bar{\alpha}} + \dots + \frac{1}{k} \delta u^{k\bar{\alpha}} \otimes \delta_{(k-1)\bar{\alpha}}$$

Theorem 5.1. *The structure $J, \bar{J}, \theta, \bar{\theta}, p, p^*$ on the subspaces are connected by:*

$$\begin{aligned} J'\theta' &= I_{m \times m} - p'_0 & \bar{J}'\bar{\theta}' &= I_{m \times m} - p^{*'}_k \\ J''\theta'' &= I_{(n-m) \times (n-m)} - p''_0 & \bar{J}''\bar{\theta}'' &= I_{(n-m) \times (n-m)} - p^{*''}_k \end{aligned} \quad (5.13)$$

Proof. From (5.4)-(5.12) it follows

$$\begin{aligned} J'\theta' &= \delta_{1\alpha} \otimes \delta u^{1\alpha} + \delta_{2\alpha} \otimes \delta u^{2\alpha} + \dots + \delta_{k\alpha} \otimes \delta u^{k\alpha}, p'_0 = \delta_{0\alpha} \otimes \delta u^{0\alpha} \\ \bar{J}'\bar{\theta}' &= \delta u^{0\alpha} \otimes \delta_{0\alpha} + \delta u^{1\alpha} \otimes \delta_{1\alpha} + \dots + \delta u^{(k-1)\alpha} \otimes \delta_{(k-1)\alpha}, p^{*'}_k = \delta u^{k\alpha} \otimes \delta_{k\alpha} \\ J''\theta'' &= \delta_{1\hat{\alpha}} \otimes \delta v^{1\hat{\alpha}} + \delta_{2\hat{\alpha}} \otimes \delta v^{2\hat{\alpha}} + \dots + \delta_{k\hat{\alpha}} \otimes \delta v^{k\hat{\alpha}}, p''_0 = \delta_{0\hat{\alpha}} \otimes \delta u^{0\hat{\alpha}} \\ \bar{J}''\bar{\theta}'' &= \delta v^{0\hat{\alpha}} \otimes \delta_{0\hat{\alpha}} + \delta v^{1\hat{\alpha}} \otimes \delta_{1\hat{\alpha}} + \dots + \delta v^{(k-1)\hat{\alpha}} \otimes \delta_{(k-1)\hat{\alpha}}, p^{*''}_k = \delta u^{k\hat{\alpha}} \otimes \delta_{k\hat{\alpha}}. \end{aligned} \quad (5.14)$$

Proposition 5.9. *The following relations are valid:*

$$\begin{aligned} J'\theta' &\text{ is the identity operator on } V_1' \oplus V_2' \oplus \dots \oplus V_k' \\ \bar{J}'\bar{\theta}' &\text{ is the identity operator on } V_0^{*'} \oplus V_1^{*'} \oplus \dots \oplus V_{k-1}^{*'}, \\ J''\theta'' &\text{ is the identity operator on } V_1'' \oplus V_2'' \oplus \dots \oplus V_k'' \\ \bar{J}''\bar{\theta}'' &\text{ is the identity operator on } V_1^{*''} \oplus V_2^{*''} \oplus \dots \oplus V_k^{*''}. \end{aligned}$$

Theorem 5.2. *The following relations are valid:*

$$\begin{aligned} \theta'J' &= I_{m \times m} - p_k' & \bar{\theta}'\bar{J}' &= I_{m \times m} - p_0^{*'} \\ \theta''J'' &= I_{(n-m) \times (n-m)} - p_k'' & \bar{\theta}''\bar{J}'' &= I_{(n-m) \times (n-m)} - p_0^{*''}. \end{aligned} \quad (5.15)$$

Proof. It is easy to see that

$$\begin{aligned} \theta'J' &= \delta_{0\alpha} \otimes \delta u^{0\alpha} + \delta_{1\alpha} \otimes \delta u^{1\alpha} + \dots + \delta_{(k-1)\alpha} \otimes \delta u^{(k-1)\alpha}, p_k' = \delta_{k\alpha} \otimes \delta u^{k\alpha} \\ \bar{\theta}'\bar{J}' &= \delta u^{1\alpha} \otimes \delta_{1\alpha} + \delta u^{2\alpha} \otimes \delta_{2\alpha} + \dots + \delta u^{k\alpha} \otimes \delta_{k\alpha}, p_0^{*'} = \delta u^{0\alpha} \otimes \delta_{0\alpha} \\ \theta''J'' &= \delta_{0\hat{\alpha}} \otimes \delta v^{0\hat{\alpha}} + \delta_{1\hat{\alpha}} \otimes \delta v^{1\hat{\alpha}} + \dots + \delta_{(k-1)\hat{\alpha}} \otimes \delta v^{(k-1)\hat{\alpha}}, p_k'' = \delta_{k\hat{\alpha}} \otimes \delta v^{k\hat{\alpha}} \\ \bar{\theta}''\bar{J}'' &= \delta v^{1\hat{\alpha}} \otimes \delta_{1\hat{\alpha}} + \delta v^{2\hat{\alpha}} \otimes \delta_{2\hat{\alpha}} + \dots + \delta v^{k\hat{\alpha}} \otimes \delta_{k\hat{\alpha}}, p_0^{*''} = \delta v^{0\hat{\alpha}} \otimes \delta_{0\hat{\alpha}}. \end{aligned} \quad (5.16)$$

Proposition 5.10. *The following relations are valid:*

$\theta' J'$ is the identity operator on $H^{*'} \oplus V_1' \oplus \dots \oplus V_{k-1}'$
 $\bar{\theta}' \bar{J}'$ is the identity operator on $V_1^{*'} \oplus V_2^{*'} \oplus \dots \oplus V_k^{*'}$,
 $\theta'' J''$ is the identity operator on $H'' \oplus V_1'' \oplus \dots \oplus V_{k-1}''$
 $\bar{\theta}'' \bar{J}''$ is the identity operator on $V_1^{*''} \oplus V_2^{*''} \oplus \dots \oplus V_k^{*''}$.

From the above we have

Theorem 5.3. For the subspaces in $Osc^k M$ the following relations are valid:

$$J'\theta' + \theta'J' = 2I_{m \times m} - p'_0 - p'_k \quad (5.17)$$

$$J''\theta'' + \theta''J'' = 2I_{(n-m) \times (n-m)} - p''_0 - p''_k$$

$$\bar{J}'\bar{\theta}' + \bar{\theta}'\bar{J}' = 2I_{m \times m} - p^{*'}_0 - p^{*'}_k$$

$$\bar{J}''\bar{\theta}'' + \bar{\theta}''\bar{J}'' = 2I_{(n-m) \times (n-m)} - p^{*''}_0 - p^{*''}_k.$$

Proposition 5.11. For the subspaces in $Osc^1 M$ we have $p'_k = p'_1$, $p''_k = p''_1$,

$p^{*'}_k = p^{*'}_1$, $p^{*''}_k = p^{*''}_1$ and

$$p'_0 + p'_1 = I_{m \times m}, \quad p''_0 + p''_1 = I_{(n-m) \times (n-m)}$$

$$p^{*'}_0 + p^{*'}_1 = I_{m \times m}, \quad p^{*''}_0 + p^{*''}_1 = I_{(n-m) \times (n-m)}.$$

From (5.16) and Proposition 5.11 we have

Theorem 5.4. For the subspaces in $Osc^1 M$ the following relations are valid:

$$J'\theta' + \theta'J' = I_{m \times m} \quad (5.18)$$

$$J''\theta'' + \theta''J'' = I_{(n-m) \times (n-m)}$$

$$\bar{J}'\bar{\theta}' + \bar{\theta}'\bar{J}' = I_{m \times m}$$

$$\bar{J}''\bar{\theta}'' + \bar{\theta}''\bar{J}'' = I_{(n-m) \times (n-m)}.$$

As $\dim E_1 = (k+1)m$, $\dim E_2 = (k+1)(n-m)$ the notion $I_{m \times m}$ is not precise, it means $(k+1)$ blocks on diagonal, each of which is of form $m \times m$.

The exact form of (5.16) and (5.17) can be obtained from (5.13) and (5.15).

Theorem 5.5. The structures J' , J'' , \bar{J}' , \bar{J}'' , θ' , θ'' , $\bar{\theta}'$, $\bar{\theta}''$, are k -tangent structures, namely

$$(J')^k = 0, \quad (J'')^k = 0, \quad (\bar{J}')^k = 0, \quad (\bar{J}'')^k = 0$$

$$(\theta')^k = 0, \quad (\theta'')^k = 0, \quad (\bar{\theta}')^k = 0, \quad (\bar{\theta}'')^k = 0$$

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RAZNE STRUKTURE U PROSTORIMA $Osc^k M$

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Geometrija prostora $Osc^k M$ je uveo R. Miron i Gh. Atanasiu u [6] i [7]. Teorija ovih prostora je razvijena od strane R. Mirona i njegovih saradnika iz Rumunije, Japana i drugih zemalja i prikazana je u mnogim člancima i knjigama. Samo neki od njih su spomenuti u literaturi. Ovde predstavljamo konstrukciju takvih adaptiranih baza u prostorima $T(Osc^k M)$ i $T^(Osc^k M)$, koje su saglasne sa J strukturom. Teorija dva komplementarna potprostora je ovde data kao u [2] i [4]. Operatori J , \bar{J} , θ , $\bar{\theta}$, p , p^* su uvedeni u okolnom prostoru, kao i u potprostorima. Među njima su uspostavljene neke nove relacije. Ispitana je akcija ovih operatora na Liouville-ova vektorska polja.*

Ključne reči i fraze: operacija projekcije, J struktura, θ struktura, potprostori u $Osc^k M$.