NEW ANALYTICAL SOLUTION FOR SOLVING STEADY-STATE HEAT CONDUCTION PROBLEMS WITH SINGULARITIES

by

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A problem of steady-state heat conduction which presents singularities is solved in this paper by using the conformal mapping method. The principle of this method is based on the Schwarz-Christoffel transformation. The considered problem is a semi-infinite medium with two different isothermal surfaces separated by an adiabatic annular disc. We show that the thermal resistance can be determined without solving the governing equations. We determine a simple and exact expression that provides the thermal resistance as a function of the ratio of annular disc radii.

Key words: *thermal resistance, heat sources, conformal mappings, Dirichlet problem*

Introduction

Steady-state heat conduction in solids is governed by Laplace's equation. Most configurations are solved in the literature by using analytical or numerical models [1-8]. The most used analytical methods are based on the integral transforms (Fourier, Hankel …). The solutions can be easily obtained when the boundary conditions on the same face are homogeneous. The mixed boundary conditions, as for example Dirichlet and Neumann on the same face, induce singularities that are often very difficult to solve by using the usual techniques. These problems are encountered in various scientists' fields as heat transfer, electromagnetism, electrostatic, *etc*. Many authors have investigated these problems since 1950. Smythe [9] has performed an ingenious analysis based on the superposition technique to determine the capacitance of a circular annulus. Cooke [10] has developed an analytical study from integral method. The author has solved the obtained integral solution by numerical way and has shown that his results are in excellent agreement with those computed by Smythe [9]. Collins [11] has studied the potential problem for a circular annulus. He performed analytical developments for the axisymmetric case using a superposition technique. The solutions are given under imbricated's Fredholm integral equations, which require iterative numerical solving. Some authors are interested on the problem of Dirichlet condition on an annular disc which involves the Bessel's functions. The solutions show three integral equations which correspond to the three parts of the surface (*i. e.* the inner surface, the annular surface, and the outer surface). Some authors call this problem: the triple integral equations. Cooke [12] has proposed different solutions for the triple integral equations. These solutions involve imbricated's Fedholm integral equations as Collins [11]. Fabrikant [13] considered the Dirichlet problem taking into account the non-axisymmetry. The obtained solution is also under Fredholm's integral equation form. The kernel of this integral equations beginning

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no-singular, it can be solved by an iterative method. No general solutions to these problems have been attempted yet. Recently, Laraqi [14] has interested in two different Dirichlet problems: (1) an annular disc subjected to a Dirichlet condition, with a uniform temperature, when the remainder is insulated, and (2) an isothermal annular disc, with zero temperature, when the inner surface is subjected to a uniform heat flux and the outer surface is insulated. The authors proposed a method to determine the resistance through the annulus. The principal of this method consists on the determination of the asymptotic behaviours under their most compact form and the use of a correlation technique to connect the asymptotic behaviours. The obtained solutions are a compact and the results are in excellent agreement with available data of Smyth [9] and Cook [10].

For some problems with singularities, it is more interesting to use the conformal mapping method which is based in the Schawrz-Christoffel transformation. The principle of this transformation is presented in some books [15, 16].

In this paper, the problem of a semi-infinite medium with two different isothermal surfaces separated by an adiabatic annular disc is considered. The conformal mapping method is used to find the thermal contact resistance. We show that the thermal resistance can be determined without solving the governing equations and we obtain a simple and exact solution that gives the thermal resistance as a function of the ratio of annular disc radii.

Governing equations

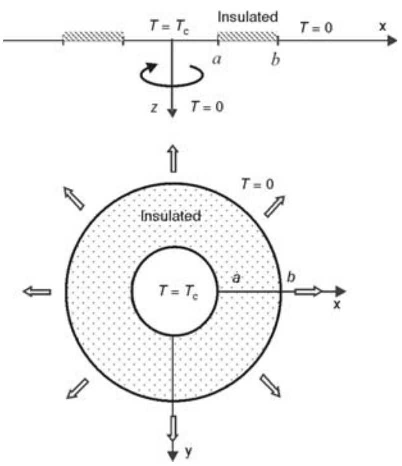

Figure 1. The studied problem

Let us consider a semi-infinite medium (fig. 1) with an axial symmetry. The surface $z = 0$ is subjected to a three mixed boundary conditions: (1) the disc $r < a$ is isothermal at $T = T_c$, (2) the annulus $a < r < b$ is adiabatic, and (3) the remained of the surface $r > b$ is isothermal at $T = 0$. The reference temperature is zero and the thermal regime is stationary.

The thermal behaviour of the considered problem is governed by the heat conduction equation (Laplace's equation) as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

with the boundary conditions:

$$T(x, y, z) = T_c, \ \sqrt{x^2 + y^2} < a, \ z = 0 \tag{2}$$

$$T(x, y, z) = 0, \ \sqrt{x^2 + y^2} > b, \ z = 0 \tag{3}$$

$$\frac{\partial T(x, y, z)}{\partial z} = 0, \ a < \sqrt{x^2 + y^2} < b, \ z = 0 \tag{4}$$

$$T(x, y, z) = 0, \ \sqrt{x^2 + y^2} \to \infty, \ z \to \infty \tag{5}$$

Analytical solution

Conformal mappings and Schwarz-Christoffel transformation

It is considered that all lengths are dimensionless with respect to the radius a, and the Schwarz-Christoffer transformation was used, which is a conformal mapping.

We use the Schwarz-Christoffel transformation which is a conformal mapping. It entails transforming a complex plan ($w = x + iz$) in another complex plan ($u = \mu + iv$) that maps the real axis into the boundary of a polygon and the half plane of the complex plan into the interior of this polygon (fig. 2).

The Schwarz-Chritoffel transformation is defined by:

$$\frac{\mathrm{d}u}{\mathrm{d}w} = \prod_{m=1}^{M} \frac{1}{(w - x_{\mathrm{m}})^{\beta}} \quad \text{and} \quad u = \int \prod_{m=1}^{M} \frac{1}{(w - x_{\mathrm{m}})^{\beta}} \mathrm{d}z \tag{6}$$

The points x_m over the real axis are mapped into u_m (fig. 2) with angles equal to $\pi/2$. This leads to $\beta = (\pi/2)/\pi = 1/2$. In the present study, we have the consecutive points: $x_1 = -b$, $x_2 = -a$, $x_3 = a$, $x_4 = b$, which correspond to the dimensionless values: $-1/k$, -1, 1, $1/k$, respectively (with $k = a/b$).

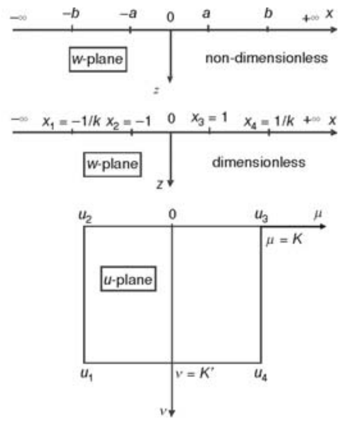

Figure 2. Rectangular and curvilinear maps

Replacing these particular values in eq. (6), we have:

$$u = \int_0^w \frac{1}{\sqrt{(1 - w^2)(1 - k^2 w^2)}} \mathrm{d}w \tag{7}$$

Equation (7) is the incomplete elliptic integral of first kind, $u = F(w, k)$.

Inversely, we have: $$w = sn(u) \tag{8}$$

where sn is the Jacobi function.

Replacing w and u by $x + iz$ and $\mu + iv$, respectively, we obtain the relationship between rectangular and curvilinear co-ordinates as (see details in Appendix):

$$\begin{cases} x = a \dfrac{sn(\mu) dn(v)}{\Delta} \cos\psi \\ y = \dfrac{sn(\mu) dn(v)}{\Delta} \sin\psi, \text{ with } \Delta = 1 - sn^2(v) dn^2(\mu) \\ z = a \dfrac{cn(\mu) dn(\mu) sn(v) cn(v)}{\Delta} \end{cases} \tag{9}$$

where cn, dn, sn are the Jacobi functions.

The elementary length which is written in Cartesian co-ordinates as:

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} \tag{10}$$

can be written in curvilinear co-ordinates as:

$$\mathrm{d}l = \sqrt{e_1^2 \mathrm{d}v^2 + e_2^2 \mathrm{d}\mu^2 + e_3^2 \mathrm{d}\psi^2} \tag{11}$$

with

$$\begin{aligned} e_1^2 &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\ e_2^2 &= \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 \\ e_3^2 &= \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2 \end{aligned} \tag{12}$$

That gives:

$$e_1^2 = e_2^2 = \frac{a^2}{\Delta^2}[dn^2(\nu) - k^2 sn^2(\mu)][1 - sn^2(\mu)\, dn^2(\nu)]$$
$$e_3^2 = \frac{a^2}{\Delta^2} sn^2(\mu) dn^2(\nu) \tag{13}$$

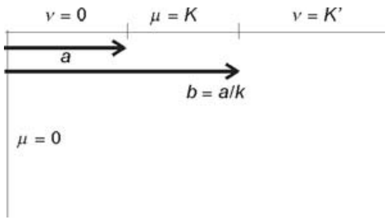

Figure 3. Boundary conditions in curvilinear co-ordinates

According to the conformal map, we have (fig. 3):

(1) The curvilinear abscissa $\nu = 0$ corresponds to the isotherm $T = T_c$, ($z = 0$, $0 < r < a$),
(2) The curvilinear abscissa $\nu = K'$ corresponds to the isotherm $T = 0$, ($z = 0$, $r > b$),
(3) The curvilinear abscissa $\mu = 0$ corresponds to the axis of symmetry ($z > 0$, $r = 0$), and
(4) The curvilinear abscissa $\mu = K$ corresponds to the adiabatic annulus ($z = 0$, $a < r < b$).

where K and K' are the complete elliptic integrals, with $K' = K(k')$.

Thermal resistance

The thermal resistance R_c is bounded by the isotherms $\nu = 0$ and $\nu = K'$, and the adiabatic surfaces $\mu = 0$ and $\mu = K$ (fig. 2).

We can write R_c as:

$$R_c = \frac{T_{\nu=0} - T_{\nu=K'}}{\phi} \tag{14}$$

where ϕ is the flux through the heat flux tube such as:

$$\phi = \int_S \varphi \mathrm{d}s \tag{15}$$

with

$$\varphi = -\lambda \frac{\partial T}{\partial z} = -\lambda \frac{\partial T}{\partial \nu}\frac{\mathrm{d}\nu}{\mathrm{d}z} = -\lambda \frac{\mathrm{d}T}{\mathrm{d}\nu}\frac{1}{e_1} \tag{16}$$

Replacing eq. (16) into eq. (15), we obtain:

$$\phi = \int_S -\lambda \frac{\mathrm{d}T}{\mathrm{d}\nu}\frac{1}{e_1}(e_2 \mathrm{d}\mu)(e_3 \mathrm{d}\psi) \tag{17}$$

Taking into account the symmetry condition we have $e_1 = e_2$. Then, the thermal resistance can be written as:

$$R_c = \frac{1}{\lambda \int\limits_{\psi=0}^{2\pi} \int\limits_{\mu=0}^{K} \dfrac{\mathrm{d}\mu \mathrm{d}\psi}{\int\limits_{\nu=0}^{\nu=K'} \dfrac{1}{e_3} \mathrm{d}\nu}} \tag{18}$$

Replacing the expression of e_3 given by (13) into eq. (18) and noting that the integration with respect to ψ is equal to 2π, we obtain:

$$R_c = \frac{1}{2\pi\lambda a \int\limits_{\mu=0}^{K} \dfrac{\mathrm{d}\mu}{\int\limits_{0}^{K'} \dfrac{1 - dn^2(\mu) sn^2(\nu)}{sn(\mu) dn(\nu)} \mathrm{d}\nu}} \tag{19}$$

$$R_c = \frac{1}{2\pi\lambda a \int_0^K \dfrac{d\mu}{\dfrac{1}{sn(\mu)} \int_0^{K'} \dfrac{1}{dn(\nu)} d\nu - \dfrac{dn^2(\mu)}{sn(\mu)} \int_0^{K'} \dfrac{sn^2(\nu)}{dn(\nu)} d\nu}} \tag{20}$$

After the integration on $\nu \in [0, K']$, we get:

$$R_c = \frac{1}{2\pi\lambda a \int_0^K \dfrac{d\mu}{\dfrac{1}{sn(\mu)}\left(\dfrac{\pi}{2k}\right) - \dfrac{dn^2(\mu)}{sn(\mu)}\left(\dfrac{\pi}{2k'^2 k} - \dfrac{\pi}{2k'^2}\right)}} \tag{21}$$

with $k = a/b$; $k' = (1 - k^2)^{1/2}$.

Equation (21) can be written as:

$$R_c = \frac{1}{4k\lambda a \int_0^K \dfrac{d\mu}{\dfrac{1}{sn(\mu)} - \dfrac{dn^2(\mu)}{sn(\mu)}\left(\dfrac{1-k}{k'^2}\right)}} \tag{22}$$

Using the relationship between Jacobi functions as: $dn^2(\mu) = 1 - k^2 sn^2(\mu)$, and replacing into eq. (22), the thermal resistance can be expressed in the form:

$$R_c^* = R_c \lambda a \frac{1}{4(1+k) \int\limits_{\mu=0}^{K} \dfrac{sn(\mu) d\mu}{k sn^2(\mu) + 1}} \tag{23}$$

We decompose the integral in the denominator in two parts:

$$\int\limits_{\mu=0}^{K} \frac{sn(\mu) d\mu}{k sn^2(\mu) + 1} = \frac{1}{k} \int\limits_{\mu=0}^{K} \frac{sn(\mu) d\mu}{sn^2(\mu) + \dfrac{1}{k}} = \frac{1}{\dfrac{2ik}{\sqrt{k}}} \left[\int\limits_{\mu=0}^{K} \frac{sn(\mu) d\mu}{sn(\mu) - \dfrac{i}{\sqrt{k}}} - \int\limits_{\mu=0}^{K} \frac{sn(\mu) d\mu}{sn(\mu) + \dfrac{i}{\sqrt{k}}} \right] \tag{24}$$

The integrals (24) are explicit, and we obtain the final expression of the dimensionless thermal resistance under a simple form:

$$R_c^* = R_c \lambda a = \frac{\sqrt{k}}{2 \ln\left[\dfrac{1+\sqrt{k}}{1-\sqrt{k}}\right]}, \quad \text{with } k = \frac{a}{b} \tag{25}$$

From eq. (25) we have:

$$\lim_{k \to 1^-} R_c^* = \frac{1}{2 \ln\left(\dfrac{2}{0^+}\right)} = \frac{1}{\infty} = 0, \quad b \to a \tag{26}$$

and

$$\lim_{k \to 0} R_c^* = \lim_{k \to 0} \frac{\sqrt{k}}{2 \ln\left[\dfrac{1+\sqrt{k}}{1-\sqrt{k}}\right]} = \lim_{k \to 0} \frac{1}{2\left[\dfrac{\ln(1+\sqrt{k})}{\sqrt{k}} - \dfrac{\ln(1-\sqrt{k})}{\sqrt{k}}\right]} = \frac{1}{4}, \quad b \to \infty \tag{27}$$

The value given by eq. (27) is in agreement with the known result for an isothermal disc [15]: $R_c = 1/(4\lambda a)$.

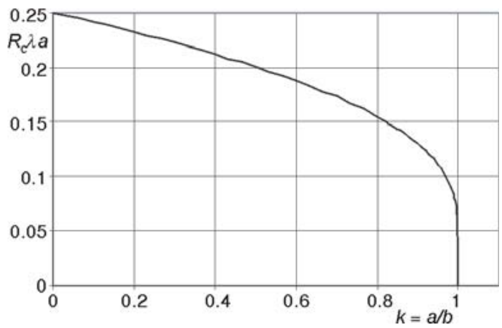

Figure 4. Changes in the thermal resistance as a function of *k = a/b*

Figure 4 shows the changes in $R_c^*=R_c\lambda a$ as a function of $k = a/b$. The thermal resistance decreases almost linearly from $k = 0$ and it suddenly goes down to zero when k is near 1.

Conclusions

The present study shows that the problem of heat conduction with singularities can be solved by using the conformal mappings. The thermal resistance is determined without solving the governing equations. This procedure provides a compact and simple expression. The proposed method can be extended to other type of boundary conditions and also to other physical problems governed by Laplace's equation.

Nomenclature

a – inner radius of the annulus, [m]
b – outer radius of the annulus, [m]
cn, dn, sn – Jacobi functions, [–]
F – incomplete elliptic integral of first kind, [–]
K – complete elliptic integral of the first kind [$= K(k)$], [–]
K' – complete elliptic integral of the first kind ($= K(k')$),
k – ratio of radii ($= a/b$), [–]
k' – parameter [$= (1 - k^2)^{1/2}$], [–]
R_c – thermal resistance, [KW^{-1}]
S – surface of the tube cross-section, [m^2]
s – elementar surface, [m^2]
T – temperature, [K]
T_c – temperature at the centre, [K]
u, w – curvilinear and Cartesian plane, – respectively
x, y, z – Cartesian co-ordinates

Greek symbols

λ – thermal conductivity, [$Wm^{-1}K^{-1}$]
μ, ν, ψ – curvilinear co-ordinates
φ – heat flux density, [Wm^{-2}]
ϕ – heat flux, [W]

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Appendix

Relation between rectangular and curvilinear co-ordinates

From eq. (8) we have:

$$w = sn(\mu + i\nu) = \frac{sn(\mu)cn(i\nu)dn(i\nu) + sn(i\nu)cn(\mu)dn(\mu)}{1 - k^2 sn^2(\mu)sn^2(i\nu)} \tag{A1}$$

with:

$$sn(i\nu) = i\frac{sn(\nu)}{cn(\nu)},\ cn(i\nu) = \frac{1}{cn(\nu)},\ dn(i\nu) = \frac{dn(\nu)}{cn(\nu)} \tag{A2}$$

Replacing (A2) into (A1), we obtain:

$$w = \frac{sn(\mu)\dfrac{1}{cn(\nu)}\dfrac{dn(\nu)}{cn(\nu)} + i\dfrac{sn(\nu)}{cn(\nu)}cn(\mu)dn(\mu)}{1 + k^2 sn^2(\mu)\dfrac{sn^2(\nu)}{cn^2(\nu)}} \tag{A3}$$

So:

$$w = \frac{sn(\mu)dn(\nu) + isn(\nu)dn(\nu)cn(\mu)dn(\mu)}{cn^2(\nu) + k^2 sn^2(\mu)sn^2(\nu)} \tag{A4}$$

Using the following relations between Jacobi functions:

$$\begin{aligned} cn^2(\nu) &= 1 - sn^2(\nu) \\ dn^2(\mu) &= 1 - k^2 sn^2(\mu) \end{aligned} \tag{A5}$$

eq. (A4) becomes:

$$w = \frac{sn(\mu)dn(\nu) + isn(\nu)dn(\nu)cn(\mu)dn(\mu)}{1 - sn^2(\nu)dn^2(\mu)} \tag{A6}$$

Separating the real and imaginary parts as $w = x + iz$, we obtain:

$$\begin{aligned} x &= a\frac{sn(\mu)dn(\nu)}{1 - sn^2(\nu)dn^2(\mu)} \\ z &= a\frac{sn(\nu)dn(\nu)cn(\mu)dn(\mu)}{1 - sn^2(\nu)dn^2(\mu)} \end{aligned} \tag{A7}$$

Tacking into account the rotation ψ with respect to $z-$axis, we have:

$$\begin{cases} x = a\dfrac{sn(\mu)dn(\nu)}{\Delta}\cos\psi \\ y = a\dfrac{sn(\mu)dn(\nu)}{\Delta}\sin\psi, \text{ with } \Delta = 1 - sn^2(\nu)dn^2(\mu) \\ z = a\dfrac{cn(\mu)dn(\mu)sn(\nu)cn(\nu)}{\Delta} \end{cases} \tag{A8}$$

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