The article is written on the memory of our friend and great mathematical physisist
Prof. Dmytro Ya. Petryna
(2006†, Kyiv, Institute of Mathematics, Ukrainian Academy of Sciences),
who had done so much for quantum statistical physics to become mature and so attractive.

Nikolai N. Bogoliubov (Jr.), Denis L. Blackmore,
Valeriy Hr. Samoylenko, Anatoliy K. Prykarpatsky

ON KINETIC BOLTZMANN EQUATIONS AND RELATED HYDRODYNAMIC FLOWS WITH DRY VISCOSITY*

Abstract. A two-component particle model of Boltzmann-Vlasov type kinetic equations in the form of special nonlinear integro-differential hydrodynamic systems on an infinite-dimensional functional manifold is discussed. We show that such systems are naturally connected with the nonlinear kinetic Boltzmann-Vlasov equations for some one-dimensional particle flows with pointwise interaction potential between particles. A new type of hydrodynamic two-component Benney equations is constructed and their Hamiltonian structure is analyzed.

Keywords: kinetic Boltzmann-Vlasov equations, hydrodynamic model, Hamiltonian systems, invariants, dynamical equivalence.

Mathematics Subject Classification: Primary 58F08, 70H35; Secondary 34B15.

1. INTRODUCTION

It is well known [1, 2] that the classical Boltzmann equation under the no correlation condition describes long waves in dense gas with short-range interaction potential. The same equation, which is called the Vlasov equation [3] in the one-dimensional case, is clearly equivalent to the hydrodynamic equations for long waves in an ideal incompressible liquid with a free surface under gravity. It is also quite easy to see that in the classical random phase approximation this equation reduces to the completely integrable nonlinear Schrödinger equation [1, 2, 7–9] on the $\mathbb{R}$ axis. These equivalences for the hydrodynamic Benney type equations can be used for studying chaos in many-particle systems and turbulence arising in fluid flow. Yet the dynamical

many-particle systems discussed in [1,2] do not possess an important intrinsic property of particle motion in a liquid – convective mass transfer of particles in a fixed volume – which is known to always accompany a transition from laminar to turbulent flow and cause convective vortex motion. Moreover, these models do not possess an intrinsic dry viscosity for the particle flows.

To partially overcome the inadequacies noted for the Benney type hydrodynamic model, in this investigation we introduce a new generalized dynamical system for the flow of particles on an axis, namely its Boltzmann equation in the Vlasov approximation with no many-particle correlation, which describes the long waves in dense gas of particles with short-range interaction potential. Then the associated Benney type system of equations contains the convective terms in a form that is especially convenient for describing turbulence [5]. Moreover, the mathematical model of interacting particles on axis $\mathbb{R}$ we choose is such that the associated Benney type system of equations is bi-Hamiltonian with an infinite hierarchy of polynomial conservation laws in involution. The approach devised in the paper is applied effectively to constructing Boltzmann-Vlasov and Benney type hydrodynamic equations describing interacting to each other two-component particle flows.

2. BOLTZMANN EQUATION
AND THE ASSOCIATED TWO-COMPONENT MOMENT PROBLEM

2.1. Let us consider a quantum two-component dynamical system on the $\mathbb{R}$ axis consisting of $N := N_x + N_y \in \mathbb{Z}_+$ identical spinless particles of two types with the singular Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2} \sum_{j=1}^{N_x} \frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{2} \sum_{k=1}^{N_y} \frac{\partial^2}{\partial y_k^2} + \beta \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \delta(x_j - y_k), \qquad (2.1)$$

where α and $\beta \in R$ are real parameters, $\hbar$ is the Planck's constant (divided by 2π) and $\delta(x-y)$, $x, y \in \mathbb{R}$, is the Dirac delta-function. Then Wigner's transformation [3,7,12] at the quasiclassical limit as $\hbar \to 0$ yields $\hat{H} \stackrel{\hbar \to 0}{\to} H_{x,y}$, where the classical Hamiltonian function $H_{x,y} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ has the form

$$H_{x,y} = \frac{1}{2} \sum_{j=1}^{N_x} p_{j,x}^2 + \frac{1}{2} \sum_{k=1}^{N_y} p_{k,y}^2 + \beta \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \delta(x_j - y_k). \qquad (2.2)$$

Here $x_j, y_k \in \mathbb{R}$, $1 \le j \le N_x, 1 \le k \le N_y$, are the corresponding coordinates of the system of two-component particles on the $\mathbb{R}$ axis. The Heisenberg commutator for dynamical observables [3–5] becomes the standard canonical Hamiltonian bracket $\{\cdot, \cdot\}$, viz.

$$[\cdot, \cdot]\frac{i}{\hbar} \stackrel{\hbar \to 0}{\to} \{\cdot, \cdot\}, \qquad (2.3)$$

in accordance with the Bohr principle. Therefore, in the phase space $M = T^*(\mathbb{R}^N)$ the Hamiltonian equations take the following form:

$$
\begin{aligned}
dx_j/dt &= \{H_{x,y}, x_j\} = \partial H/\partial p_j, & dp_{j,x}/dt &= \{H_{x,y}, p_{j,x}\} = -\partial H_{x,y}/\partial x_j, \\
dy_k/dt &= \{H_{x,y}, y_k\} = \partial H_{x,y}/\partial p_{k,y}, & dp_{k,y}/dt &= \{H_{x,y}, p_{k,y}\} = -\partial H_{x,y}/\partial y_k,
\end{aligned}
\tag{2.4}
$$

where $t \in \mathbb{R}$ is an evolution parameter and $(x_j, p_j) \in T^*(\mathbb{R})$, $1 \le j \le N_x$, $(y_k, p_k) \in T^*(\mathbb{R})$, $1 \le k \le N_y$.

In view of the singularities in (2.2), equations (2.4) cannot in general be solvable effectively for arbitrary Cauchy data and large $N_x, N_y \in \mathbb{Z}_+$. Therefore because of our hydrodynamic interest in the motion of dynamical system (2.4), we further pass to their statistical description [3], using the Boltzmann-Bogoliubov distribution function $F : (\mathbb{R}^2 \times \mathbb{R}^2) \to D'(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}_+^2)$ defined by

$$
F(x, y, p_x, p_y; t) := (\sum_{j=1}^{N_x} \delta\left(x - x_j(t)\right)\delta\left(p_x - p_{j,x}(t)\right), \sum_{k=1}^{N_y} \delta\left(y - y_k(t)\right)\delta\left(p - p_{k,y}(t)\right)).
\tag{2.5}
$$

Here $(x, y, p_x, p_y) \in \mathbb{R}^2 \times \mathbb{R}^2$ and functions $(x_j, p_{j,x}) \in T^*(\mathbb{R})$, $1 \le j \le N_x$, $(y_k, p_{k,y}) \in T^*(\mathbb{R})$, $1 \le k \le N_y$, are solutions of Hamiltonian equations (2.4). The distribution function (2.5) satisfies the standard Liouville-Hamilton equation

$$
dF/dt = \{F, H_{x,y}\},
\tag{2.6}
$$

which will be studied in detail below.

2.2. Now we apply the averaging operator $\langle \cdot \rangle$ to distribution function (2.5) assuming no many-particle correlation over all initial states of (2.6). The averaging operation on (2.6) results in kinetic Boltzmann-Vlasov equation [3,7] of the form

$$
df/dt = \langle \{F, H_{x,y}\} \rangle := \{\{f, H\}\},
\tag{2.7}
$$

where $f = (f_1(x, p; t), f_2(y, p; t))^\intercal := \langle F(x, p; t) \rangle$ is the statistically averaged distribution function (2.5) and $\{\{\cdot, \cdot\}\}$ is a new averaged Poisson bracket on the infinite-dimensional functional space $M_{(f)} \subset C^\infty(D(M); \mathbb{R}_+)$, which for a pair of functionals $\gamma, \mu \in D(M_{(f)})$ has the form [2,7]:

$$
\{\{\gamma, \mu\}\} = \sum_{j=\overline{1,2}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dp_x \langle f(x, p; t), \{grad\gamma, grad\mu\}(x, p; t)\rangle
\tag{2.8}
$$

and is called the Lie-Poisson bracket [2,8] and the operation "$grad$" is the standard Euler variational derivative on $D(M_{(f)})$. The Hamiltonian $H \in D(M_{(f)})$ in (2.7) is given by

$$
\begin{aligned}
H := &\int_{\mathbb{R}} dx \int_{\mathbb{R}} dp \frac{1}{2}\langle (p, p)^\intercal, f(x, p; t)\rangle + \\
&+ \beta \int_{\mathbb{R}} dx \int_{\mathbb{R}} dp_x \int_{\mathbb{R}} dp_y f_1(x, p_x; t) f_2(x, p_y; t)
\end{aligned}
\tag{2.9}
$$

To derive (2.8) let us consider on the phase space $M \subset T^*(\mathbb{R}) \simeq \mathbb{R}^2$ the canonical Poisson brackets $\{f, g\} = \partial_p f \partial_x g - \partial_p g \partial_x f$, where $f, g \in D(M)$ are some smooth functions. The space $D(M) \times D(M)$ of smooth functions on M has the natural Lie algebra structure: $\mathcal{G} \simeq (\mathcal{D}(M) \times \mathcal{D}(M); \{\cdot, \cdot\})$ with respect to the naturally defined canonical bracket $\{\cdot, \cdot\}$.

Let $\mathcal{G}^*$ be the adjoint or dual space to $\mathcal{G}$, i.e., the space of continuous linear functionals on $\mathcal{G}$. The space $\mathcal{G}$ is a Hilbert space with respect to the scalar product defined by

$$(f, g) := \int_{\mathbb{R}} dx \int_{\mathbb{R}} dp \langle f(x, p; t), g(x, p; t) \rangle \qquad (2.10)$$

for all $f, g \in \mathcal{G}$. Then $\mathcal{G}^* \simeq \mathcal{G}$ follows from the Riesz theorem [12] and we note that the above scalar product is invariant with respect to the Poisson bracket $\{\cdot, \cdot\}$ in the sense that

$$(f, \{g, h\}) = (\{f, g\}, h) \qquad (2.11)$$

for all $f, g, h \in \mathcal{G}$. This structure enables us to determine the map $grad : D(\mathcal{G}^*) \to \mathcal{G}$ by means of the formula $(grad\gamma(f), g) = \frac{d}{d\varepsilon}\gamma(f + \varepsilon g) \mid_{\varepsilon=0}$ for arbitrary $f, g \in \mathcal{G}^* \simeq \mathcal{G}$. Consequently, $grad\gamma(f) \in \mathcal{G}$ is completely equivalent to the variational Euler derivative of the functional $\gamma \in D(\mathcal{G}^*)$ at a point $f \in \mathcal{G}^* \simeq \mathcal{G}$. For convenience we will also denote $grad\gamma(f)$ by $\nabla\gamma(f)$.

The canonical Hamiltonian structure $\{\{\cdot, \cdot\}\}$ on the manifold $\mathcal{G}^*$ can now be expressed via the well-known Lie-Poisson formula [2, 8–10, 13]

$$\{\{\gamma, \mu\}\} = (\langle f, \{\nabla\gamma(f), \nabla\mu(f)\}\rangle), \qquad (2.12)$$

which coincides with (2.8). To reveal the essence of formula (2.7) we consider the coadjoint action of the Lie algebra $\mathcal{G}$ on $\mathcal{G}^*$ as follows: $df/dt = ad^*_{\nabla\gamma(f)}f$, where $t \in \mathbb{R}$ is a real evolution parameter and $grad\gamma(f) \in \mathcal{G}$ at $f \in \mathcal{G}^*$. Then owing to the invariance of the scalar product on $\mathcal{G}$, the above vector field is equivalent to the following Lax type representation on $\mathcal{G}$: $df/dt = \{f, \nabla\gamma(f)\}$, which in turn is equivalent to (2.8) after an identification $\gamma \equiv H \in D(M_{(f)}) \subset D(\mathcal{G}^*)$.

It follows from (2.7) that the Hamiltonian function H given by (2.9) is a conservation law for Boltzmann-Vlasov equation (2.7), i.e, $dH/dt = 0$ for all $t \in \mathbb{R}$. Apart from this conservation law, the dynamical system (2.7) possesses the following additional invariant functionals on $\mathcal{G}^* \simeq \mathcal{G}$:

$$N = \sum_{j=\overline{1,2}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dp f_j(x, p; t), \quad P = \sum_{j=\overline{1,2}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} p dp f_j(x, p; t), \qquad (2.13)$$

where $N \in \mathbb{Z}_+$ is the whole number of particles and $P \in D(M_{(f)})$ is the total particles momentum.

Below we shall show that the Boltzmann-Vlasov system (2.7) with Hamiltonian (2.9) can be represented in the equivalent commutator form

$$df/dt = \{f, gradH(f)\}, \qquad (2.14)$$

where $f \in D(M) \simeq \mathcal{G}^* \simeq \mathcal{G}$, and has an infinite involutive (with respect to the Lie-Poisson bracket) hierarchy of conservation laws yielding the expected complete integrability [8] of flow (2.14).

3. TWO COMPONENT BOLTZMANN-VLASOV TYPE KINETIC FLOW AND ITS HYDRODYNAMIC COUNTERPART

3.1. Now one can easily compute the Boltzmann-Vlasov type equation related with the Hamiltonian function (2.2):

$$df_j/dt = -p_x f_{j,x} - p_y f_{j,y} + \beta a_{(0,0),x} f_{j,p}, \tag{3.1}$$

where $a_{(0,0)}(x_1, x_2) := \int_{\mathbb{R}} dp_{x_1} \int_{\mathbb{R}} dp_{x_2} f_1(x_1, p_{x_1}) f_2(x_{x_2}, p_{x_2})$, $x_1, x_2 \in \mathbb{R}$, and $f := (f_1, f_2)^{\mathsf{T}} \in \mathcal{D}(M_{(f)})$ is a positive valued vector function being naturally interpreted as a two-component density function of particles in the phase space $T^*(\mathbb{R})$.

The set of kinetic equations (3.1) enables the description by means of the Benney type momentum vector functions

$$a_{(m,n)}(x_1, x_2) := \int_{\mathbb{R}} dp_{x_1} p_{x_1}^m \int_{\mathbb{R}} dp_{x_2} p_{x_2}^n f_1(x_1, p_{x_1}) f_2(x_{x_2}, p_{x_2}), \tag{3.2}$$

for all $m, n \in \mathbb{Z}_+$ and $x_1, x_2 \in \mathbb{R}^1$ in the following Hamiltonian form

$$da_{(m,n)}/dt = \{\tilde{H}, a_{(m,n)}\}_{\theta(a)}, \tag{3.3}$$

generalizing that before discussed in [9], making use of a completely different approach. Here the Hamiltonian function $\tilde{H} \in \mathcal{D}(M_{(\mathbb{Z}_+^2)})$ is given by the expression

$$\hat{H} := \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \{\frac{1}{2}[a_{(2,0)}(x_1, x_2) + a_{(0,2)}(x_1, x_2)] + \beta\delta(x_1 - x_2)a_{(0,0)}(x_1, x_2)\}, \tag{3.4}$$

resulting after application the mapping

$$\nu_{(f)} : \mathcal{D}(M_f) \to \mathcal{D}(M_{\mathbb{Z}_+^2}) \tag{3.5}$$

defined by (3.2), where $M_{(\mathbb{Z}_+^2)} := l_2(\mathbb{Z}_+^2; \mathbb{R}^2)$ and Lie-Poisson bracket (2.12) on $\mathcal{D}(M_{(f)})$. The equations (3.3) generalize the corresponding ones studied before for the case of one-dimensional and one-component kinetic flows. This results can be formulated as the following proposition.

Proposition 3.1. *The two-component Boltzmann-Vlasov kinetic system of equations (3.1) is a Hamiltonian flow on the functional manifold $\mathcal{D}(M_{(f)})$ with respect to the Lie-Poisson bracket (2.12) with Hamiltonian function (3.4). Moment mapping (3.5) relates this Lie-Poisson structure to that on the functional manifold $l_2(\mathbb{Z}_+^2; \mathbb{R}^2)$, giving rise to analog (3.3) of two-component Benney type moment equations.*

3.2. We now proceed to constructing the corresponding Benney type equations (3.3) to the momentum hydrodynamic counterpart. To do this let us define in a natural way the following new spatial variable

$$y := \sum_{j=\overline{1,2}} \sigma_j \int_{-\infty}^{u_j(x;y)} dp f_j(x,p), \tag{3.6}$$

where $\sigma_j \in \mathbb{R}_+^1$, $j = \overline{1,2}$, are the corresponding sizes of the ball-like particles and $u : \mathbb{R} \times \mathbb{R}_+^1 \to \mathbb{R}^2$ is an Euler field type vector of hydrodynamic horizontal velocities of the corresponding flow components. Thereby, we have defined yet another mapping of the phase space $\mathcal{D}(M_{(f)})$ into the space $\mathcal{D}(M_{(u;h)})$, where $M_{(u;h)} := C^\infty(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}^2 \times \mathbb{R}_+)$ is the corresponding hydrodynamic velocity-height space of our two interacting to each other flow components possessing a freely moving surface:

$$\nu_{(u;h)} : \mathcal{D}(M_{(u;h)}) \to \mathcal{D}(M_{(\mathbb{Z}_+^2;h)}). \tag{3.7}$$

Mapping (3.7) iin a natural way induces a new Lie-Poisson structure $\{\cdot,\cdot\}_{\theta(u;h)}$ on $\mathcal{D}(M_f)$ with respect to which one can obtain the corresponding Hamiltonian hydrodynamic type equations

$$du/dt = \{\hat{H}, u\}_{\theta(u;h)}, \quad dh/dt = \{\hat{H}, u\}_{\theta(u;h)}, \tag{3.8}$$

where $(u;h) \in M_{(u;h)}$ and the Hamiltonian function $\hat{H} \in \mathcal{D}(M_{(u,v;h)})$ is naturally obtained from expression (3.4) under the mapping (3.7), which is equivalent to the following hydrodynamic counterpart:

$$a_{(m,n)}(x_1, x_2) := \int_0^{h(x_1)} dy_1 u_1^m(x_1, y) \int_0^{h(x_2)} dy_2 u_2^n(x_2, y_2) \tag{3.9}$$

for all $m, n \in \mathbb{Z}_+$ and $x_1, x_2 \in \mathbb{R}$. By means of momentum functions (3.9) one can construct the corresponding generating vector function

$$a(\lambda, \mu)(x_1, x_2) := \sum_{m,n \in \mathbb{Z}_+} a_{(m,n)} \lambda^{m-1} \mu^{n-1} =$$

$$= \int_0^{h(x_1)} \int_0^{h(x_2)} \frac{dy_1 dy_2}{(\lambda - u_1(x_1, y_1))(\mu - u_2(x_2, y_2))} \tag{3.10}$$

for $\lambda, \mu \in \mathbb{C}$. Expression (3.10) generalizes similar results constructed in [16] for a one-component hydrodynamic flow. The one-component case was there thoroughly analyzed from the Lie-algebraic point of view, and the related Lax type representation was constructed there in the exact form. This problem, being important for analyzing moment function (3.10), needs still for its solving some development of the usual Lie algebraic scheme. By now, the results obtained above can be formulated as the following final proposition.

Proposition 3.2. *The Hamiltonian two-fluid hydrodynamical system* (3.8) *is equivalent to the infinite Benney type momentum equations* (3.3)*, whose generating function for its infinite hierarchy of conservation laws is related with the momentum expansions* (3.8).

A problem concerning the Lie-algebraic structure of the two-component hydrodynamic flow with free surface (3.6) is still under question as its solution is strongly depending on the algebraic structure of binary series like (3.10). This and related topics we plan to analyze in detail elsewhere. A description of possible solutions to (3.1) and (3.8) as well as their properties related with dry viscosity can be effectively studied by means of the related Lax type representation for them whose existence is still under search. We only mention here that these new types of two-component Boltzmann-Vlasov type kinetic equations considered above are in some sense restrictive concerning real many interparticle interaction potentials, and there is great practical interest in constructing suitable two-component coupled kinetic Boltzmann-Vlasov type equations for one and two particle distribution functions using the analytic and algebraic methods devised before in [10–16].

Acknowledgements

REFERENCES

[1] V.E. Zakharov, *The Benney equations and the quasiclassical approximation in the inverse scattering transform,* Func. Anal. **14** (1980), 15–24 (in Russian).

[2] J. Gibbon, *Collisionless Boltzmann equations and integrable moment equations,* Physica **3D** (1981), 502–511.

[3] N.N. Bogoliubov, N.N. Bogoliubov (Jr.), *Introduction to Quantum Statistical Mechanics,* Nauka, Moscow, 1984 (in Russian).

[4] N.N. Bogoliubov, *Lectures on Statistical Mechanics,* Nauka, Moscow, 1982, 270–297 (in Russian).

[5] L.M. Brehovski, V.V. Goncharov, *Introduction to Solid State Mechanics,* Nauka, Moscow, 1982 (in Russian).

[6] Yu.A. Mitropolski, N.N. (Jr.) Bogoliubov, A.K. Prykarpatsky, V.Hr. Samoilenko, *Integrable Dynamical Systems,* Nauka dumka, Kiev, 1987 (in Russian).

[7] N.N. (Jr.) Bogoliubov, A.K. Prykarpatsky, *Quantum method of Bogoliubov generating functionals: Lie algebra of currents, its representation and functional equations,* Physics of Elementary Particles and Atom Nucleus **17** (1986), 799–827 (in Russian).

[8] A.K. Prykarpatsky, I.V. Mykytiuk, *Algebraic Aspects of Integrability of Nonlinear Dynamical Systems on Manifolds,* Naukova dumka, Kiev, 1991 (in Russian).

[9] B. Kupershmidt, *Hydrodynamical Poisson brackets and local Lie algebras,* Phys. Lett. **21A** (1987), 167–174.

[10] D.R. Lebedev, Yu.I. Manin, *Benney's long wave equations: Lax representation and conservation laws,* Zapiski nauchnykh seminarov LOMI.-1980-96; Boundary Value Problems of Mathematical Physics and Adjacent Function Theory Questions. pp. 169–178 (in Russian).

[11] P.V. Malyshev, D.Y. Petrina, V.I. Gerasimenko, *Mathematical Foundations of Classical Statistical Mechanics: Continuous Systems,* Taylor & Francis, 1989

[12] N.N. Bogoliubov, A.A. Logunov, A.I. Oksak, J.T. Todorov, *The General Principles of Quantized Field Theory,* Nauka, Moscow, 1987.

[13] B.A. Dubrovin, S.P. Novikov *Hydrodynamics of weakly deformable solitonic lattices,* Russian Math. Surv. **44** (1989), 29–98.

[14] J. Cavalcante, H.P. McKean *The classical shallow water equations: symplectic geometry,* Physica **4D** (1982), 253–260.

[15] J. Verosky *Higher-order symmetries of the compressible one-dimensional isentropic fluid equations,* J. Math. Phys. **25** (1984), 884–888.

[16] N.N. Bogoliubov (jr), A.K. Prykarpatsky, V.Hr. Samoylenko *The Hamiltonian aspects of hydrodynamical Benney type equations and associated with them Boltzmann-Vlasov equations on axis,* Ukr. Phys. Journal **37** (1992), N1, 147–156

Nikolai N. Bogoliubov (Jr.)
nikolai_bogolubov@hot.mail

V.A. Steklov Mathematical Institute of RAN,
Moscow, Russia

Denis L. Blackmore
deblac@m.njit.edu

Department of Mathematical Sciences at the NJIT,
NJ, 07102 Newark, USA

Valeriy Hr. Samoylenko
svhr@mechmat.kiev.ua

Dept. of Mechanics and Mathematics at the Shevchenko National University
Kyiv, 00617, Ukraine

Anatoliy K. Prykarpatsky
prykanat@cybergal.com, prikarpa@wms.mat.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
Cracow 30-059 Poland