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Fuzzy production planning models for an unreliable production system with fuzzy production rate and stochastic/fuzzy demand rate

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A R T I C L E I N F O	A B S T R A C T
	In this article, we consider a single-unit unreliable production system which produces a single item. During a production run, the production process may shift from the in-control state to the out-of-control state at any random time when it produces some defective items. The defective item production rate is assumed to be imprecise and is characterized by a trapezoidal fuzzy number. The production rate is proportional to the demand rate where the proportionality constant is taken to be a fuzzy number. Two production planning models are developed on the basis of fuzzy and stochastic demand patterns. The expected cost per unit time in the fuzzy sense is derived in each model and defuzzified by using the graded mean integration representation method. Numerical examples are provided to illustrate the optimal results of the proposed fuzzy models.

1. Introduction

Inventory represents an important asset to any business organization. After the pioneering work by Harris (1915) who developed the classical economic order quantity (EOQ) model with known constant demand, a great deal of researches on inventory modeling have been conducted to capture many interesting and realistic situations. However, in real world inventory systems, there exist parameters and variables which are uncertain or almost uncertain. When these uncertainties are significant, they are usually treated by probability theory. Of course, to address such an uncertainty, we need to prescribe an appropriate probability distribution. In some cases, uncertainties can be defined as fuzziness or vagueness, which are characterized by fuzzy numbers of the fuzzy set theory. Zadeh (1965) introduced fuzzy set theory to deal with quality-related problems with imprecise demand. Bellman and Zadeh (1970) distinguished the difference between randomness and fuzziness by showing that the former deals with uncertainty regarding membership or non-membership of an

element in a set while later is concerned with the degree of uncertainty by which an element belongs to a set. In an inventory control model, Petrovic and Sweeney (1994) fuzzified the demand, lead time and inventory level into triangular fuzzy numbers. They used the fuzzy proposition method to obtain the optimal order quantity. Ishii and Konno (1998) introduced fuzziness in shortage cost by an L-shape fuzzy number when demand is stochastic. Gen et al. (1997) expressed the input data by fuzzy numbers, where they used interval mean value concept to solve an inventory problem. Yao and Chiang (2003) considered an inventory model with total demand and storing cost as triangular fuzzy numbers. They performed the defuzzification by centroid and signed distance methods. Mondal and Maiti (2002) applied genetic algorithms (GAs) to solve a multi-item fuzzy EOQ model. Maiti and Maiti (2006) dealt with a fuzzy inventory model with two warehouses under possibility constraints. Mahapatra and Maiti (2006) formulated a multi-item, multi-objective inventory model for deteriorating items with stock- and time-dependent demand rate over a finite time horizon in fuzzy stochastic environment. Halim et al. (2008) developed a fuzzy inventory model for perishable items with stochastic demand, partial backlogging and fuzzy deterioration rate. The model is further extended to consider fuzzy partial backlogging factor. Goni and Maheswari (2010) discussed the retailer's ordering policy under two levels of delay payments considering the demand and the selling price as triangular fuzzy numbers. They used graded mean integration representation method for defuzzification.

Lee and Yao (1998) developed an economic production quantity (EPQ) model in which the demand and the production quantity are assumed to be fuzzy. Lo et al. (2007) presented an EPQ model which includes uncertain factors like unreliability of the machineries, flaw of the products or shortage caused by reworked process. They used fuzzy analysis hierarchy procedure (AHP) to calculate the set-up, holding and internal failure costs which affect the optimum production quantity. Halim et al. (2010) addressed the lot sizing problem in an unreliable production system with stochastic machine breakdown and fuzzy repair time. They defuzzified the cost per unit time using the signed distance method. Mahata and Goswami (2006) developed a fuzzy production-inventory model with permissible delay in payment. They assumed the demand and the production rates as fuzzy numbers and defuzzified the associated cost in the fuzzy sense using extension principle. Hsieh (2002) considered two fuzzy production-inventory models: one for crisp production quantity with fuzzy parameters and the other one for fuzzy production quantity. He used the graded mean integration representation method for defuzzifying the fuzzy total cost.

 Production of defective items in any manufacturing industry is a natural phenomenon. The number of defectives may have a change from one lot to another that cannot be assessed by a crisp value. If the uncertainty of the product flaw is treated as random then the estimation from the historical data of the value(s) of the parameter(s) involved in the associated probability distribution may not always be accurate. Chen and Chang (2008) developed a fuzzy economic production quantity (EPQ) model with defective productions that cannot be repaired. In this model, they considered a fuzzy opportunity cost and trapezoidal fuzzy costs under crisp production quantity or fuzzy production. Halim et al. (2009) developed an EPQ model in which the fraction of defective items produced after process shift is characterized by a fuzzy number. The production rate and demand rate are being known constants. In another attempt, they assumed that the fraction of defective items follow an exponential probability distribution where the parameter of the distribution is a fuzzy number. Similar to the defective item production rate, it may be difficult to search for an appropriate probability distribution for the annual demand rate and also to estimate the parameter(s) involved in the probability distribution. It is rather easier to locate the annual demand in an interval. So, to capture the real situation better, this paper considers the production and demand rates as fuzzy numbers besides fuzzy defective item production rate. The paper is organized as follows. Notations and assumptions for the proposed models are given in the next Section. The crisp model is presented in Section 3 for better understanding of the production planning problem. Section 4 develops fuzzy model with fuzzy defective item production rate and stochastic demand rate. This fuzzy model is also extended to consider fuzziness in the

demand rate. Numerical examples are provided in Section 5 to illustrate the developed models and to examine the sensitivity of the model parameters. Finally, in Section 6, some concluding remarks are given.

2. Notations and Assumptions

The following notations are used throughout the paper:

$T\,(>0)$: scheduling period

$t_1\,(<T)$: production run time during the scheduling period T

$d\,(>0)$: annual demand rate

$p\,(>d)$: production rate

X : random variable denoting the time to process shift

$f_X\,(.)$: probability density function of X

$E(N)$: expected number of defective items produced during a production run

$K\,(>0)$: fixed cost per production batch

$h\,(>0)$: holding cost per unit item per unit time

$c\,(>0)$: defective item cost per unit item.

β : a constant fraction, $0 < \beta < 1$

γ : defective item production rate, $0 \leq \gamma \leq 1$

To develop the proposed models the following assumptions are made:

(1) The production system which is operated by a single unit produces a single item.

(2) The production process is always in in-control state at the beginning of each

 production run.

(3) The process may shift from the in-control state to the out-of-control state at any

    random time when some defective items are produced.

(4) The elapsed time before process shift follows an exponential distribution with

    probability density function

$$f_X\,(t) = \begin{cases} \lambda e^{-\lambda t}, & 0 \leq t \leq t_1; \ \lambda > 0, \\ 0, & t > t_1. \end{cases}$$

(5) Defective items are neither repaired nor replaced i.e. those are scrapped.

(6) Shortages are not permitted.

(7) Production rate (p) is dependent on the demand rate (d) and is connected by the relation:

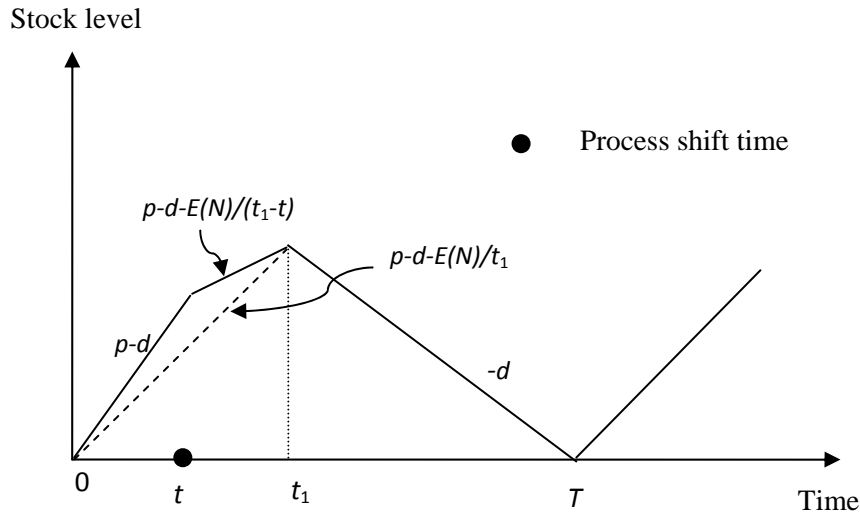$$p = \frac{1}{\beta}d, \quad 0 < \beta < 1.$$

(1)

Fig. 1. Schematic diagram of the proposed production-inventory model

3. Formulation of the Crisp Model

To derive the inventory cost function for the first scheduling period T, we divide the time interval $[0, T]$ into two parts: $[0, t_1]$ and $[t_1, T]$. The production starts at time $t = 0$ and stops at time $t = t_1$. So, stock builds up during the period $[0, t_1]$ and declines during the period $[t_1, T]$. The inventory path pattern is depicted in Fig 1. If $I_1(t)$ and $I_2(t)$ denote, respectively, the inventory levels at any time during the time periods $[0, t_1]$ and $[t_1, T]$, then the differential equations representing the inventory status are given by

$$\frac{dI_1(t)}{dt} = p - d - \frac{E(N)}{t_1}, \quad 0 \le t \le t_1 \text{ with } I_1(0) = 0,$$

(2)

$$\frac{dI_2(t)}{dt} = -d, \quad t_1 \le t \le T \text{ with } I_2(T) = 0,$$

(3)

where $E(N)$, the expected number of defective items produced during the production run is calculated as given below:

If the process shifts at time t ($0 \le t \le t_1$) then the total number of items produced after process shift is $p(t_1 - t)$. Hence, the expected number of defective items produced during the production run is

$$E(N) = \int_0^\infty \gamma p(t_1 - t) f_X(t) dt = \int_0^{t_1} \gamma p(t_1 - t) \lambda e^{-\lambda t} dt = \frac{\gamma p}{\lambda} \left[e^{-\lambda t_1} + \lambda t_1 - 1 \right].$$

(4)

Using (4) in (2) and then solving the differential equations (2) and (3), we obtain

$$I_1(t) = (p-d)t - \frac{\gamma p}{\lambda t_1}\left[e^{-\lambda t_1} + \lambda t_1 - 1\right]t, \quad 0 \le t \le t_1,$$ (5)

$$I_2(t) = d(T-t), \quad t_1 \le t \le T.$$ (6)

Therefore, inventory holding cost is as follows,

$$h\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] = \frac{h}{2\lambda}\left[\lambda p t_1^2 + \lambda dT^2 - 2\lambda d t_1 T - \gamma p t_1\left(e^{-\lambda t_1} + \lambda t_1 - 1\right)\right],$$

and defective item cost is $cE(N) = \frac{c\gamma p}{\lambda}\left(e^{-\lambda t_1} + \lambda t_1 - 1\right)$.

The total cost per unit time (W) which is the time average of the sum of set up cost, holding cost and defective item cost is given by

$$W(t_1) = \frac{K}{T} + \frac{1}{2\lambda T}[h\{\lambda p t_1^2 + \lambda dT(T - 2t_1)\} + \gamma p(2c - ht_1)(e^{-\lambda t_1} + \lambda t_1 - 1)].$$ (7)

Since $I_1(t_1) = I_2(t_1)$, therefore, we have $T = \frac{1}{d\lambda}\{\lambda p t_1 - \gamma p(e^{-\lambda t_1} + \lambda t_1 - 1)\}$.

Rearranging the terms, Eq.(7) can be rewritten as

$$W(t_1) = \frac{K}{T} + \frac{p}{2\lambda T}[\lambda h t_1^2 + \gamma(2c - ht_1)(e^{-\lambda t_1} + \lambda t_1 - 1)] + \frac{dh}{2}(T - 2t_1).$$

Now substituting $p = \frac{1}{\beta}d$ and $T = \frac{1}{d\lambda}\{\lambda p t_1 - \gamma p(e^{-\lambda t_1} + \lambda t_1 - 1)\}$ in the above equation, we get

$$W(t_1) = \frac{K\lambda\beta}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{\lambda dp}{2\lambda p\{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)\}}[-\lambda t_1(2c - ht_1) + 2c\lambda t_1 + \gamma(2c - ht_1)(e^{-\lambda t_1} + \lambda t_1 - 1)]$$

$$+ \frac{dh}{2\lambda\beta}\{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)\} - dht_1,$$

which after simplification gives

$$W(t_1) = -\frac{d}{2}(2c + ht_1) + \frac{\lambda(cdt_1 + K\beta)}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{dh}{2\lambda\beta}\{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)\}.$$ (8)

The objective of this crisp model is to find the optimal production time t_1^* which minimizes the cost per unit time W.

4. Development of Fuzzy Models

In this section, we develop two fuzzy models corresponding to the crisp model developed in the previous section. For the fundamental concept of fuzzy sets and numbers, we refer the readers to any standard text book on fuzzy set theory (e.g. Dubois and Prade (1980); Kaufmann and Gupta (1992); Zimmermann (1996); etc.). Furthermore, we introduce the following basic definitions of fuzzy sets

and numbers (Chen and Hsieh (1999); Hsieh (2002)) essential for development of the proposed fuzzy models.

Definition 1. *Generalized fuzzy number*

A generalized fuzzy number $\tilde{A}$ is a fuzzy set on $\Re = (-\infty, \infty)$ whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

(i) $\mu_{\tilde{A}}(x)$ is a continuous mapping from $\Re$ to the closed interval $[0, 1]$,

(ii) $\mu_{\tilde{A}}(x) = 0$, $-\infty < x \le a_1$,

(iii) $\mu_{\tilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$,

(iv) $\mu_{\tilde{A}}(x) = w_A$, $a_2 \le x \le a_3$,

(v) $\mu_{\tilde{A}}(x) = R(x)$ is strictly decreasing on $[a_3, a_4]$,

(vi) $\mu_{\tilde{A}}(x) = 0$, $a_4 \le x < \infty$

where $0 < w_A \le 1$ and a_1, a_2, a_3 and a_4 are real numbers. The above generalized fuzzy number is denoted by $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$. When $w_A = 1$, $\tilde{A}$ becomes $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

Definition 2. *Graded mean integration representation method* (Chen and Hsieh, 1999)

The method is based on the integral value of graded mean h-level of a generalized fuzzy number for defuzzification. The graded mean h-level value of a generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$ is given by $h(L^{-1}(h) + R^{-1}(h))/2$ where L^{-1} and R^{-1} are the inverse functions of L and R, respectively. Then, the graded mean integration representation of $\tilde{A}$ with grade w_A denoted by $P(\tilde{A})$ is defined as

$$P(\tilde{A}) = \int_0^{w_A} h(\frac{L^{-1}(h) + R^{-1}(h)}{2})dh / \int_0^{w_A} hdh,$$

where $0 < h \le w_A$ and $0 < w_A \le 1$. We use trapezoidal fuzzy numbers for all fuzzy parameters in our proposed fuzzy production inventory models. Let $\tilde{B} = (b_1, b_2, b_3, b_4)$ be a trapezoidal fuzzy number defined on $\Re = (-\infty, \infty)$. Then, for $\tilde{B}$, the graded mean integration representation is

$$P(\tilde{B}) = \int_0^1 h(\frac{b_1 + b_4 + (b_2 - b_1 - b_4 + b_3)h}{2})dh / \int_0^1 hdh = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$
(9)

Definition 3. *Fuzzy arithmetic operation under function principle*

Function principle was introduced by Chen (1986). Some fuzzy arithmetical operations of trapezoidal fuzzy numbers under function principle are described as follows:

Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers where a_is and b_i s, $i = 1, 2, 3, 4$ are real numbers. Then

(i) $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3, \ a_4 + b_4)$.

(ii) $-\tilde{B} = (-b_4, \ -b_3, \ -b_2, \ -b_1)$.

(iii) $\tilde{A} \ominus \tilde{B} = (a_1 - b_4, \ a_2 - b_3, \ a_3 - b_2, \ a_4 - b_1)$.

(iv) $\tilde{A} \otimes \tilde{B} = (c_1, \ c_2, \ c_3, \ c_4)$

where $E = \{a_1 b_1, \ a_1 b_4, \ a_4 b_1, \ a_4 b_4\}$, $E_1 = \{a_2 b_2, \ a_2 b_3, \ a_3 b_2, \ a_3 b_3\}$, $c_1 = \min E$, $c_2 = \min E_1$, $c_3 = \max E_1$, $c_4 = \max E$. If a_is and b_is, $i = 1, 2, 3, 4$ are all non zero positive real numbers, then $\tilde{A} \otimes \tilde{B} = (a_1 b_1, \ a_2 b_2, \ a_3 b_3, \ a_4 b_4)$.

(v) $1/\tilde{B} = \tilde{B}^{-1} = (1/b_4, 1/b_3, 1/b_2, 1/b_1)$, where b_1, b_2, b_3 and b_4 are all positive real numbers.

(vi)   If a_is and b_is, $i = 1, 2, 3, 4$ are all non-zero positive real numbers, then the division of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \phi \tilde{B}$ is defined as $\tilde{A} \phi \tilde{B} = (a_1 / b_4, \ a_2 / b_3, \ a_3 / b_2, \ a_4 / b_1)$.

(vii)   For $\alpha \in \Re$, $\alpha \otimes \tilde{A} = (\alpha a_1, \ \alpha a_2, \ \alpha a_3, \ \alpha a_4)$ when $\alpha \geq 0$ and $\alpha \otimes \tilde{A} = (\alpha a_4, \ \alpha a_3, \ \alpha a_2, \ \alpha a_1)$ when $\alpha < 0$ where ϕ, $\otimes$, $\ominus$ and $\oplus$ are the fuzzy arithmetical operations under function principle.

4.1. Model-I with stochastic demand and fuzzy defective rate

In this sub-section, we develop a fuzzy model with stochastic demand treating the production rate p and the defective item production rate γ as fuzzy numbers. Let the annual demand be represented by a random variable X which follows a uniform probability distribution with mean d and a range $d(1-a)$ to $d(1+a)$, $0 \leq a \leq 1$.

i.e.,
$$f_X(x) = \begin{cases} \dfrac{1}{2ad}, & d(1-a) \leq x \leq d(1+a), \\ 0, & otherwise. \end{cases}$$

In this case, Eq. (8) takes the form

$$W(t_1) = \frac{K\lambda\beta}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + [\frac{\lambda c t_1}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{h}{2\lambda\beta}\{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)\} - \frac{1}{2}(2c + h t_1)]x.$$

Therefore, the expected inventory cost per unit time $W_0(t_1)$ is given by

$$W_0(t_1) = \int_{d(1-a)}^{d(1+a)} W(t_1) f_X(x) dx$$

$$= \frac{K\lambda\beta}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + d[\frac{\lambda c t_1}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{h}{2\lambda\beta}\{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)\} - \frac{1}{2}(2c + h t_1)]$$

$$= -\frac{d}{2}(2c + h t_1) + \frac{\lambda(c d t_1 + K\beta)}{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{dh}{2\lambda\beta}\{\lambda t_1 - \gamma(e^{-\lambda t_1} + \lambda t_1 - 1)\}. \tag{10}$$

Suppose that β and γ are two generalized fuzzy numbers, say $\tilde{\beta}$ and $\tilde{\gamma}$, respectively. Then, using fuzzy arithmetical operations ϕ, Θ, $\otimes$ and $\oplus$ under function principle, we may rewrite the above equation as

$$\tilde{W_1} \equiv \tilde{W_1}(t_1) = -[(d/2)(2c + ht_1)] \oplus [\{\lambda \otimes (cdt_1 \oplus K \otimes \tilde{\beta})\}$$

$$\phi\{\lambda t_1 \Theta \tilde{\gamma} \otimes (e^{-\lambda t_1} + \lambda t_1 - 1)\}] \oplus [\{dh\phi(2\lambda \otimes \tilde{\beta})\} \otimes \{\lambda t_1 \Theta \tilde{\gamma} \otimes (e^{-\lambda t_1} + \lambda t_1 - 1)\}]. \quad (11)$$

Let us assume β and γ as two non-negative trapezoidal fuzzy numbers, i.e., $\tilde{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$ and $\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, where β_is and γ_is, $i = 1, 2, 3, 4$ are determined by the decision maker. Then, the expected cost per unit time is a fuzzy value and we obtain it by formula (11) as

$$\tilde{W_1} = [-\frac{d}{2}(2c + ht_1) + \frac{\lambda(cdt_1 + K\beta_1)}{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{dh}{2\lambda\beta_4}\{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)\},$$

$$-\frac{d}{2}(2c + ht_1) + \frac{\lambda(cdt_1 + K\beta_2)}{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{dh}{2\lambda\beta_3}\{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)\},$$

$$-\frac{d}{2}(2c + ht_1) + \frac{\lambda(cdt_1 + K\beta_3)}{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{dh}{2\lambda\beta_2}\{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)\},$$

$$-\frac{d}{2}(2c + ht_1) + \frac{\lambda(cdt_1 + K\beta_4)}{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{dh}{2\lambda\beta_1}\{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)\}]. \quad (12)$$

We defuzzify $\tilde{W_1}$ using the graded mean integration representation method (see Chen and Hsieh (1999); Hsieh (2002)) and estimate cost per unit time in the fuzzy sense by formula (9) as

$$P(\tilde{W_1}) = -\frac{d}{2}(2c + ht_1) + \frac{\lambda}{6}[\frac{cdt_1 + K\beta_1}{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{2(cdt_1 + K\beta_2)}{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)}$$

$$+ \frac{2(cdt_1 + K\beta_3)}{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{cdt_1 + K\beta_4}{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)}]$$

$$+ \frac{dh}{12\lambda}[\frac{1}{\beta_4}\{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)\} + \frac{2}{\beta_3}\{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)\}$$

$$+ \frac{2}{\beta_2}\{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)\} + \frac{1}{\beta_1}\{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)\}]. \quad (13)$$

If the objective function (13) is convex, then any suitable one dimensional search technique can be applied to find numerically the optimal value t_1^* which minimizes $P(\tilde{W_1})$.

Proposition: There exists at least one local optimal value of $P(\tilde{W_1})$ if $\beta + \gamma < 1$.

Proof: Differentiating equation (13) with respect to t_1, we obtain the optimality condition for minimization of $P(\tilde{W_1})$ as:

$$g(t_1) \equiv \frac{dP}{dt_1} = -\frac{dh}{2} + \frac{\lambda}{6}[\sum_{i=1}^{4}\frac{cdA_i - (cdt_1 + K\beta_i)\frac{dA_i}{dt_1}}{A_i^2} + \sum_{i=2}^{3}\frac{cdA_i - (cdt_1 + K\beta_i)\frac{dA_i}{dt_1}}{A_i^2}]$$

$$+\frac{dh}{12\lambda}[\sum_{i=1}^{4}\frac{1}{\beta_i}\frac{dA_i}{dt_1}+\sum_{i=2}^{3}\frac{1}{\beta_i}\frac{dA_i}{dt_1}]=0$$

where $A_i = \lambda t_1(1-\gamma_i)+\gamma_i(1-e^{-\lambda t_1})$, $i = 1, 2, 3, 4$.

Clearly, $g(0) = -\infty$

and

$$g(\infty) = \frac{dh}{12}[\frac{1-\gamma_1}{\beta_1}+\frac{2(1-\gamma_2)}{\beta_2}+\frac{2(1-\gamma_3)}{\beta_3}+\frac{1-\gamma_4}{\beta_4}-6]-\lim_{t_1\to\infty}[\sum_{i=1}^{4}\frac{(cdt_1+K\beta_i)\frac{dA_i}{dt_1}}{A_i^2}+\sum_{i=2}^{3}\frac{(cdt_1+K\beta_i)\frac{dA_i}{dt_1}}{A_i^2}].$$

Now,

$$\lim_{t_1\to\infty}\frac{(cdt_1+K\beta_1)\frac{dA_1}{dt_1}}{A_1^2} \quad (\frac{\infty}{\infty} \text{ form})$$

$$=\lim_{t_1\to\infty}\frac{cd\frac{dA_1}{dt_1}-\lambda^2\gamma_1 e^{-\lambda t_1}(cdt_1+K\beta_1)}{2A_1\frac{dA_1}{dt_1}} \quad \text{(using L' Hospital's rule)}$$

$$=\lim_{t_1\to\infty}\frac{-2cd\lambda^2\gamma_1 e^{-\lambda t_1}+\lambda^3\gamma_1 e^{-\lambda t_1}(cdt_1+K\beta_1)}{2(\frac{dA_1}{dt_1})^2-2\lambda^2\gamma_1 e^{-\lambda t_1}A_1} \quad \text{(using L' Hospital's rule)}$$

$$=\frac{0}{\infty}=0.$$

Similarly, it can be shown that

$$\lim_{t_1\to\infty}\frac{(cdt_1+K\beta_i)\frac{dA_i}{dt_1}}{A_i^2}=0 \text{ for } i = 2, 3, 4.$$

Therefore, $g(\infty)=\frac{dh}{12}[\frac{1-\gamma_1}{\beta_1}+\frac{2(1-\gamma_2)}{\beta_2}+\frac{2(1-\gamma_3)}{\beta_3}+\frac{1-\gamma_4}{\beta_4}-6]>0$ if $\frac{1-\gamma_4}{\beta_4}>1$,

because $\beta_1, \beta_2, \beta_3 < \beta_4$ and $\gamma_1, \gamma_2, \gamma_3 < \gamma_4$ implying that $\frac{1-\gamma_i}{\beta_i}>1$ for $i = 1, 2, 3$ when $\beta_4 + \gamma_4 < 1$. Hence the proposition is proved.

4.2. Model-II with fuzzy demand and fuzzy defective rate

In this sub-section, we will extend the previous model by assuming the demand rate d as fuzzy. The reason behind this assumption is that it is sometimes easier to locate annual demand rate in an interval rather than finding an appropriate probability distribution for it. We fuzzify d by assuming it to be a non-negative trapezoidal fuzzy member $\tilde{d}=(d_1, d_2, d_3, d_4)$ where d_1, d_2, d_3 and d_4 are determined by the decision maker. In this case, the expected fuzzy cost $\tilde{W}_2 \equiv \tilde{W}_2(t_1)$ per unit time can be obtained by formula (11) as:

$$\tilde{W}_2 = [-\frac{d_4}{2}(2c + ht_1) + \frac{\lambda(cd_1t_1 + K\beta_1)}{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{d_1 h}{2\lambda\beta_4}\{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)\},$$

$$-\frac{d_3}{2}(2c + ht_1) + \frac{\lambda(cd_2t_1 + K\beta_2)}{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{d_2 h}{2\lambda\beta_3}\{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)\},$$

$$-\frac{d_2}{2}(2c + ht_1) + \frac{\lambda(cd_3t_1 + K\beta_3)}{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{d_3 h}{2\lambda\beta_2}\{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)\},$$

$$-\frac{d_1}{2}(2c + ht_1) + \frac{\lambda(cd_4t_1 + K\beta_4)}{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{d_4 h}{2\lambda\beta_1}\{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)\}] . \quad (14)$$

Similar to the previous sub-section, we defuzzify $\tilde{W}_2$ using the graded mean integration representation method by formula (9) as

$$P(\tilde{W}_2) = -\frac{1}{12}(2c + ht_1)(d_1 + 2d_2 + 2d_3 + d_4)$$

$$+ \frac{\lambda}{6}[\frac{cd_1t_1 + K\beta_1}{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{2(cd_2t_1 + K\beta_2)}{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)}$$

$$+ \frac{2(cd_3t_1 + K\beta_3)}{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)} + \frac{cd_4t_1 + K\beta_4}{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)}]$$

$$+ \frac{h}{12\lambda}[\frac{d_1}{\beta_4}\{\lambda t_1 - \gamma_4(e^{-\lambda t_1} + \lambda t_1 - 1)\} + \frac{2d_2}{\beta_3}\{\lambda t_1 - \gamma_3(e^{-\lambda t_1} + \lambda t_1 - 1)\}$$

$$+ \frac{2d_3}{\beta_2}\{\lambda t_1 - \gamma_2(e^{-\lambda t_1} + \lambda t_1 - 1)\} + \frac{d_4}{\beta_1}\{\lambda t_1 - \gamma_1(e^{-\lambda t_1} + \lambda t_1 - 1)\}]] . \quad (15)$$

The objective here is to find the optimal value of t_1 which minimizes $\tilde{W}_2(t_1)$. Similar to Model-I, it is difficult to prove the convexity property of the cost function $P(\tilde{W}_2)$ analytically. However, an appropriate search technique can be applied to find the optimal solution numerically.

5. Numerical Results

In order to illustrate the numerical outcomes of the models developed in Sections 3 and 4, we consider the following input data: $K = 600$, $d = 80$, $h = 1$, $c = 5$, $\lambda = 0.5$, $\beta = 0.5$, $\gamma = 0.15$ in appropriate units. Using the numerical computational software Mathematica, we obtain the optimal crisp value of the production time t_1^* as 2.84289 units and the corresponding expected cost per unit time $W(t_{1c}^*)$ as 241.352 units. For the fuzzy models, instead of taking $\beta = 0.5$ we now take β around 0.5. Also, we consider the defective rate around $\gamma = 0.15$ i.e. good-quality rate is about 0.85. The optimal production time and the minimum expected cost per unit time in the fuzzy sense corresponding to the fuzzy Models I & II are presented in Table 1. Here, we use a general rule to transfer the linguistic data, "greater or less than Z" or "around Z", into trapezoidal fuzzy number as:

"greater or less than Z" or "around Z" $= (0.9Z, 0.95Z, 1.05Z, 1.1Z)$. Then, by the above rule, the fuzzy parameters in this example can be transferred as follows:

Fuzzy demand rate="greater or less than 80"$= \tilde{d} = (d_1, d_2, d_3, d_4) = (72, 76, 84, 88)$,

Fuzzy β = "around 0.5" $= \tilde{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4) = (0.450, 0.475, 0.525, 0.550)$,

Fuzzy γ = "around 0.15" $= \tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (0.1350, 0.1425, 0.1575, 0.1650)$.

Table 1
Optimal results of the proposed fuzzy models

Model	t_1^*	$W_1^*(t_1^*)$
I	2.8087	242.958
II	2.8067	243.848

5.1 Sensitivity Analysis

We will now perform the sensitivity analysis to examine the effects of changes in the input parameters K, d, h, c, λ, β and γ on the optimal results obtained in Model-I. At first, we find the optimal values of t_1 and W_1 by changing each of the parameters by 50%, 20%, -20% and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. Then we calculate the percentage change of t_1^* and W_1^* with the base value. The results are shown in Table 2. The following observations can be made from the sensitivity analysis:

 (i) From Table 2, we see that the percentage change in the cost is almost equal for small changes (both positive and negative) of all the parameters.

(ii) The model is moderately sensitive to the changes in the parameters K, d, h and β.

(iii) The model shows low sensitivity with respect to the parameters c, λ and γ.

(iv) t_1^* is highly sensitive with respect to the parameter β.

It is also noted that $W_1^*(t_1^*)$ increases with the increase in all the parameters K, d, h, c, λ and γ whereas it decreases with the increase in the parameter β. The reason is that when β increases, the production rate decreases (by Eq. (1)). As a result, the machine produces less in quantity. Consequently, the defective items produced are also less. Thus, the lower holding cost and defective item cost result in a decrease in the average cost. Similar characteristics are observed in Model-II.

Table 2
Sensitivity analysis with respect to the parameters in Model-I

Parameter	% change in parameter	% change in t_1^*	% change in $W_1(t_1^*)$
	+50	26.6686	20.9332
K	+20	11.1902	8.9472
	-20	- 12.1084	- 10.0026
	-50	- 32.7970	- 28.0464
	+50	- 20.8459	23.8070
d	+20	- 10.0150	10.1038
	-20	13.8690	- 11.1583
	-50	49.8861	- 30.8769
	+50	- 17.6284	18.5250
h	+20	- 8.2999	7.8046
	-20	11.0489	- 8.5031
	-50	37.8951	- 23.1707
	+50	- 5.1280	6.0327
c	+20	- 2.1120	2.4395
	-20	2.1960	- 2.4753
	-50	5.6510	- 6.2571
	+50	2.8825	2.7342
λ	+20	1.1080	1.2117
	-20	- 1.0055	- 1.3928
	-50	- 2.2512	- 3.8682
	+50	177.4230	- 27.9472
β	+20	41.3540	- 9.0481
	-20	- 28.1796	7.1708
	-50	- 60.3870	16.2361
γ	+50	0.7747	5.4754
	+20	0.4262	2.1106
	-20	- 0.5469	- 2.0119
	-50	- 1.5406	- 4.8576

6. Concluding Remarks

Uncertainties in demand, production, defective item production etc. are inherent in any unreliable manufacturing system. In this article, we have developed two production planning models for an unreliable manufacturing system. In the first model, the demand rate was assumed to be stochastic whereas in the second model, the demand rate was assumed to be fuzzy. The production rate was proportional to the demand rate where the constant of proportionality was assumed to be a fuzzy number. Well known trapezoidal membership function was used for all the fuzzy numbers. Though several approaches viz. random number technique, probability theory including fuzzy set theory have the capability to capture uncertainties arising in inventory system but it is still difficult to identify which technique performs better. However, the advantage of the fuzzy approach is that it relaxes the rigid assumptions such as constant defective rate, constant demand rate etc. Also, it eases the difficulties in searching for suitable probability distribution function to represent the random behavior of uncontrollable variables.

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