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The Influence of the Number of Finite Elements upon the Accuracy of the Results Obtained Using Discrete Models

Discrete models are often used because they require a simple mathematical approach, even if their accuracy is inferior to continues models. This paper presents a study regarding the influence of the number of used elements upon the accuracy with which the natural frequencies of straight beams can determined. The results show that, to achieve a reasonable accuracy, it is necessary to use at least ten elements, while for rigorous calculus, more than three hundred elements must be considered.

Keywords: *beam, frequency, discrete model, model accuracy*

1. Introduction

Theoretical models of technical systems whose mathematical model contains simple and/or differential equations are called discrete models. They use several numbers of elements, each of them staying for a system portion; consequently they have a finite number of degrees of freedom. Opposite to them, the so-called continuous or rheological models completely fill the space portion of the system; they are easy to be developed, but are described by complex and complicated mathematical models, difficult to be solved. Continuous models are more suitable, providing trustful results [1].

Referring to beams, usually in the continuous approach the Euler-Bernoulli or Timoshenko models are used [2], while discrete models proposed by Duncan, Rayleigh or Ritz (se [3] and [4]) are known. Obviously, using diverse models different results are obtained; this paper compares the reliability and precision level of discrete models, having as reference the continuous one.

2. Determination of the equivalent mass from dynamical conditions

By considering the behavior of a beam-like structure modeled by the Euler-Bernoulli model, the basic relation providing the natural frequencies is:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1}$$

where f is the frequency (in Hz), k is the stiffness and m is the vibrating mass.

From simple bending theory, we have the deflection at the free end:

$$\delta = \frac{FL^3}{3EI} \tag{2}$$

where E is the Young's modulus, I the inertia moment of the cross section.

The stiffness k is defined by the relation:

$$k = \frac{F}{\delta} = \frac{3EI}{L^3} \tag{3}$$

Thus, substituting relation (3) in relation (1), the frequency becomes:

$$f = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}} \tag{4}$$

Consider the case where the bar has negligible mass, and has an element with mass m_1 placed at the distance x_1. We want to find the equivalent mass m_e placed at the free end L of the bar, which produces the same dynamic effect (characteristic frequency) as the mass m_1.

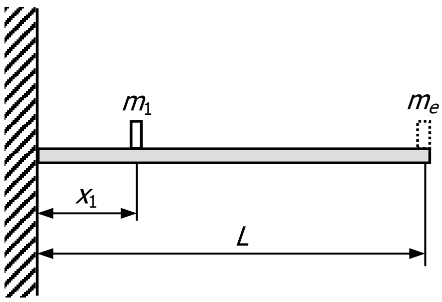

Figure 1. Cantilever bar with masses placed at certain distances

In the first case, considering only the mass m_1, from equation (4) is deduced the following relationship:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{3EI}{m_1 x_1^3}} \tag{5}$$

and for the mass m_e placed at the free end, we obtain:

$$f_e = \frac{1}{2\pi}\sqrt{\frac{3EI}{m_e L^3}} \tag{6}$$

The frequencies should be equal, so from (5) and (6) we obtain:

$$m_1 x_1^3 = m_e L^3 \tag{7}$$

or

$$m_e = \left(\frac{x_1}{L}\right)^3 m_1 \tag{8}$$

In this case, the cantilever with the mass m uniformly distributed can be analyzed using an equivalent weight bars placed at the free end. Considering an element of length dx located at a distance x from the fixed end, the mass of it is $m \cdot dx$, and the equivalent mass at the rear of this element is:

$$dm_c = \left(\frac{x}{L}\right)^3 mdx \tag{9}$$

and by integration over the entire length, one obtains:

$$m_e = \int_0^L \left(\frac{x}{L}\right)^3 mdx = \frac{m_0 L}{4} \tag{10}$$

or

$$m_e = \frac{m_0 L}{4} \tag{11}$$

where m_0 is the unit weight of the cantilever $m_0 = A \cdot \rho$

In conclusion, a cantilever with its own weight uniformly distributed along the length vibrate at the same frequency as a cantilever loaded at free end with a mass equal to ¼ of the mass of the bar, which meant that the bar loaded with equivalent weight m_e located at a certain distance of restraint, shall be calculated with the relation (11).

3. Discrete models of straight bars

Figure 2 shows two modes of mass of an element mesh of constant section beam. Duncan's model (fig. 2. b, d) has the total mass concentrated in the center of gravity; Rayleigh's model (fig. 2. c, e) has one half of the total mass concentrated at each end of the bar. With $m_0 = A \cdot \rho$ is noted the mass per unit length, where ρ is the density of the material and A is the cross-sectional area.

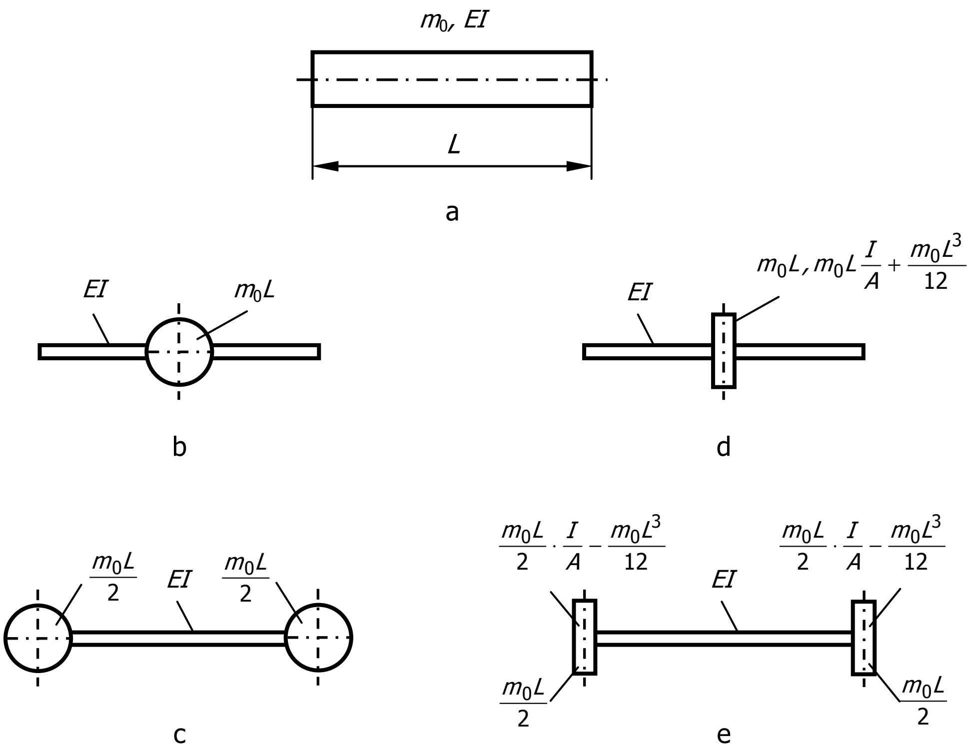

Figure 2. Mesh mass modes of a constant cross section beam element

Mesh mode, using Rayleigh model, depending on the number of items selected is exemplified for a cantilever (fig. 3) for different levels of approximation.

In [3], states that a segment model (fig. 3.a), the ratio between pulsation own and the true value is ω_1/ω_{10} = 0,7. If the beam is divided into two sections (fig. 3.b), the ratio of the first pulse is ω_1/ω_{10} = 0,9. If the beam is divided into three sections (fig. 3.c), the ratio of the first pulse is ω_1/ω_{10} = 0,95.

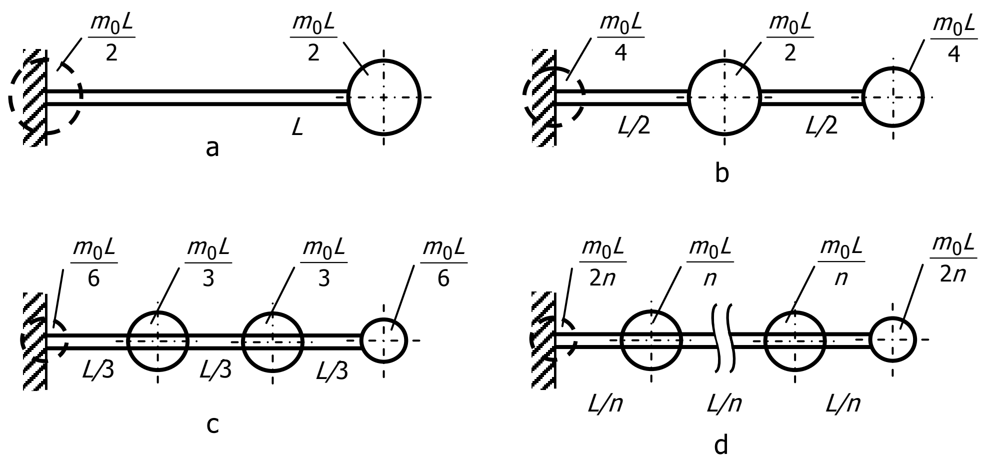

Figure 3. Rayleigh model for n levels of approximation

To start with, we consider the model of (fig. 3.a), in which the mass is distributed to the ends of the bar (m/2). As shown in the relation (11) the equivalent mass is considered concentrated in the free end of the bar:

$$m_e = \frac{m_0 L}{4} = \frac{m}{4} \tag{12}$$

where: m is the total mass of the bar, that mines $m = m_0 L$

In this case, we can write the frequency f_c:

$$f_c = \frac{1}{2\pi}\sqrt{\frac{3EI}{m_e L^3}} = \frac{1}{2\pi}\sqrt{4\frac{3EI}{mL^3}} \tag{13}$$

For the case when considering a single element, the mass is distributed equally to the two ends. In this case we have:

$$m_{eI} = \frac{1}{2}m \tag{14}$$

So we can write the frequency f_I:

$$f_I = \frac{1}{2\pi}\sqrt{\frac{3EI}{m_{eI} L^3}} = \frac{1}{2\pi}\sqrt{2\frac{3EI}{mL^3}} \tag{15}$$

The ratio of calculated frequency Rayleigh model with a single element and the continuous case is:

$$r_I = \frac{f_I}{f_c} = \frac{\frac{1}{2\pi}\sqrt{2\frac{3EI}{mL^3}}}{\frac{1}{2\pi}\sqrt{4\frac{3EI}{mL^3}}} = \frac{\sqrt{2}}{\sqrt{4}} = 0{,}70711 \tag{15}$$

If we consider two elements, the mass is distributed evenly on the ends of their elements (Fig. 3, b). Thus, at the free end we have $m_{II}^2 = m/4$ and in the middle we have $m_{II}^1 = m/2$. In this case:

$$m_{eII}^1 L^3 = \frac{m}{2} \cdot \frac{L^3}{2^3} \quad \Rightarrow \quad m_{eII}^1 = \frac{m}{16} \tag{16}$$

and

$$m_{eII}^2 = \frac{m}{4} \tag{17}$$

resulting the total equivalent weight m_{eII}:

$$m_{eII} = \frac{m}{16} + \frac{m}{4} = \frac{5m}{16} \tag{18}$$

So the frequency f_{II} is:

$$f_{II} = \frac{1}{2\pi}\sqrt{\frac{3EI}{m_{eII}L^3}} = \frac{1}{2\pi}\sqrt{\frac{16}{5}\cdot\frac{3EI}{mL^3}} \tag{19}$$

The ratio of calculated frequency Rayleigh model with two elements and continuous case is:

$$r_{II} = \frac{f_{II}}{f_c} = \frac{\frac{1}{2\pi}\sqrt{\frac{16}{5}\cdot\frac{3EI}{mL^3}}}{\frac{1}{2\pi}\sqrt{4\frac{3EI}{mL^3}}} = \frac{\sqrt{\frac{16}{5}}}{\sqrt{4}} = \sqrt{\frac{4}{5}} = 0{,}89443 \tag{20}$$

For the mesh case with three elements, the mass of each element is $m/3$, so at the very end we have $m_{III}^1 = m/6$ and in the other points we have $m_{III}^1 = m_{III}^2 = m/3$, therefore:

$$m_{eIII}^1 L^3 = \frac{m}{3} \cdot \frac{L^3}{3^3} \quad \Rightarrow \quad m_{eIII}^1 = m\frac{1}{3^4} \tag{21}$$

$$m_{eIII}^{2}L^{3}=\frac{m}{3}\cdot\frac{L^{3}\cdot 2^{3}}{3^{3}}\quad\Rightarrow\quad m_{eIII}^{2}=m\frac{2^{3}}{3^{4}}\tag{22}$$

$$m_{eIII}^{3}=\frac{m}{6}\tag{23}$$

So the total equivalent weight is placed at the free end of the bar:

$$m_{eIII}=m\left(\frac{1}{3^{4}}+\frac{2^{3}}{3^{4}}+\frac{1}{2\cdot 3}\right)=m\left(\frac{2}{2\cdot 3^{4}}+\frac{2^{4}}{2\cdot 3^{4}}+\frac{3^{3}}{2\cdot 3^{4}}\right)=\frac{2+16+27}{2\cdot 3^{4}}=\frac{45}{2\cdot 3^{4}}\tag{24}$$

And the frequency f_{III} is:

$$f_{III}=\frac{1}{2\pi}\sqrt{\frac{2\cdot 3^{4}}{45}\cdot\frac{3EI}{m_{eII}L^{3}}}\tag{25}$$

In this case the frequency ratio calculated with the Rayleigh model, is:

$$r_{III}=\frac{f_{III}}{f_{c}}=\frac{\frac{1}{2\pi}\sqrt{\frac{2\cdot 3^{4}}{45}\cdot\frac{3EI}{m_{eII}L^{3}}}}{\frac{1}{2\pi}\sqrt{4\frac{3EI}{mL^{3}}}}=\frac{\sqrt{\frac{2\cdot 3^{4}}{45}}}{\sqrt{4}}=\sqrt{\frac{3^{4}}{90}}=\sqrt{\frac{9}{10}}=0{,}94868\tag{26}$$

There is a good correlation between the results obtained with the method described above and shown in [5].

The solution is sought for meshing with n elements. In this case, the total weight is distributed in n, and on the free end have $m_{(n)}^{n}=m/2n$ and in other points $m_{(n)}^{1}=m_{(n)}^{n-1}=m/n$, thus the equivalent mass to the free end is:

$$m_{e(n)}^{n}L^{3}=\frac{mL^{3}}{2n}\quad\Rightarrow\quad m_{e(n)}^{n}=\frac{m}{2n}\tag{27}$$

For the first point the equivalent mass, is:

$$m_{e(n)}^{1}L^{3}=\frac{m}{n}\cdot\frac{L^{3}}{n^{3}}\quad\Rightarrow\quad m_{e(n)}^{1}=\frac{m}{n}\cdot\frac{1}{n^{3}}\tag{28}$$

For the second point the equivalent mass, is:

$$m_{e(n)}^{2}L^{3}=\frac{m}{n}\cdot\frac{2^{3}\cdot L^{3}}{n^{3}}\quad\Rightarrow\quad m_{e(n)}^{2}=\frac{m}{n}\cdot\frac{2^{3}}{n^{3}}\tag{29}$$

For the n - 1 point the equivalent mass, is:

$$m_{e(n)}^{n-1} L^3 = \frac{m}{n} \cdot \frac{(n-1)^3 \cdot L^3}{n^3} \quad \Rightarrow \quad m_{e(n)}^{n-1} = \frac{m}{n} \cdot \frac{(n-1)^3}{n^3} \tag{30}$$

The total equivalent mass $m_{e(n)}$, is:

$$m_{e(n)} = \frac{m}{n} \cdot \frac{1}{n^3} + \frac{m}{n} \cdot \frac{2^3}{n^3} + \ldots + \frac{m}{n} \cdot \frac{(n-1)^3}{n^3} + \frac{m}{2n} \cdot \frac{n^3}{n^3} = \frac{m}{n^4} \left(\frac{n^3}{2} + \sum_{i=1}^{n-1} i^3 \right) \tag{31}$$

Given that:

$$\sum_{i=1}^{n-1} i^3 = \left[\frac{(n-1)}{2} \right]^2 \tag{32}$$

results:

$$\begin{aligned} m_{e(n)} &= \frac{m}{n^4} \left(\frac{n^3}{2} + \left[\frac{(n-1)}{2} \right]^2 \right) = \frac{m}{4 \cdot n^4} \left[n^2 \right] \cdot \left[2n + (n-1)^2 \right] = \\ &= \frac{m}{4 \cdot n^2} \left[2n + (n-1)^2 \right] = \frac{m}{4 \cdot n^2} \left(n^2 + 1 \right) \end{aligned} \tag{33}$$

In this case the frequency for the system modeled with n elements, can be written:

$$f_{(n)} = \frac{1}{2\pi} \sqrt{\frac{4 \cdot n^2}{m(n^2+1)} \cdot \frac{EI}{L^3}} \tag{34}$$

and the ratio of the frequency for the beam modeled with n elements can be written:

$$r_{(n)} = \frac{f_{(n)}}{f_c} = \frac{\frac{1}{2\pi} \sqrt{\frac{4 \cdot n^2}{m(n^2+1)} \cdot \frac{EI}{L^3}}}{\frac{1}{2\pi} \sqrt{\frac{4}{m} \cdot \frac{EI}{L^3}}} = \sqrt{\frac{n^2}{n^2+1}} \tag{35}$$

4. Results and conclusions

Table 1 shows the frequency f_i and in Table 2 the ratio r_i for some simulated cases and their diagram (figure 4). There is a good convergence to 1 when the number of elements analyzed is grater then 300.

Table 1. Frequency f_i as the number of elements used

No. of elements i	1	2	3	4	5	6
f_i	2,221441	2,809926	2,980376	3,047793	3,080585	3,098848
No. of elements i	7	8	10	20	40	60
f_i	3,110018	3,117333	3,126002	3,137673	3,140611	3,141156
No. of elements i	80	100	200	300	500	900
f_i	3,141347	3,141436	3,141553	3,141575	3,141586	3,141591

Table 2. Ratio r_i as the number of elements used

No. of elements i	1	2	3	4	5	6
r_i	0,707107	0,894427	0,948683	0,970143	0,980581	0,986394
No. of elements i	7	8	10	20	40	60
r_i	0,989949	0,992278	0,995037	0,998752	0,999688	0,999861
No. of elements i	80	100	200	300	500	900
r_i	0,999922	0,999950	0,999988	0,999994	0,999998	0,999999

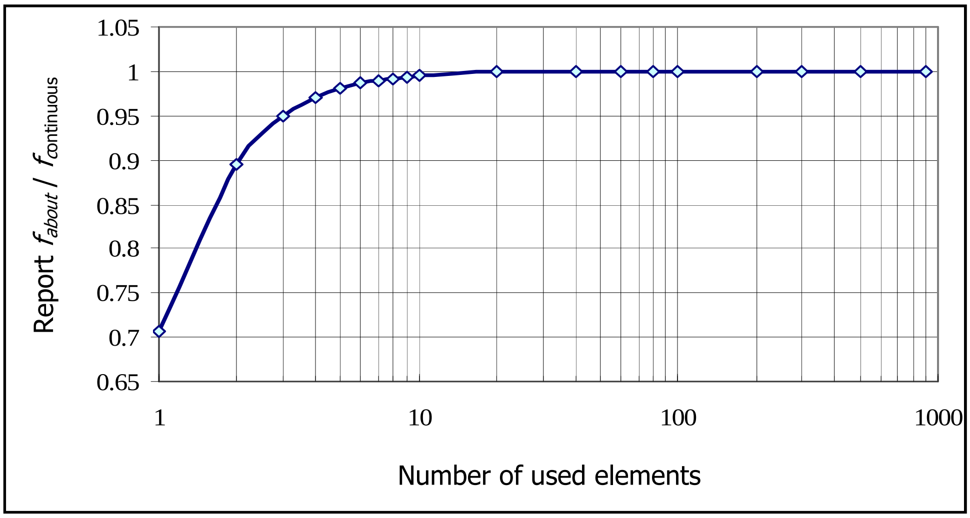

Figure 4. The ratio diagram r_i by number of used elements

For any bar with constant cross-section the results from Table 1 and 2 are valid, independent of density or geometric shape of the cross-section. That shows a good agreement between the results obtained for the model with 1, 2, or 3 elements with those presented in [3]. Technically acceptable accuracy is obtained for the model with 10 elements and consistent results with continuous system requires more then 300 elements.

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