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Professor Andrzej Lasota in Memoriam

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# ON PERIODIC AND STABLE SOLUTIONS OF THE LASOTA EQUATION IN DIFFERENT PHASE SPACES

**Abstract.** We study properties of the Lasota partial differential equation in two different spaces: $V_\alpha$ (Hölder continuous functions) and $L^p$. The aim of this paper is to generalize the results of [1].

**Keywords:** partial differential equations, periodic solutions, stable solutions.

**Mathematics Subject Classification:** 35B10, 35B35, 37C75, 47D06.

## 1. INTRODUCTION

We consider the partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda(x)u, \quad t \geq 0,\ 0 \leq x \leq 1 \tag{1.1}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leq x \leq 1, \tag{1.2}$$

where $v$ belongs to some normed vector space $V$ of functions defined on $[0,1]$ and $\lambda : [0,1] \to \mathbb{R}$ is a given continuous function. Let a semidynamical system

$$T_t : V \to V$$

be given by the formula

$$(T_t v)(x) = u(t,x),$$

where $u$ is the solution of (1.1), (1.2). It is clear that this unique solution is given by the formula

$$(T_t v)(x) = u(t,x) = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{-t})}v(x\mathrm{e}^{-t}), \quad x \in [0,1], \tag{1.3}$$

where
$$g(x) = -\int_x^1 \frac{\lambda(s)}{s} ds$$
with the condition
$$\int_0^1 \frac{\lambda(s)}{s} ds = \infty. \tag{1.4}$$
We wish to investigate some properties of system (1.3): periodic solutions, strong and exponential stability.

**Definition 1.1.** *A function $v_0 \in V$ is a* periodic point *of the semigroup $(T_t)_{t\geq 0}$, with a period $t_0 \geq 0$ iff $T_{t_0} v_0 = v_0$. A number $t_0 > 0$, is called a* principal period *of a periodic point $v_0$ iff the set of all periods of $v_0$ is equal to $\mathbb{N} t_0$.*

**Definition 1.2.** *The semigroup $(T_t)_{t\geq 0}$ is* strongly stable *in $V$ iff for every $v \in V$,*
$$\lim_{t\to\infty} T_t v = 0 \quad \textit{in } V.$$

**Definition 1.3.** *The semigroup $(T_t)_{t\geq 0}$ is* exponentially stable *iff there exist $D < \infty$ and $\omega > 0$ such that*
$$\|T_t\| \leq De^{-\omega t}, \quad \textit{for } t \geq 0.$$

The problem of the chaotic behaviour of a partial differential equation was considered by Lasota [5], Rudnicki [8], Łoskot [7] and Szarek [6]. In the papers [1–4] there were described properties of the partial differential equation, analogical to (1.1), but with a constant function $\lambda$:
$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma u, \quad t \geq 0,\ 0 \leq x \leq 1 \tag{1.5}$$
and with the initial condition
$$u(0,x) = v(x), \quad 0 \leq x \leq 1. \tag{1.6}$$
This work has been intended as an attempt at generalizing the results of [1]. In [1] there was described the chaotic and stability behaviour of the suitable semidynamical system
$$(\widetilde{T}_t v)(x) = \widetilde{u}(t,x) = \mathrm{e}^{\gamma t} v(x\mathrm{e}^{-t}), \quad x \in [0,1] \tag{1.7}$$
in different phase spaces $V$. All properties depended on the value $\gamma$. We are interested in finding a connection between this two equations. It is easy to check that if $u$ and $\widetilde{u}$ are the solutions of equation (1.1) and (1.5), respectively, then
$$\widetilde{u}(t,x) = \kappa(x) u(t,x), \tag{1.8}$$
where
$$\kappa(x) = \mathrm{e}^{\int_0^x \frac{\lambda(0)-\lambda(s)}{s} ds} \quad \text{and} \quad \gamma = \lambda(0). \tag{1.9}$$
Hence the diagram

$$\begin{array}{ccc} V & \xrightarrow{T_t} & V \\ {\scriptstyle m_\kappa}\big\downarrow & & \big\downarrow{\scriptstyle m_\kappa} \\ V & \xrightarrow[\widetilde{T}_t]{} & V \end{array}$$

This substitution will be a useful tool. It will be used in the proofs of theorems on chaos and stability of system (1.3) in the spaces $V_\alpha$ and $L^p$.

## 2. PROPERTIES OF THE DYNAMICAL SYSTEM $(T_t)_{t\geq 0}$ IN THE SPACE $V_\alpha$

Let $v$ be a continous function on $[0,1]$ such that $v(0)=0$. For every interval $A\subset[0,1]$ and for every $\alpha\in(0,1]$, define

$$H_{A,\alpha}(v)=\sup_{x,y\in A,x\neq y}\frac{|v(x)-v(y)|}{|x-y|^\alpha}.$$

A function $v$ for which $H_{A,\alpha}(v)<\infty$ is called a Hölder continuous on the interval $A$ with exponent $\alpha$. Write

$$H_\alpha=H_{[0,1],\alpha}.$$

**Definition 2.1.** *Denote by $V_\alpha$ the space of all Hölder continuous functions $v$ on $[0,1]$ with exponent $\alpha$, vanishing at zero and satisfying the following condition*

$$\lim_{x\to 0}H_{[0,x],\alpha}(v)=0.$$

Certain properties of system (1.7) in the space $V_\alpha$ have been estabilished. For $\gamma>\alpha$, there exist periodic solutions of problem (1.5) and the set of all periodic points is dense in $V_\alpha$. Strong and exponential stability take place, provided that $\gamma\leq\alpha$ and $\gamma<\alpha$, respectively (see [1] for more details).

**Theorem 2.2.** *Let*

$$|\lambda(0)-\lambda(x)|\leq Cx,\quad C>0,\quad x\in[0,1] \tag{2.1}$$

*hold. Then the function $u\in V_\alpha$ if and only if $\widetilde{u}\in V_\alpha$.*

*Proof.* The assumption $u\in V_\alpha$ means that $u$ is a Hölder continuous function with exponent $\alpha$, vanishing at zero and $\lim_{a\to 0}H_{[0,a],\alpha}(u)=0$. This gives

$$\begin{aligned} H_{[0,a],\alpha}(\widetilde{u}) &= \sup_{x,y\in[0,a],x\neq y}\frac{|\widetilde{u}(t,x)-\widetilde{u}(t,y)|}{|x-y|^\alpha}= \\ &= \sup_{x,y\in[0,a],x\neq y}\frac{|\kappa(x)u(t,x)-\kappa(y)u(t,y)|}{|x-y|^\alpha}\leq \\ &\leq \sup_{x\in[0,a]}|\kappa(x)|\cdot\sup_{x,y\in[0,a],x\neq y}\frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha}+ \\ &\quad+\sup_{y\in[0,a]}|u(t,y)|\cdot\sup_{x,y\in[0,a],x\neq y}\frac{|\kappa(x)-\kappa(y)|}{|x-y|^\alpha}. \end{aligned}$$

Using (1.9) and (2.1), we obtain

$$|\kappa(x)| \leq \mathrm{e}^{\int_0^x \frac{|\lambda(0)-\lambda(s)|}{s} ds} \leq \mathrm{e}^{\int_0^x C ds} = \mathrm{e}^{Cx}.$$

It remains to show that the function $\kappa$ is Hölder continous. If $\kappa^{\frac{1}{\alpha}}$ apears the Lipschitz function, it will mean that $\kappa$ is Hölder continuous with exponent $\alpha$. For $x \in [0,a]$

$$|(\kappa^{\frac{1}{\alpha}}(x))'| = \left| \mathrm{e}^{\frac{1}{\alpha}\int_0^x \frac{\lambda(0)-\lambda(s)}{s} ds} \cdot \frac{1}{\alpha} \cdot \frac{\lambda(0)-\lambda(x)}{x} \right| \leq \left| \mathrm{e}^{\frac{1}{\alpha}\int_0^x C ds} \cdot \frac{C}{\alpha} \right| = \\ = \left| \mathrm{e}^{\frac{Cx}{\alpha}} \cdot \frac{C}{\alpha} \right| \leq \mathrm{e}^{\frac{Ca}{\alpha}} \cdot \frac{C}{\alpha}.$$

Summarizing,

$$H_{[0,a],\alpha}(\widetilde{u}) \leq \mathrm{e}^{Ca} \left( H_{[0,a],\alpha}(u) + \sup_{y\in[0,a]} |u(t,y)| \cdot \left(\frac{C}{\alpha}\right)^{\alpha} \right).$$

Therefore, $\lim_{a\to 0} H_{[0,a],\alpha}(\widetilde{u}) = 0$ and finally $\widetilde{u} \in V_\alpha$. The rest of the proof runs as before. We can draw the same conclusion for the funcition $u$, assuming that $\widetilde{u} \in V_\alpha$. □

Assumption (2.1) will be needed throughout this section and the above theorem will be crucial for next results.

**Theorem 2.3.** *If $\lambda(0) > \alpha \in (0,1]$, then for any $t_0$ there exists such $v_0 \in V_\alpha$ that*

$$T_{t_0} v_0 = v_0 \tag{2.2}$$

*and*

$$T_t v_0 = v_0 \quad \text{if and only if} \quad t = nt_0 \text{ for some positive integer } n. \tag{2.3}$$

*Proof.* Let $w$ be an arbitrary Hölder continuous function with the exponent $\alpha$ defined on the interval $[\mathrm{e}^{-t_0}, 1]$ and satisfying the following conditions:

$$\mathrm{e}^{-g(\mathrm{e}^{-t_0})} w(\mathrm{e}^{-t_0}) = w(1), \tag{2.4}$$

$$\forall t \in (0,t_0) \quad \mathrm{e}^{-g(\mathrm{e}^{-t})} w(\mathrm{e}^{-t}) \neq w(1). \tag{2.5}$$

Consider the following function $v$ on the interval $(0,1]$ :

$$v(x) = \mathrm{e}^{g(x)} \mathrm{e}^{-g(x\mathrm{e}^{nt_0})} w(x\mathrm{e}^{nt_0}) \quad \text{for} \quad x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}].$$

The function $v$ is defined on the whole interval $(0,1] = \bigcup_{n=0}^{\infty} (\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$ and come into being by squeezing the graph of the function $w$ into each interval $(\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$.
By the assumption of the continuity of $w$ on $[\mathrm{e}^{-t_0}, 1]$, its boundedness follows, i.e.,

there is $M > 0$ such that $|w(x)| \leq M$ for each $x \in [\mathrm{e}^{-t_0}, 1]$. By the above, for $x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$, the following estimate holds:

$$|v(x)| = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{nt_0})}|w(x\mathrm{e}^{nt_0})| \leq M\mathrm{e}^{g(x)} \cdot \sup_{x\in[\mathrm{e}^{-t_0},1]} \mathrm{e}^{-g(x)} \leq M_1\mathrm{e}^{g(x)},$$

where $M_1 = M \cdot \sup_{x\in[\mathrm{e}^{-t_0},1]} \mathrm{e}^{-g(x)}$. From assumption (1.4), $\lim_{x\to 0} \mathrm{e}^{g(x)} = 0$, so we deduce that $v(0) = 0$. We obtain the continuous function $v$ defined on the whole interval $[0, 1]$. Property (2.2) follows from (2.4), while property (2.3) from (2.5).

The assumption $\lambda(0) > \alpha$ yields $\widetilde{v} \in V_\alpha$, see [1] for details. Under Theorem 2.2, we see at once that $v \in V_\alpha$. □

**Theorem 2.4.** *For $\lambda(0) > \alpha$, the set of periodic points of* (1.1) *is dense in $V_\alpha$.*

*Proof.* Let $v$ be an arbitrary function belonging to $V_\alpha$. Define

$$w(x) = \mathrm{e}^{g(x)}\left(\mathrm{e}^{-g(x\mathrm{e}^{nt_0})}v(x\mathrm{e}^{nt_0}) - \sum_{k=n+1}^{\infty} m\mathrm{e}^{-g(x\mathrm{e}^{kt_0})}\right), \tag{2.6}$$

where $m = \mathrm{e}^{-g(\mathrm{e}^{-t_0})}v(\mathrm{e}^{-t_0}) - v(1)$ and $x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$. To show the correctness of this definition, it is sufficient to make the following observation

$$\mathrm{e}^{g(\mathrm{e}^{-(n+1)t_0})}\left(\mathrm{e}^{-g(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{nt_0})}v(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{nt_0}) - \sum_{k=n+1}^{\infty} m\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{kt_0}\right)}\right) =$$

$$= \mathrm{e}^{g(\mathrm{e}^{-(n+1)t_0})}\left(v(1) + m - \sum_{k=n+1}^{\infty} m\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{kt_0}\right)}\right) =$$

$$= \mathrm{e}^{g(\mathrm{e}^{-(n+1)t_0})}\left(\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{(n+1)t_0}\right)}v(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{(n+1)t_0}) - \right.$$

$$\left. - \sum_{k=n+2}^{\infty} m\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{kt_0}\right)}\right).$$

The function $w$ is continuous and vanishes at 0, which is a consequence of (2.6). Let $\epsilon > 0$. Since $v \in V_\alpha$ and $w \in V_\alpha$, there exists such $t_0$ that $H_{[0,\mathrm{e}^{-t_0}],\alpha}(v) < \frac{\epsilon}{4}$ and $H_{[0,\mathrm{e}^{-t_0}],\alpha}(w) < \frac{\epsilon}{4}$. From (1.8) we know that $w(x) = \frac{\widetilde{w}(x)}{\kappa(x)}$, where $\widetilde{w}$ is the periodic solution of (1.5). For $\lambda(0) > \alpha$ the set of periodic points of (1.5) is dense in $V_\alpha$ (see [1]), so $H_\alpha(v - \widetilde{w}) < \frac{\epsilon}{4}$ and $H_\alpha(w - \widetilde{w}) < \frac{\epsilon}{4}$.
Thus

$$\begin{aligned} H_\alpha(v - w) &\leq H_{[\mathrm{e}^{-t_0},1],\alpha}(v - w) + H_{[0,\mathrm{e}^{-t_0}],\alpha}(v - w) \leq \\ &\leq H_\alpha(v - \widetilde{w}) + H_\alpha(\widetilde{w} - w) + H_{[0,\mathrm{e}^{-t_0}],\alpha}(v) + H_{[0,\mathrm{e}^{-t_0}],\alpha}(w) < \\ &< \epsilon. \end{aligned}$$

This completes the proof. □

**Theorem 2.5.** *If $\lambda(0) \le \alpha$ and $v \in V_\alpha$, then*

$$\lim_{t\to\infty} H_\alpha(T_t v) = 0.$$

*Moreover, if $\lambda(0) < \alpha$, then the semigroup $(T_t)_{t\ge 0}$ is exponentially stable.*

*Proof.* Take any $v \in V_\alpha$. Using Theorem 2.2 and proceeding analogously as in its proof, we compute

$$\begin{aligned} H_\alpha(T_t v) &= \sup_{x,y\in[0,1],x\neq y} \frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha} = \sup_{x,y\in[0,1],x\neq y} \frac{\left|\frac{\widetilde{u}(t,x)}{\kappa(x)} - \frac{\widetilde{u}(t,y)}{\kappa(y)}\right|}{|x-y|^\alpha} \le \\ &\le \sup_{x\in[0,1]} \left|\frac{1}{\kappa(x)}\right| \cdot \sup_{x,y\in[0,1],x\neq y} \frac{|\widetilde{u}(t,x)-\widetilde{u}(t,y)|}{|x-y|^\alpha} + \\ &\quad + \sup_{y\in[0,1]} |\widetilde{u}(t,y)| \cdot \sup_{x,y\in[0,1],x\neq y} \frac{\left|\frac{1}{\kappa(x)} - \frac{1}{\kappa(y)}\right|}{|x-y|^\alpha} \le \\ &\le \mathrm{e}^C \cdot \left( H_\alpha(T_t\widetilde{v}) + \sup_{y\in[0,1]} |\widetilde{u}(t,y)| \cdot \left(\frac{C}{\alpha}\right)^\alpha \right). \end{aligned}$$

We know that $T_t\widetilde{v} \to 0$ in $V_\alpha$ for every $\widetilde{v} \in V_\alpha$. The claim $\lim_{t\to\infty} H_\alpha(T_t\widetilde{v}) = 0$ is based on the results of paper [1]. From the same source, we have derived the estimate $\|T_t\widetilde{v}\| \le \mathrm{e}^{(\gamma-\alpha)t}\|\widetilde{v}\|$.

$$|\widetilde{u}(t,y)| = |(T_t\widetilde{v})(y)| = |(T_t\widetilde{v})(y) - (T_t\widetilde{v})(0)| \le H_\alpha(T_t\widetilde{v})y^\alpha \le H_\alpha(T_t\widetilde{v}),$$

hence, since $\lambda(0) < \alpha$, there follows the exponential stability of the semigroup $(T_t)_{t\ge 0}$ with $D = \mathrm{e}^C\left(1 + \left(\frac{C}{\alpha}\right)^\alpha\right)$ and $\omega = \alpha - \lambda(0)$. □

## 3. CHAOS AND STABILITY OF THE SYSTEM $(T_t)_{t\ge 0}$ IN THE SPACE $L^p$

**Theorem 3.1.** *Assume that*

$$\exists C, q > 0 \quad \forall x \in [0,1] \quad |\lambda(0) - \lambda(x)| \le Cx^q. \tag{3.1}$$

*The function $u$ belongs to the space $L^p$ if and only if $\widetilde{u} \in L^p$.*

*Proof.* Using (3.1), substitution (1.8) discussed in Section 1 and assuming that $u \in L^p$, we obtain

$$\begin{aligned} \|\widetilde{u}(t,x)\|^p &= \int_0^1 |\kappa(x)u(t,x)|^p dx \le \int_0^1 \mathrm{e}^{p\int_0^x \frac{|\lambda(0)-\lambda(s)|}{s}ds} |u(t,x)|^p dx \le \\ &\le \int_0^1 \mathrm{e}^{\frac{Cp}{q}x^q} |u(t,x)|^p dx \le \mathrm{e}^{\frac{Cp}{q}} \int_0^1 |u(t,x)|^p dx = \\ &= \mathrm{e}^{\frac{Cp}{q}} \|u(t,x)\|^p < \infty \end{aligned}$$

In the same manner, we can establish the inverse implication. □

The above theorem will be significant in obtaining the next one. It enables the results of [1] to be used and generalized. From now on, we assume (3.1).

**Theorem 3.2.** *For* $\lambda(0) > -\frac{1}{p}$ *there exists a periodic solution of* (1.1).

*Proof.* For any $t_0$, define the following function $v$ :

$$v(x) = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{nt_0})}w(x\mathrm{e}^{nt_0}), \quad x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}], \tag{3.2}$$

$$v(0) = 0,$$

where $w$ is an arbitrary function from the space $L^p$. We have shown that such function $v$ is a periodic solution, so it is sufficient to prove that $v \in L^p$. The function $v$ is the solution of equation (1.1), so we can express it using the function $\widetilde{v}$, the solution of (1.5), $v(x) = \frac{\widetilde{v}(x)}{\kappa(x)}$. Brzeźniak and one of the authors [1] showed that $\widetilde{v}$ is a periodic solution and belongs to $L^p$ when $\gamma > -\frac{1}{p}$. Our assumption and Theorem 3.1 guarantee the same conclusion for the function $v$. □

**Theorem 3.3.** *If* $\lambda(0) > -\frac{1}{p}$, *then the set of periodic points is dense in* $L^p$.

*Proof.* Let $w \in L^p$ and $\epsilon > 0$. Fix $t_0$ such that

$$\left[\int_0^{\mathrm{e}^{-t_0}} |w(x)|^p dx\right]^{\frac{1}{p}} < \frac{\epsilon}{2}$$

and

$$\mathrm{e}^{\frac{C}{q}}\|\widetilde{v}\| < \frac{\epsilon}{2}, \quad C > 0$$

where $\widetilde{v}$ is a periodic solution of (1.5).
The function $v$ is defined by formula (3.2). The function $v$ belongs to the set of periodic points due to Theorem 3.2. It is sufficient to estimate $||v - w||$. Since $v(x) = w(x)$ for $x \in [\mathrm{e}^{-t_0}, 1]$, it is obvious that $||v - w|| = ||(v - w)1_{[0,\mathrm{e}^{-t_0}]}||$, where $1_{[0,\mathrm{e}^{-t_0}]}$ denotes the indicator of the set $[0, \mathrm{e}^{-t_0}]$. Applying the estimate from Theorem 3.1 and substitution (1.8), we can assert that

$$\|v - w\| \le \|v1_{[0,\mathrm{e}^{-t_0}]}\| + \|w1_{[0,\mathrm{e}^{-t_0}]}\| < \epsilon.$$

□

**Theorem 3.4.** *If* $\lambda(0) \le -\frac{1}{p}$, *and* $v \in L^p$ *then*

$$\lim_{t\to\infty} \|T_t v\| = 0.$$

*Moreover, for* $\lambda(0) < -\frac{1}{p}$ *the semigroup* $(T_t)_{t\ge 0}$ *is exponentially stable on* $L^p$.

*Proof.* Let $v \in L^p$ be an arbitrary function.

$$\|T_t v\|^p = \int_0^1 |u(t,x)|^p dx = \int_0^1 \left| \frac{\widetilde{u}(t,x)}{\kappa(x)} \right|^p dx =$$
$$= \int_0^1 \left| \frac{1}{\kappa(x)} (T_t \widetilde{v})(x) \right|^p dx \leq \mathrm{e}^{\frac{Cp}{q}} \|T_t \widetilde{v}\|^p,$$

where $C > 0$. We know that $\|T_t \widetilde{v}\| \to 0$, as $t \to \infty$ (see [1]), which proves the first part of the Theorem. From [1] we know that $\|T_t \widetilde{v}\|^p \leq \mathrm{e}^{(\gamma p+1)t} \|\widetilde{v}\|^p$. It gives the exponential stability of the semigroup $(T_t)_{t \geq 0}$ with $D = \mathrm{e}^{\frac{C}{q}}$ and $\omega = -\frac{1}{p}(\lambda(0)p + 1)$. □

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