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ON RECONSTRUCTION OF STRUCTURE
OF A LINEAR SYSTEM WITH TIME DELAY

Abstract. The problem of reconstruction of a structure of a linear system with delay is considered. A solution algorithm stable with respect to the informational noise and computational errors is specified. The algorithm is based on the method of auxiliary positionally controlled models.

1. INTRODUCTION

A linear system described by the following vector equation with time delay

$$\dot{x}(s) = Ax(s) + Bx(s - \tau), \quad s \in [0, T], \tag{1}$$
$$x(\nu) = x_0(\nu), \quad \nu \in [-\tau, 0],$$

is considered. Here $x \in R^q$ is the phase vector of the system, $\tau = \text{const} \in (0, +\infty)$ is a constant time delay, A and B are $(q \times q)$-dimensional matrices. An initial state $x_0(\nu)$, $\nu \in [-\tau, 0]$, is a continuous function $x_0(\nu) \in C(-\tau, 0; R^q)$. It is assumed that the system structure (i.e., matrices A and B) is unknown. It is only known that they belong to convex, bounded, and closed sets $\mathcal{F}_1 \subset R_M^{q \times q}$ and $\mathcal{F}_2 \subset R_M^{q \times q}$, respectively.

A priori information on system (1) consists of the fixed sets $\mathcal{F}_1$ and $\mathcal{F}_2$, initial state $x_0(s)$ and time delay τ. The goal of the work is to design an algorithm of reconstruction of unknown matrices A and B with some accuracy μ through (inaccurate) measurements of the phase trajectory of system (1). Note that the same trajectory may be generated by different pairs of matrices (A, B) from the set $\mathcal{F}_1 \times \mathcal{F}_2$. In this

case, we will as a rule reconstruct not "true" matrices A and B, but another pair (A_*, B_*), $A_* \in \mathcal{F}_1$, $B_* \in \mathcal{F}_2$, such that the solution of the equation

$$\dot{y}(s) = A_* y(s) + B_* y(s - \tau), \quad s \in [0, T],$$
$$y(\nu) = x_0(\nu), \quad \nu \in [-\tau, 0],$$

coincides with $x(s)$, $s \in [0, T]$. The rule of "choice" of this pair (A_*, B_*) will be specified below. We want to construct an algorithm for calculating matrices A_1 and B_1 with the properties:

$$\|A_* - A_1\|_{q \times q} \leq \mu, \quad \|B_* - B_1\|_{q \times q} \leq \mu.$$

Input data of the algorithm are results of (inaccurate) measurements of the phase state $x(s)$, $0 \leq s \leq T$ of the system at sufficiently frequent moments $\tau_i \in [0, T]$, $\tau_{i+1} = \tau_i + \gamma$, $\tau_0 = 0$, $\gamma = \text{const} > 0$. These results are vectors $\xi_i^h \in R^q$ satisfying the inequalities

$$\left\| \xi_i^h - x(\tau_i) \right\| \leq h.$$

Here the symbol $h \in (0, 1)$ denotes the value of the measurement error.

In [1], an algorithm for dynamical reconstruction of an n-dimensional function $u(\cdots)$ (control) was indicated for a system described by the ordinary differential equation

$$\dot{x}(t) = f(t, x(t)) + f_1(t, x(t)) u(t)$$

provided that a convex, bounded, and closed set $P \subset R^n$, containing $u(t)$, is known, i.e., a set P such that

$$u(s) \in P \quad \text{for a.a.} \quad s \in [0, T] \tag{2}$$

is known. The algorithm is based on the combination of methods of the theory of guaranteed control [2] and the method of smoothing functional (Tikhonov's method), well-known in the theory of ill-posed problems [3, 4]. Then, in the case when the set of admissible disturbances has the form

$$P(\cdot) = \{u(\cdot) \in L_2(0, T; R^n) \colon u(s) \in P \quad \text{for a. e.} \quad s \in [0, T]\}, \tag{3}$$

the problems of dynamical reconstruction of inputs were also studied for other classes of systems, in particular, for those described by:

(a) ordinary differential equations,

(b) equations with time delay,

(c) parabolic and hyperbolic equations,

as well as variational inequalities with distributed and boundary control (for more details, see surveys [5–7]).

The method studied in these papers can also be used in the case when unknown parameters are subject to reconstruction. In the present paper, we will design an algorithm for reconstruction of matrices A and B with use of the ideas, developed in [8–10].

2. AUXILIARY STATEMENT

We introduce a family of continuous linear operators $S(x_T(\cdot))$ depending on elements $x_T(\cdot) \in C(0, T; R^q)$ and acting from $R^{2q \times q}$ into $L_2(0, T; R^q)$. Namely, for every $u \in R^{2q \times q}$, we define

$$(S(x_T(\cdot)))(s)u = Z(x(s), x(s - \tau))u \quad \text{for a. a.} \quad s \in [0, T].$$

Here $Z(x(s), x(s - \tau))$ is a $(2q \times q) \times q$-matrix with the following structure:

$$Z(x(s), x(s - \tau)) =$$

$$= \overbrace{\begin{pmatrix} x'(s) & 0 & \dots & 0 & x'(s - \tau) & 0 & \dots & 0 \\ 0 & x'(s) & \dots & 0 & 0 & x'(s - \tau) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x'(s) & 0 & 0 & \dots & x'(s - \tau) \end{pmatrix}}^{2q \times q \text{ columns}} \left.\vphantom{\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}}\right\} q \text{ rows.}$$

Primes denote transposition (i. e., the symbol $x'(s)$ means the vector-row corresponding to the vector-column $x(s)$). The symbol $x_T(\cdot)$ is used to recall that the function is defined on the interval $[0, T]$.

We introduce the one-to-one mapping $Q\colon R_M^{q \times q} \times R_M^{q \times q} \to R^{2q \times q}$ which transforms every matrix

$$F = (A, B), \tag{4}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \dots & a_{qq} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \dots & b_{qq} \end{pmatrix}$$

into the vector-column

$$u_F = QF =$$
$$= (a_{11}, \dots, a_{1q}, a_{21}, \dots a_{2q}, \dots a_{q1}, \dots, a_{qq}, b_{11}, \dots, b_{1q}, b_{21}, \dots b_{2q}, \dots b_{q1}, \dots, b_{qq})'.$$

In this case, the mapping Q^{-1} transforms the vector

$$u = (a_{11}, \dots, a_{1q}, a_{21}, \dots a_{2q}, \dots a_{q1}, \dots, a_{qq},$$
$$b_{11}, \dots, b_{1q}, b_{21}, \dots b_{2q}, \dots b_{q1}, \dots, b_{qq})' \in R^{2q \times q}$$

into the pair of matrices $F = F_u$ of form (4).

Equation (1) may be written in the form

$$\dot{x}(s) = S(x_T(\cdot))(s)u_F, \quad s \in [0, T],$$

i. e.,

$$\dot{x}(s) = S(x_T(\cdot))(s)QF, \quad s \in [0, T],$$

which can be written as a functional equation in the space $L_2(0, T; R^q)$:

$$b(\cdot) = S_*(x_T(\cdot))(\cdot)u_F,$$

where

$$b(s) = x(s) - x_0(s) \quad \text{for a. e.} \quad s \in [0, T].$$

The family of continouos linear operators $S_*(x_T(\cdot)) : R^{2q \times q} \to L_2(0, T; R^q)$ is defined by the rule

$$S_*(x_T(\cdot))(s)w = \left(\int\limits_0^s S(x_T(\cdot))(g) \, dg \right) w, \quad s \in [0, T], \quad (w \in R^{2q \times q}).$$

Let $\mathcal{F}_* = \mathcal{F}_1 \times \mathcal{F}_2$,

$$U_1 = \{u \in Q\mathcal{F}_* : \ b(s) = S_*(x_T(\cdot))(s)u \quad \text{for a. a.} \quad s \in [0, T]\}.$$

It is easily seen that this set is convex, bounded, and closed. Therefore, the set

$$U^* = \arg\min\{\|u\| : \ u \in U_1\}$$

is a singleton: $U^* = \{u_0\}$.

Let

$$R_j = \sup\{\|F\|_{q \times q} : \ F \in \mathcal{F}_j\}, \quad j = 1, 2,$$
$$R_3 = \sup\{\|Q(A, B)\| : \ A \in \mathcal{F}_1, B \in \mathcal{F}_2\},$$
$$R = \max\{R_1 + R_2, R_3\},$$
$$\xi^h(s) = \xi_i^h, \quad s \in [\tau_i, \tau_{i+1}), \quad i \in [0 : n], \tag{5}$$
$$\xi^h(s) = x_0(\tau_i), \quad s \in [\tau_i, \tau_{i+1}), \quad i \in [-r_n : -1], \quad \tau_i = \frac{iT}{n}, \quad r_n = [\tau/n].$$

The symbol $[a]$ denotes the integer part of a. Then the following equality holds:

$$\big(S_*(\xi_T^h(\cdot))\big)(s) = \sum_{i=0}^{i(s)-1} \gamma Z\big(\xi_i^h, \xi_{i-r_n}^h\big) + (s - \tau_{i(s)})Z(\xi_{i(s)}^h, \xi_{i(s)-r_n}^h), \quad t \in [0, T].$$

Here

$$\gamma = \gamma_n = T/n, \quad i(s) = [sn/T], \quad \tau_{i(s)} = i(s)T/n.$$

We introduce the function

$$b_{h,n}(s) = \xi_i^h - \xi_0^h, \quad s \in [\tau_i, \tau_{i+1}).$$

Lemma 1. *The following inequality is valid:*

$$\|b(\cdot) - b_{h,n}(\cdot)\|_{L_2} \le d_1(h, 1/n) = \sqrt{T}(2h + RCT/n),$$

where

$$C_1 = (\|x_0\| + R \int_{-\tau}^{0} \|x_0(\nu)\| \, d\nu)(1 + RT \exp(RT)),$$

$$C = \max\{C_1, \max_{\nu \in [-\tau, 0]} \|x_0(\nu)\|\}.$$

Proof. The following estimate is true:

$$\|x(s)\| \le \|x_0\| + \int_{0}^{s} \|Ax(y) + Bx(y - \tau)\| \, dy, \quad s \in [0, T].$$

From the inequalities $\|A\|_{q \times q} \le R_1$, $\|B\|_{q \times q} \le R_2$, we get

$$\|x(s)\| \le \|x_0\| + R \int_{-\tau}^{0} \|x_0(\nu)\| \, d\nu + R \int_{0}^{s} \|x(y)\| \, dy, \quad s \in [0, T].$$

By virtue of Gronwall's lemma, we deduce that

$$\|x(s)\| \le C_1, \quad \|\dot{x}(s)\| \le RC, \quad s \in [0, T]. \tag{6}$$

Note that the following inequality is valid for $s \in [\tau_i, \tau_{i+1})$:

$$\|x(s) - \xi_i^h\| \le h + \int_{\tau_i}^{s} \|\dot{x}(\nu)\| \, d\nu \le h + RCT/n. \tag{7}$$

Therefore, by virtue of (6), (7), there is

$$\|b(\cdot) - b_{h,n}(\cdot)\|_{L_2}^2 = \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \|x(\nu) - x_0 - \xi_i^h + \xi_0^h\|^2 \, d\nu \le$$

$$\le \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} (2h + RCT/n)^2 \, d\nu \frac{n-1}{n} T(2h + RCT/n)^2 \le T(2h + RCT/n)^2.$$

The lemma is proved. $\qquad\square$

Lemma 2. *The following inequality is true:*

$$\left\| S_*(x_T(\cdot)) - S_*(\xi_T^h(\cdot)) \right\|_{\mathcal{L}_2} \le d_2(h, 1/n) = 2T\sqrt{\frac{qT}{3}}(h + RCT/n).$$

Here the symbol $\xi_T^h(\cdot)$ means a function $\xi^h(t)$, $t \in [0, T]$, defined by (5).

Proof. From (7) there follows:

$$\|\{(S(x_T(\cdot)) - S(\xi_T^h(\cdot)))\}(s)u_F\| \leq \sqrt{2}q(h + RCT/n)\|u_F\|, \quad s \in [0, T].$$

Therefore,

$$\|S_*(x_T(\cdot)) - S_*(\xi_T^h(\cdot))\|_{\mathcal{L}_2} =$$

$$= \left(\sup_{\|u_F\| \leq 1} \int_0^T \left\| \int_0^t (S(x_T(\cdot)) - S(\xi_T^h(\cdot)))(s)\, ds u_F \right\|^2 dt \right)^{1/2} \leq$$

$$\leq \left(\int_0^T \left(\int_0^t \sqrt{2}q(h + RCT/n)\, ds \right)^2 dt \right)^{1/2} \leq \sqrt{2}q(h + RCT/n) \left(\int_0^T t^2\, dt \right)^{1/2} \leq$$

$$\leq 2T\sqrt{\frac{qT}{3}}(h + RCT/n).$$

The lemma is proved. $\square$

3. SOLUTION ALGORITHM

For solving the problem, let us take a pair of matrices of the form $Q^{-1}u_0$ as matrices A_* and B_* subject to reconstruction. Namely, we will calculate

$$(A_*, B_*) = Q^{-1}u_0.$$

Introduce a dynamical control system

$$\dot{z}(t) = v(t), \quad z(0) = 0 \tag{8}$$

on an "artificial" time interval $R_+ = [0, +\infty)$. A system state $z(t)$, $t \in R_+$, and control $v(t)$ are elements of the Euclidean vector space $R^{2q \times q}$. Our goal is to construct a control function $v(\cdot)$ such that for the corresponding trajectory $z(\cdot)$ of system (8), relation $z(t)/t$ is "close" to u_0, provided t is large enough. Controls $v(t)$ in system (8) will be formed by the feedback control rule. This rule is identified with the function

$$U \colon R_+ \times R^{2q \times q} \to Q\mathcal{F}_*.$$

For every $\delta > 0$, let us define a δ-trajectory $z_\delta(\cdot)$ generated by the rule $U(t, z)$:

$$z_\delta(0) = 0, \quad z_\delta(t) = z_\delta(t_j) + v_j^\delta(t - t_j),$$
$$t \in [t_j, t_{j+1}], \quad t_j = j\delta, \quad v_j^\delta \in U(t_j, z_\delta(t_j)).$$

Introduce the functional

$$\Lambda_\alpha\big(t \mid z_\delta(t)\big) = \|S_*(x_T(\cdot))(\cdot)z_\delta(t) - tb(\cdot)\|_{L_2}^2 + \alpha \int\limits_0^t \|\dot{z}_\delta(\nu)\|^2 \, d\nu - \alpha t J^0, \quad (9)$$

where α is an auxiliary parameter, which (along with $\int\limits_0^t \|\dot{z}_\delta(\nu)\|^2 \, d\nu$) plays the role of the "smoothing functional" [3, 4],

$$J^0 = \|u_0\|^2 \,,$$

well-known in the theory of ill-posed problems. Hereinafter, the symbol $\|\cdot\|_{L_2}$ denotes the norm in the space $L_2(0, T; R^q)$; the symbol $\|\cdot\|_{\mathcal{L}_2}$ denotes the norm in the space of bounded linear operators acting from $R^{2q \times q}$ to $L_2(0, T; R^q)$; the symbol $\langle \cdot, \cdot \rangle_{L_2}$ denotes the inner product in the space $L_2(0, T; R^q)$. The functional Λ_α is an analog of Lyapunov's functional. We will indicate a rule of choice of control by the feedback principle $U(t, z)$ such that the following inequalities hold:

$$\Lambda_\alpha(t|z_\delta(t)) \leq \Lambda_\alpha(t_j|z_\delta(t_j)) + c_1(t - t_j)\left\{(t - t_j) + t_j\left(h + \frac{1}{n}\right)\right\}, \quad (10)$$
$$t \in [t_j, t_{j+1}).$$

Here c_1 is a constant which may be written explicitly.

Let the rule of choice of control $U(t, z)$ has the form

$$U(t, z) = U_\alpha(t, z) =$$
$$= \arg\min\big\{2\left\langle S_*(\xi_T^h(\cdot))(\cdot)z - tb_{h,n}(\cdot), S_*(\xi_T(\cdot))(\cdot)u\right\rangle_{L_2} + \alpha \|u\|^2 : u \in Q\mathcal{F}_*\big\}. \quad (11)$$

Theorem 1. *The rule of feedback control $U(t, z)$ (11) guarantees that inequality (10) holds.*

Proof. For the initial time moment there is

$$\Lambda_\alpha(0 \mid z_\delta(0)) = 0. \quad (12)$$

Thus, inequality (10) is true for $t = 0$. Suppose that this inequality is true for all $t \in [0, t_j]$. Take an arbitrary number $s \in [t_j, t_{j+1}]$ and prove (10) for $t = s$. It is not difficult to see that

$$\|z_\delta(t)\| \leq tR, \quad t \geq 0.$$

Using lemmas 1, 2, we derive

$$\|s_j(x, z_\delta) - s^j(\xi, z_\delta)\|_{L_2} \leq \|S_*(x_T(\cdot))(\cdot) - S_*(\xi_T^h(\cdot))\|_{\mathcal{L}_2}\|z_\delta(t_j)\| + t_j\|b(\cdot) - b_{h,n}(\cdot)\|_{L_2} \leq$$
$$\leq t_j(d_1(h, 1/n) + Rd_2(h, 1/n)). \quad (13)$$

Here

$$s_j(x, z_\delta) = S_*(x_T(\cdot))(\cdot)z_\delta(t_j) - t_j b(\cdot) \in L_2(0, T; R^q),$$
$$s^j(\xi, z_\delta) = S_*(\xi_T^h(\cdot))(\cdot)z_\delta(t_j) - t_j b_{h,n}(\cdot).$$

Taking into account the definition of functional Λ_α (see (9)), we obtain

$$\Lambda_\alpha(s|z_\delta(s)) = \Lambda_\alpha(t_j|z_\delta(t_j)) + \mu_j + \nu_j + \alpha(\|v_j^\delta\|^2 - J^0)(s - t_j), \tag{14}$$

where

$$\mu_j = 2(s - t_j)\left\langle s_j(x, z_\delta), S_*(x_T(\cdot))(\cdot)v_j^\delta - b(\cdot)\right\rangle_{L_2},$$
$$\nu_j = \|S_*(x_T(\cdot))(\cdot)v_j^\delta - b(\cdot)\|_{L_2}^2 (s - t_j)^2.$$

Note

$$S_*(x_T(\cdot))(\cdot)u_0 - b(\cdot) = 0.$$

We conclude from (14) that

$$\Lambda_\alpha(s|z_\delta(s)) =$$
$$= \Lambda_\alpha(t_j|z_\delta(t_j)) + \nu_j + 2(s - t_j)\left\{\left[\left\langle s_j(x, z_\delta), S_*(x_T(\cdot))(\cdot)v_j^\delta - b(\cdot)\right\rangle_{L_2} + \alpha\|v_j^\delta\|^2\right] - \right.$$
$$\left. - \left[\left\langle s_j(x, z_\delta), S_*(x_T(\cdot))(\cdot)u_0 - b(\cdot)\right\rangle_{L_2} + \alpha\|u_0\|^2\right]\right\}.$$

Further we get the following estimates

$$\|S_*(x_T(\cdot))(\cdot)v_j^\delta - b(\cdot)\|_{L_2} \leq d_0 R + b_0,$$
$$\|s_j(x, z_\delta)\|_{L_2} \leq (d_0 R + b_0)t_j, \tag{15}$$
$$d_0 = \|S_*(x_T(\cdot))(\cdot)\|_{\mathcal{L}_2}, \quad b_0 = \|b(\cdot)\|_{L_2}.$$

Consequently, by virtue of (13), (15), the following inequality is valid:

$$\Lambda_\alpha(s|z_\delta(s)) \leq \Lambda_\alpha(t_j|z_\delta(t_j)) + \nu_j + 2(s - t_j) \times$$
$$\times \left\{\left[\langle s^j(\xi, z_\delta), S_*(x_T(\cdot))(\cdot)v_j^\delta - b(\cdot)\rangle_{L_2} + \alpha\|v_j^\delta\|^2\right] - \right.$$
$$\left. - \left[\langle s^j(\xi, z_\delta), S_*(x_T(\cdot))(\cdot)u_0 - b(\cdot)\rangle_{L_2} + \alpha\|u_0\|^2\right]\right\} + 4(s - t_j)t_j d_3(h, 1/n),$$
$$d_3(h, 1/n) = \left(d_1(h, 1/n) + R d_2(h, 1/n)\right)(d_0 R + b_0). \tag{16}$$

Besides, from (15) (using again (13)) it follows that

$$\|s^j(\xi, z_\delta)\|_{L_2} \leq t_j d_4(h, 1/n),$$
$$d_4(h, 1/n) = b_0 + d_0 R + d_1(h, 1/n) + R d_2(h, 1/n). \tag{17}$$

It is easily seen that

$$\nu_j \le (s - t_j)^2 (b_0 + d_0 R)^2.$$

We again use lemmas 1, 2. From (16), (17) we obtain

$$\Lambda_\alpha(s|z_\delta(s)) \le \Lambda_\alpha(t_j|z_\delta(t_j)) + 2(s - t_j)\Big\{\Big[\langle s^j(\xi, z_\delta), S_*(\xi_T^h(\cdot))(\cdot)v_j^\delta\rangle_{L_2} + \alpha\|v_j^\delta\|^2\Big] -$$
$$- \Big[\langle s^j(\xi, z_\delta), S_*(\xi_T(\cdot))(\cdot)u_0\rangle_{L_2} + \alpha\|u_0\|^2\Big]\Big\} +$$
$$+ 4(s - t_j)t_j d_5(h, 1/n) + (s - t_j)^2(b_0 + d_0 R)^2, \quad (18)$$

where

$$d_5(h, 1/n) = d_3(h, 1/n) + d_4(h, 1/n)d_2(h, 1/n)R \le c_0\left(h + \frac{1}{n}\right),$$

c_0 is a constant which may be written explicitly. By definition of mapping $U(t, z)$ (see (11)), the following inequality holds:

$$\Big[\langle s^j(\xi, z_\delta), S_*(\xi_T(\cdot))(\cdot)v_j^\delta\rangle_{L_2} + \alpha\|v_j^\delta\|^2\Big] -$$
$$- \Big[\langle s^j(\xi, z_\delta), S_*(\xi_T(\cdot))(\cdot)u_0\rangle_{L_2} + \alpha\|u_0\|^2\Big] \le 0. \quad (19)$$

In this case, from (18), (19), we obtain

$$\Lambda_\alpha(s|z_\delta(s)) \le \Lambda_\alpha(t_j|z_\delta(t_j)) + 4(s - t_j)t_j d_5(h, 1/n) + (s - t_j)^2(b_0 + d_0 R)^2$$

for $s \in [t_j, t_{j+1})$.

The theorem is proved. $\qquad\square$

Let us choose sequences of positive numbers $\{\alpha_n\}$, $\{h_n\}$ and $\{t_n\}$ such that

$$\alpha_n \to 0, \quad h_n \to 0, \quad t_n \to +\infty,$$
$$(\alpha_n + \delta_n)/t_n \to 0, \quad \delta_n/\alpha_n \to 0, \quad t_n(h_n + 1/n)/\alpha_n \to 0 \quad \text{as} \quad n \to \infty. \quad (20)$$

Then the following theorem holds.

Theorem 2. *If conditions* (20) *are fulfilled, then*

$$z_{\delta_n}(t_n)/t_n \to u_0, \quad as \quad n \to \infty. \quad (21)$$

Proof. From (10) and (12), we derive

$$\Lambda_{\alpha_n}(t_n|z_{\delta_n}(t_n)) \le c_1\left(\delta_n t_n + t_n^2(h_n + 1/n)\right).$$

Thus, the following inequalities are valid:

$$\|S_*(x_T(\cdot))(\cdot)z_{\delta_n}(t_n) - t_n b(\cdot)\|_{L_2}^2 \le c_1\left(\delta_n t_n + t_n^2(h_n + 1/n)\right) + 2\alpha_n R^2 t_n, \quad (22)$$

$$\int_0^{t_n} \|\dot{z}_{\delta_n}(\nu)\|^2 \, d\nu \le c_1 \left(\frac{\delta_n t_n}{\alpha_n} + t_n^2 \frac{h_n + 1/n}{\alpha_n} \right) + t_n J^0. \qquad (23)$$

Dividing both sides of inequality (22) by t_n^2, we obtain

$$\|S_*(x_T(\cdot))(z_{\delta_n}(t_n)/t_n) - b(\cdot)\|_{L_2}^2 \le c_2(\delta_n/t_n + h_n + 1/n) + 2\alpha_n R^2/t_n. \qquad (24)$$

By the convexity of the norm, using Jensen's inequality we get

$$\frac{1}{t} \int_0^t \|\dot{z}_{\delta_n}(\nu)\|^2 \, d\nu \ge \left\| \frac{1}{t} \int_0^t \dot{z}_{\delta_n}(\nu) \, d\nu \right\|^2 = \|z_{\delta_n}(t)/t\|^2 \quad \forall t > 0.$$

Hence, from (23) we deduce that

$$\|z_{\delta_n}(t_n)/t_n\|^2 \le c_1(\delta_n/\alpha_n + t_n(h_n + 1/n)/\alpha_n) + J^0. \qquad (25)$$

Convergence (21) follows from (24), (25). Note that the set $Q\mathcal{F}_*$ is convex, bounded, and closed. In this case the following inclusion is true:

$$z_{\delta_n}(t_n)/t_n = \sum_{j=0}^{n-1} v_j^{\delta_n} \delta_n/t_n \in Q\mathcal{F}_*.$$

The sequence $\{z_{\delta_n}(t_n)/t_n\}_{n=1}^{\infty}$ is bounded, so, without loss of generality, we may set

$$z_{\delta_n}(t_n)/t_n \to u^* \in Q\mathcal{F}_* \quad \text{as} \quad n \to \infty.$$

Taking into account (24), we conclude that

$$u^* \in U_1.$$

Besides, by (25) there is

$$\|u^*\| \le \|u_0\|.$$

However, the set U_1 contains a single element of minimal norm. Thus, $u^* = u_0$. The theorem is proved. $\qquad \square$

Theorem 3. *If conditions* (20) *are fulfilled, then*

$$Q^{-1} z_{\delta_n}(t_n)/t_n \to (A_*, B_*), \quad as \quad n \to \infty.$$

Acknowledgments

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