A Coordinate Transformation for Unsteady Boundary Layer Equations

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DOI: 10.13111/2066-8201.2011.3.4.4

Abstract: *This paper presents a new coordinate transformation for unsteady, incompressible boundary layer equations that applies to both laminar and turbulent flows. A generalization of this coordinate transformation is also proposed. The unsteady boundary layer equations are subsequently derived. In addition, the boundary layer equations are derived using a time linearization approach and assuming harmonically varying small disturbances.*

Key Words: *Boundary-layer theory, Coordinate transformation*

1. INTRODUCTION

Although the boundary layer equations have a simplified form compared to the Navier-Stokes equations, they are still a difficult mathematical problem. To simplify the boundary layer equations, several coordinate transformation have been proposed in the past. Many of these coordinate transformations were developed in the precomputer era when a significant analytical effort was put into finding simpler forms for the boundary layer equations.

Blasius [2] was the first to propose a coordinate transformation to reduce the partial differential equations that describe the incompressible boundary layer over a flat plate to an ordinary differential equation. Goldstein [6] proposed a coordinate transformation for the boundary layer equations of steady, two-dimensional flows, assuming the velocity outside of the boundary layer is $U = ce^{\alpha x}$ and $\alpha > 0$. He also showed that no similar solutions existed if α is negative.

Howarth [8] proposed a coordinate transformation of the compressible flow in a laminar boundary layer, assuming Prandtl number is unity and viscosity is proportional to the absolute temperature. This coordinate transformation, also known as Howarth-Dorodnitsyn, leads to a simplified form of the boundary layer equations that is very similar to the incompressible equation. A related coordinate transformation was proposed by Stewartson [12]. Stewartson used the same assumptions as Howarth and introduced a coordinate transformation that transformed the boundary layer equations for a compressible fluid into those for an incompressible fluid.

Illingworth [9] proposed a coordinate transformation for the flow over a porous plate with uniform suction, which reduced the governing equations to a set of ordinary differential equations. Mangler [13, p. 296] introduced a coordinate transformation that converts the axisymmetric boundary layer equations to the plane boundary layer equations. Görtler [7] proposed a coordinate transformation (ξ, η) for plane and steady laminar boundary layers in incompressible fluids with arbitrary outer pressure distribution. The solution of the boundary layer problem is given as a power series in ξ with the coefficients functions depending on η. This series is a formally exact solution of the boundary layer problem.

Fewer coordinate transformations have been proposed in the last forty years. One of the most recent of them was developed by Carter et al. [3], who introduced a composite transformation for laminar and turbulent boundary layers. This coordinate transformation was conceived to include the two transverse lengths scales of the turbulent boundary layer: the boundary-layer thickness and the wall-layer thickness.

The boundary-layer thickness is captured by using a turbulent generalization of the Mangler-Levy-Lees variables. The wall-layer thickness is captured by a coordinate transformation based on the appropriate analytical velocity profile expression proposed by Whitfield [14].

This paper presents a new coordinate transformation for unsteady, incompressible boundary layer equations that applies to both laminar and turbulent flows. Section 2 briefly presents the governing equations of the unsteady boundary layer for an incompressible fluid. The new coordinate transformation is described in Section 3. A generalization of this coordinate transformation is also proposed. The unsteady boundary layer equations written using the new coordinate transformation are subsequently derived. In addition, the boundary layer equations are derived using a time linearization approach and assuming harmonically varying small disturbances.

2. UNSTEADY BOUNDARY LAYER EQUATIONS

The boundary layer equations are obtained from the mass and momentum conservation equations by using a scale analysis [10]. Assuming that the flow is incompressible, that the viscosity does not vary with temperature and that very sudden accelerations are excluded, the Prandtl's boundary layer equations are [11, p. 130]

$$\frac{\partial \hat{u}}{\partial s} + \frac{\partial \hat{v}}{\partial n} = 0 \tag{1}$$

$$\hat{\rho}\left(\frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial s} + \hat{v}\frac{\partial \hat{u}}{\partial n}\right) = -\frac{\partial \hat{p}}{\partial s} + \mu\left(\frac{\partial^2 \hat{u}}{\partial n^2}\right) \tag{2}$$

$$-\frac{\partial \hat{p}}{\partial n} = 0 \tag{3}$$

where s and n are coordinates parallel and normal to the boundary, the symbol "^" denotes an unsteady, possible nonlinear flow quantity, $\hat{u}$ and $\hat{v}$ are the velocity components in the parallel and normal directions, $\hat{\rho}$ is the density, $\hat{p}$ is the pressure and μ is the dynamic viscosity. Note that $s = 0$ corresponds to the start of the boundary layer and $n = 0$ corresponds to the surface of the wall.

Equations (1)-(3) with the three unknowns $\hat{u}$, $\hat{v}$ and $\hat{p}$ can be reduced to a system of two equations with two unknowns $\hat{u}$ and $\hat{v}$ by eliminating the pressure $\hat{p}$. To eliminate the pressure, the s-momentum equation (2) is written at the edge of the boundary layer where the viscous term $\mu\frac{\partial^2 \hat{u}}{\partial n^2}$ and the convection term $\rho\hat{v}\frac{\partial \hat{u}}{\partial n}$ can be neglected. One obtains:

$$\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e \frac{\partial \hat{u}_e}{\partial s} = -\frac{1}{\rho} \frac{d\,\hat{p}}{ds} \tag{4}$$

where the subscript "*e*" denotes values at the edge of the boundary layer.

Note that it is not necessary to use the index *e* for the density $\hat{\rho}$ and pressure $\hat{p}$ because the density is constant and the pressure does not vary with the *n* coordinate. Substituting equation (4) into the *s*-momentum equation (2) yields:

$$\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial s} + \hat{v} \frac{\partial \hat{u}}{\partial n} = \frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e \frac{\partial \hat{u}_e}{\partial s} + \frac{\mu}{\hat{\rho}} \frac{\partial^2 \hat{u}}{\partial n^2} \tag{5}$$

Equations (1) and (5) represent the two partial differential equations which must be solved to obtain the two unknowns $\hat{u}$ and $\hat{v}$.

Due to the fact that the continuity equation is a first-order partial differential equation and the *s*-momentum equation is a second-order partial differential equation, it is necessary to impose three boundary conditions. The boundary conditions vary depending on whether the boundary layer position is along a wall or in a wake. If the boundary layer develops on a wall, the no-slip condition is:

$$\hat{u}(s, n = 0) = \hat{v}(s, n = 0) = 0,$$

At the edge of the boundary layer, the velocity of the boundary layer must match the velocity of the inviscid flow field, so that:

$$\hat{u}(s, n = \delta) = \hat{u}_e(s).$$

where δ is the thickness of the boundary layer.

Additional details about the boundary conditions are not given here because, as will be presented in the next section, the boundary layer equations will not be solved in the "physical coordinates."

A coordinate transformation or "stretching" of the governing equations will be applied prior to formulating the difference equations.

3. COORDINATE TRANSFORMATION

The main goals of the coordinate transformation are to remove the singularity in the equations at the leading edge or stagnation point and to generate a coordinate frame for computation in which the boundary layer thickness remains as constant as possible [1, p. 355]. Three coordinate transformations will be presented in the following sections. The first coordinate transformation is similar to the transformation used in the Blasius similarity solution. The second coordinate transformation generates a more compact grid than the modified Blasius transformation.

The third coordinate transformation represents a generalization of the previous two coordinate transformations.

3.1. Modified Blasius Coordinate Transformation

In the Blasius coordinate transformation the rectangular Cartesian physical coordinates (*s*, *n*) are transformed to (s, η) coordinates. The crucial element of the transformation is the definition of the η variable:

$$\eta = \frac{n}{s}\sqrt{\frac{\hat{u}_e s}{\nu}},$$

where ν is the kinematic viscosity.

The modified version of the Blasius coordinate transformation proposed herein replaces $\hat{u}_e$ by u_0, which is considered constant.

Instead of solving the boundary layer equations written with the $\hat{u}$, $\hat{v}$ variables, one follows Blasius [11, p. 136] and introduces a stream function $\hat{\psi}$. The difference from Blasius' approach is that in the present analysis, the stream function $\hat{\psi}$ varies not only in the *n*-direction but also in time and in the *s*-direction. The stream function $\hat{\psi}$ is defined by using a dimensionless stream function $\hat{f}(s,\eta,t)$:

$$\hat{\psi}(s,\eta,t) = \sqrt{\nu s u_0}\ \hat{f}(s,\eta,t)$$

By introducing the stream function $\hat{\psi}$, the continuity equation is identically satisfied. If one writes the velocities $\hat{u}$ and $\hat{v}$ using the stream function

$$\hat{u} = \frac{\partial\hat{\psi}}{\partial n} \tag{6}$$

$$\hat{v} = -\frac{\partial\hat{\psi}}{\partial s} \tag{7}$$

and one takes into account the expressions of the derivatives:

$$\begin{aligned} \left.\frac{\partial}{\partial s}\right|_n &= \left.\frac{\partial}{\partial s}\right|_\eta - \frac{\eta}{2s}\cdot\left.\frac{\partial}{\partial \eta}\right|_s \\ \left.\frac{\partial}{\partial n}\right|_s &= \sqrt{\frac{u_0}{\nu s}}\left.\frac{\partial}{\partial \eta}\right|_s \end{aligned} \tag{8}$$

one obtains for $u_0 = 1$:

$$\frac{\partial\hat{u}}{\partial t} = \frac{\partial\hat{f}'}{\partial t} \tag{9}$$

$$\hat{u}\frac{\partial\hat{u}}{\partial s} = \hat{f}'\left(\frac{\partial\hat{f}'}{\partial s} - \frac{\eta}{2s}\hat{f}''\right) \tag{10}$$

$$\hat{v}\frac{\partial\hat{u}}{\partial n} = \left(-\frac{\sqrt{\nu}}{2\sqrt{s}}\hat{f} - \sqrt{\nu s}\frac{\partial\hat{f}}{\partial s} + \frac{\eta}{2s}\sqrt{\nu s}\ \hat{f}'\right)\frac{1}{\sqrt{\nu s}}\hat{f}'' \tag{11}$$

$$\nu\frac{\partial^2\hat{u}}{\partial n^2} = \frac{1}{s}\hat{f}''' \tag{12}$$

The primes denote differentiation with respect to η. After the substitution of (9)-(12) into (5), one obtains:

$$\hat{f}''' + \frac{1}{2}\hat{f}\,\hat{f}'' + s\left(\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\frac{\partial \hat{u}_e}{\partial s}\right) = s\left(\frac{\partial \hat{f}'}{\partial t} + \hat{f}'\frac{\partial \hat{f}'}{\partial s} - \hat{f}''\frac{\partial \hat{f}}{\partial s}\right) \tag{13}$$

Note that the momentum equation (13) was derived using the assumption that the viscosity μ is constant. This hypothesis is true for laminar flows without temperature variation. For turbulent flow (with or without temperature variation) the term $\frac{\mu}{\rho}\frac{\partial^2 \hat{u}}{\partial n^2}$ must be replaced by $\frac{1}{\hat{\rho}}\frac{\partial}{\partial n}\left(\mu\frac{\partial \hat{u}}{\partial n} - \hat{\rho}\,\overline{u'v'}\right)$ where $\hat{\rho}\,\overline{u'v'}$ represents the Reynolds stresses. Assuming that the turbulence is modeled using Bousinesq's eddy viscosity, the term $\frac{1}{\hat{\rho}}\frac{\partial}{\partial n}\left(\mu\frac{\partial \hat{u}}{\partial n} - \hat{\rho}\,\overline{u'v'}\right)$ can be replaced by $\nu\frac{\partial}{\partial n}\left(\hat{b}\frac{\partial \hat{u}}{\partial n}\right)$ where

$$\hat{b} = 1 - \frac{\overline{u'v'}}{\nu}\cdot\frac{1}{\frac{\partial \hat{u}}{\partial n}}$$

For turbulent flows, equation (5) can be written as:

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial s} + \hat{v}\frac{\partial \hat{u}}{\partial n} = \frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\frac{\partial \hat{u}_e}{\partial s} + \nu\frac{\partial}{\partial n}\left(\hat{b}\frac{\partial \hat{u}}{\partial n}\right) \tag{14}$$

Using the nondimensional stream function $\hat{f}$, equation (14) becomes:

$$\left(\hat{b}\,\hat{f}''\right)' + \frac{1}{2}\hat{f}\,\hat{f}'' + s\left(\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\frac{\partial \hat{u}_e}{\partial s}\right) = s\left(\frac{\partial \hat{f}'}{\partial t} + \hat{f}'\frac{\partial \hat{f}'}{\partial s} - \hat{f}''\frac{\partial \hat{f}}{\partial s}\right) \tag{15}$$

As claimed at the beginning of this section, by considering u_0 to be constant, the transformed momentum equation (15) is simpler than for the case $u_0 = \hat{u}_e$, for which the momentum equation is [4]:

$$\left(\hat{b}\,\hat{f}''\right)' + \frac{P+1}{2}\hat{f}\,\hat{f}'' - P\left(\hat{f}'\right)^2 + s\left(\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\frac{\partial \hat{u}_e}{\partial s}\right) = s\left(\frac{\partial \hat{f}'}{\partial t} + \hat{f}'\frac{\partial \hat{f}'}{\partial s} - \hat{f}''\frac{\partial \hat{f}}{\partial s}\right)$$

where

$$P = \frac{s}{\hat{u}_e}\frac{\partial \hat{u}_e}{\partial s}$$

3.2. Novel Coordinate Transformation

The basic idea of the Blasius coordinate transformation is to define the dimensionless coordinate $\eta \sim n/\delta$, where the boundary layer thickness δ is of the order of $\sqrt{\nu\, s/\hat{u}_e}$. The variation of the boundary layer thickness as the square root of the length *s* is true for laminar flows but not for turbulent flows where boundary layer thickness varies more rapidly. For example, in the case of the turbulent flat plate flow, the boundary layer thickness δ varies as $s^{\frac{4}{5}}$ (for the $\frac{1}{7}$ th power velocity distribution law, [11, p. 638]), while for the laminar flow on the flat plate δ varies as $s^{\frac{1}{2}}$.

Most often, the flow on the airfoils has both laminar and turbulent regions. For this reason, it would be ideal if one could obtain a coordinate transformation which would accommodate both laminar and turbulent flows. The most natural way would be to define the dimensionless coordinate is $\eta(s,n) = n/\delta(s)$. The boundary layer thickness δ, however, is not easily available from the boundary layer codes. A related variable, the mean value of the displacement thickness, could be used instead, especially when the displacement thickness is a variable used in the coupling of viscous and inviscid regions.

In the proposed coordinate transformation the dimensionless coordinate is defined as $\eta(s,n) = n/\Delta^*(s)$ where Δ^* is the mean value of the displacement thickness [5, p. 48]. A nondimensional stream function $\hat{f}(s,\eta,t)$ is defined as $\hat{f}(s,\eta,t) = \hat{\psi}(s,\eta,t)/\Delta^*(s)\,u_0$. For simplicity one considers $u_0 = 1$. By using the stream function $\hat{\psi}$, the continuity equation (1) is identically satisfied. Taking into account equations (6), (7) and the expressions of the derivatives:

$$\left.\frac{\partial}{\partial s}\right|_n = \left.\frac{\partial}{\partial s}\right|_\eta - \left.\frac{\Delta^*_s}{\Delta^*}\eta\frac{\partial}{\partial \eta}\right|_s$$

$$\left.\frac{\partial}{\partial n}\right|_s = \left.\frac{1}{\Delta^*}\frac{\partial}{\partial \eta}\right|_s$$

one obtains:

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial \hat{f}'}{\partial t}$$

$$\hat{u}\,\frac{\partial \hat{u}}{\partial s} = \hat{f}'\left(\hat{f}'_s - \eta\,\frac{\Delta^*_s}{\Delta^*}\,\hat{f}''\right)$$

$$\hat{v}\,\frac{\partial \hat{u}}{\partial n} = -\left(\Delta^*_s\;\hat{f} + \Delta^*\;f_s - \Delta^*_s\,\eta\;\hat{f}'\right)\frac{\hat{f}''}{\Delta^*_s}$$

$$\frac{\partial}{\partial n}\left(\hat{b}\,\frac{\partial \hat{u}}{\partial n}\right) = \frac{1}{\Delta^{*2}}\left(\hat{b}\;\hat{f}''\right)'$$

As before, the index s denotes differentiation with respect to the *s* variable and primes denote differentiation with respect to the η variable. After substituting these terms in the momentum equation (14), one obtains:

$$\left(\hat{b}\;\hat{f}''\right)' + \Delta^*\,\Delta^*_s\;\hat{f}\;\hat{f}'' + \Delta^{*2}\left(\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\,\frac{\partial \hat{u}_e}{\partial s}\right) = \Delta^{*2}\left(\frac{\partial \hat{f}'}{\partial t} + \hat{f}'\,\frac{\partial \hat{f}'}{\partial s} - \hat{f}''\,\frac{\partial \hat{f}}{\partial s}\right) \tag{16}$$

3.3. A Generalization of the Coordinate Transformation

Comparing equations (15) and (16) one observes that both can be written in a more general form:

$$\left(\hat{b}\,\hat{f}''\right)' + \frac{1}{2}A_{1s}\,\hat{f}\,\hat{f}'' + A_1\left(\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\frac{\partial \hat{u}_e}{\partial s}\right) = A_1\left(\frac{\partial \hat{f}'}{\partial t} + \hat{f}'\frac{\partial \hat{f}'}{\partial s} - \hat{f}''\frac{\partial \hat{f}}{\partial s}\right) \tag{17}$$

For the Blasius modified coordinate transformation, $A_1 = s$, while for the novel coordinate transformation, $A_1 = \Delta^{*2}$. The similarity of the results is remarkable and suggests generalization. Note that in the Blasius modified coordinate transformation $\eta \sim 1/\sqrt{s}$ and $\hat{\psi} \sim \sqrt{s}$, while for the novel coordinate transformation $\eta \sim 1/\Delta^*$ and $\hat{\psi} \sim \Delta^*$. It results that in both cases $\eta \sim 1/\sqrt{A_1}$ and $\hat{\psi} \sim \sqrt{A_1}$. This result can be generalized as follows: for a coordinate transformation $\eta(s,n) = n/[\alpha\,\beta(s)]$ and a stream function $\hat{\psi}(s,n) = \alpha\,\beta(s)\,f(s,\eta)$ the parameter A_1 in the momentum equation (17) has the value $A_1 = [\alpha\,\beta(s)]^2$.

An immediate application of the generalized coordinate transformation is the definition of the nondimensional coordinate η using the time-average value of the boundary layer thickness $\bar{\delta}$.

By defining $\eta = n/\bar{\delta}(s)$ and $\hat{\psi}(s,n) = f(s,n)\,\bar{\delta}$ one obtains the momentum equation:

$$\left(\hat{b}\,\hat{f}''\right)' + \bar{\delta}\,\bar{\delta}_s\,\hat{f}\,\hat{f}'' + \bar{\delta}^2\left(\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e\frac{\partial \hat{u}_e}{\partial s}\right) = \bar{\delta}^2\left(\frac{\partial \hat{f}'}{\partial t} + \hat{f}'\frac{\partial \hat{f}'}{\partial s} - \hat{f}''\frac{\partial \hat{f}}{\partial s}\right)$$

4. TIME-LINEARIZED UNSTEADY BOUNDARY LAYER EQUATIONS

The boundary layer equation (17) is an unsteady nonlinear third-order parabolic partial differential equation. Two main approaches can be used to solve this equation. The first approach is to use time marching for solving the boundary layer equation [4]. Although it seems straightforward, this approach is computationally expensive. Using time marching for an unsteady two-dimensional boundary layer problem is roughly as expensive as solving a steady three-dimensional boundary layer problem.

The second possible approach for solving the unsteady boundary layer equation is to linearize the boundary layer equation about some nominal mean flow.

This is a valid approximation as long as the flow unsteadiness is small compared to the mean flow. Since up to this point no assumption was made about the variation in time of the small disturbances, one could calculate them by marching in time.

However, one can introduce a further simplification by assuming that the unsteady part of the flow is harmonic in time.

This assumption removes the explicit time dependency from the unsteady boundary layer equation.

This section derives the time-linearized unsteady boundary layer equations using the coordinate transformation proposed herein and assuming harmonically varying small disturbances.

The nondimensional stream function $\hat{f}$ and the edge velocity $\hat{u}_e$ can be expanded in the perturbation series:

$$\hat{f}(s,\eta,t) = F(s,\eta) + Re[f(s,\eta)\, e^{j\omega t}] \tag{18}$$

$$\hat{u}_e(s,t) = U_e(s) + Re[u_e(s)\, e^{j\omega t}] \tag{19}$$

Note that the amplitudes of the unsteady parts, $f(s,\eta)$ and $u_e(s)$ are complex and *Re* denotes the real part. Note also that the mean flow variables are represented by upper case characters.

The next step in the linearization process is to substitute the flow decomposition (18) and (19) into the boundary layer equation (17). To provide a better explanation of this derivation, the result for each term of equation (17) will be presented separately. The exponential $e^{j\omega t}$ which accompanies the perturbation will be omitted to clarify the explanation. The presence of the exponential $e^{j\omega t}$ is assumed for all the first-order terms.

After substituting the perturbation series (18) and (19) into (17) and neglecting the second-order terms, one obtains:

$$(\hat{b}\, \hat{f}'')' \rightarrow (B\,F'')' + (b\,F'')' + (B\, f'')'$$

$$\frac{1}{2}\hat{f}\ \hat{f}'' \rightarrow \frac{1}{2}F\,F'' + \frac{1}{2}F\ f'' + \frac{1}{2}F''\ f$$

$$\frac{\partial \hat{u}_e}{\partial t} + \hat{u}_e \frac{\partial \hat{u}_e}{\partial s} \rightarrow U_e \frac{\partial U_e}{\partial s} + j\,\omega\, u_e + U_e \frac{\partial u_e}{\partial s} + u_e \frac{\partial U_e}{\partial s}$$

$$\frac{\partial \hat{f}'}{\partial t} \rightarrow j\,\omega\, f'$$

$$\hat{f}'\ \hat{f}_s' \rightarrow F'\,F_s' + F'\ f_s' + F_s'\ f'$$

$$\hat{f}_s'\ \hat{f}'' \rightarrow F_s\,F'' + F_s\ f'' + F''\ f_s$$

Collecting the zeroth-order terms, one obtains the mean flow equation of the boundary layer:

$$(B\,F'')' + A_{1s}\frac{1}{2}F\,F'' + A_1\,U_e\,U_{es} = A_1\,(F'\,F_s' - F_s F'') \tag{20}$$

Equation (20) is identical to the steady boundary layer equation so that one can conclude that the mean flow represents in fact the steady flow.

Collecting the first-order terms one obtains the small disturbance boundary layer equations:

$$\begin{aligned}(b\,F'')' + (B\ f'')' + A_{1s}\frac{1}{2}(F\ f'' + F''\ f) + A_1\,(\,j\,\omega\, u_e + U_e\, u_{es} + u_e\, U_{es})\\ = A_1\,(\,j\,\omega\, f' + F'\ f_s' + F_s' f' - F_s\ f'' - F''\ f_s)\end{aligned}$$

5. BOUNDARY CONDITIONS

Once the governing equations have been developed, the next step in properly defining the problem is to impose the appropriate boundary conditions. The boundary conditions depend on whether the shear layer develops along a wall or in a wake. A separate treatment is necessary for the boundary layer starting point.

5.1. Wall Boundary Conditions

Because the boundary layer is modeled by a third-order parabolic partial differential equation, one needs three boundary conditions at each station in the *s*-direction. The no-slip boundary conditions at the wall are:

$$\hat{u}(s, n = 0) = \hat{v}(s, n = 0) = 0 \tag{21}$$

At the edge of the boundary one imposes the continuity of the velocity $\hat{u}$:

$$\lim_{n\to\infty} \hat{u}(s, n) = \hat{u}_e \tag{22}$$

The boundary conditions (21) and (22) must be written using the nondimensional potential function $\hat{f}$ and the coordinate system (s, η). To write the boundary condition $\hat{u} = 0$ as a function of $\hat{f}$, one uses (6) and (8):

$$\hat{u} = \frac{\partial \hat{\psi}}{\partial n} = \sqrt{\frac{u_0}{\nu s}} \cdot \frac{\partial \hat{\psi}}{\partial \eta} = \sqrt{\frac{u_0}{\nu s}} \cdot \sqrt{\nu s u_0}\, \frac{\partial \hat{f}}{\partial \eta} = u_0\, \hat{f}' \tag{23}$$

The boundary condition becomes:

$$\hat{f}'(s, \eta = 0) = 0$$

The second boundary condition $\hat{v}(s, n = 0) = 0$ states that the airfoil is a streamline.

As a result, this boundary condition written in terms of the nondimensional stream function $\hat{f}$ is:

$$\hat{f}(s, \eta = 0) = 0$$

The boundary condition for the outer edge is obtained using (23) which yields:

$$\lim_{\eta\to\infty} \hat{f}'(s, \eta) = \hat{u}_e / u_0 \tag{24}$$

The boundary layer equations are parabolic equations so that in order to solve them one needs to impose initial conditions in addition to imposing boundary conditions. An initial boundary condition is needed at the boundary layer starting points. The imposed initial condition is a similarity solution obtained by solving Blasius' equation [11, p. 136]:

$$\hat{f}\, \hat{f}'' + 2\, \hat{f}''' = 0$$

with the boundary conditions:

$$\hat{f}(s, \eta = 0) = 0$$

$$\hat{f}'(s,\eta = 0) = 0$$

$$\lim_{\eta \to \infty} \hat{f}'(s,\eta) = \hat{u}_e / u_0$$

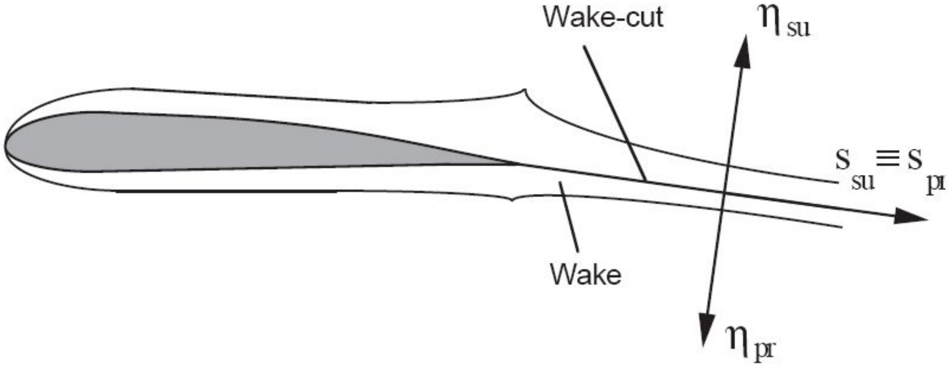

Figure 1. Airfoil wake and the wake-cut

5.2. Wake Boundary Conditions

Let us consider the flow over an airfoil. In the wake region, the two shear layers coming from the suction and pressure sides of the airfoil merge. The position of the merging line is computed by the inviscid flow solver.

The wake merging line (or wake-cut) represents the line along which the two shear layers merge, as shown in Figure 1. The wake-cut is assumed to be an impermeable line, having equal pressure on both sides. Boundary conditions must be imposed for both shear layers coming from the pressure and suction sides of the airfoil. As a result, six boundary conditions must be imposed. It is also necessary to impose that the wake-cut be a stream line at the inner edge:

$$\hat{f}(s,\eta_{su} = 0) = 0$$

$$\hat{f}(s,\eta_{pr} = 0) = 0$$

where the subscripts su and pr denote the suction and pressure side, respectively.

Along the wake-cut, the two shear layers must be continuous, that is, their velocities and slopes must be continuous:

$$\hat{f}'(s,\eta_{pr} = 0) = \hat{f}'(s,\eta_{su} = 0)$$

$$\hat{f}''(s,\eta_{pr} = 0) = -\hat{f}''(s,\eta_{su} = 0) \tag{25}$$

The minus sign in the boundary condition (25) is necessary because of the discontinuity of the η coordinate along the wake-cut. The boundary conditions at the outer edge of the wake shear layers are identical to the boundary conditions (24) at the edge of the boundary layer along the wall.

6. CONCLUSIONS

A coordinate transformation was introduced to remove the singularity in the equations at the leading edge and to generate a coordinate frame for computation in which the boundary layer thickness remains as constant as possible. A novel coordinate transformation was proposed, where the coordinate normal to the wall was nondimensionalized by the displacement thickness. A generalization of the coordinate transformation was also developed. Then, by making the assumption that the fluid flow is composed of a mean flow plus a harmonically varying small unsteady disturbance, the nonlinear unsteady viscous flow equations were linearized. The paper concluded with a presentation of the shear layer boundary conditions along the airfoil and in the wake.

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