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Research Article

Ground State Solutions for the Periodic Discrete Nonlinear Schrödinger Equations with Superlinear Nonlinearities

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We consider the periodic discrete nonlinear Schrödinger equations with the temporal frequency belonging to a spectral gap. By using the generalized Nehari manifold approach developed by Szulkin and Weth, we prove the existence of ground state solutions of the equations. We obtain infinitely many geometrically distinct solutions of the equations when specially the nonlinearity is odd. The classical Ambrosetti-Rabinowitz superlinear condition is improved.

1. Introduction

The following discrete nonlinear Schrödinger equation (DLNS):

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - \sigma\chi_n f_n(\psi_n), \quad n \in \mathbb{Z}, \tag{1}$$

where $\sigma = \pm 1$ and

$$\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n \tag{2}$$

is the discrete Laplacian operator, appears in many physical problems, like polarons, energy transfer in biological materials, nonlinear optics, and so forth (see [1]). The parameter σ characterizes the focusing properties of the equation: if $\sigma = 1$, the equation is self-focusing, while $\sigma = -1$ corresponds to the defocusing equation. The given sequences $\{\varepsilon_n\}$ and $\{\chi_n\}$ are assumed to be T-periodic in n, that is, $\varepsilon_{n+T} = \varepsilon_n$ and $\chi_{n+T} = \chi_n$. Moreover, $\{\chi_n\}$ is a positive sequence. Here, T is a positive integer. We assume that $f_n(0) = 0$ and the nonlinearity $f_n(u)$ is gauge invariant, that is,

$$f_n(e^{i\theta}u) = e^{i\theta} f_n(u), \quad \theta \in \mathbb{R}. \tag{3}$$

We are interested in the existence of solitons of (1), that is, solutions which are spatially localized time-periodic and decay to zero at infinity. Thus, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t}, \qquad \lim_{|n|\to\infty} \psi_n = 0, \tag{4}$$

where $\{u_n\}$ is a real-valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then, (1) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = \sigma\chi_n f_n(u_n), \quad n \in \mathbb{Z}, \tag{5}$$

$$\lim_{|n|\to\infty} u_n = 0 \tag{6}$$

holds. Naturally, if we look for solitons of (1), we just need to get the solutions of (5) satisfying (6).

Actually, we consider a more general equation:

$$Lu_n - \omega u_n = \sigma\chi_n f_n(u_n), \quad n \in \mathbb{Z}, \tag{7}$$

with the same boundary condition (6). Here, L is a second-order difference operator

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n, \tag{8}$$

where $\{a_n\}$ and $\{b_n\}$ are real-valued T-periodic sequences. When $a_n \equiv -1$ and $b_n = 2 + \varepsilon_n$, we obtain (5).

We consider (7) as a nonlinear equation in the space l^2 of two-sided infinite sequences. Note that every element of l^2 automatically satisfies (6).

As it is well known, the operator L is a bounded and self-adjoint operator in l^2. The spectrum $\sigma(L)$ is a union of a finite number of closed intervals, and the complement $\mathbb{R} \setminus \sigma(L)$ consists of a finite number of open intervals called spectral gaps. Two of them are semi-infinite (see [2]). If $T = 1$, then finite gaps do not exist. However, in general, finite gaps exist, and the most interesting case in (7) is when the frequency ω belongs to a finite spectral gap. Let us fix any spectral gap and denote it by (α, β).

DNLS equation is one of the most important inherently discrete models. DNLS equation plays a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology (see [1, 3–6] and references therein). In the past decade, solitons of the periodic DNLS have become a hot topic. The existence of solitons for the periodic DNLS equations with superlinear nonlinearity [7–10] and with saturable nonlinearity [11–13] has been studied, respectively. If ω is below or above the spectrum of the difference operator $-\Delta + \varepsilon_n$, solitons were shown by using the Nehari manifold approach and a discrete version of the concentration compactness principle in [14]. If ω is a lower edge of a finite spectral gap, the existence of solitons was obtained by using variant generalized weak linking theorem in [10]. If ω lies in a finite spectral gap, the existence of solitons was proved by using periodic approximations in combination with the linking theorem in [8] and the generalized Nehari manifold approach in [9], respectively. The results were extended by Chen and Ma in [7]. In this paper, we employ the generalized Nehari manifold approach instead of periodic approximation technique to obtain the existence of a kind of special solitons of (7), which called ground state solutions, that is, nontrivial solutions with least possible energy in l^2. We should emphasize that the results are obtained under more general super nonlinearity than the classical Ambrosetti-Rabinowitz superlinear condition [8, 9, 15].

This paper is organized as follows. In Section 2, we first establish the variational framework associated with (7) and transfer the problem on the existence of solutions in l^2 of (7) into that on the existence of critical points of the corresponding functional. We then present the main results of this paper and compare them with existing ones. Section 3 is devoted to the proofs of the main results.

2. Preliminaries and Main Results

The following are the basic hypotheses to establish the main results of this paper:

(V_1) $\omega \in (\alpha, \beta)$,

(f_1) $f_n \in C(\mathbb{R}, \mathbb{R})$ and $f_{n+T}(u) = f_n(u)$, and there exist $a > 0$ and $p \in (2, \infty)$ such that

$$\left|f_n(u)\right| \leq a\left(1 + |u|^{p-1}\right) \quad \forall n \in \mathbb{Z},\ u \in \mathbb{R}, \tag{9}$$

(f_2) $f_n(u) = o(|u|)$ as $u \to 0$,

(f_3) $\lim_{|u|\to\infty} F_n(u)/u^2 = \infty$, where $F_n(u)$ is the primitive function of $f_n(u)$, that is,

$$F_n(u) = \int_0^u f_n(s)\,ds, \tag{10}$$

(f_4) $u \mapsto f_n(u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

To state our results, we introduce some notations. Let

$$A = L - \omega, \qquad E = l^2(\mathbb{Z}). \tag{11}$$

Consider the functional J defined on E by

$$J(u) = \frac{1}{2}(Au, u)_E - \sigma \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n), \tag{12}$$

where $(\cdot,\cdot)_E$ is the inner product in E and $\|\cdot\|_E$ is the corresponding norm in E. The hypotheses on $f_n(u)$ imply that the functional $J \in C^1(E, \mathbb{R})$ and (7) is easily recognized as the corresponding Euler-Lagrange equation for J. Thus, to find nontrivial solutions of (7), we need only to look for nonzero critical points of J in E.

For the derivative of J, we have the following formula:

$$\left(J'(u), v\right) = (Au, v)_E - \sigma \sum_{n\in\mathbb{Z}} \chi_n f_n(u_n)\, v_n, \quad \forall v \in E. \tag{13}$$

By (V_1), we have $\sigma(A) \subset \mathbb{R} \setminus (\alpha - \omega, \beta - \omega)$. So, $E = E^+ \oplus E^-$ corresponds to the spectral decomposition of A with respect to the positive and negative parts of the spectrum, and

$$\begin{aligned} (Au, u)_E &\geq (\beta - \omega)\|u\|_E^2, \quad u \in E^+, \\ (Au, u)_E &\leq (\alpha - \omega)\|u\|_E^2, \quad u \in E^-. \end{aligned} \tag{14}$$

For any $u, v \in E$, letting $u = u^+ + u^-$ with $u^\pm \in E^\pm$ and $v = v^+ + v^-$ with $v^\pm \in E^\pm$, we can define an equivalent inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u, v) = \left(Au^+, v^+\right)_E - \left(Au^-, v^-\right)_E, \qquad \|u\| = (u, u)^{1/2}, \tag{15}$$

respectively. So, J can be rewritten as

$$J(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \sigma \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n). \tag{16}$$

We define for $u \in E \setminus E^-$, the subspace

$$E(u) := \mathbb{R}u + E^- = \mathbb{R}u^+ \oplus E^-, \tag{17}$$

and the convex subset

$$\widehat{E}(u) := \mathbb{R}^+ u + E^- = \mathbb{R}^+ u^+ \oplus E^-, \tag{18}$$

of E, where, as usual, $\mathbb{R}^+ = [0, \infty)$. Let

$$\mathscr{M} = \left\{u \in E \setminus E^- : J'(u)\,u = 0,\ J'(u)\,v = 0\ \forall v \in E^-\right\}, \tag{19}$$

$$c = \inf_{u\in\mathscr{M}} J(u). \tag{20}$$

In this paper, we also consider the multiplicity of solutions of (7).

For each $\ell \in \mathbb{Z}$, let

$$\ell * u = (u_{n+\ell T})_{n\in\mathbb{Z}}, \quad \forall u = (u_n)_{n\in\mathbb{Z}}, \tag{21}$$

which defines a $\mathbb{Z}$-action on E. By the periodicity of the coefficients, we know that both J and J' are $\mathbb{Z}$-invariants. Therefore, if $u \in E$ is a critical point of J, so is $\ell * u$. Two critical points $u_1, u_2 \in E$ of J are said to be geometrically distinct if $u_1 \neq \ell * u_2$ for all $\ell \in \mathbb{Z}$.

Now, we are ready to state the main results.

Theorem 1. *Suppose that conditions (V_1), (f_1)–(f_4) are satisfied. Then, one has the following conclusions.*

(1) *If either $\sigma = 1$ and $\beta \neq \infty$ or $\sigma = -1$ and $\alpha \neq -\infty$, then (7) has at least a nontrivial ground state solution.*

(2) *If either $\sigma = 1$ and $\beta = \infty$ or $\sigma = -1$ and $\alpha = -\infty$, then (7) has no nontrivial solution.*

Theorem 2. *Suppose that conditions (V_1), (f_1)–(f_4) are satisfied and f_n is odd in u. If either $\sigma = 1$ and $\beta \neq \infty$ or $\sigma = -1$ and $\alpha \neq -\infty$, then (7) has infinitely many pairs of geometrically distinct solutions.*

In what follows, we always assume that $\sigma = 1$. The other case can be reduced to $\sigma = 1$ by switching L to $-L$ and ω to $-\omega$.

Remark 3. In [8], the author considered (7) with f_n defined by

$$f_n(u) = |u|^2 u, \tag{22}$$

which obviously satisfies (f_1)–(f_4); the author also discussed the case where f satisfies the Ambrosetti-Rabinowitz condition; that is, there exists $\mu > 2$ such that

$$0 < \mu F_n(u) \le f_n(u)\, u, \quad u \neq 0. \tag{23}$$

Clearly, (23) implies that $F_n(u) \ge c|u|^{\mu} > 0$ for $|u| \ge 1$. So, it is a stronger condition than (f_3).

Remark 4. In [9], the author assumed that f_n satisfies the following condition: there exists $\theta \in (0,1)$ such that

$$0 < u^{-1} f_n(u) \le \theta f_n'(u), \quad u \neq 0. \tag{24}$$

Obviously, (24) implies (23) with $\mu = 1 + (1/\theta)$, so it is a stronger condition than the Ambrosetti-Rabinowitz condition. In our paper, the nonlinearities satisfy more general superlinear assumptions instead of (24) which also implies (f_4). However, we do not assume that f_n is differentiable and satisfies (24), $\mathscr{M}$ is not a C^1 manifold of E, and the minimizers on $\mathscr{M}$ may not be critical points of J. Hence, the method of [9] does not apply any more. Nevertheless, $\mathscr{M}$ is still a topological manifold, naturally homeomorphic to the unit sphere in E^+ (see in detail in Section 3). We use the generalized Nehari manifold approach developed by Szulkin and Weth which is based on reducing the strongly indefinite variational problem to a definite one and prove that the minimizers of J on $\mathscr{M}$ are indeed critical points of J.

Remark 5. In [7], it is shown that (7) has at least a nontrivial solution $u \in l^2$ if f satisfies (V_1), (f_2), (f_3), and the following conditions:

(B_1) $F_n(u) \ge 0$ for any $u \in \mathbb{R}$ and $H_n(u) := (1/2) f_n(u)u - F_n(u) > 0$ if $u \neq 0$,

(B_2) $H_n(u) \to \infty$ as $|u| \to \infty$, and there exist $r_0 > 0$ and $\gamma > 1$ such that $|f_n(u)|^{\gamma}/|u|^{\gamma} \le c_0 H_n(u)$ if $|u| \ge r_0$, where c_0 is a positive constant,

In our paper, we use (9) and (f_4) instead of (B_1) and (B_2).

3. Proofs of Main Results

We assume that (V_1) and (f_1)–(f_4) are satisfied from now on.

Lemma 6. *$F_n(u) > 0$ and $(1/2) f_n(u)u > F_n(u)$ for all $u \neq 0$.*

Proof. By (f_2) and (f_4), it is easy to get that

$$F_n(u) > 0 \quad \forall u \neq 0. \tag{25}$$

Set $H_n(u) = (1/2) f_n(u)u - F_n(u)$. It follows from (f_4) that

$$\begin{aligned} H_n(u) &= \frac{u}{2} f_n(u) - \int_0^u f_n(s)\, ds \\ &> \frac{u}{2} f_n(u) - \frac{f_n(u)}{u} \int_0^u s ds = 0. \end{aligned} \tag{26}$$

So, $(1/2) f_n(u)u > F_n(u)$ for all $u \neq 0$. □

To continue the discussion, we need the following proposition.

Proposition 7 (see [16, 17]). *Let $u, s, v \in \mathbb{R}$ be numbers with $s \ge -1$ and $w := su + v \neq 0$. Then,*

$$f_n(u)\left[s\left(\frac{s}{2} + 1\right)u + (1+s)\, v\right] + F_n(u) - F_n(u+w) < 0. \tag{27}$$

Lemma 8. *If $u \in \mathscr{M}$, then*

$$\begin{aligned} &J(u+w) < J(u) \quad \textit{for every } w \in U \\ &:= \{su + v : s \ge -1, v \in E^-\}, \quad w \neq 0. \end{aligned} \tag{28}$$

Hence, u is the unique global maximum of $J|_{\widehat{E}(u)}$.

Proof. We rewrite J by

$$J(u) = \frac{1}{2}(Au^+, u^+)_E + \frac{1}{2}(Au^-, u^-)_E - \sigma \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n). \tag{29}$$

Since $u \in \mathscr{M}$, we have

$$0 = \left(J'(u), \frac{2s+s^2}{2} u + (1+s) v \right) = \frac{2s+s^2}{2} (Au^+, u^+)_E + \frac{2s+s^2}{2} (Au^-, u^-)_E + (1+s)(Au^-, v)_E - \sum_{n\in\mathbb{Z}} \chi_n f_n(u_n) \left(\frac{2s+s^2}{2} u_n + (1+s) v_n \right). \tag{30}$$

Together with Proposition 7, we know that

$$\begin{aligned} J(u+w) - J(u) &= \frac{1}{2} \{ (A(1+s)u^+, (1+s)u^+)_E - (Au^+, u^+)_E \} \\ &\quad + \frac{1}{2} \{ (A((1+s)u^- + v), (1+s)u^- + v)_E - (Au^-, u^-)_E \} \\ &\quad + \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n) - \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n + w_n) \\ &= \frac{2s+s^2}{2} (Au^+, u^+)_E + \frac{2s+s^2}{2} (Au^-, u^-)_E + \frac{1}{2} (Av, v)_E \\ &\quad + (1+s)(Au^-, v)_E + \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n) - \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n + w_n) \\ &= \frac{1}{2} (Av, v)_E + \sum_{n\in\mathbb{Z}} \chi_n \left\{ f_n(u_n) \left[s\left(\frac{s}{2} + 1 \right) u_n + (1+s) v_n \right] \right. \\ &\qquad \left. + F_n(u_n) - F_n(u_n + w_n) \right\} < 0. \end{aligned} \tag{31}$$

The proof is complete. □

Lemma 9. (a) *There exists $\alpha > 0$ such that $c := \inf_{\mathscr{M}} J(u) \geq \inf_{S_\alpha} J(u) > 0$, where $S_\alpha := \{u \in E^+ : \|u\| = \alpha\}$.*

(b) *$\|u^+\| \geq \max\{\|u^-\|, \sqrt{2c}\}$ for every $u \in \mathscr{M}$.*

Proof. (a) By (f_1) and (f_2), it is easy to show that for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f_n(u)| \leq \varepsilon |u| + c_\varepsilon |u|^{p-1}, \qquad |F_n(u)| \leq \varepsilon |u|^2 + c_\varepsilon |u|^p. \tag{32}$$

$\|\cdot\|$ is equivalent to the E norm on E^+ and $E \subset l^q$ for $2 \leq q \leq \infty$ with $\|u\|_{l^q} \leq \|u\|_E$. Hence, for any $\varepsilon \in (0, 1/2)$ and $u \in E^+$, we have

$$J(u) \geq \frac{1}{2} \|u\|^2 - \varepsilon \|u\|^2 - c_\varepsilon \overline{\chi} \|u\|^p, \tag{33}$$

which implies $\inf_{S_\alpha} J(u) > 0$ for some $\alpha > 0$ (small enough), where $\overline{\chi} = \max\{\chi_n\}$.

The first inequality is a consequence of Lemma 8 since for every $u \in \mathscr{M}$, there is $s > 0$ such that $su^+ \in \widehat{E}(u) \cap S_\alpha$.

(b) For $u \in \mathscr{M}$, by (25), we have

$$c \leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \sum_{n\in\mathbb{Z}} \chi_n F_n(u_n) \leq \frac{1}{2} \left(\|u^+\|^2 - \|u^-\|^2 \right). \tag{34}$$

Hence, $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2c}\}$. □

Lemma 10. *Let $\mathscr{W} \subset E^+ \setminus \{0\}$ be a compact subset. Then, there exists $R > 0$ such that $J \leq 0$ on $E(u) \setminus B_R(0)$ for every $u \in \mathscr{W}$, where $B_R(0)$ denotes the open ball with radius R and center 0.*

Proof. Suppose by contradiction that there exist $u^{(k)} \in \mathscr{W}$ and $w^{(k)} \in E(u^{(k)})$, $k \in \mathbb{N}$, such that $J(w^{(k)}) > 0$ for all k and $\|w^{(k)}\| \to \infty$ as $k \to \infty$. Without loss of generality, we may assume that $\|u^{(k)}\| = 1$ for $k \in \mathbb{Z}$. Then, there exists a subsequence, still denoted by the same notation, such that $u^{(k)} \to u \in E^+$. Set $v^{(k)} = w^{(k)}/\|w^{(k)}\| = s^{(k)} u^{(k)} + v^{(k)-}$. Then,

$$0 < \frac{J(w^{(k)})}{\|w^{(k)}\|^2} = \frac{1}{2} \left(\left(s^{(k)}\right)^2 - \left\| v^{(k)-} \right\|^2 \right) - \sum_{n\in\mathbb{Z}} \chi_n \frac{F_n(w_n^{(k)})}{\left(w_n^{(k)}\right)^2} \left(v_n^{(k)}\right)^2. \tag{35}$$

By (25), we have

$$\left\| v^{(k)-} \right\|^2 \leq \left(s^{(k)}\right)^2 = 1 - \left\| v^{(k)-} \right\|^2. \tag{36}$$

Consequently, we know that $\|v^{(k)-}\| \leq 1/\sqrt{2}$ and $1/\sqrt{2} \leq s^{(k)} \leq 1$. Passing to a subsequence if necessary, we assume that $s^{(k)} \to s \in [1/\sqrt{2}, 1]$, $v^{(k)} \rightharpoonup v$, $v^{(k)-} \rightharpoonup v_*^- \in E^-$, and $v_n^{(k)} \to v_n$ for every n. Hence, $v = su + v_*^- \neq 0$ and $v_*^- = v^-$. It follows that for $n_0 \in \mathbb{Z}$ with $v_{n_0} \neq 0$, $|w_{n_0}^{(k)}| = \|w^{(k)}\| \cdot |v_{n_0}^{(k)}| \to \infty$, as $k \to \infty$. Then, by (f_3), we have

$$\sum_{n\in\mathbb{Z}} \chi_n \frac{F_n(w_n^{(k)})}{\left(w_n^{(k)}\right)^2} \left(v_n^{(k)}\right)^2 \longrightarrow \infty, \tag{37}$$

which contradicts with (35). □

Lemma 11. *For each $u \in E^+ \setminus \{0\}$, the set $\mathscr{M} \cap \widehat{E}(u)$ consists of precisely one point which is the unique global maximum of $J|_{\widehat{E}(u)}$.*

Proof. By Lemma 8, it suffices to show that $\mathscr{M} \cap \widehat{E}(u) \neq \emptyset$. Since $\widehat{E}(u) = \widehat{E}(u^+/\|u^+\|)$, we may assume that $u \in S^+$. By Lemma 10, there exists $R > 0$ such that $J \leq 0$ on $E(u) \setminus B_R(0)$ provided that R is large enough. By Lemma 9 (a), $J(tu) > 0$ for small $t > 0$. Moreover, $J \leq 0$ on $\widehat{E}(u) \setminus B_R(0)$. Hence, $0 < \sup_{\widehat{E}(u)} J < \infty$.

Let $v^{(k)} \rightharpoonup v$ in $\widehat{E}(u)$. Then, $v_n^{(k)} \to v_n$ as $k \to \infty$ for all n after passing to a subsequence if necessary. Hence, $F_n(v_n^{(k)}) \to F_n(v_n)$. Let $\varphi(v) = \sum_{n\in\mathbb{Z}} \chi_n F_n(v_n)$. Then,

$$\begin{aligned}\varphi(v) &= \sum_{n\in\mathbb{Z}} \lim_{k\to\infty} \chi_n F_n\left(v_n^{(k)}\right) \\ &\leq \liminf_{k\to\infty} \sum_{n\in\mathbb{Z}} \chi_n F_n\left(v_n^{(k)}\right) \\ &= \liminf_{k\to\infty} \varphi\left(v^{(k)}\right);\end{aligned} \tag{38}$$

that is, φ is a weakly lower semicontinuous. From the weak lower semi-continuity of the norm, it is easy to see that J is weakly upper semicontinuous on $\widehat{E}(u)$. Therefore, $J(u_0) = \sup_{\widehat{E}(u)} J$ for some $u_0 \in \widehat{E}(u) \setminus \{0\}$. By the proof of Lemma 10, u_0 is a critical point of $J|_{\widehat{E}(u)}$. It follows that $(J'(u_0), u_0) = (J'(u_0), z) = 0$ for all $z \in E$ and hence $u_0 \in \mathcal{M}$. To summarize, $u_0 \in \mathcal{M} \cap \widehat{E}(u)$. □

According to Lemma 11, for each $u \in E^+ \setminus \{0\}$, we may define the mapping $\widehat{m} : E^+ \setminus \{0\} \to \mathcal{M}$, $u \mapsto \widehat{m}(u)$, where $\widehat{m}(u)$ is the unique point of $\mathcal{M} \cap \widehat{E}(u)$.

Lemma 12. *J is coercive on $\mathcal{M}$; that is, $J(u) \to \infty$ as $\|u\| \to \infty$, $u \in \mathcal{M}$.*

Proof. Suppose, by contradiction, that there exists a sequence $\{u^{(k)}\} \subset \mathcal{M}$ such that $\|u^{(k)}\| \to \infty$ and $J(u^{(k)}) \leq d$ for some $d \in [c, \infty)$. Let $v^{(k)} = u^{(k)}/\|u^{(k)}\|$. Then, there exists a subsequence, still denoted by the same notation, such that $v^{(k)} \rightharpoonup v$ and $v_n^{(k)} \to v_n$ for every n as $k \to \infty$.

First, we know that there exist $\delta > 0$ and $n_k \in \mathbb{Z}$ such that

$$\left|v_{n_k}^{(k)+}\right| \geq \delta. \tag{39}$$

Indeed, if not, then $v^{(k)+} \to 0$ in l^∞ as $k \to \infty$. By Lemma 9(b), $1/2 \leq \|v^{(k)+}\|^2 \leq 1$, which means that $\|v^{(k)+}\|_{l^2}$ is bounded. For $q > 2$,

$$\left\|v^{(k)+}\right\|_{l^q}^q \leq \left\|v^{(k)+}\right\|_{l^\infty}^{q-2} \left\|v^{(k)+}\right\|_{l^2}^2. \tag{40}$$

Then, $v^{(k)+} \to 0$ in all $l^q, q > 2$. By (32), for any $s \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} \chi_n F_n\left(s v_n^{(k)+}\right) \leq \varepsilon s^2 \overline{\chi} \left\|v^{(k)+}\right\|_{l^2}^2 + c_\varepsilon s^p \overline{\chi} \left\|v^{(k)+}\right\|_{l^p}^q, \tag{41}$$

which implies that $\sum_{n\in\mathbb{Z}} \chi_n F_n(s v_n^{(k)+}) \to 0$ as $k \to \infty$.

Since $s v^{(k)+} \in \widehat{E}(u^{(k)})$ for $s \geq 0$, Lemma 8 implies that

$$\begin{aligned}d &\geq J\left(u^{(k)}\right) \geq J\left(s v^{(k)+}\right) \\ &= \frac{s^2}{2}\left\|v^{(k)+}\right\|^2 - \sum_{n\in\mathbb{Z}} \chi_n F_n\left(s v_n^{(k)+}\right) \\ &\geq \frac{s^2}{4} - \sum_{n\in\mathbb{Z}} \chi_n F_n\left(s v_n^{(k)+}\right) \longrightarrow \frac{s^2}{4},\end{aligned} \tag{42}$$

as $k \to \infty$. This is a contradiction if $s > \sqrt{4d}$.

Due to the periodicity of coefficients, both J and $\mathcal{M}$ are invariant under T-translation. Making such shifts, we can assume that $1 \leq n_k \leq T - 1$ in (39). Moreover, passing to a subsequence if needed, we can assume that $n_k = n_0$ is independent of k. Next, we may extract a subsequence, still denoted by $\{v^{(k)}\}$, such that $v_n^{(k)+} \to v_n^+$ for all $n \in \mathbb{Z}$. In particular, for $n = n_0$, inequality (39) shows that $|v_{n_0}^+| \geq \delta$ and hence $v^+ \neq 0$.

Since $|u_n^{(k)}| \to \infty$ as $k \to \infty$, it follows again from (f_3) and Fatou's lemma that

$$\begin{aligned}0 \leq \frac{J\left(u^{(k)}\right)}{\left\|u^{(k)}\right\|^2} &= \frac{1}{2}\left(\left\|v^{(k)+}\right\|^2 - \left\|v^{(k)-}\right\|^2\right) \\ &\quad - \sum_{n\in\mathbb{Z}} \chi_n \frac{F_n\left(u_n^{(k)}\right)}{\left(u_n^{(k)}\right)^2} \\ &\quad \times \left(v_n^{(k)}\right)^V \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty,\end{aligned} \tag{43}$$

a contradiction again. The proof is finished. □

Lemma 13. (a) *The mapping $\widehat{m} : E^+\setminus\{0\} \to \mathcal{M}$ is continuous.*

(b) *The mapping $m = \widehat{m}|_{S^+} : S^+ \to \mathcal{M}$ is a homeomorphism between S^+ and $\mathcal{M}$, and the inverse of m is given by $m^{-1}(u) = u^+/\|u^+\|$, where $S^+ := \{u \in E^+ : \|u\| = 1\}$.*

(c) *The mapping $m^{-1} : \mathcal{M} \mapsto S^+$ is the Lipschitz continuous.*

Proof. (a) Let $(u^{(k)}) \subset E^+ \setminus \{0\}$ be a sequence with $u^{(k)} \to u$. Since $\widehat{m}(w) = \widehat{m}(w^+/\|w^+\|)$, without loss of generality, we may assume that $\|u^{(k)}\| = 1$ for all k. Then, $\widehat{m}(u^{(k)}) = \|\widehat{m}(u^{(k)})^+\| u^{(k)} + \widehat{m}(u^{(k)})^-$. By Lemma 10, there exists $R > 0$ such that

$$\begin{aligned}J\left(\widehat{m}\left(u^{(k)}\right)\right) &= \sup_{E(u^{(k)})} J \leq \sup_{B_R(0)} J \\ &\leq \sup_{u\in B_R(0)} \left\|u^+\right\|^2 = R^2 \quad \text{for every } k.\end{aligned} \tag{44}$$

It follows from Lemma 12 that $\widehat{m}(u^{(k)})$ is bounded. Passing to a subsequence if needed, we may assume that

$$\begin{aligned}t^{(k)} &:= \left\|\widehat{m}\left(u^{(k)}\right)^+\right\| \longrightarrow t, \\ \widehat{m}\left(u^{(k)}\right)^- &\rightharpoonup u_*^- \quad \text{in } E \text{ as } k \to \infty,\end{aligned} \tag{45}$$

where $t \geq \sqrt{2c} > 0$ by Lemma 9(b). Moreover, by Lemma 11,

$$\begin{aligned}J\left(\widehat{m}\left(u^{(k)}\right)\right) &\geq J\left(t^{(k)} u^{(k)} + \widehat{m}(u)^-\right) \longrightarrow J\left(tu + \widehat{m}(u)^-\right) \\ &= J\left(\widehat{m}(u)\right).\end{aligned} \tag{46}$$

Therefore, using the weak lower semicontinuity of the norm and φ (defined in Lemma 11), we get

$$\begin{aligned} J(\widehat{m}(u)) &\le \lim_{k\to\infty} J\left(\widehat{m}\left(u^{(k)}\right)\right) \\ &= \lim_{k\to\infty}\left(\frac{1}{2}\left(t^{(k)}\right)^2 - \frac{1}{2}\left\|\widehat{m}\left(u^{(k)}\right)^-\right\|^2 \right.\\ &\qquad \left. - \sum_{n\in\mathbb{Z}} \chi_n F_n\left(\widehat{m}\left(u_n^{(k)}\right)\right)\right) \\ &\le \frac{1}{2}t^2 - \frac{1}{2}\left\|u_*^-\right\|^2 - \sum_{n\in\mathbb{Z}}\chi_n F_n\left(tu_n + u_{*,n}^-\right) \\ &= J\left(tu + u_*^-\right) \le J(\widehat{m}(u)), \end{aligned} \tag{47}$$

which implies that all inequalities above must be equalities and $\widehat{m}(u^{(k)})^- \to u_*^-$. By Lemma 11, $u_*^- = \widehat{m}(u)^-$ and hence $\widehat{m}(u^{(k)}) \to \widehat{m}(u)$.

(b) This is an immediate consequence of (a).

(c) For $u, v \in \mathscr{M}$, by (b), we have

$$\begin{aligned} \left\|m^{-1}(u) - m^{-1}(v)\right\| &= \left\|\frac{u^+}{\|u^+\|} - \frac{v^+}{\|v^+\|}\right\| \\ &= \left\|\frac{u^+ - v^+}{\|u^+\|} + \frac{\left(\|v^+\| - \|u^+\|\right)v^+}{\|u^+\|\,\|v^+\|}\right\| \\ &\le \frac{2}{\|u^+\|}\left\|(u-v)^+\right\| \le \sqrt{\frac{2}{c}}\,\|u - v\|. \end{aligned} \tag{48}$$

□

We will consider the functional $\widehat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}$ and $\Psi : S^+ \to \mathbb{R}$ defined by

$$\widehat{\Psi} := J(\widehat{m}(w)), \qquad \Psi := \widehat{\Psi}|_{S^+}. \tag{49}$$

Lemma 14. (a) $\widehat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$ *and*

$$\widehat{\Psi}'(w) z = \frac{\left\|\widehat{m}(w)^+\right\|}{\|w\|} J'(\widehat{m}(w)) z \quad \forall w, z \in E^+,\ w \neq 0. \tag{50}$$

(b) $\Psi \in C^1(S^+, \mathbb{R})$ *and*

$$\begin{aligned} \Psi'(w) z = \left\|\widehat{m}(w)^+\right\| J'(m(w)) z \quad \forall z \in T_w S^+ \\ = \left\{v \in E^+ : (w, v) = 0\right\}. \end{aligned} \tag{51}$$

(c) $\{w_n\}$ *is a Palais-Smale sequence for* Ψ *if and only if* $\{m(w_n)\}$ *is a Palais-Smale sequence for* J.

(d) $w \in S^+$ *is a critical point of* Ψ *if and only if* $m(w) \in \mathscr{M}$ *is a nontrivial critical point of* J. *Moreover, the corresponding values of* Ψ *and* J *coincide and* $\inf_{S^+}\Psi = \inf_{\mathscr{M}} J = c$.

Proof. (a) We put $u = \widehat{m}(w) \in \mathscr{M}$, so we have $u = (\|u^+\|/\|w\|)w + u^-$. Let $z \in E^+$. Choose $\delta > 0$ such that $w_t := w + tz \in E^+ \setminus \{0\}$ for $|t| < \delta$ and put $u_t = \widehat{m}(w_t) \in \mathscr{M}$. We may write $u_t = s_t w_t + u_t^-$ with $s_t > 0$. From the proof of Lemma 13, the function $t \mapsto s_t$ is continuous. Then, $s_0 = \|u^+\|/\|w\|$. By Lemma 11 and the mean value theorem, we have

$$\begin{aligned} \widehat{\Psi}(w_t) - \widehat{\Psi}(w) &= J(u_t) - J(u) \\ &= J\left(s_t w_t + u_t^-\right) - J\left(s_0 w + u^-\right) \\ &\le J\left(s_t w_t + u_t^-\right) - J\left(s_t w + u_t^-\right) \\ &= J'\left(s_t\left[w + \eta_t(w_t - w)\right] + u_t^-\right) s_t t z \end{aligned} \tag{52}$$

with some $\eta_t \in (0, 1)$. Similarly,

$$\begin{aligned} \widehat{\Psi}(w_t) - \widehat{\Psi}(w) &= J\left(s_t w_t + u_t^-\right) - J\left(s_0 w + u^-\right) \\ &\ge J\left(s_0 w_t + u^-\right) - J\left(s_0 w + u^-\right) \\ &= J'\left(s_0\left[w + \tau_t(w_t - w)\right] + u^-\right) s_0 t z, \end{aligned} \tag{53}$$

with some $\tau_t \in (0, 1)$. Combining these inequalities and the continuity of function $t \mapsto s_t$, we have

$$\lim_{t\to 0}\frac{\widehat{\Psi}(w_t) - \widehat{\Psi}(w)}{t} = s_0 J'(u) z = \frac{\left\|\widehat{m}(w)^+\right\|}{\|w\|} J'(\widehat{m}(w)) z. \tag{54}$$

Hence, the Gâteaux derivative of $\widehat{\Psi}$ is bounded linear in z and continuous in w. It follows that $\widehat{\Psi}$ is of class C^1 (see [15]).

(b) It follows from (a) by noting that $m(w) = \widehat{m}(w)$ since $w \in S^+$.

(c) Let $\{w_n\}$ be a Palais-Smale sequence for Ψ, and let $u_n = m(w_n) \in \mathscr{M}$. Since for every $n \in \mathbb{Z}$, we have an orthogonal splitting $E = T_{w_n}S^+ \oplus E(w_n)$; using (b), we have

$$\begin{aligned} \left\|\Psi'(w_n)\right\| &= \sup_{\substack{z\in T_{w_n}S^+\\ \|z\|=1}} \Psi'(w_n) z \\ &= \left\|m(w_n)^+\right\| \sup_{\substack{z\in T_{w_n}S^+\\ \|z\|=1}} J'(m(w_n)) z \\ &= \left\|u_n^+\right\| \sup_{\substack{z\in T_{w_n}S^+\\ \|z\|=1}} J'(u_n) z, \end{aligned} \tag{55}$$

because $J'(u_n)v = 0$ for all $v \in E(w_n)$ and $E(w_n)$ is orthogonal to $T_{w_n}S^+$. Using (b) again, we have

$$\begin{aligned} \left\|\Psi'(w_n)\right\| &\le \left\|u_n^+\right\| \left\|J'(u_n)\right\| \\ &= \left\|u_n^+\right\| \sup_{\substack{z\in T_{w_n}S^+, v\in E(w_n)\\ z+v\neq 0}} \frac{J'(u_n)(z+v)}{\|z+v\|} \\ &\le \left\|u_n^+\right\| \sup_{z\in T_{w_n}S^+\setminus\{0\}} \frac{J'(u_n)(z)}{\|z\|} = \left\|\Psi'(w_n)\right\|. \end{aligned} \tag{56}$$

Therefore,

$$\left\|\Psi'(w_n)\right\| = \left\|u_n^+\right\| \left\|J'(u_n)\right\|. \tag{57}$$

According to Lemma 9(b) and Lemma 12, $\sqrt{2c} \le \|u_n^+\| \le \sup_n \|u_n^+\| < \infty$. Hence, $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{u_n\}$ is a Palais-Smale sequence for J.

(d) By (57), $\Psi'(w) = 0$ if and only if $J'(m(w)) = 0$. The other part is clear. □

Proof of Theorem 1. (1) We know that $c > 0$ by Lemma 9(a). If $u_0 \in \mathscr{M}$ satisfies $J(u_0) = c$, then $m^{-1}(u_0) \in S^+$ is a minimizer of Ψ and therefore a critical point of Ψ and also a critical point of J by Lemma 14. We shall show that there exists a minimizer $u \in \mathscr{M}$ of $J|_{\mathscr{M}}$. Let $\{w^{(k)}\} \subset S^+$ be a minimizing sequence for Ψ. By Ekeland's variational principle, we may assume that $\Psi(w^{(k)}) \to c$ and $\Psi'(w^{(k)}) \to 0$ as $k \to \infty$. Then, $J(u^{(k)}) \to c$ and $J'(u^{(k)}) \to 0$ as $k \to \infty$ by Lemma 14(c), where $u^{(k)} := m(w^{(k)}) \in \mathscr{M}$. By Lemma 12, $\{u^{(k)}\}$ is bounded, and hence $\{u^{(k)}\}$ has a weakly convergent subsequence.

First, we show that there exist $\delta > 0$ and $n_k \in \mathbb{Z}$ such that

$$\left|u_{n_k}^{(k)}\right| \ge \delta. \tag{58}$$

Indeed, if not, then $u^{(k)} \to 0$ in l^∞ as $k \to \infty$. From the simple fact that for $q > 2$,

$$\left\|u^{(k)}\right\|_{l^q}^q \le \left\|u^{(k)}\right\|_{l^\infty}^{q-2} \left\|u^{(k)}\right\|_{l^2}^2, \tag{59}$$

we have $u^{(k)} \to 0$ in all $l^q, q > 2$. By (32), we know that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \chi_n f_n\left(u_n^{(k)}\right) u_n^{(k)+} &\le \varepsilon\overline{\chi} \sum_{n\in\mathbb{Z}} \left|u_n^{(k)}\right| \cdot \left|u_n^{(k)+}\right| \\ &\quad + c_\varepsilon \overline{\chi} \sum_{n\in\mathbb{Z}} \left|u_n^{(k)}\right|^{p-1} \cdot \left|u_n^{(k)+}\right| \\ &\le \varepsilon\overline{\chi} \left\|u^{(k)}\right\|_{l^2} \cdot \left\|u^{(k)+}\right\|_{l^2} \\ &\quad + c_\varepsilon\overline{\chi} \left\|u^{(k)}\right\|_{l^p}^{p-1} \cdot \left\|u^{(k)+}\right\|_{l^p} \\ &\le \varepsilon\overline{\chi} \left\|u^{(k)}\right\|_{l^2} \cdot \left\|u^{(k)+}\right\| \\ &\quad + c_\varepsilon\overline{\chi} \left\|u^{(k)}\right\|_{l^p}^{p-1} \cdot \left\|u^{(k)+}\right\|,\end{aligned} \tag{60}$$

which implies that $\sum_{n\in\mathbb{Z}} \chi_n f_n(u_n^{(k)}) u_n^{(k)+} = o(\| u^{(k)+} \|)$ as $k \to \infty$. Therefore,

$$\begin{aligned} o\left(\left\|u^{(k)+}\right\|\right) &= \left(J'\left(u^{(k)}\right), u^{(k)+}\right) \\ &= \left\|u^{(k)+}\right\|^2 - \sum_{n\in\mathbb{Z}} \chi_n f_n\left(u_n^{(k)}\right) u_n^{(k)+} \\ &= \left\|u^{(k)+}\right\|^2 - o\left(\left\|u^{(k)+}\right\|\right). \end{aligned} \tag{61}$$

Then, $\| u^{(k)+} \|^2 \to 0$ as $k \to \infty$, contrary to Lemma 9(b).

From the periodicity of the coefficients, we know that J and J' are both invariant under T-translation. Making such shifts, we can assume that $1 \le n_k \le T-1$ in (58). Moreover, passing to a subsequence, we can assume that $n_k = n_0$ is independent of k.

Next, we may extract a subsequence, still denoted by $\{u^{(k)}\}$, such that $u^{(k)} \rightharpoonup u$ and $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$. Particularly, for $n = n_0$, inequality (58) shows that $|u_{n_0}| \ge \delta$, so $u \ne 0$. Moreover, we have

$$\left(J'(u), v\right) = \lim_{k\to\infty} \left(J'\left(u^{(k)}\right), v\right) = 0, \quad \forall v \in E; \tag{62}$$

that is, u is a nontrivial critical point of J.

Finally, we show that $J(u) = c$. By Lemma 6 and Fatou's lemma, we have

$$\begin{aligned} c &= \lim_{k\to\infty} \left(J\left(u^{(k)}\right) - \frac{1}{2} J'\left(u^{(k)}\right) u^{(k)}\right) \\ &= \lim_{k\to\infty} \sum_{n\in\mathbb{Z}} \chi_n \left(\frac{1}{2} f_n\left(u_n^{(k)}\right) u_n^{(k)} - F_n\left(u_n^{(k)}\right)\right) \\ &\ge \sum_{n\in\mathbb{Z}} \chi_n \left(\frac{1}{2} f_n\left(u_n\right) u_n - F_n\left(u_n\right)\right) \\ &= J(u) - \frac{1}{2} J'(u) u = J(u) \ge c. \end{aligned} \tag{63}$$

Hence, $J(u) = c$. That is, u is a nontrivial ground state solution of (7).

(2) If $\beta = \infty$, by way of contradiction, we assume that (7) has a nontrivial solution $u \in E$. Then, u is a nonzero critical point of J in E. Thus, $J'(u) = 0$. But by Lemma 6,

$$\left(J'(u), u\right) = ((L-\omega)u, u) - \sum_{n\in\mathbb{Z}} \chi_n f_n(u_n) u_n < 0. \tag{64}$$

This is a contradiction, so the conclusion holds.

This completes the proof of Theorem 1. □

Now, we are ready to prove Theorem 2. From now on, we always assume that f_n is odd in u. We need some notations. For $a \ge b \ge c$, denote

$$\begin{aligned} J^a &= \{u \in \mathscr{M} : J(u) \le a\}, \\ J_b &:= \{u \in \mathscr{M} : J(u) \ge b\}, \\ J_b^a &= J^a \cap J_b, \\ \Psi^a &= \{w \in S^+ : \Psi(w) \le a\}, \\ \Psi_b &:= \{w \in S^+ : \Psi(w) \ge b\}, \\ \Psi_b^a &= \Psi^a \cap \Psi_b, \\ K &= \left\{w \in S^+ : \Psi'(w) = 0\right\}, \\ K_a &= \{w \in K : \Psi(w) = a\}, \\ \nu(a) &= \sup\left\{\|u\| : u \in J^a\right\}. \end{aligned} \tag{65}$$

It is easy to see that $\nu(a) < \infty$ for every a by Lemma 12.

Proof of Theorem 2. It is easy to see that mappings m, m^{-1} are equivariant with respect to the $\mathbb{Z}$-action by Lemma 13; hence, the orbits $\mathscr{O}(u) \subset \mathscr{M}$ consisting of critical points of J are in 1-1 correspondence with the orbits $\mathscr{O}(w) \subset S^+$ consisting of critical points of Ψ by Lemma 14(d). Next, we may choose a subset $\mathcal{F} \subset K$ such that $\mathcal{F} = -\mathcal{F}$ and $\mathcal{F}$ consists of a unique representative of $\mathbb{Z}$-orbits. So, we only need to prove that the set $\mathcal{F}$ is infinite. By contradiction, we assume that

$$\mathcal{F} \text{ is a finite set.} \tag{66}$$

Let

$$\Gamma_j = \{A \subset S^+ : A = -A,\ A \text{ is closed and } \gamma(A) \geq j\}, \tag{67}$$

where γ denotes genus and $j \in \mathbb{N}$. We consider the sequence of the Lusternik-Schnirelmann values of Ψ defined by

$$c_k = \inf\left\{d \in \mathbb{R} : \gamma\left(\Psi^d\right) \geq k,\ k \in \mathbb{N}\right\}. \tag{68}$$

Now, we claim that

$$K_{c_k} \neq \emptyset, \quad c_k < c_{k+1}. \tag{69}$$

Firstly, we show that

$$\kappa = \inf\{\|v - w\| : v, w \in K, v \neq w\} > 0. \tag{70}$$

In fact, there exist $v^{(k)}, w^{(k)} \in \mathcal{F}$, and $g_k, l_k \in \mathbb{Z}$ such that $v^{(k)} * g_k \neq w^{(k)} * l_k$ for all k and

$$\left\|v^{(k)} * g_k - w^{(k)} * l_k\right\| \longrightarrow \kappa \quad \text{as } k \longrightarrow \infty. \tag{71}$$

Let $m_k = g_k - l_k$. Passing to a subsequence, $v^{(k)} = v \in \mathcal{F}$, $w^{(k)} = w \in \mathcal{F}$, and either $m_k = m \in \mathbb{Z}$ for all k or $|m_k| \to \infty$. In the first case, $0 < \|v^{(k)} * g_k - w^{(k)} * l_k\| = \|v - w * m\| = \kappa$ for all k. In the second case, $w * m_k \rightharpoonup 0$ and therefore $\kappa = \lim_{k\to\infty}\|v - w * m_k\| \geq \|v\| = 1$. By (70), $\gamma(K_{c_k}) = 0$ or 1.

Next, we consider a pseudogradient vector field of Ψ [18]; that is, there exists a Lipschitz continuous map $V\colon S^+ \setminus K \to T_w S^+$ and for all $w \in S^+ \setminus K$,

$$\|V(w)\| < 2\left\|\Psi'(w)\right\|,$$
$$\left\langle V(w), \Psi'(w)\right\rangle > \frac{1}{2}\left\|\Psi'(w)\right\|^2. \tag{72}$$

Let $\eta : \mathcal{D} \to S^+ \setminus K$ be the corresponding Ψ-decreasing flow defined by

$$\frac{d}{dt}\eta(t,w) = -V(\eta(t,w)),$$
$$\eta(0,w) = w, \tag{73}$$

where $\mathcal{D} = \{(t,w) : w \in S^+ \setminus K, T^-(w) < t < T^+(w)\} \subset \mathbb{R} \times (S^+ \setminus K)$, and $T^-(w) < 0, T^+(w) > 0$ are the maximal existence times of the trajectory $t \to \eta(t,w)$ in negative and positive direction. By the continuity property of the genus, there exists $\delta > 0$ such that $\gamma(\overline{U}) = \gamma(K_{c_k})$, where $U = N_\delta(K_{c_k}) := \{w \in S^+ : \operatorname{dist}(w, K_{c_k}) < \delta\}$ and $\delta < \kappa/2$. Following the deformation argument (Lemma A.3), we choose $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\lim_{t\to T^+(w)} \Psi(\eta(t,w)) < c_k - \varepsilon \quad \text{for } w \in \Psi^{c_k+\varepsilon} \setminus U. \tag{74}$$

Then, for every $w \in \Psi^{c_k+\varepsilon} \setminus U$, there exists $t \in [0, T^+(w))$ such that $\Psi(\eta(t,w)) < c_k - \varepsilon$. Hence, we may define the entrance time map

$$r : w \in \Psi^{c_k+\varepsilon} \setminus U \longrightarrow [0,\infty),$$
$$r(w) = \inf\{t \in [0, T^+(w)) : \Psi(\eta(t,w)) \leq c_k - \varepsilon\}, \tag{75}$$

which satisfies $r(w) < T^+(w)$ for every $w \in \Psi^{c_k+\varepsilon} \setminus U$. Since $c_k - \varepsilon$ is not a critical value of Ψ by (74), it is easy to see that r is a continuous and even map. It follows that the map

$$g : \Psi^{c_k+\varepsilon} \setminus U \longrightarrow \Psi^{c_k-\varepsilon}, \qquad g(w) = \eta(r(w), w) \tag{76}$$

is odd and continuous. Then, $\gamma(\Psi^{c_k+\varepsilon} \setminus U) \leq \gamma(\Psi^{c_k-\varepsilon}) \leq k-1$, and consequently,

$$\gamma\left(\Psi^{c_k+\varepsilon}\right) \leq \gamma\left(\overline{U}\right) + k - 1 = \gamma\left(K_{c_k}\right) + k - 1. \tag{77}$$

So, $\gamma(K_{c_k}) \geq 1$. Therefore, $K_{c_k} \neq \emptyset$. Moreover, the definition of c_k and of c_{k+1} implies that $\gamma(K_{c_k}) \geq 1$ if $c_k < c_{k+1}$ and $\gamma(K_{c_k}) > 1$ if $c_k = c_{k+1}$. Since $\gamma(\mathcal{F}) = \gamma(K_{c_k}) \leq 1$, $c_k < c_{k+1}$. Therefore, there is an infinite sequence $\{\pm w_k\}$ of pairs of geometrically distinct critical points of Ψ with $\Psi(w_k) = c_k$, which contradicts with (66). Therefore, the set $\mathcal{F}$ is infinite.

This completes the proof of Theorem 2. □

Appendix

Here, we give a proof of (74). We state the discrete property of the Palais-Smale sequences. It yields nice properties of the corresponding pseudogradient flow.

Lemma A.1. *Let $d \geq c$. If $\{w_1^{(k)}\}$, $\{w_2^{(k)}\} \subset \Psi^d$ are two Palais-Smale sequences for Ψ, then either $\|w_1^{(k)} - w_2^{(k)}\| \to 0$ as $k \to \infty$ or $\limsup_{k\to\infty}\|w_1^{(k)} - w_2^{(k)}\| \geq \varrho(d) > 0$, where $\varrho(d)$ depends on d but not on the particular choice of the Palais-Smale sequences.*

Proof. Set $u_1^{(k)} = m(w_1^{(k)})$ and $u_2^{(k)} = m(w_2^{(k)})$. Then, $\{u_1^{(k)}\}$, $\{u_2^{(k)}\} \subset J^d$ are the bounded Palais-Smale sequences for J. We fix p in (f_2) and consider the following two cases.

(i) $\| u_1^{(k)} - u_2^{(k)}\|_{l^p} \to 0$ as $k \to \infty$.

By a straightforward calculation and (32), for any $\varepsilon > 0$, there exist $C_1, C_2 > 0$, and k_0 such that for all $k \geq k_0$,

$$\begin{aligned}
&\left\|\left(u_1^{(k)} - u_2^{(k)}\right)^+\right\|^2 \\
&\quad = J'\left(u_1^{(k)}\right)\left(u_1^{(k)} - u_2^{(k)}\right)^+ - J'\left(u_2^{(k)}\right)\left(u_2^{(k)} - u_2^{(k)}\right)^+ \\
&\qquad + \sum_{n\in\mathbb{Z}} \chi_n\left[f_n\left(u_{1n}^{(k)}\right) - f_n\left(u_{2n}^{(k)}\right)\right]\left(u_1^{(k)} - u_2^{(k)}\right)^+ \\
&\quad \leq \varepsilon\left\|\left(u_1^{(k)} - u_2^{(k)}\right)^+\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\overline{\chi}\sum_{n\in\mathbb{Z}}\Big[\varepsilon\left(\left|u_{1n}^{(k)}\right|+\left|u_{2n}^{(k)}\right|\right)\\
&\qquad+c_\varepsilon\left(\left|u_{1n}^{(k)}\right|^{p-1}+\left|u_{2n}^{(k)}\right|^{p-1}\right)\Big]\\
&\quad\times\left|\left(u_1^{(k)}-u_2^{(k)}\right)^+\right|\\
&\le\varepsilon\left\|\left(u_1^{(k)}-u_2^{(k)}\right)^+\right\|\\
&\quad+\overline{\chi}\varepsilon\left(\left\|u_1^{(k)}\right\|+\left\|u_2^{(k)}\right\|\right)\left\|\left(u_1^{(k)}-u_2^{(k)}\right)^+\right\|\\
&\quad+\overline{\chi}c_\varepsilon\left(\left\|u_1^{(k)}\right\|_{l^p}^{p-1}+\left\|u_2^{(k)}\right\|_{l^p}^{p-1}\right)\left\|\left(u_1^{(k)}-u_2^{(k)}\right)^+\right\|_{l^p}\\
&\le\varepsilon\left\|\left(u_1^{(k)}-u_2^{(k)}\right)^+\right\|+\overline{\chi}\varepsilon C_1\left\|\left(u_1^{(k)}-u_2^{(k)}\right)^+\right\|\\
&\quad+\overline{\chi}c_\varepsilon C_2\left\|u_1^{(k)}-u_2^{(k)}\right\|_{l^p}.
\end{aligned}
\tag{A.1}
$$

This implies $\limsup_{k\to\infty}\|(u_1^{(k)}-u_2^{(k)})^+\|^2\le\limsup_{k\to\infty}(1+\overline{\chi}C_1)\varepsilon\|(u_1^{(k)}-u_2^{(k)})^+\|$. Hence, $\|(u_1^{(k)}-u_2^{(k)})^+\|\to 0$. Similarly, $\|(u_1^{(k)}-u_2^{(k)})^-\|\to 0$. Therefore, $\|u_1^{(k)}-u_2^{(k)}\|\to 0$ as $k\to\infty$. By Lemma 13(c), we have $\|w_1^{(k)}-w_2^{(k)}\|=\|m^{-1}(u_1^{(k)})-m^{-1}(u_2^{(k)})\|\to 0$ as $k\to\infty$.

(ii) $\|u_1^{(k)}-u_2^{(k)}\|_{l^p}\nrightarrow 0$ as $k\to\infty$.

There exist $\delta>0$ and $n_k\in\mathbb{Z}$ such that

$$\left|u_{1n_k}^{(k)}-u_{2n_k}^{(k)}\right|\ge\delta. \tag{A.2}$$

For bounded sequences $\{u_1^{(k)}\}$, $\{u_2^{(k)}\}$, we may pass to subsequences so that

$$u_1^{(k)}\rightharpoonup u_1\in E,\qquad u_2^{(k)}\rightharpoonup u_2\in E, \tag{A.3}$$

where $u_1\ne u_2$ by (A.2) and $J'(u_1)=J'(u_2)=0$, and

$$\left\|\left(u_1^{(k)}\right)^+\right\|\longrightarrow\alpha_1,\qquad\left\|\left(u_1^{(k)}\right)^+\right\|\longrightarrow\alpha_2, \tag{A.4}$$

where $\sqrt{2c}\le\alpha_i\le\nu(d)$, $i=1,2$ by Lemma 9(b).

If $u_1\ne 0$ and $u_2\ne 0$. Then, $u_1,u_2\in\mathscr{M}$ and $w_1=m^{-1}(u_1)\in K$, $w_2=m^{-1}(u_2)\in K$, $w_1\ne w_2$. Therefore,

$$
\begin{aligned}
\liminf_{k\to\infty}\left\|w_1^{(k)}-w_2^{(k)}\right\|&=\liminf_{k\to\infty}\left\|\frac{\left(u_1^{(k)}\right)^+}{\left\|\left(u_1^{(k)}\right)^+\right\|}-\frac{\left(u_2^{(k)}\right)^+}{\left\|\left(u_2^{(k)}\right)^+\right\|}\right\|\\
&\ge\left\|\frac{u_1^+}{\alpha_1}-\frac{u_2^+}{\alpha_2}\right\|=\left\|\beta_1w_1-\beta_2w_2\right\|,
\end{aligned}
\tag{A.5}
$$

where $\beta_1=\|u_1^+\|/\alpha_1\ge\sqrt{2c}/\nu(d)$ and $\beta_2=\|u_2^+\|/\alpha_2\ge\sqrt{2c}/\nu(d)$. Since $\|w_1\|=\|w_2\|=1$, we have

$$
\begin{aligned}
\liminf_{k\to\infty}\left\|w_1^{(k)}-w_2^{(k)}\right\|&\ge\left\|\beta_1w_1-\beta_2w_2\right\|\\
&\ge\min\{\beta_1,\beta_2\}\left\|w_1-w_2\right\|\ge\frac{\sqrt{2c}\kappa}{\nu(d)}.
\end{aligned}
\tag{A.6}
$$

If $u_1=0$, then $u_2\ne 0$ and

$$
\begin{aligned}
\liminf_{k\to\infty}\left\|w_1^{(k)}-w_2^{(k)}\right\|&=\liminf_{k\to\infty}\left\|\frac{\left(u_1^{(k)}\right)^+}{\left\|\left(u_1^{(k)}\right)^+\right\|}-\frac{\left(u_2^{(k)}\right)^+}{\left\|\left(u_2^{(k)}\right)^+\right\|}\right\|\\
&\ge\frac{\|u_2^+\|}{\alpha_2}\ge\frac{\sqrt{2c}}{\nu(d)}.
\end{aligned}
\tag{A.7}
$$

Similarly, if $u_2=0$, then $u_1\ne 0$ and $\liminf_{k\to\infty}\|w_1^{(k)}-w_2^{(k)}\|\ge\sqrt{2c}/\nu(d)$.

The proof is complete. □

Lemma A.2. *For every $w\in S^+$, the limit $\lim_{t\to T^+(w)}\eta(t,w)$ exists and is a critical point of Ψ.*

Proof. Fix $w\in S^+$ and set $d=\Psi(w)$. We distinguish two cases to finish the proof.

Case 1 ($T^+(w)<\infty$). For $0\le s<t<T^+(w)$, by (72) and (73), we have

$$
\begin{aligned}
&\left\|\eta(t,w)-\eta(s,w)\right\|\\
&\quad\le\int_s^t\left\|V(\eta(\tau,w))\right\|d\tau\\
&\quad\le 2\sqrt{2}\int_s^t\sqrt{\left\langle\Psi'(\eta(\tau,w)),V(\eta(\tau,w))\right\rangle}d\tau\\
&\quad\le 2\sqrt{2(t-s)}\left(\int_s^t\left\langle\Psi'(\eta(\tau,w)),V(\eta(\tau,w))\right\rangle d\tau\right)^{1/2}\\
&\quad=2\sqrt{2(t-s)}\left[\Psi(\eta(s,w))-\Psi(\eta(t,w))\right]^{1/2}\\
&\quad\le 2\sqrt{2(t-s)}\left[\Psi(w)-c\right]^{1/2}.
\end{aligned}
\tag{A.8}
$$

Since $T^+(w)<\infty$, this implies that $\lim_{t\to T^+(w)}\eta(t,w)$ exists and is a critical point of Ψ, otherwise the trajectory $t\to\eta(t,w)$ could be continued beyond $T^+(w)$.

Case 2 ($T^+(w)=\infty$). To prove that $\lim_{t\to T^+(w)}\eta(t,w)$ exists, we claim that for every $\varepsilon>0$, there exists $t_\varepsilon>0$ such that $\|\eta(t_\varepsilon,w)-\eta(t,w)\|<\varepsilon$ for $t\ge t_\varepsilon$. If not, then there exist $0<\varepsilon_0<(1/2)\varrho(d)$ ($\varrho(d)$ is the same number in Lemma A.1) and a sequence $\{t_n\}\subset[0,\infty)$ with $t_n\to\infty$ such that $\|\eta(t_n,w)-\eta(t_{n+1},w)\|=\varepsilon_0$ for every n. Choose the smallest $t_n^1\in(t_n,t_{n+1})$ such that $\|\eta(t_n,w)-\eta(t_n^1,w)\|=\varepsilon_0/3$. Let $\iota_n=\min_{s\in[t_n,t_n^1]}\|\Psi'(\eta(s,w))\|$. By (72) and (73), we have

$$
\begin{aligned}
\frac{\varepsilon_0}{3}&=\left\|\eta\left(t_n^1,w\right)-\eta\left(t_n,w\right)\right\|\\
&\le\int_{t_n}^{t_n^1}\left\|V(\eta(\tau,w))\right\|d\tau\\
&\le 2\int_{t_n}^{t_n^1}\left\|\Psi'(\eta(\tau,w))\right\|d\tau
\end{aligned}
$$

$$\leq \frac{2}{\iota_n}\int_{t_n}^{t_n^1}\left\|\Psi'(\eta(\tau,w))\right\|^2 d\tau$$
$$\leq \frac{4}{\iota_n}\int_{t_n}^{t_n^1}\left\langle\Psi'(\eta(\tau,w)),V(\eta(\tau,w))\right\rangle d\tau$$
$$= \frac{4}{\iota_n}\left(\Psi(\eta(t_n,w)) - \Psi\left(\eta\left(t_n^1,w\right)\right)\right). \quad \text{(A.9)}$$

Since $\Psi(\eta(t_n,w)) - \Psi(\eta(t_n^1,w)) \to 0$ as $n \to \infty$, $\iota_n \to 0$ and there exist $\tilde{t}_n^1 \in [t_n, t_n^1]$ such that $\Psi'(w_n^1) \to 0$, where $w_n^1 = \eta(\tilde{t}_n^1, w)$. Similarly, we choose the largest $t_n^2 \in (t_n^1, t_{n+1})$ such that $\|\eta(t_{n+1},w) - \eta(t_n^2,w)\| = \varepsilon_0/3$. Then, there exist $\tilde{t}_n^2 \in [t_n^2, t_{n+1}]$ such that $\Psi'(w_n^2) \to 0$, where $w_n^2 = \eta(\tilde{t}_n^2, w)$. Since $\|w_n^1 - \eta(t_n,w)\| \leq \varepsilon_0/3$ and $\|w_n^2 - \eta(t_{n+1},w)\| \leq \varepsilon_0/3$, $\{w_n^1\}$, $\{w_n^2\}$ are two the Palais-Smale sequences such that

$$\frac{\varepsilon_0}{3} \leq \left\|w_n^1 - w_n^2\right\|$$
$$\leq \left\|w_n^1 - \eta(t_n,w)\right\| + \left\|\eta(t_n,w) - \eta(t_{n+1},w)\right\| + \left\|w_n^2 - \eta(t_{n+1},w)\right\| \quad \text{(A.10)}$$
$$\leq 2\varepsilon_0 < \varrho(d),$$

which contradicts with Lemma A.1. This proves the claim. Therefore, $\lim_{t\to T^+(w)}\eta(t,w)$ exists, and, obviously, it must be a critical point of Ψ. This completes the proof. □

Lemma A.3. *Let $d \geq c$. Then, for every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that*

(a) $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$,

(b) $\lim_{t\to T^+(w)}\Psi(\eta(t,w)) < d - \varepsilon$ *for* $w \in \Psi^{d+\varepsilon} \setminus N_\delta(K_d)$.

Proof. (a) According to (66), for $\varepsilon > 0$ small enough, it is easy to see that (a) is satisfied.

(b) Without loss of generality, we may assume that $N_\delta(K_d) \subset \Psi^{d+1}$ and $\delta < \varrho(d+1)$. Set

$$\tau = \inf\left\{\left\|\Psi'(w)\right\| : w \in N_\delta(K_d) \setminus N_{\delta/2}(K_d)\right\}. \quad \text{(A.11)}$$

We claim that $\tau > 0$. Indeed, if not, then there exists a sequence $\{w_1^{(k)}\} \subset N_\delta(K_d) \setminus N_{\delta/2}(K_d)$ such that $\Psi'(w_1^{(k)}) \to 0$. By the $\mathbb{Z}$-invariance of Ψ and assumption (66), we may assume $w_1^{(k)} \in N_\delta(w_0) \setminus N_{\delta/2}(w_0)$ for some $w_0 \in K_d$ after passing to a subsequence. Let $w_2^{(k)} \to w_0$. Then, $\Psi'(w_2^{(k)}) \to 0$ and

$$\frac{\delta}{2} \leq \limsup_{n\to\infty}\left\|w_1^{(k)} - w_2^{(k)}\right\| \leq \delta < \varrho(d+1), \quad \text{(A.12)}$$

which contradicts with Lemma A.1. This proves the claim.

Let

$$M = \sup\left\{\left\|\Psi'(w)\right\| : w \in N_\delta(K_d) \setminus N_{\delta/2}(K_d)\right\}. \quad \text{(A.13)}$$

Choose $\varepsilon < \delta\tau^2/8M$ such that (a) holds. By Lemma A.1 and (a), the only way that (b) can fail is that $\eta(t,w) \to \tilde{w} \in K_d$ as $t \to T^+(w)$ for some $w \in \Psi^{d+\varepsilon} \setminus N_\delta(K_d)$. In this case, we let

$$t_1 = \sup\left\{t \in [0, T^+(w)) : \eta(t,w) \notin N_\delta(\tilde{w})\right\},$$
$$t_2 = \inf\left\{t \in (t_1, T^+(w)) : \eta(t,w) \in N_{\delta/2}(\tilde{w})\right\}. \quad \text{(A.14)}$$

Then,

$$\frac{\delta}{2} = \left\|\eta(t_1,w) - \eta(t_2,w)\right\|$$
$$\leq \int_{t_1}^{t_2}\left\|V(\eta(\tau,w))\right\| d\tau$$
$$\leq 2\int_{t_1}^{t_2}\left\|\Psi'(\eta(\tau,w))\right\| d\tau$$
$$\leq 2M(t_2 - t_1),$$
$$\Psi(\eta(t_2,w)) - \Psi(\eta(t_1,w)) \quad \text{(A.15)}$$
$$= -\int_{t_1}^{t_2}\left\langle\Psi'(\eta(\tau,w)),V(\eta(\tau,w))\right\rangle ds$$
$$\leq -\frac{1}{2}\int_{t_1}^{t_2}\left\|\Psi'(\eta(s,w))\right\|^2 ds$$
$$\leq -\frac{1}{2}\tau^2(t_2 - t_1) \leq -\frac{\delta\tau^2}{8M}.$$

It follows that $\Psi(\eta(t_2,w)) \leq d + \varepsilon - (\delta\tau^2/8M) < d$ and therefore $\eta(t_2,w) \nrightarrow \tilde{w}$, a contradiction again. This completes the proof. □

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