A production inventory model with exponential demand rate and reverse logistics

Ritu Raj[a*], Naresh Kumar Kaliraman[a], Shalini Chandra[b] and Harish Chaudhary[c]

[a]Research Scholar, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Rajasthan-304022, India
[b]Associate Professor, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Banasthali – 304022, Rajasthan, India
[c]Assistant Professor, Department of Management Studies, Indian Institute of Technology, Delhi, New Delhi-110016, India

CHRONICLE	ABSTRACT
	The objective of this paper is to develop an integrated production inventory model for reworkable items with exponential demand rate. This is a three-layer supply chain model with perspectives of supplier, producer and retailer. Supplier delivers raw material to the producer and finished goods to the retailer. We consider perfect and imperfect quality products, product reliability and reworking of imperfect items. After screening, defective items reworked at a cost just after the regular manufacturing schedule. At the beginning, the manufacturing system starts produce perfect items, after some time the manufacturing system can undergo into "out-of-control" situation from "in-control" situation, which is controlled by reverse logistic technique. This paper deliberates the effects of business strategies like optimum order size of raw material, exponential demand rate, production rate is demand dependent, idle times and reverse logistics for an integrated marketing system. Mathematica is used to develop the optimal solution of production rate and raw material order for maximum expected average profit. A numerical example and sensitivity analysis is illustrated to validate the model.

1. Introduction

Mathematical modeling, analysis and calculation are important areas for research in the study of supply chain system. Supplier, producer and retailer are the members of three-layer supply chain. Supplier supplies the raw material to producer and finished goods to retailers and customers. The manufacturing process shift to out-of-control state from in-control state after some time and produced some imperfect objects. These imperfect objects reworked just after the regular manufacturing schedule. The combination among suppliers, producers and retailers is important for an integrated supply chain system. Rosenblatt and Lee (1986) considered a reverse logistic model where the probability distribution of the instant of changing from in-control situation to out-of-control situation follows an exponential distribution. They understood that the imperfect objects produced in out-of-control situation could be reworked immediately at a price and establish that the presence of imperfect products outcome in a minor set. On the beginning of reverse logistic model, Lee and Rosenblatt (1987) showed that process analysis through the manufacturing runtime could perceive the changing time and it might be restored previously.

Liu and Yang (1996) investigated a particular stage manufacturing setup with defective method of supplying of reworkable and non-reworkable objects. The impact of imperfect objects on the batch sizing procedure was renowned in the works of Sana et al. (2007, 2007a), with others. Kim and Hong (1999) considered that imperfect objects could be found after the manufacturing run and the imperfect objects were reworked at a rate. Banerjee and Kim (1995) considered the integrated inventory models in which the retailer and customer coordinate their manufacturing and ordering plan, in order to lesser the common inventory costs.

Khouja and Mehrez (1994) considered three coordination systems among the members of the supply chain and showed that several coordination systems could induce to significant decrease in total price. Cardenas-Barron (2007) extended the model of Khouja and Mehrez (1994) by mathematical method, assuming n-stage multi-customer supply chain inventory coordination. Jalbar et al. (2008) developed a multi-echelon inventory coordination in which one retailer delivers an item to many customers. Chiu (2003) developed an economic production quantity model with backorders by combining the assumptions of a symmetrical of the imperfect objects were reworked to build them superior quality objects instead of reworking on all the imperfect objects and the remaining objects are sold at a low rate. Yang and Wee (2001) developed a supply chain model by integrating producer, distributer and vendor as three members of the chain.

Cardenas-Barron (2008) presented a straightforward derivation to discover best production lot size with rework procedure at solitary phase manufacturing method. Cardenas-Barron (2009) developed an economic production quantity model with planned backorders for determining the manufacturing batch size and the size of backorders in an imperfect production process where all imperfect objects were reworked at the similar rotation. Sana and Chaudhuri (2010), Sana (2010, 2010a), Sarkar et al. (2010) and Chiu et al. (2007) showed that the imperfect objects might be reworked at a cost where overall production-inventory costs could be decreased considerably.

Chiu et al. (2007) developed the job of Chiu (2003) and considered the best run-time difficulty of economic production quantity model with scrap, reworking of imperfect objects and stochastic breakdowns. Sarker et al. (2008) addressed the problem connecting to reworking of imperfect objects in a multi-stage manufacturing method by considering two operational policies: reworking of imperfect objects within the same cycle and after N cycles. Biswas and Sarker (2008) described an inventory system of a solitary manufacturing procedure with an in-cycle rework strategy of fragment with complete analysis. Sana (2011) developed an integrated production inventory model of perfect and imperfect quality products in a three-layer supply chain.

Jamal et al. (2004) developed a manufacturing system with rework process consisting two cases of rework process to minimize the total production cost. At the first instant, they executed reworking in the same cycle. At the second instant, they executed reworking after N cycles. Chiu et al. (2007) considered a manufacturing system with rework, including optimal lot sizing decision, random scrap rate and service level constraint. They derived that the expected total cost of a manufacturing system is less if backlogging is allowed or equal if backlogging is not allowed. Hafshejani et al. (2012) presented a multi-product economic production quantity model with imperfect quality items and reworking, it is a nonlinear programming problem solved by genetic algorithm with limited warehouse space.

Krishnamoorthi and Panayappam (2013) considered a single stage manufacturing system in which imperfect quality items produced and reworked. They developed two production models, one with shortages and second without shortages. Pasandideh et al. (2010) considered a multi-product economic production quantity model with defective products, reworking and limited warehouse space. Liao et al. (2009) examined an integrated repairable and manufacturing procedure with economic production quantity model with defective maintenance and rework upon fulfillment. Haji et al. (2009) developed a model for an inventory problem considering a single machine with imperfect quality items. They considered that all imperfect quality items are to be reworked, setup cost for rework, no shortages are

allowed and waiting time for imperfect quality item. Peter Chiu et al. (2010) developed a mathematical model to establish the optimal run time for a defective fixed production rate with rework, scrap and stochastic machine wreck. Wee and Chung (2009) and Chung and Wee (2010) developed an integrated production inventory model with deterioration assuming rework and green component value design. They extended the economic production quantity model with product reliability. We developed a mathematical model of three–layer supply chain including supplier, producer and retailer. The imperfect items at retailer's level are sent back to the external supplier and the imperfect items at producer's level are reworked after regular manufacturing time. The average profit of producer at in-control situation and out-of-control situation is considered to derive the solution of the problem. The average profits of suppliers and retailers are considered also.

2. Essential assumptions and symbolizations

The mathematical model is developed on the bases of following assumptions:

1. The demand rates are exponential increasing function of time for each member of three layer supply chain.

2. Production rate is demand dependent i.e. $P(t) = \lambda D(t)$, where $D(t) = be^{at}$ *and* $\lambda > 1$

3. Production cost per unit item is production rate dependent.

4. Idle time costs are assumed at supplier's and producer's level.

5. Different probability distributions functions are considered for defective items at supplier's level.

6. Single item products are considered for joint effect of supplier, producer and retailer in a three layer

   supply chain.

7. Replenishment rate is instantly infinite but its size is finite at supplier's level.

8. Stock-out situation is not conceded.

9. No product of producer is discarded at any stage.

10. No imperfect items are manufactured during rework.

11. Reworking starts just after the regular production process in each production run.

12. The imperfect objects are restored to its original quality, after reworking.

13. Insignificant lead time.

The mathematical model is developed on the bases of following symbolizations:

$Q_s(t)$ Supplier's on-hand inventory of good items at time t,

$Q_p(t)$ Producer's on-hand inventory of good items at time t,

$Q_{p_r}(t)$ Producer's on-hand inventory of defective items which would be reworked,

$Q_r(t)$	Retailer's on-hand inventory of good items at time t,
R	Supplier's replenishment lot size,
$\lambda b e^{at}$	Producer's production rate that is equal to supplier's demand rate,
α	Supplier's proportional probability of imperfect items with probability density function $f(\alpha)$,
A_s	Supplier's set up cost,
r_s	Supplier's screening rate per unit time,
S_c	Supplier's screening cost per unit item,
h_s	Supplier's holding cost per unit per unit time,
I_s	Supplier's cost per unit idle time,
C_s	Supplier's purchasing cost per unit item,
w_s	Supplier's selling price per unit perfect items,
$\overline{w_s}$	Supplier's selling price per unit imperfect items,
$E(x)$	Expected value of variable x,
SAP	Supplier's average profit,
$ESAP$	Supplier's expected average profit,
β	The percentage of imperfect items in "out-of-control" situation at producer level,
$\lambda b e_1^{at}$	Rate of reworking per unit time,
τ	Random time with mean $\frac{1}{\theta}$ after which the production system becomes uncontrollable for producer,
$F(\tau)$	Probability distribution function of τ,
$f(\tau)$	Probability density function of τ,
A_p	Producer's set up cost,
r_p	Producer's screening rate per unit time,
S_p	Producer's screening cost per unit item,
h_p	Producer's holding cost per unit per unit time for perfect items,
h_p'	Producer's holding cost per unit per unit time for defective items, which would be reworked,
r_p'	Cost to rework for imperfect item of producer,
L	Cost of labor, energy, technology at fixed rate,
γ	Cost of tool and die is a variation constant,
p_I	Producer's cost per unit idle time,
$C(P)$	Per unit item production cost,
N	Numbers of imperfect objects in the production process,
w_p	Producer's selling price per unit perfect item,
$\overline{w_p}$	Producer's selling price per unit imperfect item,
PAP	Producer's average profit,
$EPAP$	Producer's expected average profit,
$b e_c^{at}$	Customer's demand rate,
$b e_r^{at}$	Retailer's demand rate,

A_r Retailer's set up cost,

h_r Retailer's holding cost per unit per unit time,

w_r Retailer's selling price per unit item,

RAP   Retailer's average profit,

ERAP   Retailer's expected average profit,
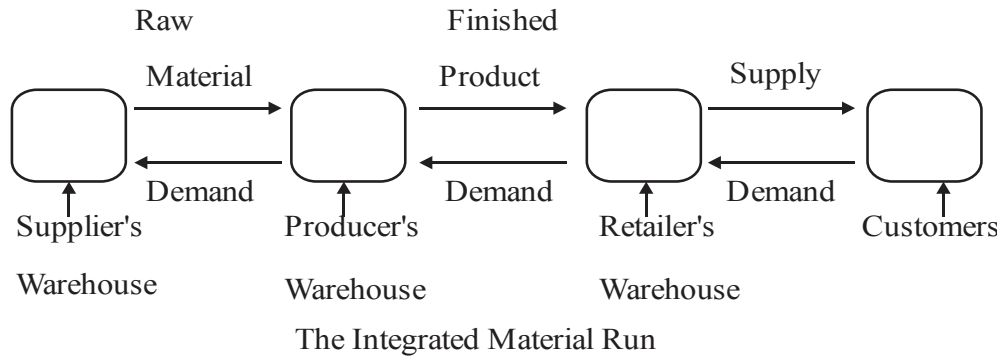
T   Retailer's cycle length.

Fig. 1. The integrated material run

3. Mathematical Model

In this projected model, supplier delivers the raw materials at rate λbe^{at} to the producer up to manufacturing run time t_1. The imperfect objects at supplier level are sent back after examine carefully with sale rate $\overline{w_s}$ per unit item to the external dealer from where the raw materials are purchased. We assumed that the manufacturing process is "in-control" state at starting stage. The process may shift to an "out-off-control" state and may produce imperfect objects after a random time (τ) with mean $\left(\dfrac{1}{\theta}\right)$.

The total imperfect objects manufactured at time t_1 are reworked with production rate λbe_1^{at}, which takes time t_r. The inventory fills up after accommodating the demand be_r^{at} of retailer during manufacturing run time t_1. The customers demand is fulfilled with rate be_c^{at} by retailer where the delivery charge of produced items is sustained up to time $kT\,(k<1)$. The assembled inventory at time kT decreases and accomplishes to zero level at time T. The leading differential equations at supplier, producer and retailer level are as follows:

3.1. Supplier's specific average profit

The leading differential equation is

$$\frac{dQ_s(t)}{dt} = -\lambda be^{at} \tag{1}$$

with $Q_S(0)=(1-\alpha)R$ and $Q_S(t_1)=0$, $0 \le t \le t_1$

From Eq. (1), we have

$$Q_s(t) = \frac{\lambda b}{a}\left(1 - e^{at}\right) + (1-\alpha)R, 0 \le t \le t_1 \tag{2}$$

We have $Q_S(t_1) = 0$

$$\Rightarrow t_1 = \frac{1}{a}\log\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) \tag{3}$$

The inventory cost of perfect objects is

$$IC = \frac{h_s}{a}\left[\left(\frac{\lambda b}{a} + (1-\alpha)R\right)\log\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) - (1-\alpha)R\right] \tag{4}$$

Inventory cost of defective objects is

$$IC_i = \frac{h_S \alpha R^2}{r_s} \tag{5}$$

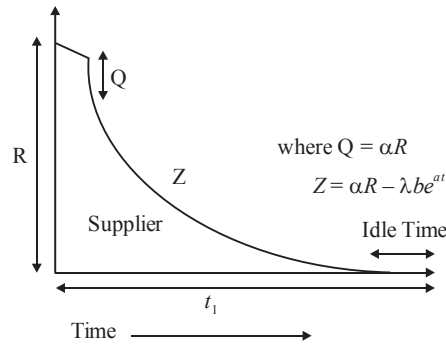where $Q = \alpha R$

$Z = \alpha R - \lambda b e^{at}$

Fig. 2. Trend on inventory change

The price of screening is $S_c R$. The earning from selling the perfect and imperfect items is $\left(w_s(1-\alpha)R + \overline{w}_s\alpha R\right)$. The buying price of R objects is $C_s R$. The price of establishment is A_s. The idle time cost is $I_s(T - t_1)$. The supplier's average profit is, using $t_1 = \frac{1}{a}\log\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right)$

$$SAP = \frac{1}{T}\left[w_s(1-\alpha)R + \overline{w}_s\alpha R - (S_c + C_s)R - I_s\left(T - \frac{1}{a}\log\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right)\right)\right]$$

$$- \frac{1}{T}\left[\frac{h_s}{a}\left\{\left(\frac{\lambda b}{a} + (1-\alpha)R\right)\log\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) - (1-\alpha)R + \frac{a\alpha R^2}{r_s}\right\} + A_s\right] \tag{6}$$

3.2. Producer's specific average profit

If a manufacturing process does not perform well according to process designer's specification then it is said to be "out-of-control". The constraint θ is a design variable and an indicator of manufactured goods consistency. Consistency of machines in a production system is usually authenticated to be an exponential function of time t which is $R(t) = e^{-\theta t}$, where $\theta = \frac{imperfect\ digits}{total\ number\ of\ working\ hours}$. Therefore, a unit either fails or carries on and one of these two states surrogates must exist. We have $R(t) = 1 - F(t)$.

Here $F(t) = \int_0^t f(u) du$, where $f(t)$ is the failure probability density function then $\int_0^\infty f(t) dt = 1$.

Thus $f(t) = \frac{d}{dt}(F(t)) = \theta e^{-\theta t}$

The unit production cost is

$$C(P) = w_s + \frac{L}{\lambda b e^{at}} + \gamma \lambda b e^{at} \tag{7}$$

We assumed that the manufacturing process is in "in-control" state at starting stage. The process may shift to an "out-off-control" state and may produce imperfect objects after a random time (τ) with mean $\left(\frac{1}{\theta}\right)$. The holding cost per unit per unit time for defective items is greater than the holding cost per unit per unit time for perfect items. Two cases are gets up in this operating system:

3.2.1 Case I: When $0 \leq \tau \leq t_1$, the "out-of-control" state exists during operations time. At the beginning, the production rate is $\lambda b e^{at}$. The inventory fills up after accommodating the demand be_r^{at} of retailer during manufacturing run time t_1. Thus (τ) exists in the time period $(0, t_1)$, the perfect items are produces during $[0, \tau]$ and it produces both perfect and imperfect items during $[\tau, t_1]$. The total numbers of items manufactured during $(0, t_1)$ are used to fulfill the demand for the time duration $(t_1 + t_2)$. Since, the total numbers of imperfect objects manufactured at time t_1 are reworked with manufacturing rate $\lambda b e_1^{at}$ which takes time t_r. The time t_3 is necessary to sell out the reworked items. On-hand inventory of perfect items at time t can be illustrated by the following differential equations:

$$\frac{dQ_p(t)}{dt} = \lambda b e^{at} - be_r^{at} \tag{8}$$

with $Q_p(0) = 0$, $0 \leq t \leq \tau$

$$\frac{dQ_p(t)}{dt} = (1-\beta)\lambda b e^{at} - be_r^{at} \tag{9}$$

with $Q_p(\tau) = (\lambda b e^{at} - be_r^{at})\tau$, $\tau \leq t \leq t_1$

$$\frac{dQ_p(t)}{dt} = -be_r^{at} \tag{10}$$

with $Q_p(0) = Q_p(t_1)$ and $Q_p(t_2) = 0$, $0 \leq t \leq t_2$

From Eq. (8), we have

$$Q_p(t) = \left\{ \frac{b}{a}(1 - e_r^{at}) - \frac{\lambda b}{a}(1 - e^{at}) \right\}, 0 \leq t \leq \tau \tag{11}$$

From Eq. (9), we have

$$Q_p(t) = b\left(\lambda e^{at} - e_r^{at}\right)\tau + \frac{(1-\beta)\lambda b}{a}\left(e^{at} - e^{a\tau}\right) + \frac{b}{a}\left(e_r^{a\tau} - e_r^{at}\right), \tau \leq t \leq t_1 \tag{12}$$

From Eq. (10), we have

$$Q_p(t) = b\left(\lambda e^{at_1} - e_r^{at_1}\right)\tau + \frac{(1-\beta)\lambda b}{a}\left(e^{at_1} - e^{a\tau}\right) + \frac{b}{a}\left(e_r^{at} + e_r^{a\tau} - e_r^{at_1} - 1\right), 0 \leq t \leq t_2 \tag{13}$$

Now, $Q_p(t_2) = 0$

We have

$$t_2 = \frac{1}{a}\log\left[(a\tau+1)e_r^{at_1} - \left(a\tau\lambda + (1-\beta)\lambda\right)e^{at_1} + (1-\beta)\lambda e^{a\tau} - e_r^{a\tau} + 1\right]$$

Using $t_1 = \frac{1}{a}\log\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right)$, we have

$$t_2 = \frac{1}{a}\log\left[\left\{(a\tau+1) - \lambda(a\tau+1-\beta)\right\}\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) + (1-\beta)\lambda e^{a\tau} - e_r^{a\tau} + 1\right] \tag{14}$$

The inventory level $Q_r(t)$ of defective items at time t liquidates the following differential equations:

$$\frac{dQ_p(t)}{dt} = \beta\lambda b e^{at} \tag{15a}$$

with $Q_p(\tau) = 0$, $\tau \leq t \leq t_1$

$$\frac{dQ_p(t)}{dt} = -b e_r^{at} \tag{15b}$$

with $Q_p(0) = Q_p(t_1) = \beta\lambda b e^{at}(t_1 - \tau)$ and $Q_p(t_3) = 0$, $0 \leq t \leq t_3$

From Eq. (15a) and Eq. (15b), we have

$$Q_p(t) = \frac{\beta\lambda b}{a}\left(e^{at_1} - e^{a\tau}\right), \tau \leq t \leq t_1 \tag{16}$$

$$Q_p(t) = \frac{\beta\lambda b}{a}\left(e^{at_1} - e^{a\tau}\right) - \frac{b}{a}\left(1 - e_r^{at}\right), 0 \leq t \leq t_3 \tag{17}$$

Thus, $Q_p(t_3) = 0$

$$t_3 = \frac{1}{a}\log\left[1 - \beta\lambda\left(1 + \frac{a(1-\alpha)R}{\lambda b} - e^{a\tau}\right)\right] \tag{18}$$

Here, $kT = t_1 + t_2 + t_3$

$$kT = \frac{1}{a} \log \left[\frac{\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) \left\{ \begin{array}{l} (a\tau + 1)\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) - \lambda(a\tau + 1 - \beta) \\ \left[\left(1 + \frac{a(1-\alpha)R}{\lambda b}\right) + (1-\beta)\lambda e^{a\tau} - e_r^{a\tau} + 1\right] \end{array} \right\}}{\left\{ 1 - \beta\lambda\left(1 + \frac{a(1-\alpha)R}{\lambda b} - e^{a\tau}\right) \right\}} \right] \tag{19}$$

Thus, the inventory cost for perfect objects is

$$HG_p = h_p \left[\begin{array}{l} \frac{b}{a^2}\left(at_1 e_r^{a\tau} - e_r^{at_1} - a\tau e_r^{a\tau} + a\tau - \lambda(1 - e^{a\tau} + a\tau)\right) + \frac{b\tau}{a}\left(\lambda e^{at_1} - e_r^{at_1} - \lambda e^{a\tau} + e_r^{a\tau}\right) \\ + \frac{(1-\beta)\lambda b}{a^2}\left(e^{at_1} - at_1 e^{a\tau} - e^{a\tau} + \tau e^{a\tau} + at_2 e^{at_1} - at_2 e^{a\tau}\right) + b\left(\lambda e^{at_1} - e_r^{at_1}\right)\tau t_2 \\ + \frac{b}{a}\left(\frac{1}{a}e_r^{at_2} + e_r^{a\tau}t_2 - e_r^{at_1}t_2 - t_2\right) \end{array} \right] \tag{20}$$

Inventory cost of defective objects is

$$HG'_p = h'_p \left[\frac{\beta\lambda b}{a^2}\left(e^{at_1} - e^{a\tau} + a\tau e^{a\tau} - at_1 e^{at_1} + at_r e^{at_1} - at_r e^{a\tau}\right) + \frac{\lambda b}{a^2}\left(e_1^{at_r} - at_r - 1\right) \right]$$
$$+ h_p \left[\begin{array}{l} \frac{b}{a^2}\left(e_r^{at_3} - 1 + \lambda - \lambda e_1^{at_r}\right) + \frac{b}{a}\left(\lambda t_r - t_3\right) + \beta\lambda b e^{at}(t_1 - \tau)(t_2 - t_r) \\ + \frac{\beta\lambda b}{a}\left(e^{at_1} - e^{a\tau}\right)t_3 \end{array} \right] \tag{21}$$

Adding Eq. (20) and Eq. (21), the total inventory cost is

$$TIC = h_p \frac{b}{a^2}\left[\begin{array}{l} a\left(\lambda e^{at_1} - e_r^{at_1} - \lambda e^{a\tau} + e_r^{a\tau}\right)\tau + (1-\beta)\lambda\left(e^{at_1} - e^{a\tau}\right) - \lambda\left(1 - e^{a\tau}\right) \\ -e_r^{at_1} + e_r^{at_2} \end{array} \right]$$
$$+ h'_p \left[\frac{\beta\lambda b}{a^2}\left(e^{at_1} - e^{a\tau} + a\tau e^{a\tau} - at_1 e^{at_1} + at_r e^{at_1} - at_r e^{a\tau}\right) + \frac{\lambda b}{a^2}\left(e_1^{at_r} - at_r - 1\right) \right]$$
$$+ h_p \left[\frac{b}{a^2}\left(e_r^{at_3} - 1 + \lambda - \lambda e_1^{at_r}\right) + \frac{b}{a}\left(\lambda t_r - t_3\right) + \beta\lambda b e^{at}(t_1 - \tau)(t_2 - t_r) + \frac{\beta\lambda b}{a}\left(e^{at_1} - e^{a\tau}\right)t_3 \right]$$

Total expected inventory cost is

$$EIC = E \begin{bmatrix} \left\{ \begin{bmatrix} h_p \dfrac{b}{a^2}\left(e_r^{at_2} - e_r^{at_1} + e_r^{at_3} - 1 + \lambda - \lambda e_1^{at_r}\right) + h'_p \dfrac{\lambda b}{a^2}\left(e_1^{at_r} - at_r - 1\right) \\ + h_p \dfrac{b}{a}\left(\lambda t_r - t_3\right) \end{bmatrix} \int_0^{t_1} dF(\tau) \right\} \\ + h_p \left(\dfrac{b(1-\beta)\lambda}{a^2} + \dfrac{\beta\lambda b t_3}{a} \right)\int_0^{t_1}\left(e^{at_1} - e^{a\tau}\right)dF(\tau) \\ + h_p \beta\lambda b e^{at}\left(t_2 - t_r\right)\int_0^{t_1}\left(t_1 - \tau\right)dF(\tau) + h_p \dfrac{b}{a}\int_0^{t_1}\begin{pmatrix} \lambda e^{at_1} - e_r^{at_1} \\ -\lambda e^{a\tau} + e_r^{a\tau} \end{pmatrix}\tau dF(\tau) \\ + h'_p \dfrac{\beta\lambda b}{a^2}\int_0^{t_1}\begin{pmatrix} e^{at_1} - e^{a\tau} + a\tau e^{a\tau} \\ -at_1 e^{at_1} + at_r e^{at_1} - at_r e^{a\tau} \end{pmatrix}dF(\tau) - h_p \dfrac{\lambda b}{a^2}\int_0^{t_1}\left(1 - e^{a\tau}\right)dF(\tau) \end{bmatrix}$$

Thus, TIC is a function of random variable τ and t_1 depends on random variable α. The expected number of imperfect objects in a manufacturing lot size $(1-\alpha)R$ is as follows,

$$E(N) = \beta\lambda b e^{at}\int_0^{t_1}(t_1 - \tau)dF(\tau) = \beta\theta\lambda b e^{at}\int_0^{t_1}(t_1 - \tau)e^{-\theta\tau}d\tau = \beta\theta\lambda b e^{at}\left(\dfrac{t_1}{\theta} + \dfrac{1}{\theta^2}e^{-\theta t_1} - \dfrac{1}{\theta^2}\right)$$

Expand the function $e^{-\theta t_1}$ up to third term, we have

$$E(N) \approx \dfrac{1}{2}\beta\theta\lambda b e^{at}t_1^2 \tag{22}$$

Total expected rework cost is

$$r_p E(N) = \dfrac{1}{2}r_p \beta\theta\lambda b e^{at}t_1^2 \tag{23}$$

The cost of arrangement is A_p

The income from selling the perfect and imperfect items is

$$\left\{w_p(1-\beta)\lambda b e^{at_1} + \overline{w_s}\beta\lambda b e^{at_1}\right\} = \left\{w_p + \overline{w_s}\dfrac{\beta}{1-\beta}\right\}b\left\{\lambda(1-\beta) - 1 + e_r^{akT}\right\}$$

where $Q = (1 - \beta)\lambda b e^{at}$
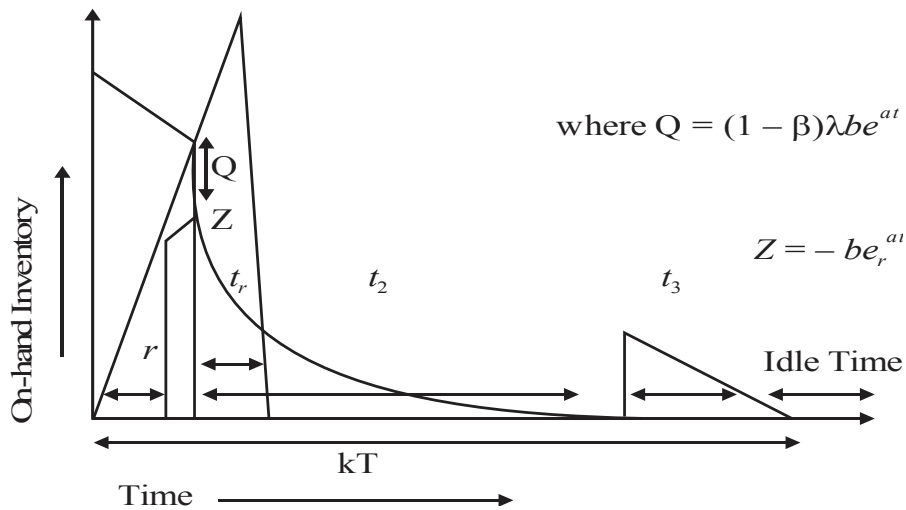
$Z = -be_r^{at}$

Fig. 3. The change in inventory

The cost of screening is $\quad S_p \lambda b e^{at_1} = \dfrac{1}{1-\beta} S_p b \left\{ \lambda(1-\beta) - 1 + e_r^{akT} \right\}$

In this case,

Total expected profit =

$$
E \begin{bmatrix}
\left\{ w_p + \overline{w_s}\,\dfrac{\beta}{1-\beta} \right\} b \left\{ \lambda(1-\beta) - 1 + e_r^{akT} \right\} \displaystyle\int_0^{t_1} dF(\tau) - p_I \left\{ (T-kT) + A_p \right\} \displaystyle\int_0^{t_1} dF(\tau) \\[2mm]
- \left\{ h_p \dfrac{b}{a^2}\left(e_r^{at_2} - e_r^{at_1} + e_r^{at_3} - 1 + \lambda - \lambda e_1^{at_r} \right) + h'_p \dfrac{\lambda b}{a^2}\left(e_1^{at_r} - at_r - 1 \right) + h_p \dfrac{b}{a}\left(\lambda t_r - t_3 \right) \right\} \\[2mm]
\displaystyle\int_0^{t_1} dF(\tau) + h_p \dfrac{\lambda b}{a^2}\int_0^{t_1}\left(1 - e^{a\tau}\right) dF(\tau) - h_p \left(\dfrac{b(1-\beta)\lambda}{a^2} + \dfrac{\beta \lambda b t_3}{a} \right)\int_0^{t_1}\left(e^{at_1} - e^{a\tau} \right) dF(\tau) \\[2mm]
- h_p \beta \lambda b e^{at}\left(t_2 - t_r \right)\displaystyle\int_0^{t_1}\left(t_1 - \tau \right) dF(\tau) - h_p \dfrac{b}{a}\int_0^{t_1}\left(\lambda e^{at_1} - e_r^{at_1} - \lambda e^{a\tau} + e_r^{a\tau} \right)\tau\, dF(\tau) \\[2mm]
- \dfrac{h'_p\,\beta \lambda b}{a^2}\displaystyle\int_0^{t_1}\left\{ (a\tau - at_r - 1)e^{a\tau} + (1 - at_1 + at_r)e^{at_1} \right\} dF(\tau) - r_p \beta \lambda b e^{at}\int_0^{t_1}\left(t_1 - \tau \right) dF(\tau)
\end{bmatrix}
$$

(24)

3.2.2 Case II:

When $t_1 \leq \tau \leq \infty$, the system is in "in-control" state during manufacturing process The leading differential equations are as follows:

$$
\frac{dQ_p(t)}{dt} = \lambda b e^{at} - b e_r^{at}
$$

(25)

with $Q_p(0) = 0$, $0 \leq t \leq t_1$

$$
\frac{dQ_p(t)}{dt} = -b e_r^{at}
$$

(26)

with $Q_p(t'_2) = 0$ and $Q_p(0) = \left(\lambda b e^{at} - b e_r^{at} \right) t_1$, $0 \leq t \leq t'_2$

From Eq. (25) and Eq. (26), we get

$$
Q_p(t) = \left\{ \frac{b}{a}\left(1 - e_r^{at}\right) - \frac{\lambda b}{a}\left(1 - e^{at}\right) \right\}, 0 \leq t \leq t_1
$$

(27)

$$
Q_p(t) = \frac{b}{a}\left(1 - e_r^{at_1}\right) - \frac{\lambda b}{a}\left(1 - e^{at_1}\right) + \frac{b}{a}\left(1 - e_r^{at}\right), 0 \leq t \leq t'_2
$$

(28)

Thus, $Q_p(t'_2) = 0$

$$
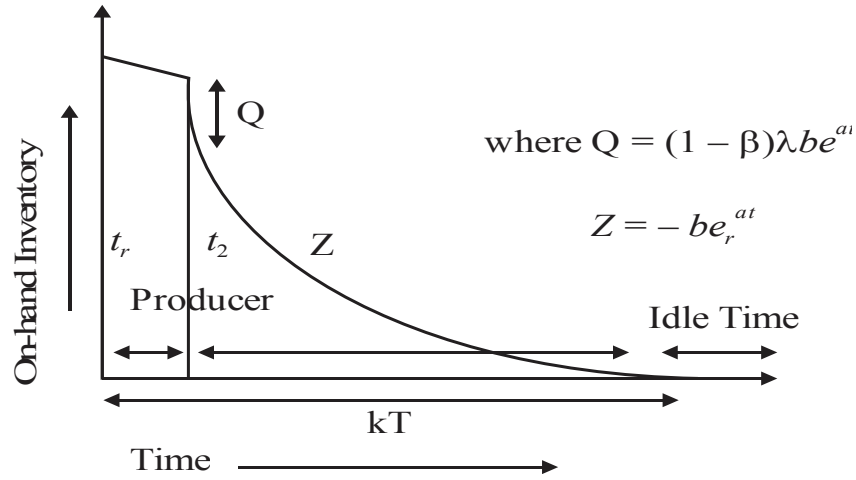t'_2 = \frac{1}{a}\log\left[2 - e_r^{at_1} - \lambda\left(1 - e^{at_1}\right) \right]
$$

(29)

Fig. 4. The change of on-hand inventory

Inventory cost for perfect objects is

$$IC = h_p \frac{b}{a}\left[(1-\lambda)\left(t_1 - \frac{1}{a}e_r^{at_1}\right) + \frac{1}{a}(2+\lambda) + \left\{2 - e_r^{at_1} - \lambda\left(1-e^{at_1}\right)\right\}t'_2 - \frac{1}{a}e_r^{at'_2}\right]$$

Using Eq. (29), we get

$$IC = h_p \frac{b}{a}\left[\begin{array}{l}(1-\lambda)\left(t_1 - \frac{1}{a}e_r^{at_1}\right) + \frac{1}{a}(2+\lambda) \\ + \left\{1 - e_r^{at_1} - \lambda\left(1-e^{at_1}\right)\right\}\frac{1}{a}\log\left[2 - e_r^{at_1} - \lambda\left(1-e^{at_1}\right)\right]\end{array}\right]$$

Here, rework cost is zero.

Total expected profit

$$= E\left[\int_{t_1}^{\infty}\left\{\begin{array}{l}\left\{w_p + \overline{w}_s \frac{\beta}{1-\beta}\right\}b\left\{\lambda(1-\beta)-1+e_r^{akT}\right\} - p_I\left\{(T-kT)+A_p\right\} \\ -h_p \frac{b}{a}\left\{\begin{array}{l}(1-\lambda)\left(t_1 - \frac{1}{a}e_r^{at_1}\right) + \frac{1}{a}(2+\lambda) \\ + \left\{1 - e_r^{at_1} - \lambda\left(1-e^{at_1}\right)\right\}\frac{1}{a}\log\left\{2 - e_r^{at_1} - \lambda\left(1-e^{at_1}\right)\right\}\end{array}\right\}\end{array}\right\}dF(\tau)\right] \tag{30}$$

Adding Eq. (24) and Eq. (30), the total expected profit is

$$
EP = E\left[
\begin{aligned}
&\left\{w_p + \frac{\overline{w_s}\beta}{1-\beta}\right\}b\left\{\lambda(1-\beta)-1+e_r^{akT}\right\}\int_0^{t_1}dF(\tau) - p_I\left\{(T-kT)+A_p\right\}\int_0^{t_1}dF(\tau) \\
&-\left\{\begin{aligned}&\frac{h_p b}{a^2}\left(e_r^{at_2}-e_r^{at_1}+e_r^{at_3}-1+\lambda-\lambda e_1^{at_r}\right)\\&+\frac{h'_p\lambda b}{a^2}\left(e_1^{at_r}-at_r-1\right)+\frac{h_p b}{a}\left(\lambda t_r - t_3\right)\end{aligned}\right\} \\
&\int_0^{t_1}dF(\tau)+\frac{h_p\lambda b}{a^2}\int_0^{t_1}\left(1-e^{a\tau}\right)dF(\tau) \\
&-h_p\left(\frac{b(1-\beta)\lambda}{a^2}+\frac{\beta\lambda b t_3}{a}\right)\int_0^{t_1}\left(e^{at_1}-e^{a\tau}\right)dF(\tau) \\
&-h_p\beta\lambda b e^{at}(t_2-t_r)\int_0^{t_1}(t_1-\tau)dF(\tau) - h_p\frac{b}{a}\int_0^{t_1}\left(\lambda e^{at_1}-e_r^{at_1}-\lambda e^{a\tau}+e_r^{a\tau}\right)\tau dF(\tau) \\
&-\frac{h'_p\beta\lambda b}{a^2}\int_0^{t_1}\left\{\begin{aligned}&(a\tau-at_r-1)e^{a\tau}\\&+(1-at_1+at_r)e^{at_1}\end{aligned}\right\}dF(\tau) - r_p\beta\lambda b e^{at}\int_0^{t_1}(t_1-\tau)dF(\tau)
\end{aligned}
\right]
$$
$$
+E\left[
\int_{t_1}^{\infty}\left\{\begin{aligned}
&\left\{w_p+\overline{w_s}\frac{\beta}{1-\beta}\right\}b\left\{\lambda(1-\beta)-1+e_r^{akT}\right\}-p_I\left\{(T-kT)+A_p\right\}\\
&-h_p\frac{b}{a}\left\{\begin{aligned}&(1-\lambda)\left(t_1-\frac{1}{a}e_r^{at_1}\right)+\frac{1}{a}(2+\lambda)\\&+\left\{1-e_r^{at_1}-\lambda\left(1-e^{at_1}\right)\right\}\frac{1}{a}\log\left\{2-e_r^{at_1}-\lambda\left(1-e^{at_1}\right)\right\}\end{aligned}\right\}
\end{aligned}\right\}dF(\tau)
\right]
\tag{31}
$$

Thus, we have $\int_0^{t_1}(t_1-\tau)^2 dF(\tau) = \theta\int_0^{t_1}(t_1-\tau)^2 e^{-\theta\tau}d\tau = t_1^2 - \frac{2t_1}{\theta}+\frac{2}{\theta^2}\left(1-e^{-\theta t_1}\right)$. Expand the function $e^{-\theta t_1}$ up to fourth term, we have $\int_0^{t_1}(t_1-\tau)^2 dF(\tau) \approx \frac{\theta t_1^3}{3}$. Thus, the average expected profit of the producer is

$$
EPAP = \frac{1}{T}\left[
\begin{aligned}
&\left\{w_p+\frac{\overline{w_s}\beta}{1-\beta}\right\}b\left\{\lambda(1-\beta)-1+e_r^{akT}\right\}\int_0^{t_1}dF(\tau)-p_I\left\{(T-kT)+A_p\right\}\int_0^{t_1}dF(\tau)- \\
&\left\{\begin{aligned}&\frac{h_p b}{a^2}\left(e_r^{at_2}-e_r^{at_1}+e_r^{at_3}-1+\lambda-\lambda e_1^{at_r}\right)+\\&\frac{h'_p\lambda b}{a^2}\left(e_1^{at_r}-at_r-1\right)+\frac{h_p b}{a}\left(\lambda t_r-t_3\right)\end{aligned}\right\}\int_0^{t_1}dF(\tau)+\frac{h_p\lambda b}{a^2}\int_0^{t_1}\left(1-e^{a\tau}\right)dF(\tau) \\
&-h_p\left(\frac{b(1-\beta)\lambda}{a^2}+\frac{\beta\lambda b t_3}{a}\right)\int_0^{t_1}\left(e^{at_1}-e^{a\tau}\right)dF(\tau)-r_p\beta\lambda b e^{at}\int_0^{t_1}(t_1-\tau)dF(\tau)- \\
&h_p\beta\lambda b e^{at}(t_2-t_r)\int_0^{t_1}(t_1-\tau)dF(\tau)-\frac{h_p b}{a}\int_0^{t_1}\left(\lambda e^{at_1}-e_r^{at_1}-\lambda e^{a\tau}+e_r^{a\tau}\right)\tau dF(\tau) \\
&-h'_p\frac{\beta\lambda b}{a^2}\int_0^{t_1}\left(e^{at_1}-e^{a\tau}+a\tau e^{a\tau}-at_1 e^{at_1}+at_r e^{at_1}-at_r e^{a\tau}\right)dF(\tau)
\end{aligned}
\right]
$$
$$
+\frac{1}{T}\left[
\int_{t_1}^{\infty}\left\{\begin{aligned}
&\left\{w_p+\overline{w_s}\frac{\beta}{1-\beta}\right\}b\left\{\lambda(1-\beta)-1+e_r^{akT}\right\}-p_I\left\{(T-kT)+A_p\right\}\\
&-h_p\frac{b}{a}\left\{\begin{aligned}&(1-\lambda)\left(t_1-\frac{1}{a}e_r^{at_1}\right)+\frac{1}{a}(2+\lambda)\\&+\left\{1-e_r^{at_1}-\lambda\left(1-e^{at_1}\right)\right\}\frac{1}{a}\log\left[2-e_r^{at_1}-\lambda\left(1-e^{at_1}\right)\right]\end{aligned}\right\}
\end{aligned}\right\}dF(\tau)
\right]
\tag{32}
$$

3.3. Retailer's specific average profit.

The leading differential equations are

$$\frac{dQ_r(t)}{dt} = b\left(e_r^{at} - e_c^{at}\right) \tag{33}$$

with $Q_r(0) = 0$, $0 \le t \le kT$

and

$$\frac{dQ_r(t)}{dt} = -be_c^{at} \tag{34}$$

with $Q_r(T) = 0$, $kT \le t \le T$

From Eq. (33) and Eq. (34), we have

$$Q_r(t) = \frac{b}{a}\left(e_r^{at} - e_c^{at}\right), 0 \le t \le kT \tag{35}$$

and using $Q_r(kT) = \frac{b}{a}\left(e_r^{akT} - e_c^{akT}\right)$

$$Q_r(t) = \frac{b}{a}\left(e_r^{akT} - e_c^{at}\right), kT \le t \le T \tag{36}$$

Thus, $Q_r(T) = 0 \Rightarrow k = \frac{be_c^{at}}{be_r^{at}} < 1$ as $e_r^{at} > e_c^{at}$

For possibility of the model, $t_1 \le kT < T$ must be gratified. As stated, $kT < T$ grasps as $k < 1$.

Now, we have $t_1 \le kT$

$$t_1 \le \frac{(1-\beta)\lambda be^{at}t_1}{be_r^{at}} \Rightarrow (1-\beta)\lambda e^{at} \ge e_r^{at} \Rightarrow \lambda E(1-\beta)e^{at} \ge e_r^{at} \tag{37}$$
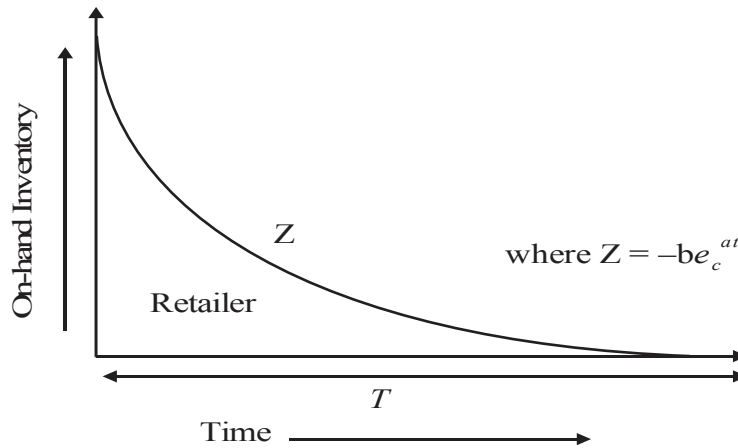

Fig. 5. The change of on-hand inventory

The cost of arrangement is A_r

The income from selling items is $w_r b e_c{}^{at} T$

The cost of buying objects is $w_p b e_c{}^{at} T$.

The cost of inventory is

$$H_r = h_r \left[\int_0^{kT} Q_r(t)\,dt + \int_{kT}^{T} Q_r(t)\,dt \right] = h_r \frac{b}{a^2} \left\{ e_c{}^{akT} - 2e_c{}^{aT} + e_r{}^{akT} (1 + aT - akT) \right\} \tag{38}$$

The retailer's average profit is

$$RAP = \frac{1}{T} \left[w_r b e_c{}^{at} T - A_r - w_p b e_c{}^{at} T - h_r \frac{b}{a^2} \left\{ e_c{}^{akT} - 2e_c{}^{aT} + e_r{}^{akT} (1 + aT - akT) \right\} \right] \tag{39}$$

3.4. Leader-Follower Association

Producer is the leader; supplier and retailer are the followers.

From Eq. (32), using $t_1 = \frac{1}{a} \log\left(1 + \frac{a(1-\alpha)R}{\lambda b} \right)$, $b e_r{}^{akt} = b e_c{}^{at}$ and $T = \frac{1}{ak} \log\left\{ 1 + \frac{a}{b}(1-\alpha)R \right\}$

The producer's expected average profit is

$$
E[PAP] = \left[\frac{ak}{\log\left\{ 1 + \frac{a}{b} E(1-\alpha)R \right\}} \right]
$$

$$
\left[
\begin{aligned}
& \left\{ \lambda b \left(w_p + \beta \overline{w_p} - S_p - C(P) - \frac{h_p \beta \lambda b}{r_p} \right) - A_p \right\} - \frac{h_p a^2 \beta E\left((1-\alpha)^2\right)}{r_p} R^2 \\
& + \left\{ \left(w_p - \frac{h_p}{a^2} \right) + \beta \left(\overline{w_p} - \frac{h_p}{a^2} - \frac{2h_p \lambda b}{r_p} \right) - (S_p + C(P)) \right\} aE(1-\alpha)R \\
& - \left(\frac{b h_p}{a^2} + \frac{h_p}{a} E(1-\alpha)R + \frac{p_I}{ak} \right) \log\left\{ 1 + \frac{a}{b} E(1-\alpha)R \right\} \\
& + \left(\frac{b h_p}{a^2} \lambda + \frac{h_p}{a} E(1-\alpha)R + \frac{h_p \beta \lambda b}{a^2} + \frac{p_I}{a} \right) \log\left(1 + \frac{aE(1-\alpha)R}{\lambda b} \right)
\end{aligned}
\right]
$$

$$E[PAP] = \left[\frac{P}{\log(1+NR)} \right] [M + KSR - HR^2 - (J + LR + G)\log(1+NR) + (F + LR + D + B)\log(1+AR)] \tag{40}$$

where $P = ak$, $N = \frac{a}{b} E(1-\alpha)$, $D = \frac{h_p \beta \lambda b}{a^2}$, $B = \frac{p_I}{a}$, $A = \frac{aE(1-\alpha)}{\lambda b}$, $J = \frac{b h_p}{a^2}$, $F = \frac{b h_p}{a^2} \lambda$

$$M = \left\{ \lambda b \left(w_p + \beta \overline{w_p} - S_p - C(P) - \frac{h_p \beta \lambda b}{r_p} \right) - A_p \right\}, \quad L = \frac{h_p}{a} E(1-\alpha), \quad S = aE(1-\alpha)$$

$$K = \left\{ \left(w_p - \frac{h_p}{a^2} \right) + \beta \left(\overline{w_p} - \frac{h_p}{a^2} - \frac{2h_p \lambda b}{r_p} \right) - \left(S_p + C(P) \right) \right\}, \quad G = \frac{p_I}{ak}, \quad H = \frac{h_p a^2 \beta E \left((1-\alpha)^2 \right)}{r_p}$$

From Eq. (6), using $t_1 = \frac{1}{a} \log \left(1 + \frac{a(1-\alpha)R}{\lambda b} \right)$, $be_r^{akt} = be_c^{at}$ and $T = \frac{1}{ak} \log \left\{ 1 + \frac{a}{b}(1-\alpha)R \right\}$

The supplier's expected average profit is

$$E[SAP] = \left[\frac{ak}{\log \left\{ 1 + \frac{a}{b} E(1-\alpha)R \right\}} \right] \left[\begin{array}{l} E(1-\alpha)Rw_s + E(\alpha)R\overline{w_s} - R(S_c + C_s) - E(1-\alpha)R\frac{h_s}{a} - A_s \\[2mm] + \frac{aE(\alpha)R^2}{r_s} - I_s \frac{1}{ak} \log \left\{ 1 + \frac{a}{b} E(1-\alpha)R \right\} \\[2mm] - \left(\frac{\lambda b}{a} h_s + E(1-\alpha)Rh_s - I_s \right) \frac{1}{a} \log \left(1 + \frac{aE(1-\alpha)R}{\lambda b} \right) \end{array} \right] \tag{41}$$

From Eq. (39), using $t_1 = \frac{1}{a} \log \left(1 + \frac{a(1-\alpha)R}{\lambda b} \right)$, $be_r^{akt} = be_c^{at}$ and $T = \frac{1}{ak} \log \left\{ 1 + \frac{a}{b}(1-\alpha)R \right\}$

The retailer's expected average profit is

$$E[RAP] = \left(w_r - w_p - \frac{h_r}{a}(1-k) \right) \left\{ b + aE(1-\alpha)R \right\} - \frac{akA_r}{\log \left\{ 1 + \frac{a}{b} E(1-\alpha)R \right\}} \tag{42}$$

Solution:
For optimum value of EPAP $[R]$, we have

$$\frac{\partial EPAP}{\partial R} = \frac{P}{\log(1+NR)} \left[\begin{array}{l} KS - 2HR - L\log(1+NR) \\ + (F + D + B + LR)\log A \\ + L\log(1+AR) - (J + LR + G)\log N \end{array} \right]$$

$$- \frac{PN}{\left(\log(1+NR)\right)^2} \left[\begin{array}{l} M + KSR - HR^2 + (F + D + B + LR) \\ \log(1+AR) - (J + LR + G)\log(1+NR) \end{array} \right]$$

We consider $\frac{\partial EPAP}{\partial R} = 0$, which give us the value of R

$$R = \frac{(H + L\log N - L\log A) - \sqrt{\begin{array}{l}(-H - L\log N + L\log A)^2 - (-NH - L\left(\log N\right)^2 + NL\log A) \\ (-NM + KG - F\log N - D\log N - B\log N \\ + K\log A + D\log A + N\log A) \end{array}}}{(-NH - L\left(\log N\right)^2 + NL\log A)}$$

Again, we have the value of

$$\frac{\partial^2 EPAP}{\partial R^2} = -2\frac{P}{\left(\log(1+NR)\right)^3}\begin{bmatrix}\left(\log(1+NR)\right)^2\left(H+L\log N-L\log A\right)+N\log(1+NR)\\[4pt]\begin{pmatrix}KS-2HR-L\log(1+NR)+(F+D+B+LR)\log A\\+L\log(1+AR)-(J+LR+G)\log N\end{pmatrix}\\[4pt]+N^2\begin{pmatrix}-M-KSR+HR^2-(K+D+B+LR)\log(1+AR)\\+(J+LR+G)\log(1+NR)\end{pmatrix}\end{bmatrix}$$

Thus $\dfrac{\partial^2 EPAP}{\partial R^2}<0$ grasps at R, then $EPAP[R]$ is maximum.

3.5. Integrated expected average profit

The integrated expected average profit of supply chain is

$$EIAP = b\left(w_r-w_p-\frac{h_r}{a}(1-k)\right)-I_s-p_I-\frac{bkh_p}{a}+a\left(w_r-w_p-\frac{h_r}{a}(1-k)-\frac{kh_p}{a}\right)E(1-\alpha)R$$

$$+\left[\frac{ak}{\log\left\{1+\frac{a}{b}E(1-\alpha)R\right\}}\right]\begin{bmatrix}\lambda bw_p+\lambda b\beta\overline{w_p}-\lambda bS_p-\lambda bC(P)-A_p-A_s-A_r-\dfrac{h_p\beta\lambda^2 b^2}{r_p}\\[6pt]+\left(\overline{w_s}E(\alpha)-S_c-C_s\right)R+a\left(\dfrac{E(\alpha)}{r_s}-\dfrac{h_p a\beta E\left((1-\alpha)^2\right)}{r_p}\right)R^2\\[6pt]+\left\{aw_p+w_s+a\beta\overline{w_p}-aS_p-aC(P)-\dfrac{h_s}{a}-\dfrac{h_p}{a}-\dfrac{\beta h_p}{a}-\dfrac{2a\beta h_p\lambda b}{r_p}\right\}E(1-\alpha)R\\[6pt]+\left(p_I+I_s+\dfrac{\lambda bh_p}{a}+\dfrac{h_p\beta\lambda b}{a}-\dfrac{\lambda bh_s}{a}+\left(h_p-h_s\right)E(1-\alpha)R\right)\dfrac{1}{a}\log\left(1+\dfrac{aE(1-\alpha)R}{\lambda b}\right)\end{bmatrix}$$

$$EIAP = A_1+B_1C_1R_1+\frac{D_1}{\log\left(1+E_1C_1R_1\right)}\left[\begin{array}{l}F_1+\left(G_1+J_1C_1\right)R_1+H_1R_1^2\\+\left(K_1+L_1C_1R_1\right)M_1\log\left(1+N_1C_1R_1\right)\end{array}\right] \tag{43}$$

Where $A_1 = b\left(w_r-w_p-\dfrac{h_r}{a}(1-k)\right)-I_s-p_I-\dfrac{bkh_p}{a}$, $\qquad B_1 = a\left(w_r-w_p-\dfrac{h_r}{a}(1-k)-\dfrac{kh_p}{a}\right)$

$C_1 = E(1-\alpha)$, $D_1 = ak$, $E_1 = \dfrac{a}{b}$, $F_1 = \lambda bw_p+\lambda b\beta\overline{w_p}-\lambda bS_p-\lambda bC(P)-A_p-A_s-A_r-\dfrac{h_p\beta\lambda^2 b^2}{r_p}$,

$G_1 = \overline{w_s}E(\alpha)-S_c-C_s$, $H_1 = a\left(\dfrac{E(\alpha)}{r_s}-\dfrac{h_p a\beta E\left((1-\alpha)^2\right)}{r_p}\right)$,

$J_1 = \left\{aw_p+w_s+a\beta\overline{w_p}-aS_p-aC(P)-\dfrac{h_s}{a}-\dfrac{h_p}{a}-\dfrac{\beta h_p}{a}-\dfrac{2a\beta h_p\lambda b}{r_p}\right\}$,

$K_1 = p_I+I_s+\dfrac{\lambda bh_p}{a}+\dfrac{h_p\beta\lambda b}{a}-\dfrac{\lambda bh_s}{a}$, $L_1 = h_p-h_s$, $M_1 = \dfrac{1}{a}$, $N_1 = \dfrac{a}{\lambda b}$

Solution: For optimum value of $EIAP[R_1]$, we have

$$\frac{\partial EIAP}{\partial R_1} = B_1 C_1 + \frac{D_1 \left(G_1 + C_1 J_1 + 2 H_1 R_1 + C_1 M_1 \left(K_1 + C_1 L_1 R_1 \right) \log N_1 + C_1 L_1 M_1 \log \left(1 + C_1 N_1 R_1 \right) \right)}{\log \left(1 + E_1 C_1 R_1 \right)}$$

$$- \frac{C_1 D_1 E_1 \left(F_1 + \left(G_1 + C_1 J_1 \right) R_1 + H_1 R_1^{\ 2} + M_1 \left(K_1 + C_1 L_1 R_1 \right) \log \left(1 + C_1 N_1 R_1 \right) \right)}{\left(\log \left(1 + E_1 C_1 R_1 \right) \right)^2}$$

Put $\dfrac{\partial EIAP}{\partial R_1} = 0$, we have

$$R_1 = \frac{-\left(D_1 H_1 + B_1 C_1^{\ 2} \log E_1 + C_1^{\ 2} D_1 L_1 M_1 \log N_1 \right) + \sqrt{\begin{pmatrix} \left(D_1 H_1 + B_1 C_1^{\ 2} \log E_1 + C_1^{\ 2} D_1 L_1 M_1 \log N_1 \right)^2 \\ - \begin{pmatrix} -C_1 D_1 E_1 F_1 + D_1 G_1 + C_1 D_1 J_1 + C_1 D_1 K_1 M_1 \log N_1 \\ -C_1 D_1 K_1 M_1 \log E_1 + \left(B_1 + D_1 L_1 M_1 \right) \log C_1 \end{pmatrix} \\ \left(C_1 D_1 E_1 H_1 + B_1 C_1^{\ 3} \left(\log E_1 \right)^2 + C_1^{\ 3} D_1 E_1 L_1 M_1 \log N_1 \right) \end{pmatrix}}}{\left(C_1 D_1 E_1 H_1 + B_1 C_1^{\ 3} \left(\log E_1 \right)^2 + C_1^{\ 3} D_1 E_1 L_1 M_1 \log N_1 \right)}$$

Again partially differentiate, we have

$$\frac{\partial^2 EIAP}{\partial R_1^{\ 2}} = -2 D_1 \left[-\frac{H_1 + C_1^{\ 2} L_1 M_1 \log N_1}{\log \left(1 + E_1 C_1 R_1 \right)} + \frac{C_1 E_1 \begin{pmatrix} G_1 + C_1 J_1 + 2 H_1 R_1 + C_1 M_1 \log N_1 \left(K_1 + C_1 L_1 R_1 \right) \\ + C_1 L_1 M_1 \log \left(1 + C_1 N_1 R_1 \right) \end{pmatrix}}{\left(\log \left(1 + E_1 C_1 R_1 \right) \right)^2} \right.$$
$$\left. - \frac{C_1^{\ 2} E_1^{\ 2} \left(F_1 + \left(G_1 + C_1 J_1 \right) R_1 + H_1 R_1^{\ 2} + M_1 \left(K_1 + C_1 L_1 R_1 \right) \log \left(1 + C_1 N_1 R_1 \right) \right)}{\left(\log \left(1 + E_1 C_1 R_1 \right) \right)^3} \right]$$

Thus, $\dfrac{\partial^2 EIAP}{\partial R_1^{\ 2}} < 0$ grasps at R_1, and then $EIAP[R_1]$ is maximum.

4. Numerical Example

The following parameters in applicable units are considered as:

$A_s = \$400$, $r_s = 185,000$ units per unit time, $C_s = \$20$ per unit, $S_c = \$0.6$ per unit, $b = 2$ per unit, $k = 0.5$ per unit, $h_s = \$3.5$ per unit per unit time, $I_s = \$30$ per unit time, $a = 0.7$ per unit, $w_s = \$70$ per unit, $\overline{w_s} = \$40$ per unit, $A_p = \$5000$, $r_p = 180,000$ units per unit time, $S_p = \$0.8$ per unit, $h_p = \$4.5$ per unit per unit time, $p_I = \$20$ per unit time, $w_p = \$600$ per unit, $\overline{w_p} = \$400$ per unit, $be_c^{\ at} = 300$ units, $A_r = \$4000$, $h_r = \$6$ per unit per unit time, $w_r = \$620$ per unit, $\gamma = \$0.02$ per unit, $\lambda = \$1.5$ per unit, $L = \$4500$, $f(\alpha) = \dfrac{1}{0.3 - 0.04}$, $0.04 < \alpha < 0.3$, $\beta = 0.05$. The ideal outcome for integrated network is $R = \$110.5$ units, $ERAP = \$3776.7$, $ESAP = \$758.1$, $EPAP = \$7638.1$. Total profit of the supply chain is $\$12172.9$.

5. Sensitivity Analysis

To analysis, how the optimal solution is influenced by the parameters, we derive the sensitivity analysis for all parameters. From the given mathematical model, we obtain the optimum result for a stable segment.

The optimum values of all parameters in the segment increases or decreases by $5\%, -5\%$ and $10\%, -10\%$. The results of total profit are presented in Table 1 as follows,

Table 1
The summary of sensitivity analysis

Parameters	Values	Values $I_s = 0$ and $p_I = 0$	Values +5%	Values +10%	Values −5%	Values −10%
A_s	400	400	420	440	380	360
r_s	185,000	185,000	194,250	203,500	175,750	166,500
C_s	20	20	21	22	19	18
S_c	0.6	0.6	0.63	0.66	0.57	0.54
b	2	2	2.1	2.2	1.9	1.8
k	0.5	0.5	0.53	0.55	0.47	0.45
h_s	3.5	3.5	3.68	3.85	3.32	3.15
I_s	30	0	31.5	33	28.5	27
a	0.7	0.7	0.74	0.77	0.66	0.63
w_s	70	70	73.5	77	66.5	63
$\overline{W}_s$	40	40	42	44	38	36
A_p	5000	5000	5250	5500	4750	4500
r_p	180000	180000	189000	198000	171000	162000
S_p	0.8	0.8	0.84	0.88	0.76	0.72
h_p	4.5	4.5	4.73	4.95	4.27	4.05
p_I	20	0	21	22	19	18
w_p	600	600	630	660	570	540
$\overline{w}_n$	400	400	420	440	380	360
be_c^{at}	300	300	315	330	285	270
A_r	4000	4000	4200	4400	3800	3600
h_r	6	6	6.3	6.6	5.7	5.4
w_r	620	620	651	682	589	558
γ	0.02	0.02	0.021	0.022	0.019	0.018
λ	1.5	1.5	1.58	1.65	1.42	1.35
L	4500	4500	4725	4950	4275	4050
E(α)	0.17	0.17	0.18	0.19	0.16	0.15
β	0.175	0.175	0.18	0.19	0.17	0.16
E($1-\alpha$)	0.83	0.83	0.83	0.91	0.79	0.75
E($1-\alpha$)^2	0.7	0.7	0.74	0.77	0.66	0.63
R	110.5	14.6	108.48	119.89	97.35	99.89
ESAP	758.1	773.8	1051.48	1476.02	359.34	280.57
EPAP	7638.1	7648.6	9708.03	23173.79	1558.17	−22.91
ERAP	3776.7	3776.7	4300.53	4977.69	3085.56	2721.49
Total Profit	12172.9	12199.1	15060.04	29627.5	5003.07	2979.15

6. Conclusion

The main conclusion draws from the sensitivity analysis is as follows: If the percentage of imperfect items β increases in the out-of-control state and the value of reliability parameter θ increases then the optimal ordering size decreases and production rate increases. Hence, the expected average integrated profit decreases. The demand rate has impact on the three layers of the supply chain. The production rate has impact on the profit of producer. The production rate is demand dependent i.e. more sensitive to increases or decreases total profit of the supply chain. The values of other parameters are less sensitive to total profit of the supply chain.

In this paper, we considered three layer supply chain including supplier, producer and retailer. For mathematical model, we considered the demand rate of supplier and retailer was exponential increasing function of time. In addition, the production rate was demand dependent. The supplier supplies the material according to the demand of retailer. We have assumed to have constant product reliability; the cycle time at each stage is equal and stock-out situation at each level is insignificant.

Future research can be accomplished for including stock-out situation, shortages are allowed, multi-supplier and multi-retailer may be considered.

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