



Artículo de investigación original

Fejér type inequalities for m -convex functions

Desigualdad del tipo Fejér para funciones m -convexas

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Abstract

In this paper we present some generalizations of the classical inequalities of Fejér for m -convex functions.

Keywords: Convex, m -convex, Fejér inequalities

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Resumen

En este artículo presentamos algunas generalizaciones de las desigualdades clásicas de Fejér para funciones m -convexas.

Palabras clave: Convexo, m -convexo, desigualdades de Fejér

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1. Introduction

The convexity of a function is a property of a wide variety of uses in several fields besides Mathematics; e.g., in Biology, Economics and particularly in (mathematical) Optimization, among others. The main reason is that if a function f is convex, then any locally optimal solution of an *extremal problem*, modeled by f , is also globally optimal (see e.g. [1]).

In recent years several generalizations and extensions of the classical notion of convex function have been introduced and the theory of inequalities has produced important contributions in that respect. This research deals with some inequalities related to the renowned works, on classical convexity, of Charles Hermite [2], Jaques Hadamard [3] and Lipót Fejér [4]. The inequalities of Hermite-Hadamard and Fejér have been object of intense investigation and have produced many applications. Many proofs of them can be found in the literature (see e.g. [5, 6, 7, 8, 9, 10] and references therein). In this paper we establish some results related with these inequalities for m -convex functions.

Definition 1.1. Let I be a real interval and let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$. If

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

for all $x, y \in I$ and $\alpha \in [0, 1]$ then f is said to be **convex on I** .

The Hermite-Hadamard inequality gives us an estimate of the (integral) mean value of a convex function; more precisely:

Theorem 1.2 (See [3]). *Let f be a convex function on $[a, b]$, with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [4], Fejér gives a generalization of (1.1) as follows:

Theorem 1.3. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and let $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx, \tag{1.2}$$

where $g : [a, b] \rightarrow \mathbb{R}$ is non negative, integrable and symmetric with respect to $(a+b)/2$, that is, $g(a+b-x) = g(x)$.

In 1984, G. Toader defined the notion of m -convex functions as follows:

Definition 1.4 (See [11]). *Let f be a real function defined on the interval $[0, b]$. f is said to be m -convex, for $m \in [0, 1]$, if for all $x, y \in [0, b]$ and $t \in [0, 1]$:*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.3}$$

Remark 1.5. Clearly a 1-convex function is a convex function in the ordinary sense. The 0-convex functions are the "starshaped" functions; that is, those functions that satisfy the inequality $f(tx) \leq tf(x)$, for $t \in [0, 1]$.

It is important to note that for $m \in (0, 1)$ there are continuous and differentiable m -convex functions which are not convex in the classical sense (see [12]).

In [13], S.S. Dragomir and G. Toader demonstrated the following Hermite-Hadamard type inequality:

Theorem 1.6. *Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be an m -convex function, with $m \in (0, 1]$. If $0 \leq a < b < +\infty$ and $f \in L^1[a, b]$ then*

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}. \tag{1.4}$$

In [14] the reader may find some others generalizations of this inequality.

Another result of this type which holds for convex functions is embodied in the following theorem, in [15],

Theorem 1.7. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a m -convex function with $m \in (0, 1]$ and that $0 \leq a < b$. If $f \in L_1[a, b]$, then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)} \int_a^b \left(f(t) + mf\left(\frac{t}{m}\right)\right) dt \\ &\leq \frac{m+1}{4} \left[\frac{f(a)+f(b)}{2} + m \frac{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)}{2} \right]. \end{aligned} \tag{1.5}$$

Remark 1.8. Notice that if we make $m = 1$ in (1.5) we get the left hand side of inequality (1.1); that is:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx.$$

2. Main Results

For the proof of our next result we use techniques similar to those used in Theorem 5 of [16].

Theorem 2.1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an m -convex function, with $m \in (0, 1]$, which is integrable in $[a, b]$, where $a, b \in [0, \infty)$, $a < b$, and let $g : [a, b] \rightarrow \mathbb{R}$ be a non negative and integrable function which is symmetric with respect to $\frac{a+b}{2}$, then

$$\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b \left(\frac{b-x}{b-a}\right)g(x)dx + \frac{m}{2} \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] \int_a^b \left(\frac{x-a}{b-a}\right)g(x)dx. \tag{2.1}$$

Proof. Since g is non negative, integrable on $[a, b]$ and symmetric with respect to $\frac{a+b}{2}$:

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ &= \frac{1}{2} \left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx \right] \\ &= \frac{1}{2} \int_a^b [f(x) + f(a+b-x)]g(x)dx \\ &= \frac{1}{2} \int_a^b \left\{ f\left(a\left(\frac{b-x}{b-a}\right) + b\left(\frac{x-a}{b-a}\right)\right) + f\left(a\left(\frac{x-a}{b-a}\right) + b\left(\frac{b-x}{b-a}\right)\right) \right\} g(x)dx. \end{aligned}$$

Hence, the m -convexity of f implies

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \leq \frac{1}{2} \int_a^b \left\{ \left(\frac{b-x}{b-a}\right)f(a) + m\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right) + m\left(\frac{x-a}{b-a}\right)f\left(\frac{a}{m}\right) + \left(\frac{b-x}{b-a}\right)f(b) \right\} g(x)dx \\ & = \frac{f(a)+f(b)}{2} \int_a^b \left(\frac{b-x}{b-a}\right)g(x)dx + \frac{m}{2} \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] \int_a^b \left(\frac{x-a}{b-a}\right)g(x)dx. \end{aligned}$$

□

Remark 2.2. Notice that if we make $m = 1$ in (2.1) we get the right hand side of inequality (1.2); that is:

$$\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x).$$

In our next result we obtain a bound for the left hand side of inequality (1.2) for m -convex functions.

Theorem 2.3. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an m -convex function, with $m \in (0, 1]$, integrable in $[a, b]$, with $0 \leq a < b$, and let $g : [a, b] \rightarrow \mathbb{R}$ be a non negative and integrable function which is symmetric with respect to $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \frac{1}{2} \int_a^b f(x)g(x)dx + \frac{m}{2} \int_a^b f\left(\frac{x}{m}\right)g(x)dx.$$

Proof. The m -convexity of f implies that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx &= \int_a^b f\left(\frac{a+b-x+x}{2}\right)g(x)dx \\ &\leq \int_a^b \left[\frac{1}{2}f(a+b-x) + \frac{m}{2}f\left(\frac{x}{m}\right) \right] g(x)dx \\ &= \frac{1}{2} \int_a^b f(a+b-x)g(x)dx + \frac{m}{2} \int_a^b f\left(\frac{x}{m}\right)g(x)dx. \end{aligned}$$

now, the hypotheses of g imply that this last expression is equal to

$$\begin{aligned} & \frac{1}{2} \int_a^b f(a+b-x)g(a+b-x)dx + \frac{m}{2} \int_a^b f\left(\frac{x}{m}\right)g(x)dx \\ &= \frac{1}{2} \int_a^b f(x)g(x)dx + \frac{m}{2} \int_a^b f\left(\frac{x}{m}\right)g(x)dx, \end{aligned}$$

which proves the result. □

Remark 2.4. If $m = 1$ in Theorem(2.3) we obtain

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx,$$

which is the left hand side of (1.2).

Now we present a generalization of (1.2). First, we prove the following result which is similar to Lemma 1 in [16].

Lemma 2.5. *If $f : [0, \infty) \rightarrow \mathbb{R}$ is an m -convex function, with $m \in (0, 1]$, then, for all $x \in [a, b] \subset [0, \infty)$ there is $\alpha = \alpha_x \in [0, 1]$ such that*

$$f(a+b-x) \leq m(1-\alpha) \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] + \alpha [f(a) + f(b)] - f(x). \tag{2.2}$$

Proof. Since any $x \in [a, b]$ can be written as

$$x = \alpha a + (1-\alpha)b,$$

for some $\alpha \in [0, 1]$, and $a+b-x = a+b-\alpha a - (1-\alpha)b = (1-\alpha)a + \alpha b$ we have

$$\begin{aligned} & f(a+b-x) \\ &= f((1-\alpha)a + \alpha b) \\ &\leq m(1-\alpha) \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) + \alpha(f(a) + f(b)) - \left[\alpha f(a) + m(1-\alpha)f\left(\frac{b}{m}\right) \right] \\ &\leq m(1-\alpha) \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) + \alpha(f(a) + f(b)) - f(\alpha a + (1-\alpha)b) \\ &= m(1-\alpha) \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) + \alpha(f(a) + f(b)) - f(x). \end{aligned}$$

□

Theorem 2.6. *Under the same hypotheses of Theorem 2.1*

$$\int_a^b f(x)g(x)dx \leq \left\{ \frac{m}{2} \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] + \frac{f(a) + f(b)}{2} \right\} \int_a^b g(x)dx.$$

Proof. By the symmetry of g with respect to $\frac{a+b}{2}$ and Lemma 2.5

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ &= \frac{1}{2} \int_a^b f(a+b-x)g(a+b-x)dx + \frac{1}{2} \int_a^b f(x)g(x)dx \\ &= \frac{1}{2} \int_a^b f(a+b-x)g(x)dx + \frac{1}{2} \int_a^b f(x)g(x)dx \\ &\leq \frac{1}{2} \int_a^b \left[m(1-\alpha_x) \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] + \alpha_x [f(a) + f(b)] - f(x) \right] g(x)dx \\ &\quad + \frac{1}{2} \int_a^b f(x)g(x)dx \\ &\leq \left\{ \frac{m}{2} \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] + \frac{f(a) + f(b)}{2} \right\} \int_a^b g(x)dx. \end{aligned}$$

□

Remark 2.7. Notice that if $m = 1$ in Theorem (2.6) we, indeed, get

$$\int_a^b f(x)g(x)dx \leq (f(a) + f(b)) \int_a^b g(x)dx.$$

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