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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE COUPLED CONDUCTION–RADIATION ENERGY TRANSFER ON DIFFUSE–GRAY SURFACES

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Abstract. This article gives very significant and up–to–date analytical results on the conductive–radiative heat transfer model containing two conducting and opaque materials which are in contact by radiation through a transparent medium bounded by diffuse–gray surfaces. Some properties of the radiative integral operator will be presented. The main emphasis of this work deals also with the question of existence and uniqueness of weak solution for this problem. The existence of weak solution will be proved by showing that our problem is pseudomonotone and coercive. The uniqueness of the solution will be proved using an idea from the analysis of nonlinear heat conduction.

1 Introduction

Radiative heat exchange plays a significant factor in modern technology. It has to be taken into account in general always, when the temperature on a visible surface of the system is high enough, or when other heat transfer mechanisms are not present (like in vacuum, for example). A part from some simple cases such as a convex radiating body with known irradiation from infinity, we have to take into account the radiative heat exchange between different parts of the surface of our system. This leads to a non–local boundary condition on the radiating part of the boundary, (see for example [13]. There, we have shown that the non–local boundary value problem has a maximum principle. Hence, we have proved the existence of a weak solution by assuming the existence of upper and lower solutions. This result is then applied to prove the existence under some hypotheses that guarantee the existence of sub– and supersolutions. In typical industrial applications involving heat radiation the material surfaces are not perfectly black, which implies that part of the radiation hitting the surface is reflected. To simplify the treatment of the reflections we shall in this work assume that the surfaces are diffuse emitters and reflectors (i.e.

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they emit and reflect radiation uniformly to all directions). Another important simplification is the assumption that the surfaces are grey, that is, they emit and absorb all wavelengths in the same manner. This means that we can forget the wavelength spectrum (color) of the radiation and model only the total intensity of the radiated waves. In our previous work on heat radiation, [1, 12, 15, 18, 17] the heat radiation integral equation

$$q_0(x) \;=\; \varepsilon(x)\,\sigma\,T^4(x) \;+\; (1-\varepsilon(x)) \int_\Gamma G(x,y)\,\beta(x,y)\,q_0(y)\,d\Gamma_y$$

has received very much attention. There, we have focused on both theoretical and numerical aspects of this equation. Moreover, the problem of coupling radiation with other heat modes (conduction and convection) was also studied by many authors. Concerning the simplest nontrivial case of conductive body with nonconvex opaque radiating surface, we are aware of the work [7, 20] and our previous work [16, 14]. They all studied some properties of the operators related to the radiative transfer and showed the existence of a weak solution under some restrictions (no enclosed surfaces, limitations to material properties). The basic case has been extended to cover several conductive bodies and time dependent problems [10]. In the case of semitransparent material the analysis has been carried out in one dimensional case with nonreflecting surfaces [7] and in two and three–dimensional with diffusively reflecting surfaces [14]. The main goal of the present work is to study and analyze a model that has not been considered before. Such model is considered as an abstraction of contactless heat transfer in protected environment arising for example in semiconductor applications. This mathematical model describing heat transfer by conduction and radiation will be illustrated in section 2. The main part of this work is to prove the existence and the uniqueness of a weak solution for this problem. The existence of a solution will be proved by showing that our problem is pseudo-monotone and coercive. The uniqueness of the solution will be proved using an idea borrowed from the analysis of nonlinear heat conduction. Throughout this work we will use the following notations:

(i) The duality between L^p_μ and L^q_μ for a Borel measure μ is defined as

$$\langle\, f,\; g \,\rangle_\mu \;=\; \int f\,g\,d\mu \quad , \quad f \,\in\, L^p_\mu \quad \text{and} \quad g \,\in\, L^q_\mu$$

with $1 \leq\; p \;\leq \infty$, p and q are conjugate exponents, that is, $\frac{1}{p} \,+\, \frac{1}{q} \;=\; 1$.

(ii) An operator K is positive if $f \geq 0$ implies $K\,f \geq 0$. We denote the positive and negative parts of a function by either sub–or superscript:

$$f^+ \;=\; f_+ \;=\; \max\;(\,f,\,0\,) \quad \text{and} \quad f^- \;=\; f_- \;=\; \min\;(\,-\,f,\,0\,)\,.$$

(iii) Let Γ be a subset of $\partial\,\Omega$ where local heat transfer occurs and define an operator A through $\langle\, A f,\, g\,\rangle \;=\; \int_\Omega\; a_{ij}\,\partial_i\, f\,\partial_i\, g\;\; dx \;+\; \int_\Gamma\; \xi\;\; |f|^{p-1}\, f\, g\;\, ds,\; p > 1$ The coefficients a_{ij} and $\xi \geq 0$ are bounded. The domain of A is $H^1(\Omega) \bigcap L^{p+1}_\gamma(\,\Gamma\,)$ where the measure γ is the surface measure of Γ weighed with the coefficient ξ. The null space of A is denoted by

$$N(A) \;=\; \{\; f \in H^1(\Omega) \bigcap L^{p+1}_\gamma(\Gamma) \;:\; A f \;=\; 0\,\}\,.$$

(iv) $\{\, a_{ij}\}$ is strictly elliptic, that is, there exists a constant C > 0 such that

$$\langle\, A f,\; f\,\rangle \;\;\geq\;\; C \int_\Omega |\nabla f|^2\; dx \quad \text{for all} \quad f \in H^1(\Omega)\,.$$

2 The mathematical model

Suppose that $\Omega \;=\; \Omega_1 \bigcup \Omega_2 \;\subset\; R^3$ is a union of two disjoint, conductive and opaque bodies surrounded by transparent and non–conductive medium. Moreover, we suppose that the radiative surfaces Γ_1 and Γ_2 are diffuse and grey, that is, the emissivity ε of the surfaces does not depend on the wavelength of the radiation. Under the above assumptions the boundary value problem reads as

$$-\,\nabla\,.\,(\,k\,\nabla\, T\,) \;\;=\;\; g \quad \text{in } \Omega \tag{2.1}$$

$$-\,k\,\frac{\partial T}{\partial n} \;\;=\; \varepsilon\,\sigma\,(\,T^4 - T_0^4\,) \quad \text{on } \Gamma_1 \tag{2.2}$$

$$-\,k\,\frac{\partial T}{\partial n} \;\;=\; q \;\;=\;\; q_0 - q_i \quad \text{on } \Gamma_2 \tag{2.3}$$

where k is the heat conductivity, n is the outward unit normal, g is the given heat generation distribution and q is the radiative heat flux, which is defined as the difference between the outgoing radiation q_0 and the incoming radiation q_i. ε is the emissivity coefficient ($0{=}\varepsilon{<}1$), σ is the Stefan–Boltzman constant which has the value $5.669996 \times 10^{-8}\;\; W/(m^2K^4)$, T is the absolute temperature and T_0 is the effective external radiation temperature. The outgoing radiation q_0 and the incoming radiation q_i are related by the relation

$$q_i \;\;=\;\; K\; q_0 \qquad \text{on} \;\; \Gamma_2. \tag{2.4}$$

Moreover, the outgoing radiation q_0 on Γ_2 is a combination of the emitted and reflected energy [19]. This yields

$$q_0 \;\;=\;\; \varepsilon\,\sigma\, T^4 \;+\; (1-\varepsilon)\, q_i \;\;=\;\; \varepsilon\,\sigma\, T^4 \;+\; (1-\varepsilon)\, K\, q_0 \tag{2.5}$$

The integral operator $K: \; L^{\infty}(\Gamma_2) \rightarrow L^{\infty}(\Gamma_2)$ appearing in (2.4) and (2.5) has the explicit form

$$K\, q_0(x) \;=\; \int_{\Gamma_2} G(x,y)\, \beta(x,y)\, q_0(y)\, d\Gamma_2(y)\,, \quad x \in \Gamma_2 \tag{2.6}$$

where $G(\, x,\, y\,)$ is called the view factor between x and y on Γ_2 and is defined as (see, e.g., [21]).

$$G(x,y) \;=\; \frac{\cos\,\theta_x\;\cos\theta_y}{\pi\,|x-y|^2} \tag{2.7}$$

This can also be written in Cartesian form as

$$G(x,y) \;=\; \frac{[\,n(y)\,.\,(x-y)\,]\;.\;[\,n(x)\,.\,(y-x)\,]}{\pi\,|x-y|^4} \tag{2.8}$$

The Boolean function β appearing in equation (2.6) takes account of the shadow zones. This function, termed the obstructing (shadow) function, is defined as

$$\beta(x\,,\,y) \;=\; \begin{cases} 1 \;\;, & \text{if a point } x \text{ can be seen when} \\ & \text{looking from point } y \\[2ex] 0 \;\;, & \text{otherwise.} \end{cases} \tag{2.9}$$

In the following we recall some properties of the operator K defined in (2.6) and the corresponding kernel $G(x,\,y)$ defined in (2.7)–(2.8). These properties have already been investigated in [18, 17]. Therefore, we will state some of these results without proof unless there is a new approach for the proof.

Lemma 1. *Assume that Γ_2 is the boundary of a convex open set Ω, and assume that Γ_2 is a surface to which the divergence theorem can be applied. Let $x \in \Gamma_2$, and let Γ_2 be smooth in an open neighborhood of Γ_2. Then $G(x,y) \geq 0$ for $y \in \Gamma_2$, and*

$$\int_{\Gamma_2} G(x,y)\, d\,\Gamma_2(y) \;=\; 1 \tag{2.10}$$

Proof. This lemma has been proved in [18], however, we will develop here a different approach to prove it. In [18] we have shown that by choosing a local coordinate system in the point $x \in \Gamma_2$ and using the assumption that $\Gamma_2 \in C^{1,\delta}$ with $\delta \in [\,0\,,\,1)$ is a Ljapunow surface together with the Taylor expansion of y in the local coordinate system, the kernel $G(x,y)$ is a weakly singular kernel of type $|\,x-y\,|^{-2(1-\delta)}$ and hence is integrable. Then we used the Stoke's theorem to show that $\int_{\Gamma_2} G(x,y)\; d\Gamma_2(y){=}1$. In this new approach the positivity of $G(x,y)$ follows from the inequalities $0 \,\leq\, \theta_x$, $\theta_y \,\leq\, \pi/2$ which follow also from the convexity of the region Ω. Let $x \in \Gamma_2$, and let δ be a sufficiently small number. Exclude an δ–neighborhood of x from Ω, and denote the remaining set by Ω_δ: $\Omega_\delta \ = \Omega \setminus \{ \, y : \ |y-x| \leq \delta \, \}$. Let Γ_δ denote the boundary of Ω_δ; and $\widetilde{\Gamma}_\delta$ denote the boundary of $\Omega \backslash \Omega_\delta$, the δ–neighborhood of x that was excluded from Ω. Then

$$\int_{\Gamma_2} G(x,y)\, d\,\Gamma_2(y) \ = \ \int_{\Gamma_\delta} G(x,y)\, d\,\Gamma_2(y) \ + \ \int_{\widetilde{\Gamma}_\delta} G(x,y)\, d\,\Gamma_2(y) \tag{2.11}$$

For a continuously differentiable vector $V(y)$ defined Ω_δ, the divergence theorem states

$$\int_{\Gamma_\delta} V(y)\,.\,n_y\, d\,\Gamma_2(y) \ = \ -\int_{\Omega_\delta} \nabla\,.\,V(y)\, d\,y$$

We apply this with

$$V(y) \ = \ \frac{[\,(y-x)\,.\,n_x\,]}{\pi\,|x-y|^4}\,(x-y)$$

A straightforward computation shows

$$\nabla\,.\,V(y) \ = \ 0 \quad , \quad y \in \Omega_\delta$$

Hence

$$\int_{\Gamma_\delta} G(x,y)\, d\,\Gamma_2(y) \ = \ \int_{\Gamma_\delta} V(y)\,.\,n_y\, d\,\Gamma_2(y) \ = \ 0$$

Decomposing $\widetilde{\Gamma}_\delta$ into two parts:

$$\widetilde{\Gamma}_\delta \ = \ W_\delta \bigcup H_\delta$$

with

$$W_\delta \ = \ \{\, y \in \Gamma_2 \ | \ \ |y-x| \leq \ \delta \,\}$$
$$H_\delta \ = \ \{\, y \in \Omega \ | \ \ |y-x| \ = \ \delta \,\}$$

Then

$$\int_{\widetilde{\Gamma}_\delta} G(x,y)\, d\,\Gamma_2(y) \ = \ \int_{W_\delta} G(x,y)\, d\,\Gamma_2(y) \ + \ \int_{H_\delta} G(x,y)\, d\,\Gamma_2(y) \tag{2.12}$$

We examine separately each of these two right–hand integrals. Since $G(x,y) \leq C$ [18], then we can write

$$0 \leq \int_{W_\delta} G(x,y)\, d\,\Gamma_2(y) \ \leq \ C \int_{W_\delta} d\,\Gamma_2(y) \ = \ \mathrm{O}\,(\delta^2) \tag{2.13}$$

This integral goes to zero as $\delta \to 0$. For the last integral in (2.12), we can simplify $G(x,\, y)$ and estimate the integral. For $y \in H_\delta$, $n_y \;=\; \frac{x-y}{|x-y|}$, $n_y \,.\, \frac{x-y}{|x-y|} \;=\; 1$ Then

$$\int_{H_\delta} G(x,y)\, d\,\Gamma_2(y) \;=\; \int_{H_\delta} \frac{n_x \,.\, (y-x)}{|x-y|^3}\, d\,\Gamma_2(y) \;=\; \frac{1}{C^3}\int_{H_\delta} n_x \,.\, (y-x)\, d\,\Gamma_2(y) \tag{2.14}$$

The set H_δ is approximately a hemisphere of radius δ. We change the variable of integration in the later integral to r, with $y - x \;=\; \delta\, r$, so that $|r| \;=\; 1$. Further, if we re–orient the set in such a manner that the unit normal n_x becomes the unit vector k directed along the positive r_3 − axisin R^3. Then the integral in (2.14) becomes

$$\int_{H_\delta} G(x,y)\, d\,\Gamma_2(y) \;=\; \int_{H_1} \mathrm{k} \,.\, r\; d\Gamma_r \;+\; \mathrm{o}\,(\delta)$$

with $H_1 \;=\; \{\, r \in R^3 \;\mid\; r_3 > 0\}$. This yields

$$\int_{H_\delta} G(x,y)\, d\,\Gamma_2(y) \;=\; 1 \;+\; \mathrm{o}\,(\delta) \tag{2.15}$$

Combining this with (2.11)–(2.13) and taking limits $\delta \to 0$, we obtain (2.10). □

Lemma 2. *Assume Γ_2 is a surface of the class $C^{1,\delta}$ with $\delta \in [\,0,\, 1\,)$. Then for any arbitrary point $x \in \Gamma_2$,*

$$\int_{\Gamma_2} G(x,y)\; \beta(x,y)\; d\,\Gamma_2(y) \quad=\quad 1$$

where $G(x\;,\;y)$ and $\beta(x\;,\;y)$ are given in (2.8) and (2.9) respectively.

Proof. See [17]. □

Lemma 3. *For the integral kernel $G(x\;,\;y)$, it holds that $G(x,y) \geq 0$. The mapping*

$$K:\; L^p(\Gamma_2) \to L^P(\Gamma_2)$$

is compact for $1 \leq p \leq \infty$. Furthermore,

(a) K is symmetric and positive

(b) K1 = 1 and $\|K\| \;=\; 1$ in L^p for $1 \leq p \leq \infty$

(c) The eigenvalue 1 of K is simple.

(d) The spectral radius $\rho(K) \;=\; 1$.

Proof. See [18, 17]. □

Lemma 4. *For $1 \leq p \leq \infty$ and$0 \leq \varepsilon < 1$ the operator $I - (1-\varepsilon)\; K$ from $L^p(\Gamma_2)$ into itself is invertible and this inverse is positive.*

Proof. See [18, 17]. □

3 Variational form

In order to write (2.1)–(2.5) into variational form, we first assume that $T \in L^5(\Gamma_2)$, and solving for q_0 from equation (2.5), we have

$$q \;=\; (I-K)\,q_0 \;=\; (I-K)\;(\,I-(1-\varepsilon)\,K\,)^{-1}\;\varepsilon\,\sigma\,T^4 \;=\; E\,\sigma\,T^4 \tag{3.1}$$

where E is a linear operator from L^p_μ to itself for $1 \le p \le \infty$. Next, we define the mapping A from $H^1(\Omega)\bigcap L^5_\gamma(\Gamma_2)$ to $(\,H^1(\Omega)\bigcap L^5_\gamma(\Gamma_2)\,)^*$ by

$$\langle\, A\,T,\;\psi\,\rangle \;=\; \int_\Omega k\,\nabla T\,.\,\nabla\psi\,dx \;+\; \int_{\Gamma_1} \varepsilon\,\sigma\,|T|^3\,T\,\psi\,ds \tag{3.2}$$

Note that since the Stefan–Boltzmann law is physical only for non–negative value of temperature we can replace T^4 by $|T|^3\,T$ for mathematical convenience. Finally, by setting $d\mu \;=\; \sigma\,ds$, we can write our problem in variational form as

$$\langle\, A\,T,\;\psi\,\rangle \;+\; \int_{\Gamma_2} E\;|T|^3\;T\;\psi\;d\mu \;=\; \langle\;\widetilde{g},\;\psi\;\rangle\,, \quad \forall\,\psi\in X \;=\; H^1(\Omega)\bigcap L^5_\mu\bigcap L^5_\gamma \tag{3.3}$$

where $\widetilde{g}$ now contains also the data term on Γ_1.

Lemma 5. *The operator E is self–adjoint. As a mapping from L^2_μ into itself, E is positive semidefinite with respect to $\langle\;.\;,\;.\;\rangle_\mu$ inner product.*

Proof. The self–adjointness of E is a consequence of (3.1). Let $q \in L^2_\mu$ be arbitrary and denote by $\;q$ the solution of $\;(\,I-(1-\varepsilon)\,K\,)\;q \;=\; \varepsilon\,q$. Then

$$\langle\;q\,,\;E\,q\,\rangle \;=\; \langle\;\varepsilon^{-1}\;(\,I-(1-\varepsilon)\,K\,)\;q\,,\;(I-K)\,q\,\rangle_\mu$$

$$=\;\langle\;q\,,\,(I-K)\,(\varepsilon^{-1}-1)\,(I-K)\,q\,\rangle_\mu \;+\; \langle\;q\,,\,(I-K)\;q\,\rangle_\mu \;\ge\; 0$$

$\square$

Lemma 6. *The operator E can be written as $E = I - F$, where F is self–adjoint positive and $\|F\|_p \;\le\; 1$. Moreover, every nonzero constant is an eigenfunction of F with eigenvalue $\lambda \;=\; 1$.*

Proof. One can write

$$E \;=\; I-F \;=\; I-\;[\,(1-\varepsilon)\,+\,\varepsilon\,K\;(\,I-(1-\varepsilon)\,K\,)^{-1}\;\varepsilon\,] \tag{3.4}$$

where F is self–adjoint. The inverse term in F can be written as

$$(\,I-(1-\varepsilon)\,K\,)^{-1} \;=\; \sum_{i=0}^{\infty}(\,(1-\varepsilon)\,K\,)^i\,.$$

As K is positive, all terms in the series are also positive. This implies that F is positive. Since E is self–adjoint, then we can write

$$F = I - E = I - \varepsilon \left(I - K(1-\varepsilon) \right)^{-1} (I - K) \tag{3.5}$$

Next, we show that $\|F\|_1 \leq 1$ and $\|F\|_\infty \leq 1$. From Riesz–Thorin theorem [3, 4] it follows that $\|F\|_p \leq 1$ for $1 < p < 8$. Since F is positive we have $F \ (1 - q/\ \|q\|_\infty) \geq 0$, for all $q \in L^\infty_\mu$, $q \neq 0$. Hence

$$\|F\|_\infty = \sup \frac{\|F\,q\|}{\|q\|} \leq \|F(1)\|_\infty = \|1\|_\infty = 1$$

as $F(a) = a$ for every constant a. Moreover, self–adjointness implies that

$$\|F\|_1 = \|F^*\|_\infty = \|F\|_\infty \leq 1.$$

□

4 Existence results

In order to prove that the original boundary value problem has a solution, it is sufficient to prove that our problem is pseudo-monotone and coercive [23, 24]. To do that we introduce next the operator $R : X \to X^*$ defined by

$$\begin{aligned} \langle R\,T, \psi \rangle = \langle A\,T, \psi \rangle + \int_{\Gamma_2} E\ |T|^3\, T\, \psi\, d\mu = \langle\, \widetilde{g},\ \psi \,\rangle\,, \\ \forall\, \psi \in X = H^1(\Omega) \bigcap L^5_\mu \bigcap L^5_\gamma \end{aligned} \tag{4.1}$$

Note that the space X is reflexive by the arguments given in [3]. To show that R is pseudo-monotone we consider the following Lemma:

Lemma 7. *The operator $R : X \to X^*$ is pseudo-monotone, that is, $T_i \rightharpoonup T$ weakly in X and $\overline{\lim_{i\to\infty}} \langle\, R\,T_i\,,\, T_i - T \,\rangle \leq 0$, imply that*

$$\langle\, R\,T\,,\ T - \psi \,\rangle \leq \lim_{i\to\infty} \langle\, R\,T_i,\ T_i - \psi \,\rangle \quad \forall\, \psi \in X \tag{4.2}$$

Proof. One can write $E = M - S$ where M is a multiplication operator $(M\,T)\ (x) = m\,(x)\,T(x)$ with $0 \leq m_0 \leq m(x) \leq 1$ and S is a compact operator in $L^{5/4}_\mu$. Since the operator

$$\langle\, \widetilde{A}\,T\,,\ \psi \,\rangle = \langle\, A\,T,\ \psi \,\rangle + \langle\, M\ |T|^3\,T\,,\ \psi \,\rangle_\mu\,, \quad \forall \psi \in X \tag{4.3}$$

is monotone then it is sufficient to prove that the mapping $T \mapsto S\ |T|^3\,T$ is pseudo–monotone in X. Let $T_i \rightharpoonup T$ weakly in X . Then $T_i \rightharpoonup T$ weakly in L^5_μ and $T_i \rightharpoonup T$ weakly in $H^1(\Omega)$. Thus $T_i \to T$ strongly in L^2_μ as the embedding $H^1(\Omega) \subset$

$L^2(\Gamma_2)$ is compact [2, 4]. Consequently, $T_i \to T \;\; \mu - \text{a.e.}$ in Γ_2 and hence also $|T_i|^3\, T_i \to |T|^3 T \;\; \mu - a.e.$ Hence $|T_i|^3\, T_i \rightharpoonup |T_i|^3\, T$ weakly in $L_\mu^{5/4}$ as the sequence $\left\{\, |T_i|^3\, T_i \right\}$ is bounded in $L_\mu^{5/4}$. Finally the compactness of S implies that

$$\langle S\ |T|^3 T,\, T-\psi \rangle - \langle S\ |T_i|^3 T_i,\, T_i-\psi \rangle_\mu \;=\; \langle S\,(\,|T|^3 T - \,|T_i|^3 T_i\,)\,,\, T_i-\psi \rangle_\mu$$

$$-\langle S\ |T|^3 T\,,\; T - T_i \rangle_\mu \;\to\; 0\,, \quad \forall \psi \in X \tag{4.4}$$

□

The coercivity in L_μ^5 can be proved through the following two Lemmas:

Lemma 8. *For $1 \le p \le \infty$ and $T \in L_\mu^5$, it holds $\|F\|_{L_\mu^p} \le 1$ and $\langle E\ |T|^3 T\,,\, T \rangle_\mu \ge 0$.*

Proof. Let $T \in L_\mu^1$ be positive. Then

$$\int F\,T\,d\mu \;=\; \int T\,F^*1\,d\mu \;\le\; \int T\,d\mu$$

Since F is positive, this implies that $\|F\|_{L_\mu^1} \le 1$. On the other hand $F\left(1 - \psi/\,\|\psi\|_{L_\mu^p}\right) \;\ge\; 0$ and thus $\|F\|_{L_\mu^\infty} \le \;\; \|F1\|_{L_\mu^\infty} \;\le\; 1$. Using Riesz interpolation theorem [24] it follows that $\|F\|_{L_\mu^p} \le 1$, $1 \le p \le \infty$. To show the second part of this Lemma we use the Hölder inequality $\langle E\ |T|^3\,,\, T \rangle_\mu \;\ge\; \|T\|_{L_\mu^5}^5 \;-\; \|F\ |T|^3\, T\,\|_{L_\mu^{5/4}}\, \|T\|_{L_\mu^5} \;\ge\; (\,1 - \|F\|\,)\;\|T\|_{L_\mu^5}^5 \;\ge\; 0\,.$ □

Lemma 9. *For $T \in L_\mu^5$, $T \notin N(E)$ implies that $\langle E\ |T|^3\,,\, T \rangle_\mu > 0$.*

Proof. Since F is positive, then we have

$$\langle E\ |T|^3\,,\, T \rangle_\mu \;\ge\; \langle E\,T_+^4\,,\, T_+ \rangle_\mu \,+\, \langle E\,T_-^4\,,\, T_- \rangle_\mu \tag{4.5}$$

Under the assumption that $T \ge 0$ and $\|T\|_{L^5} = 1$, we can use the Riesz interpolation theorem [3, 4] to show that

$$\langle\, F\,T^4,\, T\,\rangle_\mu \langle\, T^4,\, T\,\rangle_\mu \;=\; \|T\|_{L_\mu^5}^5 \;\;\text{if}\;\; T \notin N(E) \tag{4.6}$$

where $N(E)$is the null space of E defined as $N(E) \;=\; \{\, T \in L_\mu^1\,,\; E\,T \;=\; 0\,\}$. As S is compact then it can be expressed as an integral operator [4, 6]. Moreover, one can write FT as

$$F\,T \;=\; (1-m)\,T \;+\; S\,T \;=\; \lim_{\varepsilon \to 0} \int f_\varepsilon(x,y)\,T(y)\,d\mu_y \quad \text{for} \quad f_\varepsilon \;\ge\; 0.$$

Next, we let $p = 5/4$, $p_1 = 6/5$, $p_2 = 2$ and let q, q_1, q_2 be the corresponding conjugate exponents. Further, let $\delta = 9/10$ so that $\frac{1}{p} = \frac{\delta}{p_1} + \frac{1-\delta}{p_2}$. Hence for $T,\ \psi \geq 0$ we can write

$$\int T\ (\int f_\varepsilon \psi\, d\mu\ )\ d\mu = \int T\ (\int f_\varepsilon^{\delta+(1-\delta)}\ \psi^{p\,(\frac{\delta}{p_1}+\frac{1-\delta}{p_2})}\ d\mu\ )\ d\mu$$

Using Hölder inequality we get

$$\leq \int T\ (\int f_\varepsilon\, \psi^{\frac{p}{p_1}}\ d\mu\ )^\delta\ (\int f_\varepsilon\, \psi^{\frac{p}{p_2}}\, d\mu\ )^{1-\delta}\ d\mu$$

$$\leq (\int T^{\frac{q}{q_1}} \int f_\varepsilon\, \psi^{\frac{p}{p_1}}\ d\mu\, d\mu\ )^\delta\ (\int T^{\frac{q}{q_2}} \int f_\varepsilon\, \psi^{\frac{p}{p_2}}\ d\mu\, d\mu\ )^{1-\delta}$$

let $\varepsilon \to 0$ we obtain

$$\langle\ T,\ F\,\psi\ \rangle \leq \langle\ T^{\frac{q}{q_1}},\ F\,\psi^{\frac{p}{p_1}}\ \rangle_\mu^\delta\ \langle\ T^{\frac{q}{q_2}},\ F\,\psi^{\frac{p}{p_2}}\ \rangle_\mu^{1-\delta} \tag{4.7}$$

For $\psi = T^4$ (4.7) yields

$$\langle\ T,\ F\,T^4\ \rangle_\mu \leq \langle\ T^{5/2},\ F\,T^{5/2}\ \rangle_\mu \tag{4.8}$$

Finally, assume $\langle\ T^{5/2},\ F\,T^{5/2}\ \rangle = \|T\|^5_{L^5_\mu}$. Then, letting $\psi = T^{5/2}$ we have $0 = \langle\ \psi,\ \psi - F\,\psi\ \rangle_\mu \geq \|\psi\|^2_{L^2_\mu} - \|F\,\psi\|_{L^2_\mu}\ \|\psi\|_{L^2_\mu}$ so that $\|F\,\psi\|_{L^2_\mu} = \|\psi\|_{L^2_\mu}$. Since

$$\langle\ \psi,\ (I - F^*F\,\psi)\ \rangle_\mu = \langle\ \psi,\ \psi\ \rangle_\mu - \langle\ F\,\psi,\ F\,\psi\ \rangle_\mu = 0$$

we have

$$\|E\,\psi\|^2_{L^2_\mu} = \langle\,\psi\,,\ E^*\,E\,\psi\,\rangle_\mu = \langle\,\psi\,,\ (I-F)\,\psi + (I-F^*)\,\psi - (I-F^*\,F)\,\psi\,\rangle_\mu = 0$$

This implies that $T^{5/2} = \psi \in N(E)$ and hence $T \in N(E)$. Therefore, if $T \notin N(E)$ then inequalities (4.6) and (4.8) are strict . $\square$

Theorem 10. *There exists a constant C such that*

$$\langle\,RT\,,\,T\,\rangle \geq C\ \min\ \left\{\ \|T\|^2_X\ ,\ \|T\|^q_X\ \right\}\quad \forall\,T \in X \tag{4.9}$$

where $q = \max\ \{p+1\,,\,5\,\}$.

Proof. To give a sketch of the proof we follow an idea from [9, 19] and some analysis of the nonlinear heat conduction. Suppose that (4.9) is not true if $\|T\|_X \geq 1$. Then for each integer i there is $\widetilde{T}_i \in X$ such that $\left\|\widetilde{T}_i\right\| \geq 1$ and $\left\|\widetilde{T}_i\right\|_X > i\,\langle\,R\widetilde{T}_i\,,\,\widetilde{T}_i\,\rangle$. The sequence $\widetilde{T}_i = \widetilde{T}_i \Big/ \left\|\widetilde{T}_i\right\|_X$ satisfies $\|\,T_i\|_X = 1$ and

$$\|T_i\|_X^2 \geq i\,C\left\{\int_\Omega |\nabla T_i|^2\,dx + \left\|\widetilde{T}_i\right\|_X^{p-1}\int_{\Gamma_1}|T_i|^{p+1}\,ds + \left\|\widetilde{T}_i\right\|_X^3 \langle E\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu\right\}$$
$$\geq i\,C\left\{\int_\Omega |\nabla T_i|^2\,dx + \int_{\Gamma_1}|T_i|^{p+1}\,ds + \langle E\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu\right\} \tag{4.10}$$

Since $\{T_i\}$ is bounded in X , there is a subsequence T_i and $T \in X$ such that T_i converges weakly to T in the spaces L^5_μ, $H^1(\Omega)$ and $L^{p+1}(\Gamma_1)$. Moreover, $T_i \to T$ strongly in $L^2(\Omega)$, as the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact. Now (4.10) implies that

$$\|T\|_{L^{p+1}(\Gamma_1)} \leq \varliminf_{i\to\infty} \|T_i\|_{L^{P+1}(\Gamma_1)} = \lim_{i\to 0}\|T_i\|_{L^{p+1}(\Gamma_1)} = 0.$$

Thus $T\,|_{\Gamma_1} = 0$ and Radon–Riesz Theorem implies that $T_i \to T$ strongly in $L^{p+1}(\Gamma_1)$. Furthermore, (4.10) implies that

$$\langle E\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu \to 0 \tag{4.11}$$

Since $T_i \rightharpoonup T$ weakly in L^5_μ, we have $|T_i|^3\,T_i \rightharpoonup |T|^3\,T$ weakly in $L^{5/4}_\mu$ and hence

$$\langle S\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu - \langle S\,|T|^3\,T\,,\,T\rangle_\mu \to 0 \tag{4.12}$$

as S is compact (hence also S^* is compact). $T_i \rightharpoonup T$ weakly in L^5_μ implies also $\left\|m^{1/5}\,T\right\|_{L^5_\mu} \leq \varliminf_{i\to\infty}\left\|m^{1/5}\,T_i\right\|_{L^5_\mu}$ and hence from (4.11) and (4.12) we obtain

$$0 = \lim_{i\to\infty}\langle E\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu \geq \varliminf_{i\to\infty}\langle M\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu - \varliminf_{i\to\infty}\langle S\,|T_i|^3\,T_i\,,\,T_i\rangle_\mu$$

$$\geq \langle M\,|T|^3\,T\,,\,T - \langle S\,|T|^3\,T\,,T\rangle_\mu = \langle E\,|T|^3\,T\,,T\rangle_\mu \tag{4.13}$$

Hence from Lemma 9 this implies that $T \in N(E)$. Furthermore, we have

$$\left\|m^{1/5}\,T_i\right\|^5_{L^5_\mu} = \langle E\,|T_i|^3\,T_i\,,T_i\rangle_\mu - \langle S\,|T_i|^3\,T_i\,,T_i\rangle_\mu \to 0 \tag{4.14}$$

Hence $T_i \to 0$ strongly in L^5_μ . Since $T_i \to 0$ strongly also in $H^1(\Omega)$ and $L^{p+1}(\Gamma_1)$, we have $T_i \to 0$ strongly in X. This is a contradiction as $\|T_i\|_X = 1$ for every i. The proof for $\|T\|_X < 1$ is similar. In fact we only need to replace the left–hand side of (4.10) with $\|T\|^q_X$. □

5 Uniqueness of the solution

Theorem 11. *Let T_1 and T_2 be solutions of (3.3), corresponding to the right hand sides g_1 , $g_2 \in X^*$, and suppose that*

$$\langle \, g_1 - g_2 \, , \, \psi \, \rangle \;\; \geq \;\; 0 \, , \qquad \forall \, \psi \;\; \geq \;\; 0 \, , \qquad \psi \in X.$$

Then $T_1 \geq T_2$ $L-a\,.\,e\,.\,in\,\Omega$, $\gamma-a\,.\,e\,.\,on\,\Gamma_1$ and $\mu-a\,.\,e\,.\,in\,\Gamma_2$. Consequently, the solution of (3.3) is unique.

Proof. Before sketching the main ingredients of the proof of the uniqueness of the solution of (3.3) we need to introduce the following notations: For $\varepsilon > 0$ we denote $\Omega_0 \; = \; \{ \, x \in \bar{\Omega} \; : \; T_1(x) < T_2(x) \, \} \;\; \Omega_\varepsilon \; = \; \{ \, x \in \Omega_0 \; : \; T_2(x) \; - \; T_1(x) > \varepsilon \, \}$

$$\psi_\varepsilon \;\; = \;\; \min \;\; \{ \, \varepsilon \, , \, (\, T_2 - T_1 \,)^+ \, \} \, .$$

We will also denote the Lebesque measure in R^n by L . In order to prove this theorem we follow the idea from [8]. We need to show that $\mu(\Omega_0) \; + \; L(\Omega_0) \; + \; \gamma(\Omega_0) \;\; = \;\; 0.$ We argue by contradiction and assume first that $\mu(\Omega_0) > 0$. First Theorem 10 imply that

$$\| \psi_\varepsilon \|^2_{L^5_\mu} \;\; \leq \;\; C \;\; \{ \, \int_\Omega \, a_{ij} \, \partial_i \, \psi_\varepsilon \partial_j \psi_\varepsilon \, dx + (\, \int_{\Gamma_1} \;\; \xi \; | \psi_\varepsilon |^{p+1} \;\; ds \;)^{\frac{2}{p+1}} + (\, \int_{\Gamma_2} \;\; E \, \psi^4_\varepsilon \psi_\varepsilon \; d\mu \;)^{2/5} \; \}$$

The next step is to estimate

$$\int_\Omega \, a_{ij} \, \partial_i \, \psi_\varepsilon \partial_j \psi_\varepsilon \, dx \;\;\; \leq \;\;\; \varepsilon \;\; \| \psi_\varepsilon \|_{L^5_\mu} \; g - f_\varepsilon, \tag{5.1}$$

and

$$(\, \int_{\Gamma_1} \;\; \xi \; | \psi_\varepsilon |^{p+1} \;\; ds \;)^{\frac{2}{p+1}} \;\; + \;\; (\, \int_{\bar{\Omega}} \;\; E \, \psi^4_\varepsilon \psi_\varepsilon \; d\mu \;)^{2/5} \;\; \leq \;\; \varepsilon \;\; \| \psi_\varepsilon \|_{L^5_\mu} \; g_\varepsilon \;\; + \;\; h_\varepsilon \tag{5.2}$$

where $g_\varepsilon \;\; \to \;\; 0$ *as* $\varepsilon \;\; \to \;\; 0$ *and* $h_\varepsilon - f_\varepsilon$ can be ignored when ε is small enough. Finally, these estimates give

$$\mu(\, \Omega_\varepsilon) \;\; \leq \;\; \varepsilon^{-1} \;\; (\, \int_{\Omega_\varepsilon} \;\; \varepsilon^5 \; d\mu \;)^{1/5} \;\; \leq \;\;\; \varepsilon^{-1} \;\; \| \psi_\varepsilon \|_{L^5_\mu} \;\; \leq \;\; g_\varepsilon \;\; \to \;\; 0 \tag{5.3}$$

This leads to a contradiction. Similarly we can prove that $L(\Omega_0) \;\; = \;\; \gamma(\Omega_0) \;\; = \;\; 0.$ In the following we give a sketch for the derivation of (5.1)–(5.3). To derive the estimate (5.1) we can write

$$\begin{aligned} \int_\Omega \, a_{ij} \, \partial_i \, \psi_\varepsilon \partial_j \psi_\varepsilon \, dx \;\; &= \;\; \int_\Omega \, a_{ij} \, \partial_i \, (\, T_2 - T_1 \,) \, \partial_j \, \psi_\varepsilon \, dx \\ &= \;\; \langle \, g_2 - g_1 \, , \, \psi_\varepsilon \, \rangle \;\; - \;\; \int_{\Gamma_1} \;\; \xi \; (\, | T_2 |^{p-1} \, T_2 \;\; - \;\; | T_1 |^{p-1} \, T_1 \;) \;\; \psi_\varepsilon \, ds \\ &\quad + \; \int_{\Gamma_2} \;\; E \;\; (\, | T_1 |^3 \, T_1 \;\; - \;\; | T_2 |^3 \, T_2 \;) \;\; \psi_\varepsilon \, d\mu \quad . \end{aligned} \tag{5.4}$$

The last term in (5.4) can be decomposed as

$$\int_{\Gamma_2} E\,(\,|T_1|^3\,T_1\;-\;|T_2|^3\,T_2\,)\;\psi_\varepsilon\,d\mu\;=\;\int_{\Gamma_2\setminus\,\Omega_0}\;(\,|T_1|^3\,T_1\;-\;|T_2|^3\,T_2\,)\,E^*\,\psi_\varepsilon\,d\mu$$
$$+\int_{\Omega_0\setminus\,\Omega_\varepsilon}\;(\,|T_1|^3\,T_1\;-\;|T_2|^3\,T_2\,)\;E^*\,\psi_\varepsilon\,d\mu$$
$$+\int_{\Omega_\varepsilon}\;(\,|T_1|^3\,T_1\;-\;|T_2|^3\,T_2\,)\;E^*\,\psi_\varepsilon\,d\mu\;.$$

In fact the first term on the right–hand side is negative as $|T_1|^3\,T_1\;-\;|T_2|^3\,T_2\;\geq\;0$ and $E^*\,\psi_\varepsilon\;=\;0\;-\;F^*\,\psi_\varepsilon\;\leq\;0\quad in\;\Gamma_2\backslash\Omega_0$. To investigate the second term we observe

$$|T_2|^3\,T_2\;-\;|T_1|^3\,T_1\;\leq\;(\,T_2-T_1\,)\;Q\;(\,|T_2|\;,\;|T_1|\,)$$

where $Q\,(x,y)\;=\;x^3\;+\;x^2\,y\;+x\,y^2\;+\;y^3$. Then

$$\int_{\Omega_0\setminus\,\Omega_\varepsilon}\;(\,|T_1|^3\,T_1\;-\;|T_2|^3\,T_2\,)\;E^*\,\psi_\varepsilon\,d\mu\;\leq\;\int_{\Omega_0\setminus\,\Omega_\varepsilon}\;(\,|T_2|^3\;T_2\;-\;|T_1|^3\;T_1\,)\;F^*\,\psi_\varepsilon\,d\mu$$
$$\leq\;\int_{\Omega_0\setminus\,\Omega_\varepsilon}\;(\;T_2\;-\;T_1\;)\;\;Q\;\;(\,|T_2|\;,\;|T_1|\,)\;F^*\psi_\varepsilon\,d\mu$$
$$\leq\;\;\varepsilon\;\|\psi_\varepsilon\|_{L^5_\mu}\;g_\varepsilon\quad,$$

where

$$g_\varepsilon\;=\;\|\;F\;(\;Q\;(\;\;|T_2|\;\;,\;\;|T_1|\;)\;\;\|_{L^{5/4}_\mu}\;\rightarrow\;0\quad as\;\;\varepsilon\;\rightarrow\;0$$

Thus

$$\int_\Omega a_{i\,j}\,\partial_i\,\psi_\varepsilon\,\partial_j\,\psi_\varepsilon\;dx\quad\leq\quad\varepsilon\;\|\psi_\varepsilon\|_{L^{5/4}_\mu}\;\;g_\varepsilon-f_\varepsilon$$

where

$$g_\varepsilon\;\;=\;\int_{\Gamma_1}\xi\;(\,|T_2|^{p-1}\,T_2-\,|T_1|^{p-1}\,T_1\,)\;\psi_\varepsilon\;ds+\int_{\Omega_\varepsilon}\;(\,|T_2|^3\,T_2-|T_1|^3\,T_1\,)\;E^*\psi_\varepsilon\;d\mu\;\;.$$

To derive (5.2) we observe that $E^*\,\psi_\varepsilon\;\;=\;\;\varepsilon\;\;-\;\;F^*\,\psi_\varepsilon\;\;\geq\;\;\varepsilon-F^*\varepsilon\;\;\geq\;0\quad in\;\;\Omega_\varepsilon\;.$ Moreover, we can show that

$$\int_{\Gamma_2} E\,\psi_\varepsilon^4\psi_\varepsilon\,d\mu\;\leq\;\int_{\Omega_0\setminus\Omega_\varepsilon}\;\psi_\varepsilon^4\;|E^*\,\psi_\varepsilon|\;\;d\mu\;\;+\;\;\int_{\Omega_\varepsilon}\;\psi_\varepsilon^4\;E^*\,\psi_\varepsilon\,d\mu$$
$$\leq\quad\varepsilon^{5/2}\;\;\|\psi_\varepsilon\|^{5/2}_{L^5_\mu}\;\;g_\varepsilon\;\;+\;\;\varepsilon^4\;\int_{\Omega_\varepsilon}\;E^*\;\psi_\varepsilon\,d\mu$$

where $g_\varepsilon\;\;\rightarrow\;\;0\quad as\quad\;\varepsilon\;\;\rightarrow\;\;0$. Thus we conclude that

$$(\,\int_{\Gamma_1}\;\xi\;|\psi_\varepsilon|^{p+1}\;\;ds\;)^{\frac{2}{p+1}}\;\;+\;\;(\,\int_{\Gamma_2}\;(E\,\psi_\varepsilon^4)\;\psi_\varepsilon\;d\mu\;)^{2/5}\;\;\leq\;\varepsilon\;\|\psi_\varepsilon\|_{L^5_\mu}\;\;g_\varepsilon\;\;+\;\;h_\varepsilon$$

where

$$h_\varepsilon \; = \; (\int_{\Gamma_1} \xi \, |\psi_\varepsilon|^{p+1} \, ds \,)^{\frac{2}{p+1}} \; + \; (\, \varepsilon^4 \int_{\Omega_\varepsilon} E^* \, \psi_\varepsilon \, d\mu \,)^{2/5} \, .$$

Finally, we show that $\mu(\Omega_0) \; + \; L(\Omega_0) \; + \; \gamma(\Omega_0) \; = \; 0$. The steps above imply that

$$\| \psi_\varepsilon \|_{L^5_\mu} \; \leq \; \varepsilon \, g_\varepsilon$$

when ε is small enough. Hence

$$\mu(\Omega_\varepsilon) \; = \; \varepsilon^{-1} \, (\int_{\Omega_\varepsilon} \varepsilon^5 \, d\mu \,)^{1/5} \; \leq \; \varepsilon^{-1} \; \| \psi_\varepsilon \|_{L^5_\mu} \; \leq \; g_\varepsilon \; \to \; 0.$$

This is a contradiction, since also $\mu(\Omega_\varepsilon) \; \to \; \mu(\Omega_0) > 0$. Therefore $\mu(\Omega_0) \; = \; 0$. From this fact it is straight forward to conclude $L(\Omega_0) \; = \; \gamma(\Omega_0) \; = \; 0$. □

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