Equivalence of three-particle scattering formalisms

A. W. Jackura,[1,2,*] S. M. Dawid,[1,2,†] C. Fernández-Ramírez,[3] V. Mathieu,[4] M. Mikhasenko,[5] A. Pilloni,[6,7] S. R. Sharpe,[8] and A. P. Szczepaniak[1,2,9]

(Joint Physics Analysis Center)

[1]*Physics Department, Indiana University, Bloomington, Indiana 47405, USA*
[2]*Center for Exploration of Energy and Matter, Indiana University, Bloomington, Indiana 47403, USA*
[3]*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Ciudad de México 04510, Mexico*
[4]*Departamento de Física Teórica, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[5]*CERN, 1211 Geneva 23, Switzerland*
[6]*European Centre for Theoretical Studies in Nuclear Physics and Related areas (ECT*) and Fondazione Bruno Kessler, Villazzano (Trento) I-38123, Italy*
[7]*INFN Sezione di Genova, Genova I-16146, Italy*
[8]*Physics Department, University of Washington, Seattle, Washington 98195-1560, USA*
[9]*Theory Center, Thomas Jefferson National Accelerator Facility, Newport News, Virginia 23606, USA*

(Received 3 June 2019; published 13 August 2019)

In recent years, different on-shell $\mathbf{3} \to \mathbf{3}$ scattering formalisms have been proposed to be applied to both lattice QCD and infinite-volume scattering processes. We prove that the formulation in the infinite volume presented by Hansen and Sharpe in [M. T. Hansen and S. R. Sharpe, Phys. Rev. D **92**, 114509 (2015).] and subsequently Briceño *et al.* in [R. A. Briceño, M. T. Hansen, and S. R. Sharpe, Phys. Rev. D **95**, 074510 (2017).] can be recovered from the B-matrix representation, derived on the basis of S-matrix unitarity, presented by Mai *et al.* in [M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak, Eur. Phys. J. A **53**, 177 (2017).] and Jackura *et al.* in [A. Jackura, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, J. Nys, A. Pilloni, K. Saldaña, N. Sherrill, and A. P. Szczepaniak (JPAC Collaboration), Eur. Phys. J. C **79**, 56 (2019).] Therefore, both formalisms in the infinite volume are equivalent and the physical content is identical. Additionally, the Faddeev equations are recovered in the nonrelativistic limit of both representations.

DOI: 10.1103/PhysRevD.100.034508

I. INTRODUCTION

Considerable progress has been achieved recently in determination of the hadron spectrum from first principles QCD [1–8]. Comparison of experimental data or lattice results with theoretical models involves analysis of partial wave amplitudes in which resonances appear as pole singularities in the complex energy and/or angular momentum planes [9]. Thus, a proper description of resonances requires knowledge of analytic properties of the scattering amplitude. Specifically, the determination of the hadron spectrum from lattice calculations is done using a *quantization condition* [10], which relates discrete energy levels in the finite volume to the infinite volume, partial waves evaluated at real energy values and later analytically continued to the complex energy plane. The quantization condition has been extensively studied for systems with strong two-particle interactions (see, e.g., Ref. [11] and references therein). However, most of resonances of current interest decay to three and more particles.

Quantization conditions for three hadrons have been derived by various groups using different approaches [12–23], as recently reviewed in Ref. [24]. If differences exist between formalisms, this could indicate that important physical content is missing, and that results based on them will lead to unknown systematic errors. Therefore, it is important to unify our understanding of these approaches and establish relationships between all formalisms. In addition to quantization conditions, analytic representations of the infinite volume $\mathbf{3} \to \mathbf{3}$ amplitudes are required to be

*ajackura@iu.edu
†sdawid@iu.edu

able to identify pole positions of resonances. In this context we discuss two seemingly different approaches and demonstrate their equivalence.

The first approach is referred to as the B-matrix representation, and was studied in Refs. [25,26]. Motivated by unitarity of the S matrix, the B matrix refers to a kernel in a linear integral equation for an elastic $\mathbf{3} \to \mathbf{3}$ connected amplitude. The B matrix contains both the known long-range one-particle exchange (OPE) contributions and any short-range interactions. The latter play a similar role to the K matrix in $\mathbf{2} \to \mathbf{2}$ scattering amplitudes [27]. Aspects of its analytic properties were studied in Ref. [26], showing how, besides the unitarity branch point, there are other singularities near the physical region generated by the one-particle exchange, e.g., triangle singularities. Applying the B-matrix formalism in a finite volume leads to the quantization condition of Ref. [17].

The alternative approach was first derived in Ref. [12], and subsequently generalized to allow for $\mathbf{2} \leftrightarrow \mathbf{3}$ transitions in Ref. [13]. Hereafter we refer to it as the HS-BHS approach (for the authors initials). It is a bottom-up construction of the $\mathbf{3} \to \mathbf{3}$ amplitude starting from a generic, relativistic effective field theory in a finite volume [14]. In Ref. [12], the corresponding infinite-volume limit of the HS-BHS formalism was derived explicitly, providing an expression for the $\mathbf{3} \to \mathbf{3}$ scattering amplitude in terms of a $\mathbf{3} \to \mathbf{3}$ analog of the K matrix, referred to here as $\mathcal{K}_{\mathrm{df}}$.[1] This HS-BHS representation is written in terms of two integral equations, the first summing one-particle exchanges between $\mathbf{2} \to \mathbf{2}$ subprocesses, and the second involving all orders in $\mathcal{K}_{\mathrm{df}}$. Since this approach is based on Feynman diagrams, one expects that the result is consistent with unitarity, and, indeed, very recently this has been shown explicitly [28].

In the HS-BHS representation, the kernel $\mathcal{K}_{\mathrm{df}}$ appears to play a similar role to that of the short-range part of the B matrix, but it is actually quite different. It is the main purpose of this work to show that, nevertheless, the two representations are equivalent. Specifically, we derive an integral equation relating the R matrix of Ref. [26] and the $\mathbf{3} \to \mathbf{3}$ K-matrix of Ref. [29]. Furthermore, we show that the reason for the superficial difference lies in the organization of the short-range rescattering effects and difference in the order in which symmetrization of the amplitude is applied.

The paper is organized as follows. Section II summarizes definitions of on-shell $\mathbf{3} \to \mathbf{3}$ amplitudes and the relevant kinematic variables. Section III reviews the B-matrix and HS-BHS on-shell representations for the $\mathbf{3} \to \mathbf{3}$ amplitude. In Sec. IV, we derive the relationship between these two representations, proving their equivalence. In Sec. V we show that in the nonrelativistic limit the B matrix can be reduced to the Faddeev equations. Our findings and outlook are summarized in Sec. VI. We include three technical appendixes. Appendix A reviews the unitarity relation for $\mathbf{3} \to \mathbf{3}$ amplitudes, and Appendix B shows how to rewrite the B-matrix representation in a form analogous to that of the HS-BHS representation, which is used in the demonstration of Sec. IV. Finally, Appendix C proves a crucial relation discussed in Sec. IV.

[1]This quantity is denoted $\mathcal{K}_{\mathrm{df},3}$ in Ref. [12].

II. $\mathbf{3} \to \mathbf{3}$ AMPLITUDES

We consider the elastic scattering of three spinless identical particles of mass m, e.g., $3\pi^+ \to 3\pi^+$ scattering. Note that Ref. [26] considered distinguishable particles, while here we consider identical particles to compare with Refs. [12,13]. Internal symmetries such as isospin are not considered, but can be included in a straightforward manner. The initial and final three-particle state have a total energy momentum $P = (E, \mathbf{P})$ and $P' = (E', \mathbf{P}')$, respectively. This exemplifies a convention we use throughout, namely, that primed (unprimed) variables denote quantities in the final (initial) state. Total energy momentum is conserved, as is the three-particle invariant mass squared,

$$s \equiv P^2 = E^2 - \mathbf{P}^2, \tag{1}$$

which lies in the range $(3m)^2 \le s < s_{\mathrm{inel}}$, where s_{inel} the first inelastic threshold. It is convenient to split the three-particle kinematics into a pair[2] and a spectator. The spectator is a single particle that has energy momentum $p = (\omega_{\mathbf{p}}, \mathbf{p})$, where $\omega_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$ is the on-shell energy. The pair then consists of the other two particles with total energy momentum given by

$$P_{\mathbf{p}} \equiv P - p = (E_{\mathbf{p}}, \mathbf{P}_{\mathbf{p}}) = (E - \omega_{\mathbf{p}}, \mathbf{P} - \mathbf{p}), \tag{2}$$

where the subscript labels a momentum of the spectator associated with the pair. The invariant mass squared of the pair is

$$\sigma_{\mathbf{p}} \equiv P_{\mathbf{p}}^2 = (P - p)^2 = (E - \omega_{\mathbf{p}})^2 - (\mathbf{P} - \mathbf{p})^2, \tag{3}$$

for which the physical region is $(2m)^2 \le \sigma_{\mathbf{p}} \le (\sqrt{s} - m)^2$. In the helicity frame of the two-particle subsystem, i.e., the pair rest frame where $\mathbf{P}^\star - \mathbf{p}^\star = \mathbf{0}$, the three momenta of particles inside the pair are $\mathbf{q}_{\mathbf{p}}^\star$ and $-\mathbf{q}_{\mathbf{p}}^\star$, respectively. In this frame, the spectator defines the $-z$ axis, and the y axis is perpendicular to the plane formed by the three particles. Kinematic quantities without a $\star$ are taken to be in the total CMF, i.e., where $\mathbf{P} = \mathbf{0}$. Figure 1 illustrates the momenta of

[2]Other commonly used terms for the pairs found in the literature are *dimers* and *isobars*. More precisely, however, isobars refer to partial wave amplitudes of the $\mathbf{2} \to \mathbf{2}$ subsystem in a definite partial wave with only the unitarity branch cut [26]. We refrain from this terminology in an attempt to unify different approaches and avoid confusion.

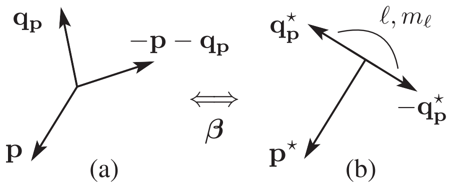


FIG. 1. A three-particle state in the (a) center of momentum frame (CMF) with fixed total momentum $\mathbf{P}=\mathbf{0}$ and (b) in the helicity frame at fixed $\mathbf{P}^\star - \mathbf{p}^\star = \mathbf{0}$. Standard Lorentz transformation with the boost $\boldsymbol{\beta} = -(\mathbf{P}-\mathbf{p})/(E-\omega_\mathbf{p})$ transforms the system from the total CMF to the pair rest frame.

the three particles in these frames. The magnitude of the spectator momentum in the CMF is given by

$$p \equiv |\mathbf{p}| = \frac{1}{2s}\lambda^{1/2}(s, m^2, \sigma_\mathbf{p}), \tag{4}$$

where $\lambda(x,y,z) = x^2+y^2+z^2-2(xy+yz+zx)$ is the Källén triangle function, while the relative momentum of the pair in the helicity frame is given by

$$q^\star_\mathbf{p} = \frac{1}{2\sqrt{\sigma_\mathbf{p}}}\lambda^{1/2}(\sigma_\mathbf{p}, m^2, m^2) = \frac{1}{2}\sqrt{\sigma_\mathbf{p} - 4m^2}. \tag{5}$$

The final state variables have similar expressions with $\mathbf{p}$ replaced by $\mathbf{p}'$.

The elastic $\mathbf{3}\to\mathbf{3}$ scattering amplitude, $\mathcal{M}$, is a Lorentz scalar that depends on eight kinematic variables, and is defined via

$$\langle \text{out}|T|\text{in}\rangle = (2\pi)^4\delta^{(4)}(P'-P)\mathcal{M}, \tag{6}$$

where the T matrix is as usual given by $S = \mathbb{1} + iT$. The Dirac δ-function ensures total energy-momentum conservation. Because of Bose statistics, the amplitude is symmetric under interchange of any pair of particles in the initial or final state. In the following, we express $\mathcal{M}$ in terms of an unsymmetrized amplitude $[\mathcal{M}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_{\ell'};\ell m_\ell}$, which is expressed in the mixed $\mathbf{p}\ell m_\ell$-basis; i.e., it depends on the spectator momenta, $\mathbf{p}$, and the angular momentum of the pair, (ℓ, m_ℓ). The fully symmetric amplitude is then obtained by replacing the dependence on spin[3] by that of the corresponding spherical angles (through multiplication by spherical harmonics[4]), and by symmetrizing with respect to particle permutations, an operation that we denote by $\mathcal{S}$,

$$\mathcal{M} = \mathcal{S}\left\{4\pi \sum_{\substack{\ell', m_{\ell'}\\ \ell, m_\ell}} Y_{\ell' m_{\ell'}}(\hat{\mathbf{q}}^\star_{\mathbf{p}'})[\mathcal{M}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_{\ell'};\ell m_\ell} Y^*_{\ell m_\ell}(\hat{\mathbf{q}}^\star_\mathbf{p})\right\}. \tag{7}$$

[3] By spin, we mean the angular momentum of the pair.

[4] Note that Ref. [12] defines their partial wave expansion with the complex conjugate on the final state spherical harmonic.

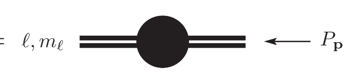
The unsymmetrized amplitudes are infinite-dimensional matrices in the (ℓ, m_ℓ)-space. Note that since the particles are identical, due to Bose symmetry, all odd-ℓ amplitudes must be 0. We often leave the indices implicit and consider amplitudes as matrices in the ℓm_ℓ-space. The $\mathbf{p}$-dependence is left explicit unless otherwise noted. The isobar representation of Ref. [26] is identical to the symmetrization operation in Eq. (7). However, unlike in Ref. [26], here we do not truncate the spin of the pair to some maximum value, and instead work formally with infinite-dimensional matrices. In practice, one must truncate the partial waves, in which the resummation strategy presented in Eq. (17) of Ref. [26] can be used to recover cross channel effects.

The scattering amplitude $\mathcal{M}$ contains disconnected and connected contributions. The disconnected terms in $\mathcal{M}$, denoted hereafter by $\mathcal{F}$,[5] are associated with $\mathbf{2}\to\mathbf{2}$ process in which the spectator particle does not participate,

$$\mathcal{M}_{\mathbf{p}'\mathbf{p}} = \delta_{\mathbf{p}'\mathbf{p}}\mathcal{F}_\mathbf{p} + \mathcal{A}_{\mathbf{p}'\mathbf{p}}, \tag{8}$$

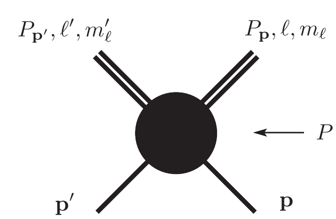
with the spectator momentum conserving δ-function, $\delta_{\mathbf{p}'\mathbf{p}} \equiv (2\pi)^3 2\omega_\mathbf{p}\delta^{(3)}(\mathbf{p}'-\mathbf{p})$, written explicitly in front of $\mathcal{F}_\mathbf{p}$. The amplitude $\mathcal{F}_\mathbf{p}$ is diagonal in the spin variables, and depends solely on the single, scalar variable, the pair's invariant mass, $\sigma_\mathbf{p} = \sigma_{\mathbf{p}'}$,

$$[\mathcal{F}_\mathbf{p}]_{\ell' m'_\ell;\ell m_\ell} = \delta_{\ell'\ell}\delta_{m'_\ell m_\ell}\mathcal{F}_\ell(\sigma_\mathbf{p}) \tag{9}$$

$=$ 

The second term, $\mathcal{A}_{\mathbf{p}'\mathbf{p}}$ in Eq. (8), is the connected $\mathbf{3}\to\mathbf{3}$ amplitude and it contains off-diagonal contributions in spin indices,

$$[\mathcal{A}_{\mathbf{p}'\mathbf{p}}]_{\ell' m'_\ell;\ell m_\ell} = \mathcal{A}_{\ell' m'_\ell;\ell m_\ell}(\mathbf{p}', s, \mathbf{p}) \tag{10}$$

$=$ 

[5] For convenience, we collect here the correspondences between our amplitude definitions and those of Ref. [12]. The $\mathbf{2}\to\mathbf{2}$ amplitude is $\mathcal{F} = \mathcal{M}_2$ (Ref. [12]), the connected $\mathbf{3}\to\mathbf{3}$ amplitude is $\mathcal{A} = \mathcal{M}_3^{(u,u)}$ (Ref. [12]), and (as already noted above) $\mathcal{K}_{\text{df}} = \mathcal{K}_{\text{df},3}$ (Ref. [12]). In addition, the ladder series encountered below is $\mathcal{D} = \mathcal{D}^{(u,u)}$ (Ref. [12]).

In addition to its implicit dependence on ℓ, m_ℓ, ℓ', and m'_ℓ, $\mathcal{A}_{\mathbf{p}'\mathbf{p}}$ depends on the remaining four independent variables. Convenient choices are s, the initial and final pair invariant mass squares, $\sigma_{\mathbf{p}}$ and $\sigma_{\mathbf{p}'}$, and the scattering angle between spectators in the CMF, $\Theta_{\mathbf{p}'\mathbf{p}}$, defined as $\cos\Theta_{\mathbf{p}'\mathbf{p}} \equiv \hat{\mathbf{p}}' \cdot \hat{\mathbf{p}} = \hat{\mathbf{P}}_{\mathbf{p}'} \cdot \hat{\mathbf{P}}_{\mathbf{p}}$.

III. ON-SHELL REPRESENTATIONS

Our interest is in constructing on-shell representations for the connected $\mathbf{3} \to \mathbf{3}$ scattering amplitude. Here we review the relevant features of the B-matrix representation discussed in Ref. [26] and the HS-BHS representation of Ref. [12].

A. B-matrix representation

As discussed in Refs. [25,26], the B matrix is an on-shell representation for the connected $\mathbf{3} \to \mathbf{3}$ amplitude that was constructed to satisfy elastic $\mathbf{3} \to \mathbf{3}$ unitarity. In the $p\ell m_\ell$-basis, the B-matrix representation leads to the integral equation

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\mathcal{B}_{\mathbf{p}'\mathbf{p}}\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\mathcal{B}_{\mathbf{p}'\mathbf{k}}\mathcal{A}_{\mathbf{k}\mathbf{p}}, \tag{11}$$

where $\mathcal{B}_{\mathbf{p}'\mathbf{p}} = \mathcal{G}_{\mathbf{p}'\mathbf{p}} + \mathcal{R}_{\mathbf{p}'\mathbf{p}}$ is the B-matrix driving term, with $\mathcal{G}_{\mathbf{p}'\mathbf{p}}$ being the OPE contribution[6] and $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ a real function called the R matrix. Figure 2 shows a diagrammatic representation of Eq. (11). By construction, Eq. (11) satisfies the $\mathbf{3} \to \mathbf{3}$ unitarity relation given that $\mathcal{F}_{\mathbf{p}}$ is known, as demonstrated in Appendix A. Equation (11) is an infinite-dimensional matrix equation in (ℓ, m_ℓ)-space, and the integration over the spectator momenta includes the measure,

$$\int_{\mathbf{k}} \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}. \tag{12}$$

The integration ranges over all momenta, or equivalently in $-\infty \le \sigma_{\mathbf{k}} \le (\sqrt{s} - m)^2$ and over the entire solid angle of the spectator. The $|\mathbf{k}| \to \infty$ ($\sigma_{\mathbf{k}} \to -\infty$) limit is divergent and needs to be regulated. The preferred option is to restrict the integration region to $4m^2 \le \sigma_{\mathbf{k}} \le (\sqrt{s} - m)^2$, which is the only domain of $\sigma_{\mathbf{k}}$ that is actually restricted by $\mathbf{3} \to \mathbf{3}$ unitarity [26]. Beyond this region, one deals with unphysical (off-mass shell) amplitudes, which depend on unknown parameters, e.g., subtraction constants.

[6]In Ref. [26] we denoted the OPE contribution by the symbol $\mathcal{E}$, while here we use $\mathcal{G}$ to provide a closer connection to the notation of Ref. [12].

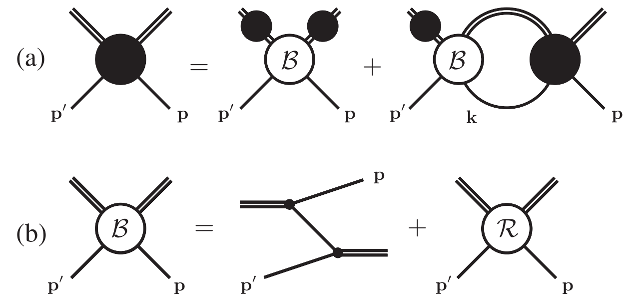
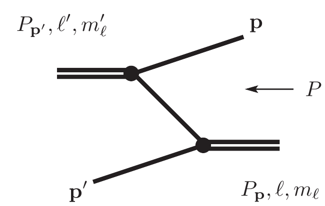


FIG. 2. Diagrammatic representation of (a) the B-matrix representation for the on-shell amplitude, Eq. (11), and (b) the B matrix that is composed of the OPE $\mathcal{G}_{\mathbf{p}'\mathbf{p}}$, Eq. (13), and the R matrix.

The OPE amplitude is given by

$$[\mathcal{G}_{\mathbf{p}'\mathbf{p}}]_{\ell' m'_\ell;\ell m_\ell} = \left(\frac{p^\star_{\mathbf{p}'}}{q^\star_{\mathbf{p}'}}\right)^{\ell'} \frac{4\pi\, Y^*_{\ell' m'_\ell}(\hat{\mathbf{p}}^\star_{\mathbf{p}'}) Y_{\ell m_\ell}(\hat{\mathbf{p}}'^\star_{\mathbf{p}})}{m^2 - (P_{\mathbf{p}} - p')^2 - i\epsilon} \left(\frac{p'^\star_{\mathbf{p}}}{q^\star_{\mathbf{p}}}\right)^{\ell}$$

$$=$$



(13)

where $\hat{\mathbf{p}}^\star_{\mathbf{p}'}$ is the direction of momentum of the initial state spectator in the final state pair rest frame. Similarly, $\hat{\mathbf{p}}^\star_{\mathbf{p}'}$ is the orientation of the final state spectator in the initial state pair rest frame. The magnitudes of these momenta are

$$p^\star_{\mathbf{p}'} = \frac{1}{2\sqrt{\sigma_{\mathbf{p}'}}}\lambda^{1/2}((P_{\mathbf{p}'} - p)^2, \sigma_{\mathbf{p}'}, m^2),$$

$$p'^\star_{\mathbf{p}} = \frac{1}{2\sqrt{\sigma_{\mathbf{p}}}}\lambda^{1/2}((P_{\mathbf{p}} - p')^2, \sigma_{\mathbf{p}}, m^2). \tag{14}$$

Note that energy-momentum conservation gives $P_{\mathbf{p}} - p' = P_{\mathbf{p}'} - p$. The normalization of the barrier factors is chosen such that they are equal to 1 when the exchanged particle is on its mass shell, $(P_{\mathbf{p}} - p')^2 = m^2$.

Our definition of $\mathcal{G}$ differs from the corresponding quantity in Ref. [12], denoted $\mathcal{G}^\infty$, in three ways. First, there is a difference in overall sign. We find the choice in Eq. (13) more convenient since it has a positive imaginary part, which avoids several minus signs in expressions. Second, $\mathcal{G}^\infty$ contains a cutoff function, which serves to cut off the integrals over spectator momenta, which in Ref. [12] run over all values. Third, the form given in Ref. [12] has the nonrelativistic form of the pole in the denominator, in contrast to the relativistic form in Eq. (13). However, in recent applications of the BH + BHS formalism, e.g., in Refs. [13,30], the relativistic form is used. We also note that the barrier factors in $\mathcal{G}$ are not required from unitarity,

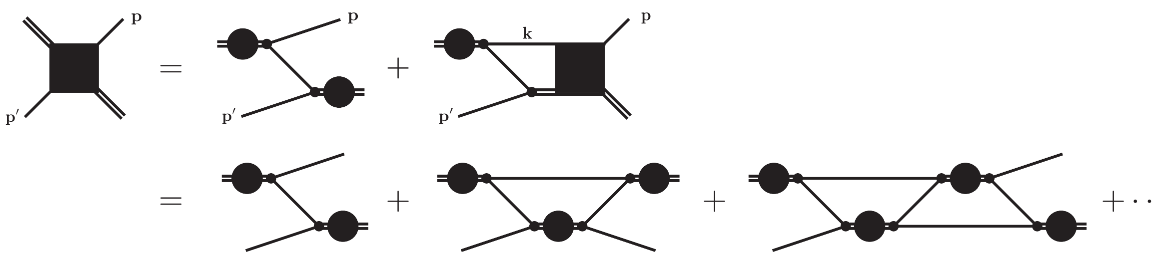


FIG. 3. Diagrammatic representation for the ladder series generated by particle exchanges, Eq. (15), where the black box represents $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$.

but are included so as to match those in $\mathcal{G}^{\infty}$, where these factors are included since they are needed in the finite-volume analysis.

As with the K-matrix representation for $\mathbf{2} \to \mathbf{2}$ scattering (see Appendix A), the R matrix is a real function that represents the dynamical content of the three-particle system, e.g., the short-range forces between pions in elastic 3π scattering. Included in this term are virtual exchange processes giving rise to left-hand cuts, and higher multi-particle thresholds, e.g., 5π production, which are off shell in the kinematic domain of elastic 3π scattering. In principle, given a specific theory, $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ can be computed. Alternatively, given some data, e.g., from lattice QCD calculations, $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ can be determined via the quantization condition of Ref. [17]. In the limit where there are no short-range three-body interactions, i.e., when $\mathcal{R}_{\mathbf{p}'\mathbf{p}} = 0$, Eq. (11) reduces to a solution composed of entirely exchanges between $\mathbf{2} \to \mathbf{2}$ subprocesses. We denote this solution as $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$, which is called the ladder series and is the solution of the integral equation,

$$\mathcal{D}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{p}}\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}}\mathcal{D}_{\mathbf{k}\mathbf{p}}. \tag{15}$$

Although the ladder series is an explicit solution of the $\mathbf{3} \to \mathbf{3}$ unitarity relations, it is dynamically controlled by long-range exchanges between $\mathbf{2} \to \mathbf{2}$ subsystems. Once the $\mathbf{2} \to \mathbf{2}$ amplitudes are known, solutions of Eq. (15) are completely fixed. Figure 3 shows a diagrammatic representation of the ladder series solution.

As noted in Ref. [26], and further explored in Ref. [31] (see also Appendix B), the B-matrix representation can be rewritten into a form where the ladder series is explicitly separated from the remaining three-particle interaction. This separation is useful in comparing to the HS-BHS equations as they schematically follow the same procedure. Genuine three-body interactions are introduced through an additional term to the ladder series solution, known in Ref. [12] as divergence-free amplitudes.[7] Following the derivation in Appendix B, the resulting $\mathbf{3} \to \mathbf{3}$ amplitude has the form

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}\tilde{\mathcal{T}}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{k}\mathbf{p}}, \tag{16}$$

where the first term is the ladder series that satisfies Eq. (15) and the second term is the divergence-free amplitude. The second term contains the amputated $\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}}$ amplitude, which is determined by the integral equation

$$\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}} = \mathcal{R}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{R}_{\mathbf{p}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{k}\mathbf{k}'}\tilde{\mathcal{T}}_{\mathbf{k}\mathbf{p}}, \tag{17}$$

as well as a $\mathbf{2} \to \mathbf{2}$ rescattering function, $\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}}$,

$$\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}}\delta_{\mathbf{p}'\mathbf{p}} + \mathcal{D}_{\mathbf{p}'\mathbf{p}} \tag{18}$$

$$= \mathcal{F}_{\mathbf{p}}\delta_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{k}\mathbf{p}}. \tag{19}$$

In the second line we used the fact that $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$ satisfies Eq. (15) to write the rescattering dressing function as an integral equation. Tildes on $\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}}$ and $\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}}$ are used to distinguish these quantities from the corresponding HS-BHS amplitudes, which, though similar, have different definitions. We discuss these differences later, when we perform the direct comparison. The interpretation of the divergence-free amplitude is now straightforward: $\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}}$ is an amplitude that involves $\mathbf{3} \to \mathbf{3}$ interactions via short-range dynamics. Rescattering functions then dress the initial and final states with all rescatterings that involve $\mathbf{2} \to \mathbf{2}$ processes, i.e., with direct $\mathbf{2} \to \mathbf{2}$ amplitudes or exchanges in the ladder series. The original B matrix, Eq. (11), explicitly shows only the direct $\mathbf{2} \to \mathbf{2}$ amplitudes dressing the initial and final state, while the ladder series remains hidden. Figure 4 illustrates diagrammatically the relations given in Eqs. (16), (17), and (19).

In Ref. [31], the authors proposed an initial-final-state factorization model of the short-range amplitude, $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$, under which Eq. (17) becomes algebraic, with the ladder series having a different definition for the real part. This construction may prove practical for analysis of data relevant to resonance phenomena.

[7] Divergence free means that the kinematic singularities from all long-range exchanges are not included, as they are contained in $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$ in Eq. (15).

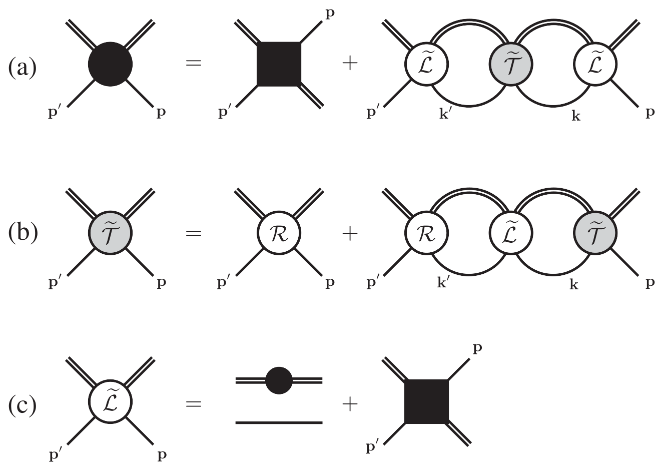


FIG. 4. Diagrammatic representation of (a) the $\mathbf{3}\to\mathbf{3}$ amplitude Eq. (16) in terms of the ladder series and the B-matrix divergence-free amplitude dressed by initial and final state rescatterings, (b) the integral equation for the B-matrix divergence-free amplitude Eq. (17), and (c) the B-matrix rescattering function Eq. (19).

B. HS-BHS representation

We now turn to the definitions of the on-shell $\mathbf{3}\to\mathbf{3}$ scattering equations of HS-BHS as given in Ref. [12]. We remind the reader that the unsymmetrized elastic $\mathbf{3}\to\mathbf{3}$ amplitude, $\mathcal{A}_{\mathbf{p}'\mathbf{p}}$, is a matrix in the angular momentum space of the pair labeled by the spectator. The unsymmetrized elastic $\mathbf{3}\to\mathbf{3}$ amplitude in HS-BHS representation is given via the integral equation

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{L}_{\mathbf{p}'\mathbf{k}'}\mathcal{T}_{\mathbf{k}'\mathbf{k}}\mathcal{L}^{\top}_{\mathbf{k}'\mathbf{p}}. \tag{20}$$

The symmetrized amplitude can be recovered as in Eq. (7). The ladder series, $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$, is defined exactly like in Eq. (15), and the end cap operators, $\mathcal{L}_{\mathbf{p}'\mathbf{p}}$, are defined by[8]

$$\mathcal{L}_{\mathbf{p}'\mathbf{p}} = \left(\frac{1}{3} + \mathcal{F}_{\mathbf{p}'} i\rho_{\mathbf{p}'}\right)\delta_{\mathbf{p}'\mathbf{p}} + \mathcal{D}_{\mathbf{p}'\mathbf{p}} i\rho_{\mathbf{p}}, \tag{21}$$

with $\mathcal{L}^{\top}_{\mathbf{p}'\mathbf{p}}$ defined with $i\rho_{\mathbf{p}}$ on the left of $\mathcal{F}_{\mathbf{p}'}$ and $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$. The quantity $\rho_{\mathbf{p}}$ is the phase-space factor for the two particles in the pair,[9]

$$[\rho_{\mathbf{p}}]_{\ell' m_{\ell}';\ell m_{\ell}} = \delta_{\ell'\ell}\delta_{m_{\ell}' m_{\ell}} \frac{1}{2!}\frac{1}{16\pi}\sqrt{1-\frac{4m^2}{\sigma_{\mathbf{p}}}}, \tag{22}$$

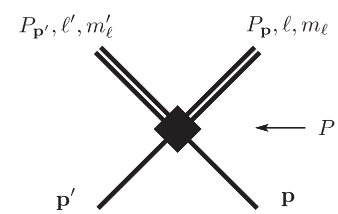
where the 2! is the symmetry factor. The constant term can be understood as the propagation of three particles without pairwise interactions, whereas the second two terms are the rescattering terms that appeared in Eq. (19). This reflects an important difference between the two formalisms, that the HS-BHS equations explicitly allow the possibility of no rescatterings. Finally, $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ is defined via the integral equation

$$\mathcal{T}_{\mathbf{p}'\mathbf{p}} = \mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{k}'} i\rho_{\mathbf{k}'}\mathcal{L}_{\mathbf{k}'\mathbf{k}}\mathcal{T}_{\mathbf{k}\mathbf{p}}, \tag{23}$$

where $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ is the three-particle K matrix, which contains all the short-range dynamical content of the scattering. The meaning of $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ is similar to that of $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$; however, we see in the next section that the difference between the two lies in how rescattering contributions are included. The amplitudes $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ are matrices in angular momenta, e.g.,

$$[\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}]_{\ell' m_{\ell}';\ell m_{\ell}} = \mathcal{K}_{\mathrm{df},\ell' m_{\ell}';\ell m_{\ell}}(\mathbf{p}', s, \mathbf{p}),$$

$$= \tag{24}$$



We note that both $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ and $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ have different symmetry properties then their counterparts in the B-matrix representation. As defined in Ref. [12], $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ are symmetric under interchange of any pair of initial or final state particles, after we sum over the product of the amplitude and its spherical harmonics of the pair orientations. Thus, the symmetric divergence-free K matrix is given by

$$\mathcal{K}_{\mathrm{df}} = 4\pi \sum_{\substack{\ell', m_{\ell}' \\ \ell, m_{\ell}}} Y_{\ell' m_{\ell}'}(\hat{\mathbf{q}}^{\star}_{\mathbf{p}'})\mathcal{K}_{\mathrm{df},\ell' m_{\ell}';\ell m_{\ell}}(\mathbf{p}', s, \mathbf{p}) Y^{*}_{\ell m_{\ell}}(\hat{\mathbf{q}}^{\star}_{\mathbf{p}}), \tag{25}$$

with a similar expression for $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$. Note that Eq. (25) is different from Eq. (7) since the latter requires a further symmetrization operation. The K matrix on the left-hand side in Eq. (25) is fully symmetric under interchange of any pair of particles in either the initial or final state.

$\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ is viewed as an amputated amplitude for which, in addition to all $\mathbf{2}\to\mathbf{2}$ rescatterings being removed, the

[8] $\mathcal{L}$ is the same as the quantity $\mathcal{L}^{(u,u)}$ of Ref. [12]. To see this requires accounting for the different integration measures used in the two works: our measure includes a factor of $1/(2\omega_{\mathbf{p}})$ that is not present in the measure of Ref. [12].

[9] In Ref. [12], the two-body phase space is defined slightly differently, $\rho_{\mathbf{p}}(\text{Ref. [13]}) = -i\rho_{\mathbf{p}}H(\mathbf{p})$. Thus there are no explicit factors of i in the expression for $\mathcal{L}$ in Ref. [12], whereas we prefer here to keep such factors explicit. The object $H(\mathbf{p})$ is a cutoff function, absent here because our momentum integrals implicitly include an ultraviolet cutoff, as discussed above.

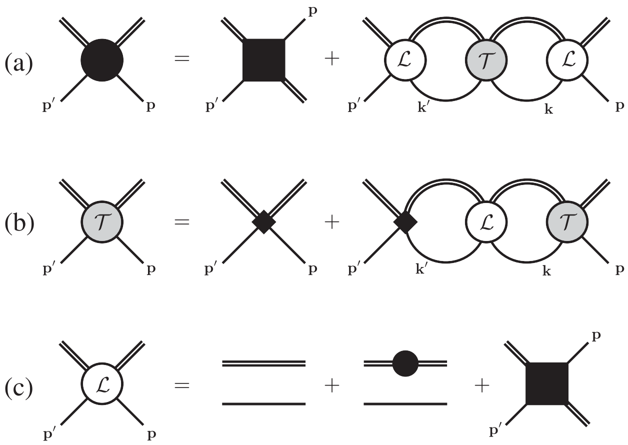


FIG. 5. Diagrammatic representation of (a) the **3 → 3** amplitude Eq. (20) in terms of the ladder series and the HS-BHS divergence-free amplitude dressed by initial and final state rescatterings, (b) the integral equation for the HS-BHS divergence-free amplitude Eq. (23), and (c) the HS-BHS rescattering function Eq. (21).

possibility of no rescattering in either initial or final states is included. This possibility is allowed by the term involving the constant $1/3$ in Eq. (21). The factor of $1/3$ is due to the partial wave definitions of $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$, Eq. (25), which is different than Eq. (7). Therefore, when we symmetrize the amplitude in Eq. (20), we would overcount the terms with no rescatterings if the $1/3$ were not present.

Figure 5 illustrates diagrammatically the relations given in Eqs. (20), (21), and (23). It is useful to compare Fig. 4 to 5, from which we see the primary difference is how the rescattering contributions are organized. In the next section, we explore this comparison in detail, and show how $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ are related.

IV. EQUIVALENCE OF THE *B*-MATRIX AND HS-BHS REPRESENTATIONS

Having established the *B*-matrix and HS-BHS equations, we now show that they are equivalent. To do so we assume that the **3 → 3** amplitudes in both representations are equal, and search for a relation between $\mathcal{R}$ and $\mathcal{K}_{\mathrm{df}}$. We first express the HS-BHS end caps, $\mathcal{L}_{\mathbf{p}'\mathbf{p}}$, in terms of the *B*-matrix rescattering functions, $\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}}$,

$$\mathcal{L}_{\mathbf{p}'\mathbf{p}} = \frac{1}{3}\delta_{\mathbf{p}'\mathbf{p}} + \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}} i\rho_{\mathbf{p}}. \tag{26}$$

The result for $\mathcal{L}^{\top}_{\mathbf{p}'\mathbf{p}}$ simply has $i\rho_{\mathbf{p}'}$ and $\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}}$ interchanged. We find that the first term of Eq. (26) can be traced to the differences in symmetrization and removal of **2 → 2** rescatterings between $\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$, while the $i\rho_{\mathbf{p}}$ factor in the second term is due to a difference in the definition of on-shell amputation.

To proceed, we rewrite Eq. (26) as

$$\mathcal{L}_{\mathbf{p}'\mathbf{p}} = \int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}} \mathcal{U}_{\mathbf{k}\mathbf{p}}, \tag{27}$$

where $\mathcal{U}_{\mathbf{p}'\mathbf{p}}$ is the "conversion factor"

$$\mathcal{U}_{\mathbf{p}'\mathbf{p}} = i\rho_{\mathbf{p}'}\delta_{\mathbf{p}'\mathbf{p}} + \frac{1}{3}\tilde{\mathcal{L}}^{-1}_{\mathbf{p}'\mathbf{p}} \tag{28}$$

$$= i\rho_{\mathbf{p}'}\delta_{\mathbf{p}'\mathbf{p}} + \frac{1}{3}\mathcal{F}^{-1}_{\mathbf{p}'}\delta_{\mathbf{p}'\mathbf{p}} - \frac{1}{3}\mathcal{G}_{\mathbf{p}'\mathbf{p}}, \tag{29}$$

where the second line follows from the inverse of $\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}}$ obtained from Eq. (19). In a similar manner, the transpose is given by $\mathcal{L}^{\top}_{\mathbf{p}'\mathbf{p}} = \int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{k}\mathbf{p}}$. Now, equating the expressions for $\mathcal{A}$ in the two formalisms, Eqs. (16) and (20), and using Eq. (27), we find the equivalence if the following relation holds,

$$\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}} = \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{T}_{\mathbf{k}'\mathbf{k}}\mathcal{U}_{\mathbf{k},\mathbf{p}}. \tag{30}$$

The amplitudes $\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$ can be formally solved in terms of $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$, respectively, as one does in matrix equations, e.g., $\tilde{\mathcal{T}} = [1 - \mathcal{R}\cdot\tilde{\mathcal{L}}]^{-1}\mathcal{R}$, which is a matrix in both angular and spectator momenta. Combining the formal solutions for $\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{T}_{\mathbf{p}'\mathbf{p}}$, the relation Eq. (30), and using the definition of $\mathcal{U}_{\mathbf{p}'\mathbf{p}}$ in Eq. (28), we arrive at an integral equation relating $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$,

$$\begin{aligned}\mathcal{R}_{\mathbf{p}'\mathbf{p}} = {}& \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathcal{U}_{\mathbf{k}\mathbf{p}} \\ & - \frac{1}{3}\int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathcal{R}_{\mathbf{k}\mathbf{p}}.\end{aligned} \tag{31}$$

When $\mathcal{K}_{\mathrm{df}}$ and the $\mathcal{R}$ matrix satisfy the above relation, the two representations of the amplitude $\mathcal{A}$ are equivalent; i.e., they both describe the same physics. Knowing $\mathcal{K}_{\mathrm{df}}$ one can solve Eq. (31) for the corresponding three-body matrix $\mathcal{R}$ in the *B*-matrix representation,

$$\int_{\mathbf{k}}\left[\delta_{\mathbf{p}'\mathbf{k}} + \frac{1}{3}\int_{\mathbf{k}'} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\right]\mathcal{R}_{\mathbf{k}\mathbf{p}} = \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathcal{U}_{\mathbf{k}\mathbf{p}}, \tag{32}$$

as long as $\det[1 + \frac{1}{3}\mathcal{U}\cdot\mathcal{K}_{\mathrm{df}}] \neq 0$. Since $\mathcal{U}$ describes a general **2 → 2** subprocess and $\mathcal{K}_{\mathrm{df}}$ an independent **3 → 3** interaction this condition should be always satisfied, unless some pathological cases for specific momenta are considered.

The final step is to show that Eq. (31) is consistent with the reality of both $\mathcal{R}$ and $\mathcal{K}_{\mathrm{df}}$. This result is not manifest, as $\mathcal{U}$ is complex. Its imaginary part is readily found to be

$$\mathrm{Im}\mathcal{U}_{\mathbf{p}'\mathbf{p}} = \frac{1}{3}(2\bar{\rho}_{\mathbf{p}}\delta_{\mathbf{p}'\mathbf{p}} - \mathcal{C}_{\mathbf{p}'\mathbf{p}}), \tag{33}$$

where we have used the **2 → 2** unitarity relation for the inverse amplitude,

$$\mathrm{Im}\mathcal{F}_{\mathbf{p}}^{-1} = -\bar{\rho}_{\mathbf{p}} = -\rho_{\mathbf{p}}\Theta(\sigma_{\mathbf{p}} - 4m^2), \tag{34}$$

which follows from Eq. (A2), as well as the result $\mathcal{C}_{\mathbf{p}'\mathbf{p}} = \mathrm{Im}\mathcal{G}_{\mathbf{p}'\mathbf{p}}$. To proceed we need the important results

$$\int_{\mathbf{k}} \mathrm{Im}\mathcal{U}_{\mathbf{p}'\mathbf{k}}\mathcal{K}_{\mathrm{df},\mathbf{kp}} = \int_{\mathbf{k}} \mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{k}}\mathrm{Im}\mathcal{U}_{\mathbf{kp}} = 0, \tag{35}$$

which are demonstrated in Appendix C. Essentially, the action of $\mathcal{C}_{\mathbf{p}'\mathbf{p}}$ on an object with the symmetry properties of Eq. (25) yields a phase-space factor that cancels the first term of Eq. (33). Combining these results, we find that $\mathcal{R}$ is real if $\mathcal{K}_{\mathrm{df}}$ is

$$\begin{aligned}\mathrm{Im}\mathcal{R}_{\mathbf{p}'\mathbf{p}} = &\int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathrm{Im}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathcal{U}_{\mathbf{kp}} \\ &- \frac{1}{3}\int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathrm{Im}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathcal{R}_{\mathbf{kp}} \\ &- \frac{1}{3}\int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathrm{Im}\mathcal{R}_{\mathbf{kp}}.\end{aligned} \tag{36}$$

By construction, $\mathrm{Im}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}} = 0$, eliminating the first two terms, leaving the equation

$$\int_{\mathbf{k}} \left[\delta_{\mathbf{p}'\mathbf{k}} + \frac{1}{3}\int_{\mathbf{k}'} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\right]\mathrm{Im}\mathcal{R}_{\mathbf{kp}} = 0. \tag{37}$$

Assuming $\mathcal{R}$ is a solution to Eq. (32), i.e., the condition $\det[1 + \frac{1}{3}\mathcal{U}\cdot\mathcal{K}_{\mathrm{df}}] \neq 0$ is satisfied, we conclude that $\mathrm{Im}\mathcal{R}_{\mathbf{p}'\mathbf{p}} = 0$, thus proving the reality condition required by unitarity, and that the B-matrix and HS-BHS representations are equivalent. A similar argument can be made assuming $\mathcal{R}$ is real and solving for the condition on $\mathcal{K}_{\mathrm{df}}$.

The relationship between $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ in Eq. (31) can be better understood if rewritten as

$$\begin{aligned}\int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}\mathcal{R}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}} = &\int_{\mathbf{k}'}\int_{\mathbf{k}} \left(\frac{1}{3}\delta_{\mathbf{p}'\mathbf{k}'} + \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}i\rho_{\mathbf{k}'}\right)\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\left(\frac{1}{3}\delta_{\mathbf{kp}} + i\rho_{\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}}\right) \\ &- \frac{1}{3}\int_{\mathbf{k}'}\int_{\mathbf{k}}\int_{\mathbf{k}''} \left(\frac{1}{3}\delta_{\mathbf{p}'\mathbf{k}'} + \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}i\rho_{\mathbf{k}'}\right)\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}\mathcal{R}_{\mathbf{kk}''}\tilde{\mathcal{L}}_{\mathbf{k}''\mathbf{p}}.\end{aligned} \tag{38}$$

We now assume that $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ is momentum independent, $\mathcal{K}_{\mathrm{df}} = \lambda$, with λ being a small constant. This isotropic form is the leading contribution in an expansion about threshold [30]. Truncating the series solution of Eq. (31) at leading order in $\mathcal{K}_{\mathrm{df}}$, we obtain

$$\begin{aligned}\int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}\mathcal{R}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}} = &\frac{1}{9}\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}} + \frac{1}{3}\int_{\mathbf{k}'} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}i\rho_{\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{p}} + \frac{1}{3}\int_{\mathbf{k}} \mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{k}}i\rho_{\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}} \\ &+ \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}i\rho_{\mathbf{k}'}\mathcal{K}_{\mathrm{df},\mathbf{k}'\mathbf{k}}i\rho_{\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}} + \mathcal{O}(\mathcal{K}_{\mathrm{df}}^2),\end{aligned} \tag{39}$$

which is represented diagrammatically in Fig. 6. Since $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ represents three-body interactions such as contact interactions, the right-hand side shows that there is a possibility that the interaction is not dressed by **2 → 2** rescatterings on either the initial or final state (or both). Contrarily, the $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ matrix is always dressed by **2 → 2** interactions in both the initial and final state. Thus, the $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ matrix represents a different organization of amplitudes. The factors $1/9$ and $1/3$ reflect the fact that $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ has a decomposition given by Eq. (25), which does not include summing over all spectator momenta, and are needed to remove the overcounting when we sum over all spectator

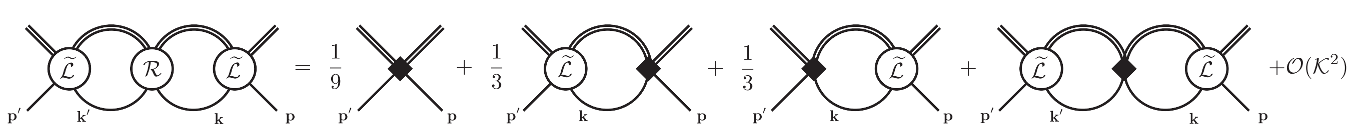


FIG. 6. Diagrammatic representation of the leading order relation between $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{K}_{\mathrm{df}}(\mathbf{p}', \mathbf{p})$, Eq. (39), where the black diamond represents $\mathcal{K}_{\mathrm{df}}(\mathbf{p}', \mathbf{p})$.

momenta in the initial or final state. We conclude that both functions represent short-range dynamics. Finally, the left-hand side has no $i\rho_{\mathbf{p}}$ factors, whereas the right-hand side does. This is due to the differences in how the amplitudes are amputated. For the B matrix, which is based on satisfying the unitarity relations, the amputation was made by removing the partial wave amplitudes via $\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}}\mathcal{F}_{\mathbf{p}}$, where the $\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}}$ is the amputated partial wave amplitude. This was convenient as it simplified the unitarity relation (see Appendix A and Ref. [26] for details). This amputation is not unique, as we can freely remove any real quantity from $\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}}$, including $(i\rho_{\mathbf{p}})^2$. The HS-BHS equations involve an all-orders summation of amplitudes in an effective field theory, which includes loop integrals over momenta of intermediate states. When the two-particle loop integral is put on shell, the $i\rho_{\mathbf{p}}$ factors naturally emerge.

V. NONRELATIVISTIC LIMIT AND FADDEEV EQUATIONS

In the nonrelativistic limit, which is relevant for near threshold processes, we can investigate the relation between these representations to the Faddeev equations. If we assume that the three-body interactions are negligible compared to the two-body interactions, then we can set the $\mathcal{K}_{\mathrm{df}}$ matrix [or equivalently the R-matrix by Eq. (31)] to 0 at leading order, leaving only the ladder rescattering solutions,

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \mathcal{O}(\mathcal{K}_{\mathrm{df}}). \tag{40}$$

The three-body amplitudes are dominated by exchanges between $\mathbf{2}\to\mathbf{2}$ processes, as in typical Faddeev-type approximations. In that case, as can be seen from Eqs. (8) and (19), the unsymmetrized scattering amplitude $\mathcal{M}$ becomes $\tilde{\mathcal{L}}$, which gives

$$\mathcal{M}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}}\delta_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}}\mathcal{M}_{\mathbf{k}\mathbf{p}}. \tag{41}$$

In the CMF, the nonrelativistic limit of the $[\mathcal{G}_{\mathbf{p}'\mathbf{p}}]_{l'm_l';lm_l}$ denominator in Eq. (13) becomes

$$\begin{aligned}(P_{\mathbf{p}} - p')^2 - m^2 &= (E_{\mathbf{p}} - \omega_{\mathbf{p}'})^2 - (\mathbf{p}+\mathbf{p}')^2 - m^2\\ &= 2m\left[\Delta E - \frac{1}{m}(\mathbf{p}'^2 + \mathbf{p}^2 + \mathbf{p}\cdot\mathbf{p}')\right] + \mathcal{O}(\mathbf{p}^4).\end{aligned} \tag{42}$$

We approximated $\omega_{\mathbf{p}} = m + \mathbf{p}^2/2m + \mathcal{O}(\mathbf{p}^4)$ and used the fact that, close to threshold, $E = 3m + \Delta E$, where ΔE is a nonrelativistic energy of three particles. Finally, we also neglected terms of the order $\mathbf{p}^4$,$\mathbf{p}'^4$, and $\mathbf{p}^2\mathbf{p}'^2$. The factor $2\omega_{\mathbf{k}}$ in the integration measure of Eq. (41) becomes $2m$. Putting everything together, and writing angular momentum indices explicitly, we obtain

$$\begin{aligned}[\mathcal{M}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_\ell';\ell m_\ell} &= \delta_{\mathbf{p}'\mathbf{p}}\delta_{\ell' m_\ell';\ell m_\ell}[\mathcal{F}_{\mathbf{p}}]_{\ell m_\ell} - \frac{4\pi}{(2\pi)^3 4m^2}\sum_{\ell'',m_{\ell''}}\int \mathrm{d}^3\mathbf{k}[\mathcal{F}_{\mathbf{p}'}]_{\ell' m_\ell'}\left(\frac{k^\star}{q^\star_{\mathbf{p}'}}\right)^{\ell'}\\ &\quad\times\frac{Y^*_{\ell' m_{\ell'}}(\hat{\mathbf{k}}^\star_{\mathbf{p}'})Y_{\ell'' m_\ell''}(\hat{\mathbf{p}}_{\mathbf{k}}{}'^\star)}{\Delta E - \frac{1}{m}(\mathbf{p}'^2 + \mathbf{k}^2 + \mathbf{k}\cdot\mathbf{p}') + i\epsilon}\left(\frac{p'^\star}{q^\star_{\mathbf{k}}}\right)^{\ell}[\mathcal{M}_{\mathbf{k}\mathbf{p}}]_{\ell'' m_\ell'';\ell m_\ell}.\end{aligned} \tag{43}$$

Following the conventions of Ref. [32], we define the nonrelativistic scattering amplitude as ${}_{\mathrm{NR}}\langle\mathrm{out}|T|\mathrm{in}\rangle_{\mathrm{NR}} = -2\pi\delta^{(4)}(P'-P)\mathcal{M}^{\mathrm{NR}}$, which is different by a factor of $-(2\pi)^3$ to our definition in Eq. (6). Moreover, nonrelativistic particle states are defined as $|\mathbf{p}\rangle = (2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}}}|\mathbf{p}\rangle_{\mathrm{NR}}$, such that the momentum eigenstates are normalized as ${}^{\mathrm{NR}}\langle\mathbf{p}'|\mathbf{p}\rangle_{\mathrm{NR}} = \delta^{(3)}(\mathbf{p}'-\mathbf{p})$. Thus, the relation between the nonrelativistic and relativistic $\mathbf{3}\to\mathbf{3}$ amplitude is

$$\mathcal{M}^{\mathrm{NR}} = -\left(\frac{1}{(2\pi)^3 2m}\right)^3\mathcal{M}, \tag{44}$$

where we have taken the nonrelativistic limit for the particle energies, e.g., $\omega_{\mathbf{p}} = m + \mathcal{O}(\mathbf{p}^2)$. The $\mathbf{2}\to\mathbf{2}$ subprocesses contains an extra $\delta_{\mathbf{p}'\mathbf{p}} = (2\pi)^3 2\omega_{\mathbf{p}}\delta^3(\mathbf{p}'-\mathbf{p})$, which conserves spectator momenta, so the nonrelativistic $\mathbf{2}\to\mathbf{2}$ amplitude $\mathcal{F}^{\mathrm{NR}}_{\mathbf{p}}$ is related to its relativistic counterpart by an equation similar to Eq. (44), except that the conversion factor is squared instead of cubed. Therefore, after inclusion of spherical harmonics to recover full amplitudes, we arrive at

$$\begin{aligned}&\mathcal{M}^{\mathrm{NR}}_{\mathbf{p}'\mathbf{p}}(\mathbf{q}_{\mathbf{p}'},\mathbf{q}_{\mathbf{p}})\\ &= \delta^3(\mathbf{p}'-\mathbf{p})\mathcal{F}^{\mathrm{NR}}_{\mathbf{p}}(\mathbf{q}_{\mathbf{p}'},\mathbf{q}_{\mathbf{p}}) + \int \mathrm{d}^3\mathbf{k}\mathcal{F}^{\mathrm{NR}}_{\mathbf{p}'}(\mathbf{q}_{\mathbf{p}'},\mathbf{k})\\ &\quad\times\frac{1}{\Delta E - \frac{1}{m}(\mathbf{p}'^2 + \mathbf{k}^2 + \mathbf{k}\cdot\mathbf{p}') + i\epsilon}\mathcal{M}^{\mathrm{NR}}_{\mathbf{k}\mathbf{p}}(\mathbf{p}',\mathbf{q}_{\mathbf{p}}),\end{aligned} \tag{45}$$

where

$$\mathcal{M}^{\mathrm{NR}}_{\mathbf{p}'\mathbf{p}}(\mathbf{q}_{\mathbf{p}'},\mathbf{q}_{\mathbf{p}}) = 4\pi \sum_{\substack{\ell',m_{\ell}'\\ \ell,m_{\ell}}} Y_{\ell' m_{\ell}'}(\hat{\mathbf{q}}_{\mathbf{p}'})[\mathcal{M}^{\mathrm{NR}}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_{\ell}';\ell m_{\ell}} Y^{*}_{\ell m_{\ell}}(\hat{\mathbf{q}}_{\mathbf{p}}) \tag{46}$$

and similarly for $\mathcal{F}^{\mathrm{NR}}_{\mathbf{p}'}(\mathbf{q}_{\mathbf{p}'},\mathbf{q}_{\mathbf{p}})$. The barrier factors have been implicitly absorbed in $\mathcal{M}^{\mathrm{NR}}_{\mathbf{p}'\mathbf{p}}$ and $\mathcal{F}^{\mathrm{NR}}_{\mathbf{p}}$.

Equation (45) is equivalent to the Faddeev equations, as can be seen, for example, by comparing to Eq. (11) of Ref. [33]. There, a symmetrized two-body matrix t_s plays a role of $\mathcal{F}^{\mathrm{NR}}_{\mathbf{p}}$ and T matrix plays a role of $\mathcal{M}^{\mathrm{NR}}_{\mathbf{p}'\mathbf{p}}$. Note that both equations are not symmetrized with respect to interchanges of three interacting bosons as in Eq. (7).

VI. CONCLUSIONS

We have shown that the relativistic on-shell representation of the $\mathbf{3}\to\mathbf{3}$ scattering amplitude of Hansen and Sharpe [12] and Briceño *et al.* [13], and the B-matrix representation presented by Mai *et al.* [17] and Jackura I [26] are equivalent, and their physical content is identical. The results of the present work are consistent with the conclusions of Ref. [28] that the HS-BHS approach is unitary. The difference in these representations is how the formalism incorporates rescattering effects. In the B-matrix representation, the $\mathbf{3}\to\mathbf{3}$ amplitude is always dressed by $\mathbf{2}\to\mathbf{2}$ rescatterings in both the initial and final states, as shown by Eq. (16). Contrarily, the representation by Briceño *et al.* allows the possibility of no initial/ final state rescatterings. It was shown in Sec. IV that the differences between these rescattering functions manifest themselves as differences in the real part of the on-shell equations, giving the integral equation (31). The nonrelativistic limit of both formalisms reproduced the Faddeev equations, providing a consistency check to well-known low-energy approaches. As was discussed in Ref. [26], the B-matrix representations of Refs. [25,26] differ only in the real part as a result of the latter approach using a cutoff on the integration range that eliminated unphysical modes.

All of the proposed formalisms require regulation of the high-energy modes in order to arrive at a convergent solution to the integral equations. Regulating the divergent behavior introduces additional cutoff dependence in the equations. Physical quantities must, however, be cutoff independent, and this is achieved by introducing cutoff dependence into the real, K-matrixlike quantities in the formalisms (i.e., $\mathcal{R}$ and $\mathcal{K}_{\mathrm{df}}$). For example, as was discussed in Ref. [26], the B-matrix representations of Refs. [25,26] differ only in their real parts, as a result of the latter using a cutoff in the integration range that eliminated unphysical modes.

It remains to be seen if the quantization conditions corresponding to the different formalisms are also identical. Naively, one might assume that, since the infinite volume equations are identical, the quantization conditions must also be, at least up to exponentially suppressed corrections. However, the details of transitioning from infinite to finite volume, e.g., the handling of angular momentum mixing, are nontrivial and have not yet been worked out. This is an interesting area of study and must be completed to ensure consistency.

An interesting direction for future studies is comparing numerical results from each representation. Although equivalent, parametrizations using the R matrix of Ref. [26] or the K matrix of Ref. [12] may turn out to be advantageous for particular numerical analyses.

ACKNOWLEDGMENTS

We thank Raúl Briceño and Maxwell Hansen for many useful discussions. This work was supported by the U.S. Department of Energy under Grants No. DE-SC0011637 (S. R. S.), No. DE-AC05-06OR23177, and No. DE-FG02-87ER40365, U.S. National Science Foundation under Award No. PHY-1415459, PAPIIT-DGAPA (UNAM, Mexico) Grant No. IA101819, and CONACYT (Mexico) Grants No. 251817 and No. A1-S-21389. V. M. acknowledges support from Comunidad Autónoma de Madrid through Programa de Atracción de Talento Investigador 2018 (Modalidad 1). The work of S. R. S. was partly supported by the International Research Unit of Advanced Future Studies at Kyoto University.

APPENDIX A: UNITARITY RELATIONS

Unitarity of the S matrix constrains the imaginary part of on-shell scattering amplitudes. Given the unitarity constraints, one can construct an on-shell representation for scattering amplitudes in terms of real quantities and kinematic functions. We present here a brief summary of the unitarity relations for identical particles. The relations for distinguishable particles have been discussed in detail in Ref. [26]. Elastic three-particle scattering satisfies the unitarity relation

$$2\mathrm{Im}\mathcal{M} = \frac{1}{3!}\prod_{j=1}^{3}\int_{\mathbf{k}_j}(2\pi)^4\delta^{(4)}\left(\sum_{j=1}^{3}k_j - P\right)\mathcal{M}^{*}\mathcal{M}, \tag{A1}$$

where the integration is over the on-shell intermediate state momenta. Writing $\mathcal{M}$ in terms of the unsymmeterized amplitudes, Eq. (7), and separating the disconnected $\mathbf{2}\to\mathbf{2}$ amplitude from the connected $\mathbf{3}\to\mathbf{3}$ amplitude via Eq. (8), we arrive at two unitarity equations. The first is the well-known $\mathbf{2}\to\mathbf{2}$ unitarity relation in angular momentum space,

$$\mathrm{Im}\mathcal{F}_{\mathbf{p}} = \mathcal{F}^{\dagger}_{\mathbf{p}}\bar{\rho}_{\mathbf{p}}\mathcal{F}_{\mathbf{p}}, \tag{A2}$$

where $\bar{\rho}$ is the two-body phase space defined in Eq. (34). Equation (A2) admits the on-shell K-matrix representation for $\mathcal{F}_{\mathbf{p}}$,

$$\mathcal{F}_{\mathbf{p}} = \mathcal{K}_{\mathbf{p}} + \mathcal{K}_{\mathbf{p}} i\bar{\rho}_{\mathbf{p}} \mathcal{F}_{\mathbf{p}} = [1 - \mathcal{K}_{\mathbf{p}} i\bar{\rho}_{\mathbf{p}}]^{-1} \mathcal{K}_{\mathbf{p}}, \tag{A3}$$

where $[\mathcal{K}_{\mathbf{p}}]_{\ell' m_{\ell'};\ell m_\ell} = \delta_{\ell'\ell}\delta_{m_{\ell'} m_\ell}\mathcal{K}_\ell(\sigma_{\mathbf{p}})$ is the $\mathbf{2}\to\mathbf{2}$ K-matrix, which is a real function of $\sigma_{\mathbf{p}}$ in the elastic kinematic region, and diagonal in angular momenta space. Since the phase-space factor contains the kinematic information of two on-shell propagating particles, the K matrix represents all the dynamical content of the two-particle system, e.g., the short-range forces between pions in elastic $\pi\pi$ scattering. This can, in principle, include virtual exchanges leading to left-hand cuts or higher multiparticle thresholds, e.g., four-particle production, which do not give singular contributions in the elastic domain. Since $\mathbf{2}\to\mathbf{2}$ amplitudes are diagonal in angular momentum space, Eq. (A3) reduces to a simple algebraic relation. It is straightforward to verify that Eq. (A3) satisfies Eq. (A2).

The second unitarity relation is for the connected $\mathbf{3}\to\mathbf{3}$ amplitude, which in the $\mathbf{p}\ell m_\ell$-basis is

$$\begin{aligned}\mathrm{Im}\mathcal{A}_{\mathbf{p}'\mathbf{p}} = {}& \int_{\mathbf{k}} \mathcal{A}^\dagger_{\mathbf{p}'\mathbf{k}}\bar{\rho}_{\mathbf{k}}\mathcal{A}_{\mathbf{k}\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{A}^\dagger_{\mathbf{p}'\mathbf{k}'}\mathcal{C}_{\mathbf{k}'\mathbf{k}}\mathcal{A}_{\mathbf{k}\mathbf{p}} \\ &+ \mathcal{F}^\dagger_{\mathbf{p}'}\bar{\rho}_{\mathbf{p}'}\mathcal{A}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}^\dagger_{\mathbf{p}'}\mathcal{C}_{\mathbf{p}'\mathbf{k}}\mathcal{A}_{\mathbf{k}\mathbf{p}} \\ &+ \mathcal{A}^\dagger_{\mathbf{p}'\mathbf{p}}\bar{\rho}_{\mathbf{p}}\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{A}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{C}_{\mathbf{k}\mathbf{p}}\mathcal{F}_{\mathbf{p}} \\ &+ \mathcal{F}^\dagger_{\mathbf{p}'}\mathcal{C}_{\mathbf{p}'\mathbf{p}}\mathcal{F}_{\mathbf{p}},\end{aligned} \tag{A4}$$

where $\mathcal{C}_{\mathbf{p}'\mathbf{p}}$ is the recoupling coefficient between a pair in one state to a different pair in the same state, e.g., from an angular momentum coupling (12)3 to (23)1, which is defined as the imaginary part of the amputated OPE amplitude, Eq. (13),

$$\begin{aligned}[\mathcal{C}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_{\ell'};\ell m_\ell} &\equiv \mathrm{Im}[\mathcal{G}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_{\ell'};\ell m_\ell} \\ &= \pi\delta((P_{\mathbf{p}} - p')^2 - m^2) \\ &\quad \times 4\pi Y^*_{\ell' m_{\ell'}}(\hat{\mathbf{p}}^\star_{\mathbf{p}'}) Y_{\ell m_\ell}(\hat{\mathbf{p}}_{\mathbf{p}}{'^\star}).\end{aligned} \tag{A5}$$

The recoupling coefficients are an additional feature of three-body scattering that can be seen in Fig. 7 when a diagram with a crossed exchange in the intermediate state is cut. Diagrams that are cut where no exchange occurs give rise to the conventional two-body phase space. One may be concerned that the complexity of spherical harmonics is not taken into account. The phases in the unitarity relation cancel since the intermediate state sums over all possibilities. To avoid this bookkeeping during intermediate calculations, one can use real spherical harmonics, which have the same completeness and orthonormality relations as the usual ones, to formally manipulate the expressions. Since the final results do not depend on the choice of harmonics, we are guaranteed the validity of the unitarity relations and the solutions.

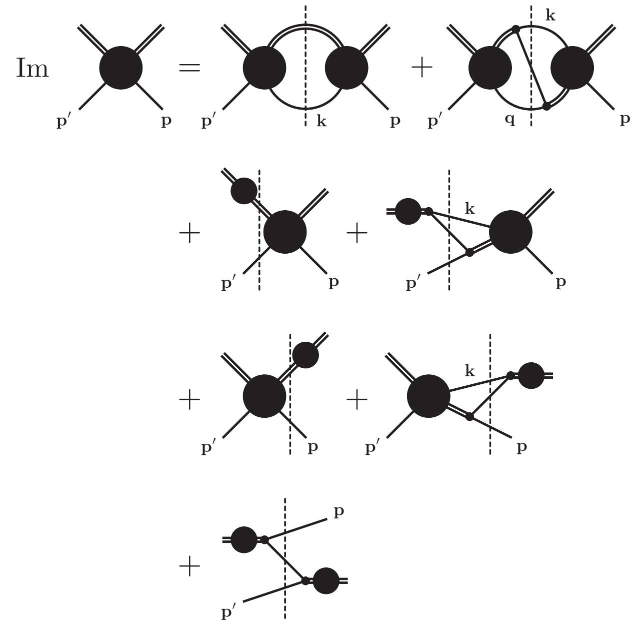


FIG. 7. Diagrammatic representation for the $\mathbf{3}\to\mathbf{3}$ unitarity relation for the amplitude $\mathcal{A}_{\ell' m_{\ell'};\ell m_\ell}(\mathbf{p}', s, \mathbf{p})$. Closed loops yield three-dimensional integrations over the labeled spectator momentum, and the dashed vertical lines represent placing all three intermediate state particles on their mass shell. Momentum flow is from right to left, as before, and each amplitude on the left of the dashed line is Hermitian conjugated.

Equation (A4) admits the on-shell representation given by Eq. (11), which we now verify. We find the following demonstration more direct than the one presented in Ref. [26]. First, let us introduce amplitudes that have the final state $\mathbf{2}\to\mathbf{2}$ amplitudes amputated, i.e., $\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}}\mathcal{F}_{\mathbf{p}}$. Equation (A4) then simplifies to

$$\begin{aligned}\mathrm{Im}\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}} = {}& \int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{F}^\dagger_{\mathbf{k}}\bar{\rho}_{\mathbf{k}}\mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} \\ &+ \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}'}\mathcal{F}^\dagger_{\mathbf{k}'}\mathcal{C}_{\mathbf{k}'\mathbf{k}}\mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} \\ &+ \int_{\mathbf{k}} \mathcal{C}_{\mathbf{p}'\mathbf{k}}\mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} + \int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{F}^\dagger_{\mathbf{k}}\mathcal{C}_{\mathbf{k}\mathbf{p}} + \mathcal{C}_{\mathbf{p}'\mathbf{p}},\end{aligned} \tag{A6}$$

and the corresponding amputated B-matrix representation is

$$\begin{aligned}\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}} &= \mathcal{B}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}} \mathcal{B}_{\mathbf{p}'\mathbf{k}}\mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} \\ &= \int_{\mathbf{k}} \mathcal{B}_{\mathbf{p}'\mathbf{k}}(\delta_{\mathbf{k}\mathbf{p}} + \mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}}),\end{aligned} \tag{A7}$$

where we remind the reader that $\mathcal{B}_{\mathbf{p}'\mathbf{p}} = \mathcal{G}_{\mathbf{p}'\mathbf{p}} + \mathcal{R}_{\mathbf{p}'\mathbf{p}}$, where $\mathcal{G}_{\mathbf{p}'\mathbf{p}}$ is given in Eq. (13) and $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ is a real function that contains the unconstrained three-body dynamics. It is straightforward to verify that Eq. (A7) satisfies Eq. (A6) directly by taking the difference between the amplitude and its Hermitian conjugate. Note that if the matrix elements of $\mathcal{A}_{\mathbf{p}'\mathbf{p}}$ are $\mathcal{A}_{\ell' m_{\ell}';\ell m_\ell}(\mathbf{p}', s, \mathbf{p})$, then the Hermitian conjugate, $\mathcal{A}^\dagger_{\mathbf{p}'\mathbf{p}}$, has elements $\mathcal{A}^*_{\ell m_\ell;\ell' m_{\ell}'}(\mathbf{p}, s, \mathbf{p}')$, since it acts on both the angular momentum space and the spectator space. The Hermitian analytic properties [34] of amplitudes then state $\mathcal{A}^*_{\ell m_\ell;\ell' m_{\ell}'}(\mathbf{p}, s, \mathbf{p}') = \mathcal{A}^*_{\ell' m_{\ell}';\ell m_\ell}(\mathbf{p}', s, \mathbf{p})$, so that $\mathcal{A}_{\mathbf{p}'\mathbf{p}} - \mathcal{A}^\dagger_{\mathbf{p}'\mathbf{p}} = \mathcal{A}_{\mathbf{p}'\mathbf{p}} - \mathcal{A}^*_{\mathbf{p}'\mathbf{p}} = 2i\mathrm{Im}\mathcal{A}_{\mathbf{p}'\mathbf{p}}$.

We begin by rewriting the difference by adding and subtracting a judiciously chosen term, leading to

$$\begin{aligned}\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}} - \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{p}} = &\int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}(\mathcal{F}_{\mathbf{k}} - \mathcal{F}^\dagger_{\mathbf{k}})\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} \\ &+ \int_{\mathbf{k}} (\delta_{\mathbf{p}'\mathbf{k}} + \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{F}^\dagger_{\mathbf{k}})\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} \\ &- \int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}(\delta_{\mathbf{k}\mathbf{p}} + \mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}}). \end{aligned} \tag{A8}$$

Next we insert Eq. (A7) into $\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}}$ on the second line of Eq. (A8), and its Hermitian conjugate,

$$\begin{aligned}\tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{p}} &= \mathcal{B}^\dagger_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{F}^\dagger_{\mathbf{k}}\mathcal{B}^\dagger_{\mathbf{k}\mathbf{p}} \\ &= \int_{\mathbf{k}} (\delta_{\mathbf{p}'\mathbf{k}} + \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{F}^\dagger_{\mathbf{k}})\mathcal{B}^\dagger_{\mathbf{k}\mathbf{p}}, \end{aligned} \tag{A9}$$

into the $\tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}$ on the third line of Eq. (A8). This gives

$$\begin{aligned}\tilde{\mathcal{A}}_{\mathbf{p}'\mathbf{p}} - \tilde{\mathcal{A}}^\dagger_{\mathbf{p}\mathbf{p}'} = &\int_{\mathbf{k}} \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}(\mathcal{F}_{\mathbf{k}} - \mathcal{F}^\dagger_{\mathbf{k}})\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}} \\ &+ \int_{\mathbf{k}'}\int_{\mathbf{k}} (\delta_{\mathbf{p}'\mathbf{k}'} + \tilde{\mathcal{A}}^\dagger_{\mathbf{p}'\mathbf{k}}\mathcal{F}^\dagger_{\mathbf{k}'}) \\ &\times (\mathcal{B}_{\mathbf{k}'\mathbf{k}} - \mathcal{B}^\dagger_{\mathbf{k}'\mathbf{k}})(\delta_{\mathbf{k}\mathbf{p}} + \mathcal{F}_{\mathbf{k}}\tilde{\mathcal{A}}_{\mathbf{k}\mathbf{p}}), \end{aligned} \tag{A10}$$

which can then be simplified using $\mathcal{F}_{\mathbf{p}} - \mathcal{F}^\dagger_{\mathbf{p}} = 2i\mathrm{Im}\mathcal{F}_{\mathbf{p}}$ and Eq. (A2), as well as the result that $\mathcal{B}_{\mathbf{p}'\mathbf{p}} - \mathcal{B}^\dagger_{\mathbf{p}'\mathbf{p}} = 2i\mathrm{Im}\mathcal{G}_{\mathbf{p}'\mathbf{p}}$ since $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ is real. Then, since the recoupling coefficients are $\mathcal{C}_{\mathbf{p}'\mathbf{p}} = \mathrm{Im}\mathcal{G}_{\mathbf{p}'\mathbf{p}}$, we arrive at Eq. (A6), thus proving that the B-matrix representation satisfies the unitary condition.

APPENDIX B: EXPRESSING THE B MATRIX IN TERMS OF THE LADDER

In this appendix, we show how the B-matrix representation can be expressed in terms of the full OPE ladder summation and a remaining piece containing genuine three-body interactions (see also Ref. [31]). The B-matrix representation for the full amplitude is given in Eq. (11). In the limit that the scattering is dominated by $\mathbf{2}\to\mathbf{2}$ interactions, and three-body interactions are negligible ($\mathcal{R}_{\mathbf{p}'\mathbf{p}} \to 0$), the system is controlled by successive particle exchanges between the $\mathbf{2}\to\mathbf{2}$ amplitudes. We defined this process as the ladder amplitude, $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$, which satisfies Eq. (15). We now want to remove the ladder solution from the general three-body system. Following a similar approach as HS-BHS, we define the divergence-free amplitude, $\mathcal{A}_{\mathrm{df},\mathbf{p}'\mathbf{p}} \equiv \mathcal{A}_{\mathbf{p}'\mathbf{p}} - \mathcal{D}_{\mathbf{p}'\mathbf{p}}$, with the $\mathbf{3}\to\mathbf{3}$ amplitude free from the ladder diagram and its singularities. We can then separate the ladder solution from the B-matrix representation and are left with an equation for $\mathcal{A}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$,

$$\begin{aligned}\mathcal{A}_{\mathrm{df},\mathbf{p}'\mathbf{p}} = &\int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\mathcal{R}_{\mathbf{p}'\mathbf{k}}(\mathcal{F}_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{p}} + \mathcal{D}_{\mathbf{k}\mathbf{p}}) \\ &+ \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}(\mathcal{R}_{\mathbf{p}'\mathbf{k}} + \mathcal{G}_{\mathbf{p}'\mathbf{k}})\mathcal{A}_{\mathrm{df},\mathbf{k}\mathbf{p}}. \end{aligned} \tag{B1}$$

Now define the $\mathbf{2}\to\mathbf{2}$ rescattering function Eq. (19), and amputate the end caps from the divergent-free amplitude,

$$\mathcal{A}_{\mathrm{df},\mathbf{p}'\mathbf{p}} = \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}\mathbf{k}}\mathcal{T}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{k}\mathbf{p}}. \tag{B2}$$

Substituting Eq. (B2) into Eq. (B1) and collecting terms, we arrive at

$$\begin{aligned}&\int_{\mathbf{k}'}\int_{\mathbf{k}} [\delta_{\mathbf{p}'\mathbf{k}'} - \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}'}]\tilde{\mathcal{L}}_{\mathbf{k}'\mathbf{k}}\mathcal{T}_{\mathbf{k}\mathbf{p}} \\ &\quad = \mathcal{F}_{\mathbf{p}'}\mathcal{R}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\mathcal{R}_{\mathbf{p}'\mathbf{k}'}\tilde{\mathcal{L}}_{\mathbf{k}'\mathbf{k}}\mathcal{T}_{\mathbf{k}\mathbf{p}}, \end{aligned} \tag{B3}$$

where we have removed the rightmost rescattering function, and collected all terms with $\mathcal{R}_{\mathbf{p}'\mathbf{p}}$ on the right-hand side. Finally, the combination on the left-hand side simplifies to

$$\begin{aligned}\int_{\mathbf{k}'} [\delta_{\mathbf{p}'\mathbf{k}'} - \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}'}]\tilde{\mathcal{L}}_{\mathbf{k}'\mathbf{k}} = &\mathcal{F}_{\mathbf{p}'}\delta_{\mathbf{p}'\mathbf{k}} \\ &+ \left(\mathcal{D}_{\mathbf{p}'\mathbf{k}} - \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}}\mathcal{F}_{\mathbf{k}} - \int_{\mathbf{k}'} \mathcal{F}_{\mathbf{p}'}\mathcal{G}_{\mathbf{p}'\mathbf{k}'}\mathcal{D}_{\mathbf{k}'\mathbf{k}}\right), \end{aligned} \tag{B4}$$

where the term inside the parenthesis is 0 from Eq. (15). Factorizing the final $\mathbf{2}\to\mathbf{2}$ amplitude from the left-hand side, we arrive at the resummed $\mathbf{3}\to\mathbf{3}$ amplitude,

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{k}'}\mathcal{T}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{k}\mathbf{p}}, \tag{B5}$$

with the new amputated amplitude satisfying

$$\mathcal{T}_{\mathbf{p}'\mathbf{p}} = \mathcal{R}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{R}_{\mathbf{p}'\mathbf{k}'}\tilde{\mathcal{L}}_{\mathbf{k}'\mathbf{k}}\mathcal{T}_{\mathbf{k}\mathbf{p}}. \tag{B6}$$

Equation (B5), along with Eqs. (15) and (B6), is an alternative on-shell representation for the $\mathbf{3}\to\mathbf{3}$ scattering amplitude that satisfies unitarity. We now proceed with similar manipulations on the unitarity relation, Eq. (A4), allowing one to derive Eq. (B6) directly from unitarity.

It is clear from the demonstration in Appendix A that $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$ satisfies the same unitarity relation as Eq. (A4). Therefore, the unitarity relation for the divergence-free amplitude states

$$\begin{aligned}\mathrm{Im}\mathcal{A}_{\mathrm{df},\mathbf{p}'\mathbf{p}} = &\int_{\mathbf{k}}\int_{\mathbf{q}} \mathcal{A}^{\dagger}_{\mathrm{df},\mathbf{p}'\mathbf{q}}(\bar{\rho}_{\mathbf{q}}\delta_{\mathbf{qk}} + \mathcal{C}_{\mathbf{qk}})\mathcal{A}_{\mathrm{df},\mathbf{kp}} \\ &+ \int_{\mathbf{k}}\int_{\mathbf{q}} \mathcal{A}^{\dagger}_{\mathrm{df},\mathbf{p}'\mathbf{q}}(\bar{\rho}_{\mathbf{q}}\delta_{\mathbf{qk}} + \mathcal{C}_{\mathbf{qk}})\tilde{\mathcal{L}}_{\mathbf{kp}} \\ &+ \int_{\mathbf{k}}\int_{\mathbf{q}} \tilde{\mathcal{L}}^{\dagger}_{\mathbf{p}'\mathbf{q}}(\bar{\rho}_{\mathbf{q}}\delta_{\mathbf{qk}} + \mathcal{C}_{\mathbf{qk}})\mathcal{A}_{\mathrm{df},\mathbf{kp}}. \end{aligned} \tag{B7}$$

Since $\mathcal{F}_{\mathbf{p}}$ obeys the $\mathbf{2}\to\mathbf{2}$ unitarity relation, Eq. (A2), and $\mathcal{D}_{\mathbf{p}'\mathbf{p}}$ satisfies Eq. (A4), we can see that the rescattering function $\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}}$ satisfies the relation

$$\mathrm{Im}\tilde{\mathcal{L}}_{\mathbf{p}'\mathbf{p}} = \int_{\mathbf{k}} \tilde{\mathcal{L}}^{\dagger}_{\mathbf{p}'\mathbf{k}}\bar{\rho}_{\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{L}}^{\dagger}_{\mathbf{p}'\mathbf{k}'}\mathcal{C}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{L}}_{\mathbf{kp}}. \tag{B8}$$

The amputated divergence-free amplitude can be defined as in Eq. (B2), so that the unitarity relation becomes

$$\begin{aligned}\mathrm{Im}\tilde{\mathcal{T}}_{\mathbf{p}'\mathbf{p}} = &\int_{\mathbf{k}'}\int_{\mathbf{k}}\int_{\mathbf{q}} \tilde{\mathcal{T}}^{\dagger}_{\mathbf{p}'\mathbf{k}'}\tilde{\mathcal{L}}^{\dagger}_{\mathbf{k}'\mathbf{q}}\bar{\rho}_{\mathbf{q}}\tilde{\mathcal{L}}_{\mathbf{qk}}\tilde{\mathcal{T}}_{\mathbf{kp}} \\ &+ \int_{\mathbf{k}'}\int_{\mathbf{k}}\int_{\mathbf{q}'}\int_{\mathbf{q}} \tilde{\mathcal{T}}^{\dagger}_{\mathbf{p}'\mathbf{k}'}\tilde{\mathcal{L}}^{\dagger}_{\mathbf{k}'\mathbf{q}'}\mathcal{C}_{\mathbf{q}'\mathbf{q}}\tilde{\mathcal{L}}_{\mathbf{qk}}\tilde{\mathcal{T}}_{\mathbf{kp}} \\ &= \int_{\mathbf{k}'}\int_{\mathbf{k}} \tilde{\mathcal{T}}^{\dagger}_{\mathbf{p}'\mathbf{k}'}\mathrm{Im}\tilde{\mathcal{L}}_{\mathbf{k}'\mathbf{k}}\tilde{\mathcal{T}}_{\mathbf{kp}}. \end{aligned} \tag{B9}$$

Using similar manipulations as in Appendix A, it is straightforward to verify that Eq. (B6) satisfies the unitarity relation Eq. (B9).

APPENDIX C: PROOF OF EQ. (35)

In Sec. IV, we showed that the R matrix and three-body K matrix are related by an integral equation, Eq. (31). Proving the reality of the Eq. (31) relied on the claim Eq. (35), which we now prove.

From the definition, Eq. (28), we find that

$$\begin{aligned} &3\sum_{\ell'',m_{\ell}''}\int_{\mathbf{k}} \mathrm{Im}[\mathcal{U}_{\mathbf{p}'\mathbf{k}}]_{\ell' m_{\ell}';\ell'' m_{\ell}''}[\mathcal{K}_{\mathrm{df},\mathbf{kp}}]_{\ell'' m_{\ell}'';\ell m_{\ell}} \\ &= \sum_{\ell'',m_{\ell}''}\int_{\mathbf{k}} [2\bar{\rho}_{\mathbf{p}'}\delta_{\mathbf{p}'\mathbf{k}} - \mathcal{C}_{\mathbf{p}'\mathbf{k}}]_{\ell' m_{\ell}';\ell m_{\ell}}[\mathcal{K}_{\mathrm{df},\mathbf{kp}}]_{\ell'' m_{\ell}'';\ell m_{\ell}} \\ &= \sum_{\ell'',m_{\ell}''} \delta_{\ell'\ell''}\delta_{m_{\ell}'m_{\ell}''}2\bar{\rho}_{\mathbf{p}'}[\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}]_{\mathrm{df},\ell'' m_{\ell}'';\ell m_{\ell}} \\ &\quad - \sum_{\ell'',m_{\ell}''}\int_{\mathbf{k}} \pi\delta((P-k-p')^2 - m^2)4\pi Y^{*}_{\ell' m_{\ell}'}(\hat{\mathbf{k}}^{\star})Y_{\ell'' m_{\ell}''}(\hat{\mathbf{p}}'^{\star})[\mathcal{K}_{\mathrm{df},\mathbf{kp}}]_{\ell'' m_{\ell}'';\ell m_{\ell}}, \end{aligned} \tag{C1}$$

where in the second term, Eq. (A5) was used. We leave the first term as is, and focus on the second term. According to Ref. [12], $\mathcal{K}_{\mathrm{df},\mathbf{kp}}$ is defined as a symmetric object after acting with spherical harmonics of the pair orientations on $\mathcal{K}_{\mathrm{df},\mathbf{kp}}$, and summing over all angular momenta. We use this property to combine the product of the final spherical harmonic $Y_{\ell'' m_{\ell}''}(\hat{\mathbf{p}}'^{\star})$ and $\mathcal{K}_{\mathrm{df},\mathbf{kp}}$, and then switch the role of $\mathbf{p}'$ and $\mathbf{k}$, finally expanding in spherical harmonics of $\hat{\mathbf{k}}^{\star}$. This allows us to write

$$\sum_{\ell'',m_{\ell}''} Y_{\ell'' m_{\ell}''}(\hat{\mathbf{p}}'^{\star})[\mathcal{K}_{\mathrm{df},\mathbf{kp}}]_{\ell'' m_{\ell}'';\ell m_{\ell}} = \sum_{\ell'',m_{\ell}''} Y_{\ell'' m_{\ell}''}(\hat{\mathbf{k}}^{\star})[\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}]_{\ell'' m_{\ell}'';\ell m_{\ell}}. \tag{C2}$$

Now, $\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}$ is independent of $\mathbf{k}$; thus we can perform the integrations

$$\begin{aligned} &\sum_{\ell'',m_{\ell}''}\int_{\mathbf{k}} \pi\delta((P-k-p')^2 - m^2)4\pi Y^{*}_{\ell' m_{\ell}'}(\hat{\mathbf{k}}^{\star})Y_{\ell'' m_{\ell}''}(\hat{\mathbf{k}}^{\star})[\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}]_{\ell'' m_{\ell}'';\ell m_{\ell}} \\ &= \frac{1}{4\pi}\int_0^{\infty} \mathrm{d}k^{\star}\,\frac{k^{\star 2}}{4\omega^2_{\mathbf{k}^{\star}}}\delta(\omega_{\mathbf{k}^{\star}} - E^{\star}_{\mathbf{p}'}/2)\sum_{\ell'',m_{\ell}''}\int \mathrm{d}\hat{\mathbf{k}}^{\star}\, Y^{*}_{\ell' m_{\ell}'}(\hat{\mathbf{k}}^{\star})Y_{\ell'' m_{\ell}''}(\hat{\mathbf{k}}^{\star})[\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}]_{\ell'' m_{\ell}'';\ell m_{\ell}} \\ &= 2\bar{\rho}_{\mathbf{p}'}\sum_{\ell'',m_{\ell}''} \delta_{\ell'\ell''}\delta_{m_{\ell}'m_{\ell}''}[\mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{p}}]_{\ell'' m_{\ell}'';\ell m_{\ell}}, \end{aligned} \tag{C3}$$

where we converted to spherical coordinates in the final pair rest frame, and used the composition properties of Dirac delta functions to convert to the on-shell energy $\omega_{\mathbf{k}^\star}$. Orthogonality properties of spherical harmonics allow the angular integration to be done, showing that the second term is identical to the first of Eq. (C1). Thus, we conclude that

$$\int_{\mathbf{k}} \mathrm{Im}\mathcal{U}_{\mathbf{p}'\mathbf{k}}\mathcal{K}_{\mathrm{df},\mathbf{kp}} = 0, \tag{C4}$$

as claimed. The relation, $\int_{\mathbf{k}} \mathcal{K}_{\mathrm{df},\mathbf{p}'\mathbf{k}}\mathrm{Im}\mathcal{U}_{\mathbf{kp}} = 0$, is verified in an identical manner.

[1] S. Durr *et al.*, Science **322**, 1224 (2008).
[2] S. R. Beane, E. Chang, W. Detmold, H. W. Lin, T. C. Luu, K. Orginos, A. Parreno, M. J. Savage, A. Torok, and A. Walker-Loud (NPLQCD Collaboration), Phys. Rev. D **85**, 054511 (2012).
[3] J. J. Dudek, R. G. Edwards, C. E. Thomas, and D. J. Wilson (Hadron Spectrum Collaboration), Phys. Rev. Lett. **113**, 182001 (2014).
[4] R. Williams, C. S. Fischer, and W. Heupel, Phys. Rev. D **93**, 034026 (2016).
[5] M. E. Carrillo-Serrano, W. Bentz, I. C. Cloët, and A. W. Thomas, Phys. Rev. C **92**, 015212 (2015).
[6] D. J. Wilson, R. A. Briceño, J. J. Dudek, R. G. Edwards, and C. E. Thomas, Phys. Rev. D **92**, 094502 (2015).
[7] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, and C. S. Fischer, Prog. Part. Nucl. Phys. **91**, 1 (2016).
[8] G. Eichmann and G. Ramalho, Phys. Rev. D **98**, 093007 (2018).
[9] V. N. Gribov, *Strong Interactions of Hadrons at High Energies: Gribov Lectures on*, edited by Y. L. Dokshitzer and J. Nyiri (Cambridge University Press, Cambridge, England, 2012).
[10] M. Lüscher, Commun. Math. Phys. **105**, 153 (1986).
[11] R. A. Briceño, J. J. Dudek, and R. D. Young, Rev. Mod. Phys. **90**, 025001 (2018).
[12] M. T. Hansen and S. R. Sharpe, Phys. Rev. D **92**, 114509 (2015).
[13] R. A. Briceño, M. T. Hansen, and S. R. Sharpe, Phys. Rev. D **95**, 074510 (2017).
[14] M. T. Hansen and S. R. Sharpe, Phys. Rev. D **90**, 116003 (2014).
[15] M. T. Hansen and S. R. Sharpe, Phys. Rev. D **95**, 034501 (2017).
[16] R. A. Briceño, M. T. Hansen, and S. R. Sharpe, Phys. Rev. D **98**, 014506 (2018).
[17] M. Mai and M. Döring, Eur. Phys. J. A **53**, 240 (2017).
[18] M. Mai and M. Döring, Phys. Rev. Lett. **122**, 062503 (2019).
[19] K. Polejaeva and A. Rusetsky, Eur. Phys. J. A **48**, 67 (2012).
[20] H. W. Hammer, J. Y. Pang, and A. Rusetsky, J. High Energy Phys. 10 (2017) 115.
[21] H. W. Hammer, J. Y. Pang, and A. Rusetsky, J. High Energy Phys. 09 (2017) 109.
[22] M. Döring, H. W. Hammer, M. Mai, J. Y. Pang, A. Rusetsky, and J. Wu, Phys. Rev. D **97**, 114508 (2018).
[23] J.-Y. Pang, J.-J. Wu, H. W. Hammer, U.-G. Meißner, and A. Rusetsky, Phys. Rev. D **99**, 074513 (2019).
[24] M. T. Hansen and S. R. Sharpe, arXiv:1901.00483.
[25] M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak, Eur. Phys. J. A **53**, 177 (2017).
[26] A. Jackura, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, J. Nys, A. Pilloni, K. Saldaña, N. Sherrill, and A. P. Szczepaniak (JPAC Collaboration), Eur. Phys. J. C **79**, 56 (2019).
[27] A. D. Martin and T. D. Spearman, *Elementary-Particle Theory* (North-Holland, Amsterdam, 1970).
[28] R. A. Briceño, M. T. Hansen, S. R. Sharpe, and A. P. Szczepaniak, arXiv:1905.11188.
[29] M. T. Hansen and S. R. Sharpe, Phys. Rev. D **93**, 014506 (2016).
[30] T. D. Blanton, F. Romero-López, and S. R. Sharpe, J. High Energy Phys. 03 (2019) 106.
[31] M. Mikhasenko, Y. Wunderlich, A. Jackura, V. Mathieu, A. Pilloni, B. Ketzer, and A. P. Szczepaniak, arXiv:1904.11894.
[32] T. Lin, C. Elster, W. N. Polyzou, and W. Glöckle, Phys. Rev. C **76**, 014010 (2007).
[33] C. Elster, T. Lin, and W. N. Polyzou, Few Body Syst. **45**, 157 (2009).
[34] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, England, 1966).