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A Simplified Analytical Approach to Calculation of the Electromagnetic Behavior of Left-Handed Metamaterials with a Graded Refractive Index Profile

N. Dalarsson[1], M. Maksimović[2], Z. Jakšić[2*]
[1]Royal Institute of Technology, Stockholm, Sweden
[2]IHTM – Centre of Microelectronic Technologies and Single Crystals, Belgrade, Serbia

Abstract:
We investigated the spectral properties of a new class of nanostructured artificial composite materials with tailored electromagnetic response, i.e. negative refractive index materials, also known as "left-handed" metamaterials. We analyzed structures incorporating both ordinary positive index media and negative refractive index metamaterials where the interface may be graded to an arbitrary degree. Utilizing a modified version of the Rosen-Morse function, we derived analytical expressions for the field intensity and spectral reflection and transmission through a graded interface between positive and negative index materials. We compared our results to numerical solutions obtained using the transfer matrix technique.
Keywords: *Electromagnetic metamaterials, Double negative materials, Left-handed metamaterials, LHM, Gradient index profile.*

Introduction

Negative refractive index metamaterials (NRM) 1, also known as left-handed metamaterials (LHM) are artificial composites structured at subwavelength level, furnishing a negative value of refractive index in a certain wavelength range. The direction of the Pointing vector in an NRM is opposite to that of the wavevector, i.e. the vectors of the electric and magnetic field and the wavevector form a left-oriented set, contrary to conventional materials ("right-handed" – RHM). The first theoretical publication on the topic appeared in 2. In his seminal works Pendry 3, 4, 5 reinvented and generalized the concept. The first experimental confirmations were presented in 6.

Structures containing negative index metamaterials with a gradient refractive index promise practical usability in various applications, for instance in lensing and filtering, for antireflection coatings, etc. Also, any realistic structures containing positive and negative index materials are likely to have a graded profile instead of an abrupt one. Thus these have been extensively studied – e.g. Ramakrishna described a metamaterial lens composed of gradient index media 7. Smith et al 8 proposed the use of metamaterial lenses for the coupling with radiative elements in high-gain antenna applications. Ref. [8] also handles graded index metamaterials experimentally. A numerical study of gradient index structures containing

*) Corresponding author: jaksa@nanosys.ihtm.bg.ac.yu

metamaterials was presented in 9.

The determination of spectral parameters of metamaterial structures is currently mostly done by numerical simulation, and typically either by the finite difference time domain method (FDTD) 10 or by the finite elements method (FEM) 11.

Approximate analytical solutions of the Helmholtz equation for the electric field in conventional materials in the case of graded refractive index were done for some special gradient index profiles (e.g. linear or exponential dependences) 12. As far as the authors know, no analytical calculations of graded metamaterial structures were published until now.

In this paper we solve the Helmholtz equation for a structure with a graded interface between negative and positive refractive index region utilizing a modified version of the Rosen-Morse dependence. In this way we obtain an analytical solution applicable for different practical situations of graded index metamaterial-containing structures. We also utilized the transfer matrix technique to numerically determine the transmittance of graded structures in order to compare it with the analytical solution.

2. Theory

We consider a plane (time-harmonic) electromagnetic wave (~exp($-i\omega t$)) incident to a graded interface between two isotropic and linear media characterized by their spatially-dependent magnetic permeability, dielectric permittivity and refractive index $\mu_1(\vec{r}), \varepsilon_1(\vec{r}), n_1(\vec{r})$ and $\mu_2(\vec{r}), \varepsilon_2(\vec{r}), n_2(\vec{r})$. Starting from the Maxwell equations one can write the Helmholtz wave equation either for an electric or magnetic field as:

$$\nabla^2 \vec{E}(\vec{r}) + \omega^2 \mu(\vec{r})\varepsilon(\vec{r})\vec{E}(\vec{r}) = \nabla\left(\frac{1}{\varepsilon(\vec{r})}\vec{E}(\vec{r})\nabla\varepsilon(\vec{r})\right) \tag{1}$$

$$\nabla^2 \vec{H}(\vec{r}) + \omega^2 \mu(\vec{r})\varepsilon(\vec{r})\vec{H}(\vec{r}) = -\frac{1}{\varepsilon(\vec{r})}\nabla\varepsilon(\vec{r}) \times \left(\nabla \times \vec{H}(\vec{r})\right) \tag{2}$$

We assume that the Poynting vector is directed along the x-axis (i.e. $\vec{H}$ is oriented along the z-axis, and $\vec{E}$ along the x-axis) and thus we can consider a one-dimensional problem. Further, if the length of the graded refractive index region is much smaller than the wavelength, we may write (1) in the following approximate form

$$\frac{d^2E}{dx^2} + \frac{\omega^2 n^2}{c^2}E = 0 \tag{3}$$

Equations (1) to (3) are equivalent in form to the Schrödinger equation, with $n = \sqrt{\mu\varepsilon}$ corresponding to the potential. We search for the solution of (3) by replacing a spatial dependence of refractive index in a modified form of the Rosen-Morse potential 13, 14. Such a potential is widely used in different fields of physics, e.g. in particle theory 15. We write the refractive index profile as:

$$n(\omega) = \frac{1}{2}(n_1 + n_2) - \frac{1}{2}(n_1 - n_2)\tanh\frac{x}{x_0} \tag{4}$$

where x_0 is a parameter describing the slope of the positive/negative index material interface. The dependence (4) is shown in Fig. 1.

It can be seen that for $x_0 \to 0$ the profile would approximate an abrupt one (not used here because of the approximation of slowly varying gradient), while for large values of x_0 it tends to a linear approximation near the interface point.

A conveniently normalized solution for the electrical field is obtained in the following form 15

$$E(u) = E_0 \frac{\Gamma(p+q+1)\Gamma(p+q)}{\Gamma(2p+1)\Gamma(2q)} u^p (1-u)^q F(p+q+1, p+q, 2p+1, u) \tag{5}$$

where E_0 is the initial amplitude of the plane wave of electric field, while $F(p+q+1, p+q, 2p+1, u)$ denotes a hypergeometric function of the variable u with p, q as parameters:

$$u = \frac{1}{1+\exp(2x/x_0)},\ p = -\frac{\omega x_0 n_2}{2c},\ q = -\frac{\omega x_0 n_1}{2c}. \tag{6}$$

Here c denotes the speed of light in vacuum, while ω is angular frequency. At distances on both sides of the interface far enough from the interface itself ($x \to \pm\infty$) the hypergeometric function may be approximated by a gamma-function and we can write

$$E(x \to -\infty) = E_0 \exp\left(i\,|\,q\,|\,\frac{x}{x_0}\right) + E_0 \frac{\Gamma(p+q+1)\,\Gamma(p+q)}{\Gamma(p-q+1)\,\Gamma(p-q)} \frac{\Gamma(-2q)}{\Gamma(2q)} \exp\left(-i\,|\,q\,|\,\frac{x}{x_0}\right) \tag{7}$$

$$E(x \to +\infty) = E_0 \frac{\Gamma(p+q+1)\Gamma(p+q)}{\Gamma(2p+1)\Gamma(2q)} \exp\left(ia\,|\,p\,|\,\frac{x}{x_0}\right),\ a = \begin{cases} 1,\ n_2 > 0 \\ -1,\ n_2 < 0 \end{cases} \tag{8}$$

The energy transmission coefficient of a graded positive-negative structure is finally calculated as the ratio of the squared values of the amplitudes of the transmitted wave and that of the incident wave.

$$T = \left|\frac{\Gamma(p+q+1)\Gamma(p+q)}{\Gamma(2p+1)\Gamma(2q)}\right|^2 \tag{9}$$

Alternatively, one may write

$$T = \frac{\sinh\left[2\pi\frac{\omega x_0 |n_2|}{2c}\right]\sinh\left[2\pi\frac{\omega x_0 |n_1|}{2c}\right]}{\sinh^2\left[\pi\left(\frac{\omega x_0 |n_2|}{2c} + \frac{\omega x_0 |n_1|}{2c}\right)\right]} \tag{10}$$

The above solution is valid for the case $n_1 \neq n_2$.

3. Calculation

We calculated the transmission of our gradient index structure described by (4) both by using (6) and (10) and numerically, utilizing the transfer matrix method (TMM) 12. To do the latter, we divided the calculation region into $N = 800$ layers with a constant refractive index throughout a single layer. The refractive index in the layers was determined as the value in the midline of a layer.

It was assumed that the metamaterial was weakly lossy, $\mathrm{Im}(n_2) = 0.001$, which is acceptable even from the experimental point of view 6. We assumed a frequency dispersion of the complex refractive index in the Drude form

$$\varepsilon(\omega) = 1 - \frac{\omega_{pe}^2}{\omega(\omega + i\Gamma_e)}, \quad \mu(\omega) = 1 - \frac{\omega_{pm}^2}{\omega(\omega + i\Gamma_m)}, \quad n_2(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2} - i\Gamma\frac{\omega_p^2}{\omega^3} \tag{11}$$

where $\omega_{pe} = \omega_{pm} = \omega_p$ is plasma frequency and $\Gamma_{pe} = \Gamma_{pm} = \Gamma_p$ is dumping constant. For low-loss model we consider here $\omega_p >> \Gamma$.

4. Results and Discussion

The Drude model was used to describe negative refractive index dispersion $n_2(\omega)$. We assumed $\mathrm{Re}[n_2(\omega)]=1-\omega_p^2/\omega^2$, where $\omega_p=1.5\cdot 10^{15}$ Hz. The incident medium had a constant and always real non-dispersive refractive index of $n_1=1$. The profiles of the utilized graded refractive index are shown in Fig. 2 for a slowly varying and abrupt spatial dependence.

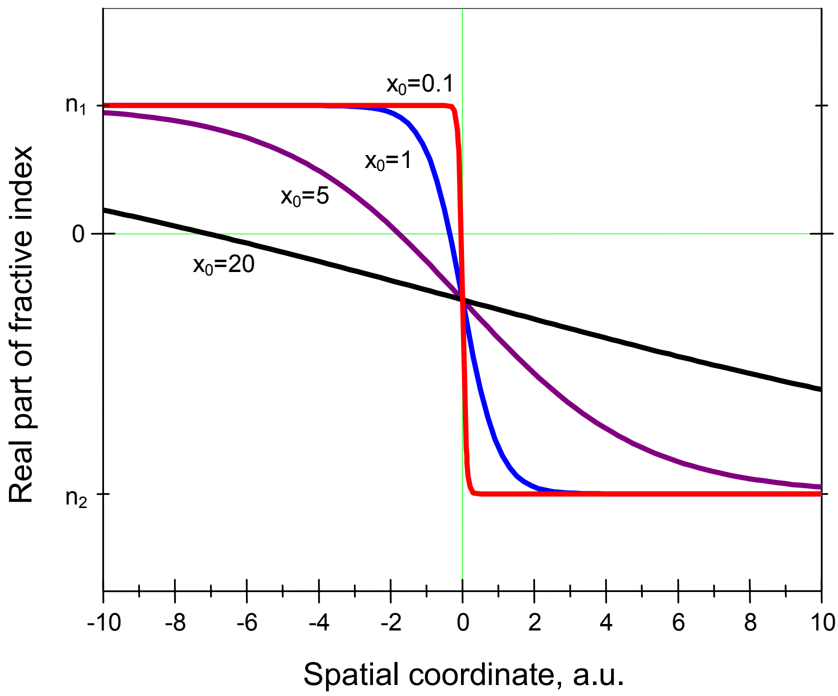

Fig. 1 Spatial profiles of refractive index for different parameters according to eq. (3)

We calculated transmission for the situation shown in Fig. 1 both by the analytical expression (10) and numerically by the TMM. Fig. 3 shows spectral transmission of a graded positive-negative index interface if the grading parameter x_0 was 10^{-4} μm.

It can be seen that the TMM results oscillate around the mean value which is very close to the analytical solution. The reason is that the division of the graded region into homogeneous slabs introduces spurious additional reflections between slabs which are strongly frequency-dependent. An averaged TMM dependence agrees relatively well with the analytical solution (and the agreement is better for larger wavelengths).

A point of discontinuity of the first kind can be seen at the crossover wavelength (near 1.5 μm) where the refractive index and transmission reach zero value.

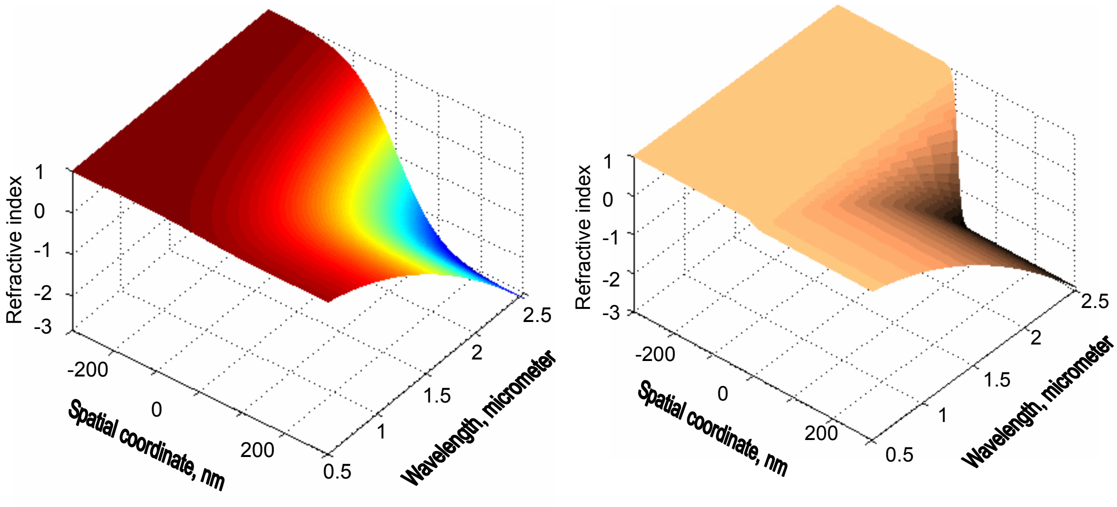

Fig. 2 Profiles of the graded refractive index interfaces for slowly varying (left) and abrupt spatial dependence (right). In both cases Drude model is assumed.

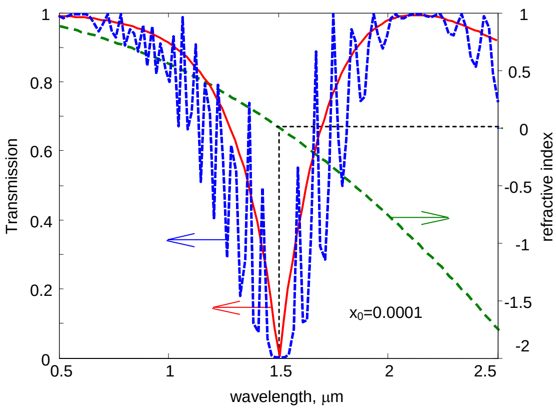

Fig. 3 Comparison of approximate analytical solution (dotted) and numerical TMM (solid) results. Dispersion of negative refractive index is also shown in the same diagram (dashed, right-hand axis)

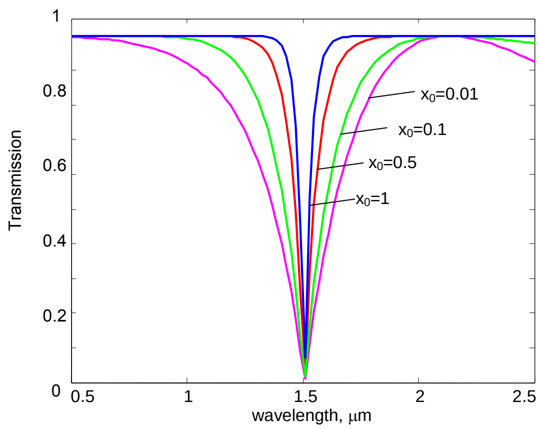

Fig. 4 Transmission of a graded positive-negative index interface for different values of x_0.

Fig. 4 shows the analytical spectral transmissions for different values of x_0 parameter. It can be seen that the spectral dependence becomes higher and less sensitive to wavelength with an increase of the grading parameter, i.e. with a lower gradient of the refractive index. This means that more gradual transitions mean a stronger antireflection behavior of the interface, an effect also met in all-positive index structures. With a higher value of the grading parameter the agreement between numerical and analytical solutions becomes worse, but even for relatively high values of x0 (0.1-0.5 μm) still allows a qualitative prediction of the graded interface behavior.

5. Conclusion

We considered a time-harmonic electromagnetic wave propagation through NRM-containing slabs with a graded refractive index. We derived approximate analytical formulas assuming a graded region thickness much smaller than the operating wavelength. For comparison, we also utilized numerical calculation using the transfer matrix method. The calculations were done for lossy media and for the dispersion in the negative part assuming the Drude form. The approximation is valid in a relatively narrow range of small graded region thickness values, but at the same time that range corresponds to realistic profiles of experimental NRM-containing structures.

Acknowledgment

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6. Literature

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Садржај: *Истраживали смо спектрална својства нове класе наноструктурних вештачких композитних материјала с пројектованим електромагнетским одзивом, материјала с негативних индексом преламања, познатих и као "леворуки" материјали. Анализирали смо структуре с градијентом индекса преламања које садржe и обичан материјал с позитивним индексом и метаматеријале с негативним индексом преламања. Користећи модификовани облик Розен-Морзеове функције извели смо аналитичке изразе за интензитет поља и спектралну рефлексију и трансмисију кроз градирани спој измежђу материјала с позитивним и негативним индексом. Наше* *резултате упоредили смо с нумеричким решењима добијеним методом матрица преноса.*

Кључне речи*: Електромагнетски метаматеријали, двоструко негативни материјали, леворуки метаматеријали, LHM, градијентни профил индекса преламања.*