Yugoslav Journal of Operations Research
14 (2004), Number 1, 83-97

THREE DIMENSIONAL FIXED CHARGE BI-CRITERION INDEFINITE QUADRATIC TRANSPORTATION PROBLEM*

S.R. ARORA

Department of Mathematics, Hans Raj College, University of Delhi
Delhi-110007, India.
srarora@yahoo.com

Archana KHURANA

Department of Mathematics, University of Delhi, Delhi-110007, India
archana2106@rediffmail.com, archana@du.ac.in

Received: October 2002 / Accepted: August 2003

Abstract: The three-dimensional fixed charge transportation problem is an extension of the classical three-dimensional transportation problem in which a fixed cost is incurred for every origin. In the present paper three-dimensional fixed charge bi-criterion indefinite quadratic transportation problem, giving the same priority to cost as well as time, is studied. An algorithm to find the efficient cost-time trade off pairs in a three dimensional fixed charge bi-criterion indefinite quadratic transportation problem is developed. The algorithm is illustrated with the help of a numerical example.

Keywords: Three dimensional quadratic transportation problem, cost-time trade-off pairs, fixed charge, bi-criterion indefinite quadratic transportation problem

1. INTRODUCTION

In the classical transportation problem the cost of transportation is directly proportional to the number of units of the commodity transported. But in real world situations when a commodity is transported, a fixed cost is incurred in the objective function. The fixed cost may represent the cost of renting a vehicle, landing fees in an airport, set up costs for machines in a manufacturing environment etc.

---

[*] 1991 Mathematics Subject Classification: Primary: 90B06; Secondary: 90C08

The fixed charge transportation problem was originally formulated by G.B.Dantzig and W. Hirsch [9] in 1954. Then in 1968 K.G.Murty [11] solved the fixed charge problem by ranking the extreme points. After that several procedures for solving fixed charge transportation problems were developed.

Sometimes there may exist emergency situations such as fire services, ambulance services, police services etc. when the time of transportation is of greater importance than the cost of transportation. Several methods [5, 12] for minimizing the time of transportation are also developed.

In 1976 Bhatia [4] et.al. provided the time-cost trade-off pairs in a linear transportation problem. Also in 1994 Basu et. al.[2] developed an algorithm for the optimum time-cost trade-off in a fixed charge linear transportation problem giving same priority to cost and time.

The transportation problem considered in the classical transportation problem is generally a two-dimensional linear transportation problem. Haley [6] in 1962 described the solution of a linear multi-index transportation problem where there are three indices. The method for solution presented by Haley is an extension of MODI method. In 1994, Basu et al. [3] provided an algorithm for finding the optimum solution of the solid fixed charge linear transportation problem.

In this paper three dimensional fixed charge bi-criterion **indefinite quadratic** transportation problem, giving the same priority to cost and time, is studied. An algorithm to identify the efficient cost-time trade-off pairs for the problem is developed.

2. PROBLEM FORMULATION

Suppose $i = 1, 2, ..., m$ are the origins
$j = 1, 2, ..., n$ are the destinations
and $k = 1, 2, ..., p$ are the various types of commodities to be transported in a three dimensional transportation problem.

Let

x_{ijk} = the amount of *k*th type of commodity transported form the *i*th origin to the *j*th destination

c_{ijk} = the variable cost per unit amount of the *k*th type of commodity transported from the *i*th origin to the *j*th destination which is independent of the amount of the commodity transported, so long as $x_{ijk} > 0$

d_{ijk} = the per unit depreciation cost (wear and tear or damaged cost) of the *k*th type of commodity transported from the *i*th origin to the *j*th destination, which is independent of the amount of commodity transported, so long as $x_{ijk} > 0$.

A_{jk} = the total quantity of *k*th type of the commodity received by *j*th destination from all the sources

B_{ki} = the total quantity of the *k*th type of the commodity available at the *i*th origin to be supplied to all destinations

$E_{ij} =$ the total quantity of all types of commodities to be supplied from ith origin to the jth destination.

Then the three dimensional transportation problem is defined as

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk} \qquad (P_0)$$

subject to

$$\sum_{i=1}^{m} x_{ijk} = A_{jk}, \quad j = 1,2,...,n, \; k = 1,2,...,p$$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}, \quad k = 1,2,...,p, \; i = 1,2,...,m$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}, \quad i = 1,2,...,m, \; j = 1,2,...,n$$

$$x_{ijk} \geq 0, \quad i = 1,2,...,m, \; j = 1,2,...,n, \; k = 1,2,...,p$$

Here, there are m origins, n destinations and p types of commodities to be transported.

Also,

$$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki}, \; k = 1,2,...,p \qquad \text{(i)}$$

$$\sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij}, \; i = 1,2,...,m \qquad \text{(ii)}$$

$$\sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}, \; j = 1,2,...,n \qquad \text{(ii)}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p} \sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij} \qquad \text{(iv)}$$

(i) implies kth type of commodity received by all destinations = kth type of commodity supplied from all origins.

(ii) implies different types of commodities supplied by the ith source = amount of commodities received by all destinations from the ith source

(iii) implies amount of commodities supplied from all sources to jth destination = different types of commodities received by the jth destination.

(iv) implies amount of commodities received by all destinations of different types of commodities = amount of commodities supplied from all origins to all destinations = amount of different types of commodities supplied from all origins.

Note: (i) to (iv) indicates that the transportation problem (P_0) considered is a balanced transportation problem.

Now let, $F_{ik} =$ the fixed cost associated with origin i and the kth type of commodity. We define F_{ik} according to the amount supplied as

$$F_{ik} = \sum_{j=1}^{n} F_{ijk}\delta_{ijk}, \ i = 1,2,...,m, \ k = 1,2,..., p$$

where $\delta_{ijk} = \begin{cases} 1 \ \text{if} \ x_{ijk} > 0, \\ 0 \ \text{if} \ x_{ijk} = 0, \end{cases}, \quad i = 1,2,...,m, \ j = 1,2,...,n, \ k = 1,2,..., p$

Now, consider the three dimensional fixed charge bi-criterion indefinite quadratic transportation problem as

$$\min \left\{ \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}x_{ijk} \right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} d_{ijk}x_{ijk} \right) + \sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik}, \ \max_{\substack{1\le i\le m \\ 1\le j\le n \\ 1\le k\le p}} [t_{ijk} / x_{ijk} > 0] \right\} \qquad (P_1)$$

subject to

$$\left.\begin{aligned}
&\sum_{i=1}^{m} x_{ijk} = A_{jk}, \ j = 1,2,...,n, \ k = 1,2,..., p \\
&\sum_{j=1}^{n} x_{ijk} = B_{ki}, \ k = 1,2,..., p, \ i = 1,2,...,m \\
&\sum_{k=1}^{p} x_{ijk} = E_{ij}, \ i = 1,2,...,m, \ j = 1,2,...,n \\
&x_{ijk} \ge 0, \ i = 1,2,...,m, \ j = 1,2,...,n, \ k = 1,2,..., p \\
&\text{Also,} \\
&\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki}, \ k = 1,2,..., p \\
&\sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij}, \ i = 1,2,...,m \\
&\sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}, \ j = 1,2,...,n \\
&\sum_{j=1}^{n}\sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p}\sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}
\end{aligned}\right\} \qquad (1)$$

In the problem (P_1), we need to minimize the transportation cost and depreciation cost simultaneously of the *k*th type of the product to be transported from the *i*th origin to *j*th destination. Also we need to minimize the total cost (variable cost + fixed cost) and the total time of transportation. Therefore we have considered the objective function of the form as in problem (P_1).

3. THEORETICAL DEVELOPMENT:

To solve the problem (P_1) we separate it into two problems (P'_1) and (P''_1) as

$$\min Z = \left\{ \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} d_{ijk} x_{ijk} \right) + \sum_{i=1}^{m} \sum_{k=1}^{p} F_{ik} \right\} \text{ subject to (1)} \qquad (P'_1)$$

$$\min T = \left\{ \max_{\substack{1 \le i \le m \\ 1 \le j \le n \\ 1 \le k \le p}} [t_{ijk} / x_{ijk} > 0] \right\} \text{ subject to (1)} \qquad (P''_1)$$

To obtain the set of efficient cost-time trade off pairs, we first solve (P'_1) and read the time with respect to the minimum cost Z where time T is given by the problem (P''_1).

At the first iteration, let Z_1^* be the minimum total cost of the problem (P'_1) and T_1^* be the optimal time of the problem (P''_1) with respect to Z_1^*, then any schedule which is completed earlier than T_1^* would cost more than Z_1^*. So (Z_1^*, T_1^*) is called the time-cost trade off pair at the first iteration.

After modifying the costs with respect to the time obtained, a new optimal cost is obtained and time is read with respect to the new optimal cost. This procedure is called re-optimization procedure. Let after *q*th iteration, the solution be infeasible. Thus we get the following complete set of time – cost trade off pairs,

$$(Z_1^*, T_1^*),\ (Z_2^*, T_2^*),\ (Z_3^*, T_3^*), \ldots, (Z_q^*, T_q^*)$$

where

$$Z_1^* < Z_2^* < Z_3^* < \ldots < Z_q^*$$

and

$$T_1^* > T_2^* > T_3^* > \ldots > T_q^* .$$

The pairs so defined are pareto-optimal solutions of the given problem.

Then we identify the minimum cost Z_1^* and minimum time T_q^* among the above trade-off pairs. The pair (Z_1^*, T_q^*) with minimum cost and minimum time is termed as the ideal solution which can not be achieved in practical situations.

Consider a three dimensional quadratic transportation problem as

$$\min Z = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} d_{ijk} x_{ijk} \right) \qquad (\mathrm{P}_2)$$

subject to

$$\sum_{i=1}^{m} x_{ijk} = A_{jk}, \quad j = 1,2,...,n, \; k = 1,2,...,p$$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}, \quad k = 1,2,...,p, \; i = 1,2,...,m$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}, \quad i = 1,2,...,m, \; j = 1,2,...,n$$

and $x_{ijk} \geq 0, \; i = 1,2,...,m, \; j = 1,2,...,n, \; k = 1,2,...,p$

Theorem 1. *Let* $X = \{x_{ijk}\}$ *be a basic feasible solution of problem* (P_2) *with basis matrix B. Then it will be an optimal basic feasible solution if*

$$R_{ijk} \geq 0, \; \forall \text{ cells } (i,j,k) \notin B$$

$$= 0, \; \forall \text{ cells } (i,j,k) \in B$$

where

$$R_{ijk} = \theta_{ijk} (z'_{ijk} - d_{ijk})(z_{ijk} - c_{ijk}) - Z_1 (z'_{ijk} - d_{ijk}) - Z_2 (z_{ijk} - c_{ijk})$$

$$u_{jk} + v_{ki} + w_{ij} = z_{ijk}, \; \forall \text{ cells } (i,j,k) \notin B$$

$$u'_{jk} + v'_{ki} + w'_{ij} = z'_{ijk}, \; \forall \text{ cells } (i,j,k) \notin B$$

Also

$$\left.\begin{aligned} u_{jk} + v_{ki} + w_{ij} &= c_{ijk}, \; \forall \text{ cells } (i,j,k) \in B \\ u'_{jk} + v'_{ki} + w'_{ij} &= d_{ijk}, \; \forall \text{ cells } (i,j,k) \in B \end{aligned}\right\} \qquad (2)$$

Z_1 *be the value of* $\sum_i \sum_j \sum_k c_{ijk} x_{ijk}$ *at the current basic feasible solution corresponding to basis matrix B.*

Z_2 *be the value of* $\sum_i \sum_j \sum_k d_{ijk} x_{ijk}$ *at the current basic feasible solution corresponding to basis matrix B.*

θ_{ijk} *is the level at which a non-basic cell* (i,j,k) *enters the basis replacing some basic cell of B.*

Note: $u_{jk}, v_{ki}, w_{ij}, u'_{jk}, v'_{ki}, w'_{ij}$ are determined by using equations (2) and taking $m+n+p-1$ of the u_{jk} 's or v_{ki} 's or w_{ij} 's and u'_{jk} 's or v'_{ki} 's or w'_{ij} 's as zero.

Proof: Let Z^0 be the objective function value of the problem (P_2).

Let $Z^0 = Z_1 Z_2$

Let $\hat{Z}$ be the value of the objective function at the current basic feasible solution $\hat{X} = \{x_{ijk}\}$ corresponding to the basis **B** obtained on entering the cell (i, j, k) into the basis. Then $\hat{Z} = [Z_1 + \theta_{ijk}(c_{ijk} - z_{ijk})][Z_2 + \theta_{ijk}(d_{ijk} - z'_{ijk})]$

Now,

$$\begin{aligned}\hat{Z} - Z^0 &= [Z_1 + \theta_{ijk}(c_{ijk} - z_{ijk})][Z_2 + \theta_{ijk}(d_{ijk} - z'_{ijk})] - Z_1 Z_2 \\ &= Z_1 Z_2 + Z_1\theta_{ijk}(d_{ijk} - z'_{ijk}) + Z_2\theta_{ijk}(c_{ijk} - z_{ijk}) + \theta_{ijk}^2(c_{ijk} - z_{ijk})(d_{ijk} - z'_{ijk}) - Z_1 Z_2 \\ &= Z_1\theta_{ijk}(d_{ijk} - z'_{ijk}) + Z_2\theta_{ijk}(c_{ijk} - z_{ijk}) + \theta_{ijk}^2(c_{ijk} - z_{ijk})(d_{ijk} - z'_{ijk}) \\ &= \theta_{ijk}[Z_1(d_{ijk} - z'_{ijk}) + Z_2(c_{ijk} - z_{ijk}) + \theta_{ijk}(c_{ijk} - z_{ijk})(d_{ijk} - z'_{ijk})]\end{aligned}$$

This basic feasible solution will give an improved value of Z if $\hat{Z} < Z^0$. i.e., if $\hat{Z} - Z^0 < 0$ i.e., if $\theta_{ijk}[Z_1(d_{ijk} - z'_{ijk}) + Z_2(c_{ijk} - z_{ijk}) + \theta_{ijk}(c_{ijk} - z_{ijk}) < (d_{ijk} - z'_{ijk})] < 0$ since $\theta_{ijk} \geq 0$

$$\therefore \; [Z_1(d_{ijk} - z'_{ijk}) + Z_2(c_{ijk} - z_{ijk}) + \theta_{ijk}(c_{ijk} - z_{ijk})(d_{ijk} - z'_{ijk})] < 0 \tag{3}$$

$\Rightarrow$ One can move from one basic feasible solution to another basic feasible solution on entering the cell (i, j, k) into the basis for which condition (3) is satisfied.

It will be an optimal basic feasible solution if

$$\theta_{ijk}(z'_{ijk} - d_{ijk})(z_{ijk} - c_{ijk}) - Z_1\,(z'_{ijk} - d_{ijk}) - Z_2(z_{ijk} - c_{ijk}) \geq 0$$

or $\quad R_{ijk} \geq 0 \;\; \forall$ cells $(i, j, k) \notin B$

where $R_{ijk} = \theta_{ijk}(z'_{ijk} - d_{ijk})(z_{ijk} - c_{ijk}) - Z_1\,(z'_{ijk} - d_{ijk}) - Z_2(z_{ijk} - c_{ijk})$

Also, it can easily be seen that $R_{ijk} = 0 \;\; \forall$ cells $(i, j, k) \in B$

ALGORITHM:

Step 1: Find the initial basic feasible solution of the problem (P'_1).

Step 2: Calculate the fixed cost of the current basic feasible solution and denote this by F^1 (current), where F^1 (current) = $\sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik}$

Step 3: Calculate $R_{ijk}^1 \;\; \forall$ cells $(i, j, k) \notin B$, B is the current basis.

where $\quad R_{ijk}^1 = \theta_{ijk}(z'_{ijk} - d_{ijk})(z_{ijk} - c_{ijk}) - Z_1\,(z'_{ijk} - d_{ijk}) - Z_2(z_{ijk} - c_{ijk})$

and $u_{jk}+v_{ki}+w_{ij}=z_{ijk}\quad \forall$ cells $(i,j,k)\notin B$

$u'_{jk}+v'_{ki}+w'_{ij}=z'_{ijk}\quad \forall$ cells $(i,j,k)\notin B$

Step 4: Find $A^1_{ijk} = R^1_{ijk} \times E^1_{ijk}$

where A^1_{ijk} is the change in cost that occurs on introducing a non-basic cell (i,j,k) with value E^1_{ijk} $(\forall\,(i,j,k)\notin B$) into the basis.

Step 5: Find F^1_{ijk} (Difference) $=F^1_{ijk}$ (NB) $-F^1$ (current) where F^1_{ijk} (NB) is the total fixed cost involved on introducing the variable x_{ijk} with values $(E_{ijk})_1$ $(\forall\,(i,j,k)\notin B$) into the current basis to form a new basis.

Step 6: Calculate $\Delta^1_{ijk} = F^1_{ijk}$ (Difference) + A^1_{ijk} $(\forall\,(i,j,k)\notin B$)

Step 7: If all $\Delta^1_{ijk}\geq 0$ then go to step 8, otherwise find $\min\{\Delta^1_{ijk}/\Delta^1_{ijk}<0,\ (i,\ j,\ k)\ \notin B\}$. Let its minimum be Δ^1_{pqr}. Then cell (p,q,r) enters the basis. Go to step 2.

Step 8: Let Z^*_1 be the optimal cost of (P'$_1$) and X^*_1 be the optimal solution of (P'$_1$) corresponding to Z^*_1.

Step 9: Find $T^*_1=\max\{t_{ijk}/x_{ijk}>0\}$

Step 10: Define $c^1_{ijk}=\begin{cases}M & \text{if}\ \ t_{ijk}\geq T^*_1\\ c_{ijk} & \text{if}\ \ t_{ijk}<T^*_1\end{cases}$, where M is a sufficiently large positive number.

Step 11: Find a basic feasible solution of the problem (P'$_1$) with respect to the new variable costs c^1_{ijk}. Go to step 2 and repeat the process.

Step 12: Let after the qth iteration, the solution is infeasible. Then identify the complete set of efficient cost time trade off pairs.

4. NUMERICAL ILLUSTRATION

Consider a three dimensional fixed charge bi-criterion transportation problem.

$$\min\left\{\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}c_{ijk}x_{ijk}\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}d_{ijk}x_{ijk}\right)+\sum_{i=1}^{3}\sum_{k=1}^{3}F_{ik},\ \max_{\substack{1\leq i\leq 3\\ 1\leq j\leq 3\\ 1\leq k\leq 3}}[t_{ijk}/x_{ijk}>0]\right\} \qquad \text{(P)}$$

subject to

$$\sum_{i=1}^{3}x_{ijk}=A_{jk}$$

$$\sum_{j=1}^{3}x_{ijk}=B_{ki}$$

$$\sum_{k=1}^{3} x_{ijk} = E_{ij}$$

$$x_{ijk} \geq 0$$

where

$$\sum_{j=1}^{3}\sum_{k=1}^{3} A_{jk} = \sum_{k=1}^{3}\sum_{i=1}^{3} B_{ki} = \sum_{i=1}^{3}\sum_{j=1}^{3} E_{ij}$$

The data of variable cost c_{ijk} and time t_{ijk} is given Table 1 and Table 1' respectively.

Table 1.

	j =1	j =2	j =3	B_{ki} ↓
c_{111} ← d_{111} ← i =1	7 4 6 8 5 3 E_{11} = 10	9 4 4 2 6 1 E_{12} = 6	5 6 8 7 3 6 E_{13} = 9	6 9 10
i =2	3 8 6 4 10 6 E_{21} = 21	9 8 7 2 5 2 E_{22} = 9	4 7 7 6 5 4 E_{23} = 14	13 14 17
i =3	5 10 6 8 7 1 E_{31} = 21	8 3 2 9 11 8 E_{32} = 13	5 9 7 8 6 6 E_{33} = 12	15 13 18
A_{jk} →	15 17 20	8 11 9	11 8 16	

The fixed costs are

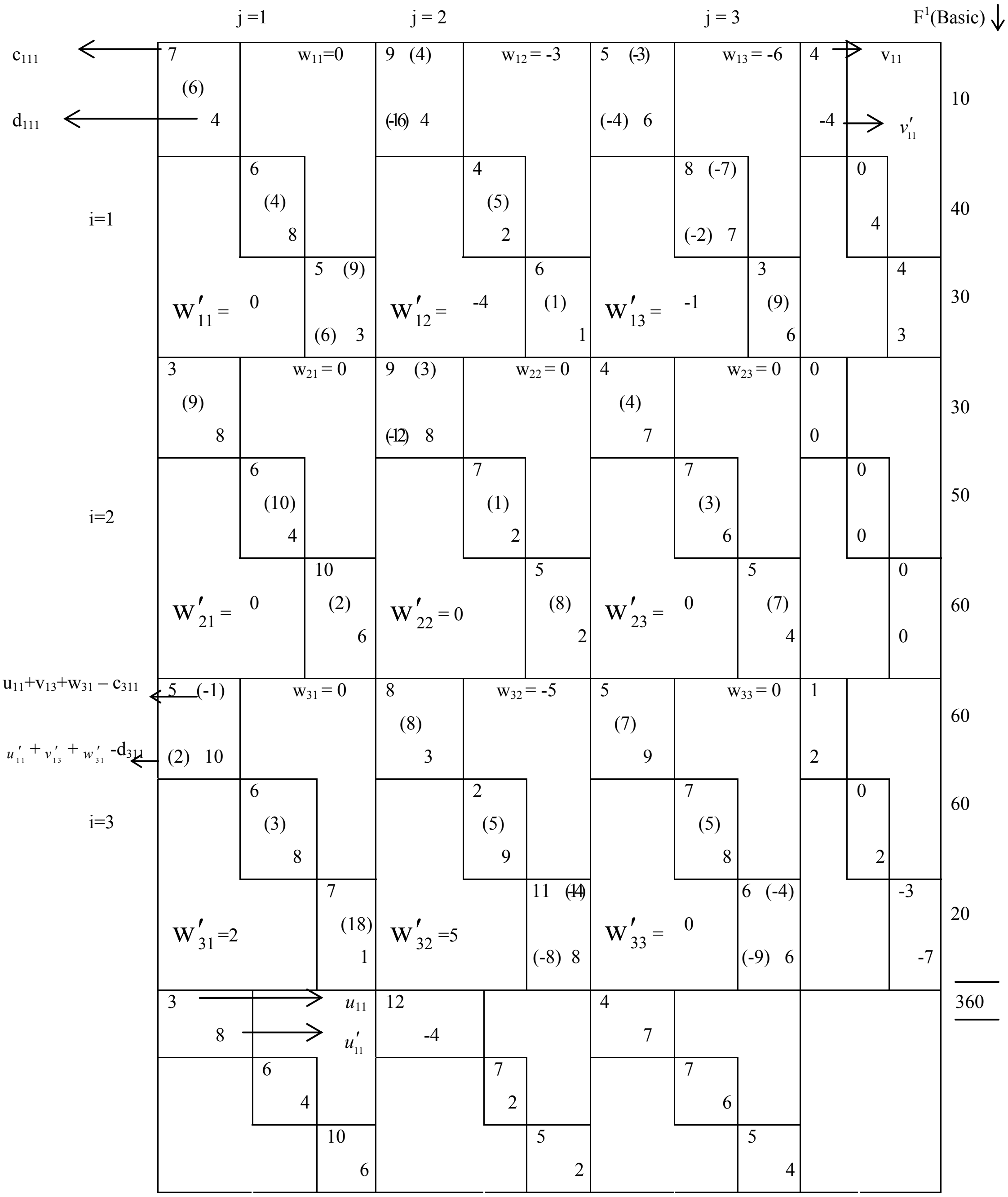
F_{111}= 10, F_{121}=30, F_{131}=20
F_{112}=20, F_{122}=20, F_{132}=20
F_{113}=30, F_{123}=20, F_{133}=10
F_{211}=10, F_{221}=20, F_{231}=20
F_{212}=10, F_{222}=10, F_{232}=30
F_{213}=40, F_{223}=10, F_{233}=10
F_{311}=10, F_{321}=40, F_{331}=20
F_{312}=20, F_{322}=10, F_{332}=30
F_{313}=20, F_{323}=10, F_{333}=10

Table 1'.

	j=1			j=2			j=3		
t_{111} →	3			8			7		
i =1		5			5			8	
			8			3			5
	2			5			6		
i =2		1			7			4	
			6			2			1
	7			3			4		
i = 3		6			2			2	
			1			8			5

Using the North-West Corner Rule, we find the initial basic feasible solution of problem (P) as given in Table 2.

Table 2.



c_{111}

d_{111}

$u_{11}+v_{13}+w_{31}-c_{311}$

$u'_{11}+v'_{13}+w'_{31}$ -d_{311}

	j =1	j = 2	j = 3		F¹(Basic) ↓
i=1	7 (6) 4 w_{11}=0	9 (4) (-16) 4 w_{12} = -3	5 (-3) (-4) 6 w_{13} = -6	4 → v_{11} -4 → v'_{11}	10
	6 (4) 8	4 (5) 2	8 (-7) (-2) 7	0 4	40
	W'_{11} = 0 5 (9) (6) 3	W'_{12} = -4 6 (1) 1	W'_{13} = -1 3 (9) 6	4 3	30
i=2	3 (9) 8 w_{21} = 0	9 (3) (-12) 8 w_{22} = 0	4 (4) 7 w_{23} = 0	0 0	30
	6 (10) 4	7 (1) 2	7 (3) 6	0 0	50
	W'_{21} = 0 10 (2) 6	W'_{22} = 0 5 (8) 2	W'_{23} = 0 5 (7) 4	0 0	60
i=3	5 (-1) (2) 10 w_{31} = 0	8 (8) 3 w_{32} = -5	5 (7) 9 w_{33} = 0	1 2	60
	6 (3) 8	2 (5) 9	7 (5) 8	0 2	60
	W'_{31} =2 7 (18) 1	W'_{32} =5 11 (-14) (-8) 8	W'_{33} = 0 6 (-4) (-9) 6	-3 -7	20
	3 → u_{11} 8 → u'_{11}	12 -4	4 7		360
	6 4	7 2	7 6		
	10 6	5 2	5 4		

Here, $Z_1 = 633,\ Z_2 = 551$

Table 3.

(i,j,k)	(1,1,3)	(1,2,1)	(1,3,1)	(1,3,2)	(2,2,1)	(3,1,1)	(3,2,3)	(3,3,3)
$A^1_{ijk} = R^1_{ijk} \times E^1_{ijk}$	8703×1 = -8703	7668×4 =30672	No loop	2549×3 =7647	5907×1 =5907	-721×3 = -2163	No loop	No loop
F^1_{ijk} (Diff.)	0	10		-10	10	-40		
$\Delta^1_{ijk} = A^1_{ijk} + F^1_{ijk}$ (Diff.)	-8703	30682		7637	5917	-2203		

Here, $\min\{\Delta^1_{ijk} / \Delta^1_{ijk} < 0,\ (i,j,k) \notin B\} = \min\{-8703, -2203\} = -8703$

$\therefore$ cell (1,1,3) enters the basis. We find the new solution.

Repeat the process. The optimal basic feasible solution is given in Table 4.

Table 4.

	j =1	j =2	j = 3	F^4(Basic) ↓	
i=1	7 (6) 4 w_{11}= 0	9 (3) (-14) 4 w_{12}=2	5 (3) (-4) 6 w_{13}=1	2 ← v_{11} -6 ← v'_{11}	10
	6 (2) 8	4 (6) 2	8 (1) 7	0 0	60
	5 (2) 3 w'_{11} = 0	6 (-6) (0) 1 w'_{12} = -7	3 (8) 6 w'_{13} = -1	-2 2	40
i=2	3 (7) 8 w_{21}= -1	9 (2) (-10) 8 w_{22}= 4	4 (6) 7 w_{23}= 0	-1 -2	30
	6 (14) 4	7 (0) 2	7 (1) (-2) 6	1 -4	10
	10 (-3) (-6) 6 w'_{21} = 0	5 (9) 2 w'_{22} = -3	5 (8) 4 w'_{23} =0	1 -1	20
i=3	5 (2) 10 w_{31}= 0	8 (8) 3 w_{32}= 0	5 (5) 9 w_{33}= 0	0 0	70
	6 (1) 8	2 (5) 9	7 (7) 8	0 0	60
	7 (18) 1 w'_{31} = 0	11 (-14) (-8) 8 w'_{32} = 0	6 (-2) (-1) 6 w'_{33} = 0	0 0	20
	5 ← u_{11} 10 ← u'_{11}	8 3	5 9		320
	6 8	2 9	7 8		
	7 1	0 6	4 5		

Here, $Z_1 = 624,\ Z_2 = 533$

Table 5.

(i,j,k)	(1,2,1)	(1,2,3)	(1,3,1)	(2,1,3)	(2,2,1)	(2,3,2)	(3,2,3)	(3,3,3)
$A^4_{ijk} = R^4_{ijk} \times E^4_{ijk}$	5373×1 =5373	No loop	885×1 =885	No loop	0	711×2=1422	No loop	1694×2 =3388
F^4_{ijk} (Diff.)	10		-20		0	20		-10
$\Delta^4_{ijk} = A^4_{ijk} + F^4_{ijk}$ (Diff.)	5374		865		0	1442		3378

Since $\Delta^4_{ijk} \geq 0$, $\forall\ (i,j,k) \notin B\ \therefore$ we stop.

The optimal value of $Z = Z_1^* = 624 \times 533 + 320 = 332912$ and the corresponding time $T_1^* = 8$. The first cost-time trade off pair is (332912, 8).

Define

$$c^1_{ijk} = \begin{cases} M, & t_{ijk} \geq T_1^* = 8 \\ c_{ijk}, & t_{ijk} < T_1^* = 8. \end{cases}$$

and form the new three dimensional quadratic transportation problem. On solving this problem the next trade off pair is $(Z_2^*, T_2^*) = (336940,\ 7)$.

Proceeding like this, we get the third cost-time trade-off pair as $(Z_3^*, T_3^*) = (349143,\ 6)$.

After that the problem defined at time T_3^* becomes infeasible.

Hence the time-cost trade-off pairs are (332912, 8), (346940, 7) and (349143, 6).

Conclusion 1. For finding efficient cost-time trade off pairs in a three-dimensional indefinite quadratic transportation problem **(A)** First we minimize total cost (variable cost + fixed cost) and then minimize time with respect to minimum cost obtained and form the first cost-time trade-off pair.

(B) Next after modifying cost with respect to the time obtained in the last result we again minimize cost, read the corresponding time and form the next cost-time trade-off pair.

(C) Therefore we keep on increasing the cost and reading the time on each step and find the efficient cost-time trade off pairs. (B) is repeated until the problem becomes infeasible.

Remark. 1. An alternative approach to solve problem (P_1) is to first minimize the time function and then read the corresponding cost (variable + fixed) from the solution which gives the minimum time. Thereafter, we keep on increasing the time steadily and reading the cost at each step. We continue till the solution becomes infeasible.

2. Problem (P'_1) cannot be solved by the method developed for linear transportation problem as the variable δ_{ijk} takes the value 0 or 1. So we first find the basic feasible solution of the problem (P_2) and then calculate the corresponding fixed charge.

Acknowledgements:

1. The authors are grateful to referees for their valuable comments and suggestions.

2. The author Ms. Archana Khurana is also grateful to Council of Scientific and Industrial Research, New Delhi for the financial assistance.

REFERENCES

[1] Ahuja, A., and Arora, S.R., “Multi-index fixed charge bi-criterion transportation problem”, *Indian Journal of Pure and Applied Mathematics*, 32(5) (2001) 739-746.

[2] Basu, M., Pal, B.B., and Kundu, A., “An algorithm for optimum time-cost trade-off in fixed charge bi-criterion transportation problem”, *Optimization*, 30 (1994) 53-68.

[3] Basu, M., Pal, B.B., and Kundu, A., “An algorithm for finding the optimum solution of solid fixed-charge transportation problem”, 31 (1994) 283-291.

[4] Bhatia, H.L., Swaroop, K., and Puri, M.C., “Time-cost trade-off in a transportation problem”, *Opsearch*, 13 (1976) 129-142.

[5] Bhatia, H.L., Swaroop, K., and Puri, M.C., “A procedure for time minimization transportation problem”, *Indian Journal of Pure and Applied Mathematics*, 8 (1977) 920- 929.

[6] Haley, K.B, “The solid transportation problem”, *Operations Research*, (1962) 448-463.

[7] Haley, K.B., “The multi-index problem”, *Operations Research*, (1963) 368-379.

[8] Hammer, Peter, L., “Time-minimizing transportation problems”, *Naval Research Logistics Quarterly*, 16 (1969) 345- 357.

[9] Hirsch, W.M., and Dantzig, G.B., “Notes on linear programming: Part XIX, The fixed charge problem”, Rand Research Memorandum No. 1383, Santa Monica, California, 1954.

[10] Kuhn, H., and Baumol, W., "An approximate algorithm for the fixed-charge transportation problem", *Naval Research Logistics Quarterly*, 9 (1962) 1-15.

[11] Murty, K. G., “Solving the fixed-charge problem by ranking the extreme points”, *Operations Research*, 16 (1968) 268-279.

[12] Ramakrishnan, C.S, “A note on the time minimizing transportation problem, *Opsearch*, 14 (1977) 207-209.

[13] Sadagopan, S., and Ravindran, A., “A vertex ranking algorithm for the fixed-charge transportation problem”, *Journal of Optimization Theory and Applications*, 37 (1982) 221-230.

[14] Sandrock, K., “A simple algorithm for solving small fixed-charge transportation problems”, *Journal of the Operational Research Society*, 39 (1988) 467-475.

[15] Steinberg, D.I., “The fixed-charge problem”, *Naval Research Logistics Quarterly*, 17 (1970) 217-236.

[16] Szwarc, W., “Some remarks on the time transportation problem”, *Naval Research Logistics Quarterly*, 18 (1971) 473 - 485.