Dolomites Research Notes on Approximation

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Interpolation on Real Algebraic Curves to Polynomial Data

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Abstract

We discuss a polynomial interpolation problem where the data are of the form of a set of algebraic curves in $\mathbb{R}^2$ on each of which is prescribed a polynomial. The object is then to construct a global bivariate polynomial that agrees with the given polynomials when restricted to the corresponding curves.

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1 Interpolation on Lines

We first discuss the simplest case – that of interpolation on lines. This, and also a more general Hermite case, was first described in Hakopian and Shakian [9] (in the general setting of $\mathbb{R}^d$). Later de Boor, Dyn and Ron [4] gave a considerably simplified exposition. However, we prefer to place the problem in the proper context of projective space as this is also a simplifying concept in the curve case.

Let Γ be a set of $n+2$ *different* lines in $\mathbb{R}^2$. For each $\gamma \in \Gamma$ we assume that we are given a polynomial P_γ of degree at most n. The problem is to find a global bivariate polynomial P, $\deg(P) \le n$, such that

$$P|_\gamma = P_\gamma, \qquad \forall \gamma \in \Gamma. \tag{1}$$

Now, if two (or more) lines intersect at a point $u \in \mathbb{R}^2$ then there are necessarily consistency conditions imposed. First of all,

$$P_\gamma(u) = P_{\gamma'}(u), \quad \forall \gamma, \gamma' \in \Gamma \text{ s.t. } u \in \gamma \cap \gamma'. \tag{2}$$

But more is true. The interpolation condition $P|_\gamma = P_\gamma$ implies that all the directional derivatives of P along γ are determined. Consequently, for example, any two lines intersecting at $u \in \mathbb{R}^2$ determine the gradient of P at u and the directional derivative along any other line passing through u must be consistent with this gradient (see Figure 1).

In order to make this more precise, we first introduce some notation. For $u \in \mathbb{R}^2$, let

$$\Gamma_u := \{\gamma \in \Gamma : u \in \gamma\}.$$

For a direction vector $v \in \mathbb{R}^2$ we let

$$D_v^j f(u) := \left\{ \frac{d^j}{dt^j} f(u+tv) \right\}\Bigg|_{t=0}$$

be the jth directional derivative of f, in the direction v, at the point u. Of course

$$D_v f(u) = \bigtriangledown f(u) \cdot v$$

and, more generally, we may write

$$\frac{1}{j!} D_v^j f(u) = \sum_{|\alpha|=j} \left(\frac{D^\alpha f(u)}{\alpha!} \right) v^\alpha \tag{3}$$

where we have used standard multinomial notation (including for the partial derivative $D^\alpha f$). The vector $[D^\alpha f(u)]_{|\alpha|\le k}$ ordered in some degree-consistent manner is known as the k-jet of f at the point u.

By an abuse of notation we will write

$$D_\gamma f(u)$$

to denote the directional derivative of f at u in the direction of the line γ. Normally the choice of the orientation of the direction vector of a line, nor its length, will not matter, as long as it is done consistently.

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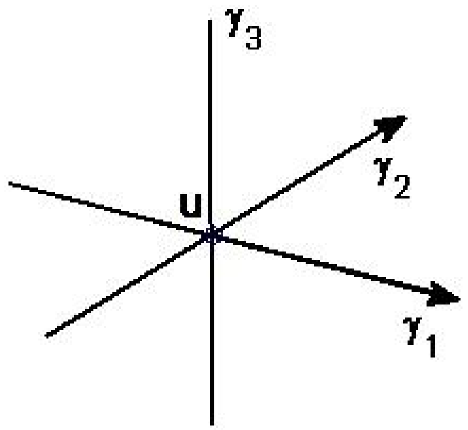

Figure 1: Two lines determine the gradient; the others must be consistent with them

Lemma 1. Let $k := \#\Gamma_u \geq 1$ and suppose that there exists $f \in C^{k-1}(\mathbb{R}^2)$ such that

$$D^j_\gamma f(u) = D^j_\gamma P_\gamma(u), \quad 0 \leq j \leq k-1, \ \forall \gamma \in \Gamma_u. \tag{4}$$

Then this information uniquely determines the $(k-1)$-jet of f at u.

Proof. If $k=1$ there is nothing to do so we may assume that $k \geq 2$. We must show that the conditions (4) determine the $\binom{k+1}{2}$ entries in the $(k-1)$–jet of f at $u \in \mathbb{R}^2$, i.e., of the vector

$$[D^\alpha f(u)]_{|\alpha|<k} \in \mathbb{R}^{\binom{k+1}{2}},$$

or, equivalently, of the vector

$$\left[\frac{D^\alpha f(u)}{\alpha!}\right]_{|\alpha|<k} \in \mathbb{R}^{\binom{k+1}{2}}. \tag{5}$$

Now, the right side of equation (3) for directional derivatives may be interpreted in matrix-vector form as the *row* vector

$$[v^\alpha]_{|\alpha|=j} = [v^{(j,0)}, v^{(j-1,1)}, v^{(j-2,2)}, \ldots, v^{(0,j)}] \in \mathbb{R}^{j+1}$$

times the *column* vector

$$\left[\frac{D^\alpha f(u)}{\alpha!}\right]_{|\alpha|=j} = \begin{bmatrix} \frac{1}{j!0!}D^{(j,0)}f(u) \\ \frac{1}{(j-1)!1!}D^{(j-1,1)}f(u) \\ \cdot \\ \cdot \\ \frac{1}{0!j!}D^{(0,j)}f(u) \end{bmatrix} \in \mathbb{R}^{j+1}.$$

Thus for $j+1$ directions $v_0, v_1, \ldots, v_j \in \mathbb{R}^2$ we have the square linear system

$$\frac{1}{j!}\begin{bmatrix} D^j_{v_0}f(u) \\ D^j_{v_1}f(u) \\ \cdot \\ \cdot \\ D^j_{v_j}f(u) \end{bmatrix} = \begin{bmatrix} v_0^{(j,0)} & v_0^{(j-1,1)} & v_0^{(j-2,2)} & \cdot & \cdot & v_0^{(0,j)} \\ v_1^{(j,0)} & v_1^{(j-1,1)} & v_1^{(j-2,2)} & \cdot & \cdot & v_1^{(0,j)} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ v_j^{(j,0)} & v_j^{(j-1,1)} & v_j^{(j-2,2)} & \cdot & \cdot & v_j^{(0,j)} \end{bmatrix} \left[\frac{D^\alpha f(u)}{\alpha!}\right]_{|\alpha|=j}. \tag{6}$$

We claim that (6) determines the partial derivatives $\left[\frac{D^\alpha f(u)}{\alpha!}\right]_{|\alpha|=j}$ given the data of directional derivatives on the left. In fact, this follows easily from the fact that the homogeneous Vandermonde matrix

$$V=\begin{bmatrix} v_0^{(j,0)} & v_0^{(j-1,1)} & v_0^{(j-2,2)} & \cdot & \cdot & v_0^{(0,j)} \\ v_1^{(j,0)} & v_1^{(j-1,1)} & v_1^{(j-2,2)} & \cdot & \cdot & v_1^{(0,j)} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ v_j^{(j,0)} & v_j^{(j-1,1)} & v_j^{(j-2,2)} & \cdot & \cdot & v_j^{(0,j)} \end{bmatrix}$$

is non-singular for distinct directions $v_0, v_1, \ldots v_j \in \mathbb{R}^2$. To see this, first write the directions in coordinates as $v_i=(x_i,y_i)$ so that V becomes

$$V=\begin{bmatrix} x_0^j y_0^0 & x_0^{j-1} y_0^1 & x_0^{j-2} y_0^2 & \cdot & \cdot & x_0^0 y_0^j \\ x_1^j y_1^0 & x_1^{j-1} y_1^1 & x_1^{j-2} y_1^2 & \cdot & \cdot & x_1^0 y_1^j \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ x_j^j y_j^0 & x_j^{j-1} y_j^1 & x_j^{j-2} y_j^2 & \cdot & \cdot & x_j^0 y_j^j \end{bmatrix}.$$

It is easily seen that

$$\det(V)=\prod_{s<t}(x_s y_t - x_t y_s).$$

These calculations are valid as long as the number of directions used, $j+1 \le k$, the total number of directions available, i.e., for $j \le k-1$. ■

We will keep track of this derivative information by means of a finite Taylor series, i.e., a polynomial with precisely those derivative values at u.

Definition 1. (Affine Consistency, first version) If $k=\#\Gamma_u \ge 2$ lines intersect at the point $u \in \mathbb{R}^2$ we say that the data are consistent at u if there exists a bivariate polynomial G_u, of degree at most $k-2$ such that for all $\gamma \in \Gamma_u$, $P_\gamma - G_u$ has a zero of order $k-1$ at u along γ.

There is a simple test for Affine Consistency. Let $R := P_\gamma - G_u$. Then, we are asking that the bivariate polynomial R, when restricted to the line γ, have a zero of order $k-1$ at a point $u=(x_0,y_0)\in R$, say. But a bivariate polynomial restricted to a line is a univariate polynomial. Specifically, suppose that we parameterize the line $\gamma:\ a_\gamma x + b_\gamma y + c_\gamma = 0$ by

$$(x,y)=t(b_\gamma,-a_\gamma)+(x_0,y_0). \tag{7}$$

Then the restriction of R to γ becomes

$$r(t):=R(tb_\gamma+x_0,-ta_\gamma+y_0).$$

That this has a zero of order $k-1$ at (x_0,y_0) is equivalent to $r(t)$ having a zero of order $k-1$ at $t=0$, i.e.,

$$r(t)=t^{k-1}q(t) \tag{8}$$

for some polynomial $q(t)$.

However, we may calculate, for $(x,y)\in\gamma$,

$$\begin{aligned}
\begin{vmatrix} x & y & 1 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix} &= \begin{vmatrix} x-x_0 & y-y_0 & 0 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix} \quad \text{(subtracting 3rd row from 1st)} \\
&= \begin{vmatrix} tb_\gamma & -ta_\gamma & 0 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix} \\
&= t\begin{vmatrix} b_\gamma & -a_\gamma & 0 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix} \\
&= t(a_\gamma^2+b_\gamma^2+c_\gamma^2)
\end{aligned}$$

as an easy calculation shows. In particular, we have

$$t=\frac{1}{a_\gamma^2+b_\gamma^2+c_\gamma^2}\begin{vmatrix} x & y & 1 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix}$$

and the condition (8) may be expressed as

$$r(t) = \begin{vmatrix} x & y & 1 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix}^{k-1} q(t)$$

(for $(x,y)\in\gamma$ given by (7)).

Hence we may reformulate our definition as

Definition 2. (Affine Consistency, second version) If $k=\#\Gamma_u\ge 2$ lines intersect at the point $u=(x_0,y_0)\in\mathbb{R}^2$ we say that the data are consistent at u if there exists a bivariate polynomial G_u (of degree at most $k-2$) such that for all $\gamma\in\Gamma_u$, there exists a polynomial $Q_\gamma(x,y)$ such that

$$P_\gamma(x,y)-G_u(x,y) = \begin{vmatrix} x & y & 1 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix}^{k-1} Q_\gamma(x,y)$$

for all $(x,y)\in\gamma$. Or, in other words,

$$P_\gamma(x,y)-G_u(x,y) - \begin{vmatrix} x & y & 1 \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & 1 \end{vmatrix}^{k-1} Q_\gamma(x,y)\in\langle\gamma\rangle,$$

with $\langle\gamma\rangle$ the ideal generated by γ.

Remark. This condition for consistency depends only on the partial derivatives of G_u at u up to order $k-2$. Hence one may use any other polynomial G'_u say, provided G_u and G'_u have the same $(k-2)$-jet at u. In other words, there is not really a constraint on the degree of G_u; we include it only because degree $k-2$ is the minimal necessary and most efficient to use in practice.

One problem remains – some of the lines in Γ could be parallel and intersect at "infinity". The natural setting for this is projective space, $\mathbb{RP}^2$ and we introduce the following notation.

We will use standard homogeneous coordinates for

$$\mathbb{RP}^2 := \{[x_0:y_0:z_0] \,:\, x_0,y_0,z_0\in\mathbb{R}\}$$

(with $x_0^2+y_0^2+z_0^2\neq 0$ and $[tx_0:ty_0:tz_0]\equiv[x_0:y_0:z_0]$ for all $t\in\mathbb{R}$). The points $[x:y:1]\in\mathbb{RP}^2$ correspond to $\mathbb{R}^2$ while the points $[x:y:0]\in\mathbb{RP}^2$ form the line at infinity.

If the affine line γ has equation

$$\gamma=\{(x_1,x_2)\in\mathbb{R}^2 \,:\, a_\gamma x_1+b_\gamma x_2+c_\gamma=0\}$$

we will let

$$\widetilde{\gamma}=\{[x_1,x_2,z]\in\mathbb{RP}^2 \,:\, a_\gamma x_1+b_\gamma x_2+c_\gamma z=0\}$$

denote its *projectivization*. It is easy to verify that two affine lines are parallel if and only if their projectivizations intersect at a point at infinity.

For a polynomial of degree at most n,

$$P(\mathbf{x})=\sum_{|\alpha|\le n} a_\alpha \mathbf{x}^\alpha,$$

we will let

$$\widetilde{P}(\mathbf{x},z) := \sum_{|\alpha|\le n} a_\alpha \mathbf{x}^\alpha z^{n-|\alpha|}$$

denote the homogenization (of degree n) of P. We emphasize here that the homogenization depends on the degree n used. For example the homogenization of $P(x,y)=1+x+y$ considered as a polynomial of degree 1 is $\widetilde{P}(x,y,z)=z+x+y$ whereas when it is considered as a polynomial of degree 3, then $\widetilde{P}(x,y,z)=z^3+xz^2+yz^2$. In general two homogenizations of different degrees will differ only by a compensating power of z. However, to avoid confusion, the homogenization degrees of the data polynomials P_γ will always be assumed to be n (the parameter of the interpolation problem).

As before, for $u\in\mathbb{RP}^2$, we let

$$\Gamma_u := \{\gamma\in\Gamma \,:\, u\in\widetilde{\gamma}\}.$$

The extension of consistency, Definition 1, is straightforward.

Definition 3. (Projective Consistency) If $k:=\#\Gamma_u\ge 2$ lines intersect at the point $u=[x_0:y_0:z_0]\in\mathbb{RP}^2$ we say that the data are consistent at u if there exists a homogeneous polynomial $\widetilde{G}_u$, of degree n, such that for all $\gamma\in\Gamma_u$, there exists a homogeneous polynomial $\widetilde{Q}_\gamma(x,y,z)$ such that

$$\widetilde{P}_\gamma(x,y,z)-\widetilde{G}_u(x,y,z) = \begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix}^{k-1} \widetilde{Q}_\gamma(x,y,z)$$

for all $[x:y:z] \in \widetilde{\gamma}$. Or, in other words,

$$\widetilde{P}_\gamma(x,y,z) - \widetilde{G}_u(x,y,z) - \begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix}^{k-1} \widetilde{Q}_\gamma(x,y,z) \in \langle \widetilde{\gamma} \rangle,$$

the ideal generated by $\widetilde{\gamma}$.

Remark. Clearly this definition is independent of the particular chart for $\mathbb{RP}^2$ that might be used to check for consistency. The points at infinity are of the form $[x_0 : y_0 : 0]$ with $x_0^2 + y_0^2 \neq 0$. Hence x_0 and y_0 are not both zero and it follows that every point of infinity is in one of the *allowable* charts $U_x := \{[1 : y : z]\}$ or $U_y := \{[x : 1 : z]\}$. If we pass to such an allowable chart then the condition of projective consistency reduces to that of affine consistency, i.e., for all $\gamma \in \Gamma_u$, $P_\gamma - G_u$ has a zero of order $k-1$ at u along γ (in that chart). From now on by consistent we will mean *projectively consistent*.

Example 1. Consider the data

$$\begin{aligned} \gamma_1 &: \quad x - y + 1 = 0, \quad P_{\gamma_1} = 12 + 26x - 4y + 5x^2 - 2xy + y^2 \\ \gamma_2 &: \quad x - y + 0 = 0, \quad P_{\gamma_2} = 1 - x + 5y + x^2 + 2xy + y^2 \\ \gamma_3 &: \quad x - y - 1 = 0, \quad P_{\gamma_3} = 4x - 9x^2 + 8xy + 5y^2. \end{aligned}$$

Here $k = 3$ and we take $n = 2$.

It is easy to check that

$$u := \widetilde{\gamma}_1 \cap \widetilde{\gamma}_2 \cap \widetilde{\gamma}_3 = [1:1:0],$$

a point at infinity, reflecting the fact that the three lines are parallel. To check for consistency we have a choice. We may work with homogeneous polynomials as in the Definition 1.4, or else we may restrict to a chart and do our calculations there.

Let us first do our calculations in the chart $U_y := \{[x : 1 : z]\}$.

Take $\widetilde{G}_u(x,y,z) = 14y^2 - 10xy + 4yz$ so that (by an abuse of notation) $G_u(x,z) = \widetilde{G}_u(x,1,z) = 14 - 10x + 4z$ (in the chart U_y).

γ_1 :

Equation: $x - y + 1 = 0$
Homogenized: $x - y + z = 0$
Restricted to $U_y : x - 1 + z = 0$
Homogenization of $P_{\gamma_1} : \widetilde{P_{\gamma_1}}(x,y,z) = 12z^2 + 26xz - 4yz + 5x^2 - 2xy + y^2$
Restricted to $U_y : P_1(x,z) = 12z^2 + 26xz - 4z + 5x^2 - 2x + 1$
Determinantal Factor: $\begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix} = \begin{vmatrix} x & 1 & z \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (1 - x + 2z)$
Test:

$$\begin{aligned} P_1(x,z) - G_u(x,z) &= (12z^2 + 26xz - 4z + 15x^2 - 2x + 1) - (14 - 10x + 4z) \\ &= (1 - x + 2z)^2(-1) + (12 + 6x + 16z)(x + z - 1) \\ &= (1 - x + 2z)^2(-1) \quad (\text{on } \gamma_1) \\ &= \begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix}^2 (-1) \quad (\text{on } \gamma_1) \end{aligned}$$

γ_2 :

Equation: $x - y = 0$
Homogenized: $x - y = 0$
Restricted to $U_y : x - 1 = 0$
Homogenization of $P_{\gamma_2} : \widetilde{P_{\gamma_2}}(x,y,z) = z^2 - xz + 5yz + x^2 + 6xy + y^2$
Restricted to $U_y : P_2(x,z) = z^2 - xz + 5z + x^2 + 2x + 1$
Determinantal Factor: $\begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix} = \begin{vmatrix} x & 1 & z \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = -2z$

Test:

$$\begin{aligned}
P_2(x,z)-G_u(x,z) &= (z^2-xz+2z+x^2+2x+1)-(14-10x+4z)\\
&= z^2+(x^2-xz+z+12x-13)\\
&= (-2z)^2\frac{1}{4}+(x-1)(x-z+13)\\
&= (-2z)^2\frac{1}{4} \quad (\text{on } \gamma_2)\\
&= \begin{vmatrix} x & y & z\\ a_\gamma & b_\gamma & c_\gamma\\ x_0 & y_0 & z_0\end{vmatrix}^2 \frac{1}{4} \quad (\text{on } \gamma_2)
\end{aligned}$$

γ_3 :

Equation: $x-y-1=0$
Homogenized: $x-y-z=0$
Restricted to U_y : $x-1-z=0$
Homogenization of P_{γ_3} : $\widetilde{P_{\gamma_3}}(x,y,z)=4xz-9x^2+8xy+5y^2$
Restricted to U_y : $P_3(x,z)=4xz-9x^2+8x+5$
Determinantal Factor: $\begin{vmatrix} x & y & z\\ a_\gamma & b_\gamma & c_\gamma\\ x_0 & y_0 & z_0\end{vmatrix} = \begin{vmatrix} x & 1 & z\\ 1 & -1 & -1\\ 1 & 1 & 0\end{vmatrix} = x+2z-1$
Test:

$$\begin{aligned}
P_3(x,z)-G_u(x,z) &= (4xz-9x^2+8x+5)-(14-10x+4z)\\
&= 4xz-9x^2+18x-9-4z\\
&= -\frac{5}{9}(x+2z-1)^2-\frac{4}{9}(x-1-z)(19x+5z-19)\\
&= -\frac{5}{9}(x+2z-1)^2 \quad (\text{on } \gamma_3)\\
&= \begin{vmatrix} x & y & z\\ a_\gamma & b_\gamma & c_\gamma\\ x_0 & y_0 & z_0\end{vmatrix}^2 (-\frac{5}{9}) \quad (\text{on } \gamma_3)
\end{aligned}$$

It follows that the data on these three lines are (projectively) consistent at u.

Secondly, for comparison's sake, let us verify consistency for the first line γ_1 by working directly with the homogeneous polynomials. As before $\widetilde{G}_u(x,y,z)=14y^2-10xy+4yz$.

γ_1 :

Equation: $x-y+1=0$
Homogenized: $x-y+z=0$
Homogenization of P_{γ_1} : $\widetilde{P_{\gamma_1}}(x,y,z)=12z^2+26xz-4yz+5x^2-2xy+y^2$
Determinantal Factor: $\begin{vmatrix} x & y & z\\ a_\gamma & b_\gamma & c_\gamma\\ x_0 & y_0 & z_0\end{vmatrix} = (-x+y+2z)$
Test:

$$\begin{aligned}
\widetilde{P}_{\gamma_1}(x,y,z)-\widetilde{G}_u(x,y,z) &= (12z^2+26xz-4yz+15x^2-2xy+y^2)\\
&\quad -(14y^2-10xy+4yz)\\
&= (-x+y+2z)^2(-1)\\
&\quad +(12y^2+6xy+16yz)(x-y+z)\\
&= (-x+y+2z)^2(-1) \quad (\text{on } \widetilde{\gamma_1})\\
&= \begin{vmatrix} x & y & z\\ a_\gamma & b_\gamma & c_\gamma\\ x_0 & y_0 & z_0\end{vmatrix}^2 (-1) \quad (\text{on } \widetilde{\gamma_1})
\end{aligned}$$

The calculations for the other lines are similar. ■

Example 2. Consider the family of parallel lines given by $\gamma_i : y - i = 0$, $0 \le i \le k-1$, with polynomial data $P_{\gamma_i}(x) := \sum_{j=0}^{n} a_j^{(i)} x^j$. The following Lemma shows that consistency at infinity is perhaps more complicated than one may have thought.

Lemma 2. The above data are (projectively) consistent iff there are $k-1$ univariate polynomials q_r of degree at most r, $0 \le r \le k-2$, such that

$$a_{n-r}^{(i)} = q_r(i), \quad 0 \le r \le k-2 \text{ and } 0 \le i \le k-1.$$

Proof. The homogenization of γ_i is $\widetilde{\gamma_i}(x,y,z) = y - iz$. Clearly these lines intersect at $u := [1:0:0] \in \mathbb{R}\mathrm{P}^2$.

Suppose first that the data are consistent, i.e., there exists a homogeneous polynomial $\widetilde{G}_u(x,y,z)$ such that for each line γ there is a polynomial $Q_\gamma(x,y)$ such that

$$\widetilde{P}_\gamma(x,y,z) - \widetilde{G}_u(x,y,z) = \begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix}^{k-1} \widetilde{Q}_\gamma(x,y,z)$$

for all $[x:y:z] \in \widetilde{\gamma}$. We will work in the chart $U_x := \{[1:y:z]\}$. Then, for γ_i, we have

$$\begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix} = \begin{vmatrix} 1 & y & z \\ 0 & 1 & -i \\ 1 & 0 & 0 \end{vmatrix} = -z - iy = -(z + iy).$$

But, on $\widetilde{\gamma_i}$, $y = iz$ and hence

$$\begin{vmatrix} x & y & z \\ a_\gamma & b_\gamma & c_\gamma \\ x_0 & y_0 & z_0 \end{vmatrix} = -(1+i^2)z, \ i = 0, \ldots, k-1.$$

It follows that

$$\widetilde{P_{\gamma_i}}(1,iz,z) - \widetilde{G}_u(1,iz,z) = z^{k-1}\widetilde{Q}_{\gamma_i}(1,iz,z)$$

for some polynomials $\widetilde{Q}_{\gamma_i}$. In particular,

$$\frac{d^r}{dz^r}\left\{\widetilde{P_{\gamma_i}}(1,iz,z)\right\}\Big|_{z=0} = \frac{d^r}{dz^r}\left\{\widetilde{G_u}(1,iz,z)\right\}\Big|_{z=0}$$

for $r = 0, 1, \ldots, (k-2)$.

But,

$$\widetilde{P_{\gamma_i}}(x,y,z) = \sum_{j=0}^{n} a_j^{(i)} x^j z^{n-j}$$

so that

$$\widetilde{P_{\gamma_i}}(1,iz,z) = \sum_{j=0}^{n} a_j^{(i)} z^{n-j} = \sum_{j=0}^{n} a_{n-j}^{(i)} z^j.$$

Hence, in other words, we have

$$r!\,a_{n-r}^{(i)} = \frac{d^r}{dz^r}\left\{\widetilde{G_u}(1,iz,z)\right\}\Big|_{z=0}$$

for $r = 0, 1, \ldots, (k-2)$.

But we may write $\widetilde{G_u}(x,y,z) = \sum_{s+t\le n} g_{st} x^s y^t z^{n-s-t}$ for some coefficients g_{st}, and hence

$$\begin{aligned} \widetilde{G_u}(1,iz,z) &= \sum_{s+t\le n} g_{st} i^t z^{n-s} \\ &= \sum_{s=0}^{n} z^{n-s}\left(\sum_{t=0}^{n-s} g_{st} i^t\right) \\ &= \sum_{s=0}^{n} z^{s}\left(\sum_{t=0}^{s} g_{n-s,t} i^t\right) \\ &=: \sum_{s=0}^{n} z^s q_s(i) \end{aligned}$$

where the last equation defines the polynomials q_s (of degree at most s).

It follows that
$$r!a_{n-r}^{(i)} = \frac{d^r}{dz^r}\left\{\widetilde{G_u}(1,iz,z)\right\}\Big|_{z=0} = r!q_r(i)$$
for $r = 0,1,\ldots,(k-2)$, as required.

Conversely, suppose that there are $k-1$ univariate polynomials q_r of degree at most s, $0 \le r \le k-2$, such that
$$a_{n-r}^{(i)} = q_r(i), \quad 0 \le r \le k-2 \text{ and } 0 \le i \le k-1.$$
We need only construct the polynomial G_u. But note that since the degree of q_r is at most $r \le k-2$, the k values of $q_r(i)$, $0 \le i \le k-1$ determine q_r and, in particular, all the coefficients of q_r. It is easy to see then that the polynomial $G_u(x,y) = \sum\limits_{s+t\le k-2} g_{st}x^s y^t$ with $g_{k-2-s,t}$ defined to be the coefficient of x^t in $q_r(x)$, appropriately homogenized, has the desired properties. ■

Remark. Note that these consistency conditions for a point at infinity are on the coefficients $a_{n-r}^{(i)}$, $0 \le r \le k-2$, i.e., on the $k-1$ *highest* order coefficients. In contrast, consistency at a finite point (e.g. 0) are on the *lower* order coefficients. This "inversion" is caused essentially by the fact that in the homogenization of a polynomial a degree j term is multiplied by z^{n-j}, i.e., there is an inversion in the degrees of the powers of z.

Example 2 Continued. Consider, for simplicity, $k = 2$. The conditions above become more explicitly:

For $r = 0 : a_n^{(i)} = q_0(i)$, for q_0 a polynomial of degree 0, i.e., a constant. In other words
$$a_n^{(0)} = a_n^{(1)} = a_n^{(2)}.$$

For $r = 1 : a_{n-1}^{(i)} = q_1(i)$ for $q_1(x) = Bx + A$, some polynomial of degree at most 1. In other words $a_{n-1}^{(0)} = q_1(0) = A$, $a_{n-1}^{(1)} = q_1(1) = B + A$ and $a_{n-1}^{(2)} = q_1(2) = 2B + A$. This is easily seen to be equivalent to the second difference condition
$$a_{n-1}^{(0)} - 2a_{n-1}^{(1)} + a_{n-1}^{(2)} = 0.$$

The point of intersection is $u = [1:0:0] \in \mathbb{RP}^2$. Hence it is convenient to work in the chart $U_x := \{[1:y:z]\}$ where u is the *finite* point $u = (y,z) = (0,0)$. The homogenized lines are $\widetilde{\gamma}_i = y - iz = 0$ which in U_x have the same equation and direction vector $\langle i,1\rangle$. The data polynomials homogenize to $\widetilde{P}_{\gamma_i}(x,y,z) = \sum_{j=0}^n a_j^{(i)} x^j z^{n-j}$ which in U_x become
$$\widetilde{P}_{\gamma_i}(1,y,z) = \sum_{j=0}^n a_j^{(i)} z^{n-j}.$$
The consistency conditions in U_x are then:

(a) Function value condition: $\widetilde{P}_{\gamma_0}(1,0,0) = \widetilde{P}_{\gamma_1}(1,0,0) = \widetilde{P}_{\gamma_2}(1,0,0)$, i.e.,
$$a_n^{(0)} = a_n^{(1)} = a_n^{(2)}.$$
(b) First derivative condition: there exists a gradient $\langle A,B\rangle$ such that
$$D_{\langle i,1\rangle}\widetilde{P}_{\gamma_i}(1,y,z)\Big|_{(y,z)=(0,0)} = \langle A,B\rangle \cdot \langle i,1\rangle, \quad i = 0,1,2.$$
In other words
$$\begin{aligned}
&\iff \quad \frac{\partial \widetilde{P}_{\gamma_i}}{\partial z}(1,0,0) = Ai + B, \quad i = 0,1,2\\
&\iff \quad a_{n-1}^{(i)} = Ai + B, \quad i = 0,1,2\\
&\iff \quad a_{n-1}^{(0)} - 2a_{n-1}^{(1)} + a_{n-1}^{(2)} = 0,
\end{aligned}$$
as before.

Alternatively we may perform a projective change of coordinates
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$
with $A \in \mathbb{C}^{3\times 3}$ that moves the point at infinity to a finite point where we may apply the usual consistency condtions. Specifically take
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix},$$

which has the effect of interchanging x and z.

In the new coordinates γ_i becomes $y'-ix'=0$ which intersect at the *finite* point $[0:0:1]\in\mathbb{RP}^2$. Moreover, the data polynomials convert to

$$\widetilde{P}_{\gamma_i}(x',y',z')=\sum_{j=0}^{n}a_j^{(i)}(z')^j(x')^{n-j}.$$

Working in the chart $z'=1$ we then have

$$\widetilde{P}_{\gamma_i}(x',y',1)=\sum_{j=0}^{n}a_j^{(i)}(x')^{n-j}.$$

The consistency conditions at $(x',y')=(0,0)$ then become:
(a) Function value condition: $\widetilde{P}_{\gamma_0}(0,0,1)=\widetilde{P}_{\gamma_1}(0,0,1)=\widetilde{P}_{\gamma_2}(0,0,1)$, i.e.,

$$a_n^{(0)}=a_n^{(1)}=a_n^{(2)}.$$

(b) First derivative condition: there exists a gradient $\langle A,B\rangle$ such that
for γ_0 (i.e. $y'=0$), $D_{\langle 1,0\rangle}P_{\gamma_0}(0,0,1)=\langle A,B\rangle\cdot\langle 1,0\rangle$, i.e., $a_{n-1}^{(0)}=A$;
for γ_1 (i.e. $y'-x'=0$), $D_{\langle 1,1\rangle}P_{\gamma_1}(0,0,1)=\langle A,B\rangle\cdot\langle 1,1\rangle$, i.e., $a_{n-1}^{(1)}=A+B$;
for γ_2 (i.e. $y'-2x'=0$), $D_{\langle 1,1\rangle}P_{\gamma_1}(0,0,1)=\langle A,B\rangle\cdot\langle 1,2\rangle$, i.e., $a_{n-1}^{(1)}=A+2B$.
Clearly these are exactly the same conditions as above, equivalent to

$$a_{n-1}^{(0)}-2a_{n-1}^{(1)}+a_{n-1}^{(2)}=0.$$

We would like to emphasize that, as this example shows, viewing projective space as a "whole", the points at infinity are really no different than any other point. ■

We are now ready to state and prove the main theorem of this section.

Theorem 1.1. *Suppose that Γ is a set of $n+2$ different lines in $\mathbb{R}^2$ and that to each line $\gamma\in\Gamma$ is associated a (bivariate) polynomial P_γ of degree at most n. Then there exists an interpolant of these data, (i.e., a polynomial P of degree at most n such that $(P-P_\gamma)\big|_\gamma=0$ for all $\gamma\in\Gamma$), if and only if the data are consistent at all points of intersection of the lines.*

Proof. Suppose first that there exists an interpolant P. Consider a point of intersection $u\in\mathbb{RP}^2$. By passing to an appropriate chart, or else by a projective change of coordinates, we may assume that $u\in\mathbb{R}^2$ is a *finite* point. Let, as before, $\Gamma_u=\{\gamma\in\Gamma\,:\,u\in\gamma\}$ with $k=k(u):=\#\Gamma_u$. If $k\le 1$ there is nothing to do so we may assume that $k\ge 2$. Clearly we may take $G_u=T_u^{k-2}P$, the Taylor polynomial of P of degree at most $k-2$ based at u, showing that then the data are consistent.

Conversely, suppose that the data are consistent. We begin with a Lemma.

Lemma 3. Suppose that the data are consistent and that P is some (bivariate) polynomial of degree at most n. Then $(P-P_\gamma)\big|_\gamma=0$ for all $\gamma\in\Gamma$, if and only $P-G_u$ has a zero of order $k(u)-1$ at each point (of intersection).

Proof of Lemma. Suppose first that $(P-P_\gamma)\big|_\gamma=0$ for all $\gamma\in\Gamma$. Then for each u, the polynomial G_u collects all the derivative information determined by the lines intersecting at u. Since P and P_γ agree along γ, (and the data are assumed to be consistent) P must also share this derivative information, i.e., P and G_u have the same Taylor polynomial of degree $k(u)-2$ (the degree of G_u) based at u. In other words, $P-G_u$ has a zero of order $k(u)-1$ at u.

Conversely, suppose that $P-G_u$ has a zero of order $k(u)-1$ at each point (of intersection). For $\gamma\in\Gamma$, let

$$X_\gamma:=\{u\in\mathbb{RP}^2\,:\,\exists\gamma'\in\Gamma\text{ such that }u=\gamma\cap\gamma'\}$$

denote the set of intersection points on the line γ. For $u\in X_\gamma$, $(P_\gamma-G_u)\big|_\gamma$ has a zero of order $k(u)-1$ at u by consistency and $\big(P-G_u\big)\big|_\gamma$ has a zero there of order $k(u)-1$ by assumption. Hence $(P-P_\gamma)\big|_\gamma$ also has a zero of order $k(u)-1$ at u. It follows that $(P-P_\gamma)\big|_\gamma$ has a total of $\sum_{u\in X_\gamma}(k(u)-1)$ zeros (on γ). But each line in $\Gamma\backslash\{\gamma\}$ contributes exactly one point of intersection to γ so that

$$\sum_{u\in X_\gamma}(k(u)-1)=\#(\Gamma\backslash\{\gamma\})=\#\Gamma-1=n+1.$$

It follows that the univariate polynomial of degree at most n, $(P-P_\gamma)\big|_\gamma$, has $n+1$ zeros and must be identically zero. ■

Continuing with the proof of the converse, by the Lemma it is sufficient to construct a bivariate polynomial P of degree at most n such that $P-G_u$ has a zero of order $k(u)-1$ at each point of intersection u, or, in other words, that P and G_u have the same Taylor polynomial of degree $k(u)-2$ at u. But since G_u is of degree at most $k(u)-2$ this is equivalent to

$$T_u^{k(u)-2}P=G_u,\quad\forall u.$$

At each u this imposes $\binom{k(u)-2+2}{2}=\binom{k(u)}{2}$ linear conditions on P for a total of

$$\sum_{k(u)\geq 2}\binom{k(u)}{2}$$

linear conditions. But $\sum_{k(u)\geq 2}\binom{k(u)}{2}$ is the number of intersections (including multiplicity) of pairs of $k(u)$ lines. Since two lines intersect at only one point, it follows that $\sum_{k(u)\geq 2}\binom{k(u)}{2}$ is the number of intersections of the $n+2$ lines of Γ, i.e.,

$$\sum_{k(u)\geq 2}\binom{k(u)}{2}=\binom{n+2}{2}.$$

Hence we have $\binom{n+2}{2}$ linear conditions on the $\binom{n+2}{2}$ coefficients of P, a square linear system.

Consider the homogeneous system, i.e., when $G_u\equiv 0$ for all u. Let P be any solution. In this case consistency implies that $P_\gamma=0$ for all $\gamma\in\Gamma$ and hence that $P|_\gamma=0$ for all $\gamma\in\Gamma$. Hence P (of degree at most n) has $n+2$ linear factors and must be zero.

Since the zero polynomial is the only solution of the homogeneous linear system the corresponding matrix is non-singular and every such system has a unique solution. ■

2 Interpolation on Algebraic Curves

Suppose that $\gamma(x,y)$ is a polynomial. Its zero set

$$V_{\mathbb{R}}(\gamma):=\{(x,y)\in\mathbb{R}^2\,:\,\gamma(x,y)=0\}$$

is an *algebraic* curve. For simplicity's sake, we will typically speak of the curve γ instead of the curve $V_{\mathbb{R}}(\gamma)$. The interpolation problem is as follows. Fix a degree $n\geq 0$. Let $\Gamma=\{\gamma\}$ be a set of distinct algebraic curves in $\mathbb{R}^2$ with $d_\gamma:=\deg(\gamma)$. We will insist that no two of the curves $\gamma\in\Gamma$ have a common component, i.e., that the defining polynomials are pairwise relatively prime (have no common factor). On each γ we are given a (bivariate) polynomial P_γ of degree at most n. We look for a global polynomial P, also of degree at most n, such that

$$(P-P_\gamma)\big|_\gamma=0,\quad \forall\gamma\in\Gamma. \tag{9}$$

(In Proposition 2 below we will give a condition on n in terms of the $\deg(\gamma)$ that guarantees uniqueness of such a P, if it exists.)

The algebraic curve case presents us with a number of technical problems. First of all, for any polynomial γ, γ^k defines the exact same curve for any positive integer k. Or even worse, if $\gamma=\gamma_1\gamma_2$ can be factored, $\gamma_1^k\gamma_2^j$ define the same curve. There would also be a redundancy if γ_1 and γ_2 had a common factor. We need to avoid these situations.

Definition 4. A polynomial $\gamma(x,y)$ is said to be square-free if it can be written

$$\gamma=\gamma_1\gamma_2\cdots\gamma_s$$

where the γ_j are mutually coprime, i.e., have no common (complex) factors.

There is also in several variables a problem of *degeneracy*. For example, for the polynomial $\gamma=x^2+y^2$, the associated "curve"

$$V_{\mathbb{R}}(\gamma):=\{(x,y)\in\mathbb{R}^2\,:\,\gamma(x,y)=0\}=\{(0,0)\},$$

a single point. In particular, if a polynomial P has the property of being zero on this particular "curve" $V_{\mathbb{R}}(\gamma)$, it is *not* the case that there is a factorization $P=\gamma Q$ for some quotient polynomial Q.

Here is a condition for such a factorization to exist.

Proposition 1. ([10, Thm. 4.3, p. 48]) Suppose that γ is a square-free bivariate real polynomial such that

$$\dim_{\mathbb{R}}(V_{\mathbb{R}}(\gamma))=\dim_{\mathbb{C}}(V_{\mathbb{C}}(\gamma)).$$

Then, if the polynomial P is zero on $V_{\mathbb{R}}(\gamma)$, there exists a polynomial Q with $\deg(Q)=\deg(P)-\deg(\gamma)$ such that

$$P=\gamma Q.$$

Remark. Here $V_{\mathbb{C}}(\gamma) = \{(z_1, z_2) \in \mathbb{C}^2 : \gamma(z_1, z_2) = 0\}$. Roughly speaking this Proposition says that if $V_{\mathbb{R}}(\gamma)$ is really a curve and not degenerate, it has the desired factorization property.

Proof of the Proposition. Clearly $V_{\mathbb{R}}(\gamma) \subset V_{\mathbb{C}}(\gamma)$. Moreover, $V_{\mathbb{C}}(\gamma)$ is the smallest complex variety with this property. To see this, suppose that V is another variety in $\mathbb{C}^2$ such that $V_{\mathbb{R}}(\gamma) \subset V$ but $V_{\mathbb{C}}(\gamma) \not\subset V$. Then $V_{\mathbb{C}}(\gamma) \cap V$ is a proper subvariety of $V_{\mathbb{C}}(\gamma)$ and hence (cf. [CLO, Prop. 10, p. 443]),

$$\dim_{\mathbb{C}}(V \cap V_{\mathbb{C}}(\gamma))) < \dim_{\mathbb{C}}(V_{\mathbb{C}}(\gamma)).$$

But, on the other hand, since $V \cap V_{\mathbb{C}}(\gamma)$ contains $V_{\mathbb{R}}(\gamma)$, it follows (cf. [CLO, Prop. 1, p. 438]) that

$$\dim_{\mathbb{C}}(V \cap V_{\mathbb{C}}(\gamma))) \geq \dim_{\mathbb{R}}(V_{\mathbb{R}}(\gamma)) = \dim_{\mathbb{C}}(V_{\mathbb{C}}(\gamma)),$$

a contradiction.

Now suppose that P is a real polynomial that is zero on $V_{\mathbb{R}}(\gamma)$. Then $V_{\mathbb{C}}(P)$ is a complex variety that contains $V_{\mathbb{R}}(\gamma)$ and hence, by the minimality property discussed above, $V_{\mathbb{C}}(\gamma) \subset V_{\mathbb{C}}(P)$. But since γ is square-free, we have (cf. [CLO, p. 178])

$$I(V_{\mathbb{C}}(\gamma)) = \langle \gamma \rangle$$

where

$$I(V) := \{p : p\,|_V = 0\}$$

is the ideal of polynomials which are zero on the variety V. Consequently we have that $P \in \langle \gamma \rangle$ and hence γ divides P over the complexes. But since both P and γ are real polynomials, it follows that γ divides P over the reals. ■

From now on we will assume that all the curves $\gamma \in \Gamma$ are square-free and have the factorization property guaranteed, for example, by Proposition 1.

The interpolation problem may, in principle, be stated for any degree n. However, uniqueness of the interpolant is only guaranteed if n is sufficiently small.

Proposition 2. Consider the interpolation problem (9). Suppose that the curves of the set Γ are square-free and have the factorization property of Propostion 1 and that

$$n \leq \left(\sum_{\gamma \in \Gamma} d_\gamma \right) - 1. \tag{10}$$

Then, if a solution of the interpolation problem (9) exists, it must be unique.

Proof. Suppose that P and Q are two polynomials of degree at most n, given by (10), that satisfy the interpolation conditions (9). Then, by the factorization property,

$$(P - Q)|_\gamma = 0, \ \gamma \in \Gamma \quad \Longrightarrow \quad (P - Q) = A \prod_{\gamma \in \Gamma} \gamma$$

for some polynomial A. But

$$\deg(\prod_{\gamma \in \Gamma} \gamma) = \sum_{\gamma \in \Gamma} d_\gamma \geq n + 1$$

and $\deg(P - Q) \leq n$. Hence $A = 0$ and $P = Q$. ■

Remark. If Γ consists only of lines, i.e., $d_\gamma = 1$, $\forall \gamma \in \Gamma$, then the maximal degree n from (10) becomes $n = \#\Gamma - 1$, so that $\#\Gamma = n + 1$. However, for the case of lines, discussed in the first section, we used $\#\Gamma = n + 2$. In fact, any line intersects $n + 1$ other lines in $n + 1$ points (counting multiplicity). A polynomial of degree at most n is determined by its values at these $n + 1$ points and hence the extra line is really redundant. We use $n + 2$ in order to be consistent with the presentation of Hakopian and Sahakian [9].

Just as for the line case, the interpolation data must be consistent at points of intersection of the curves. However, the curve case is rather more complicated. By Bezout's Theorem, two curves of degree n, in general, intersect at n^2 points, some of which could be complex, and some of which could be at infinity. Moreover, two curves can intersect at a point in a much more complicated way than two (or even many) lines. Here are some illustrative examples. For the sake of simplicity we will write P_j for P_{γ_j} etc., when no confusion is possible.

Example 1. Consider $\Gamma = \{\gamma_1 = y, \gamma_2 = y - x^2\}$. The two curves intersect (only) at the origin and are tangent there (see Figure 2). Take the data polynomials $P_1(x, y) = \sum_{i+j \leq n} a_{ij} x^i y^j$ and $P_2(x, y) = \sum_{i+j \leq n} b_{ij} x^i y^j$. We claim that the consistency conditions are that $P_1(0, 0) = P_2(0, 0)$ (the function values agree at the point of intersection) and $\dfrac{\partial P_1}{\partial x}(0, 0) = \dfrac{\partial P_2}{\partial x}(0, 0)$ (the derivatives in the direction of the common tangent agree). Clearly these are necessary for there to be an interpolant in the sense of (9). To see that they are also sufficient, note that they hold iff $a_{00} = b_{00}$ and $a_{10} = b_{10}$. Then $P_1(x, 0) - P_2(x, 0)$ has a zero of order 2 at the origin and hence there exists a polynomial $Q(x)$ (of degree at most $n - 2$) such that

$$P_1(x, 0) - P_2(x, 0) = x^2 Q(x). \tag{11}$$

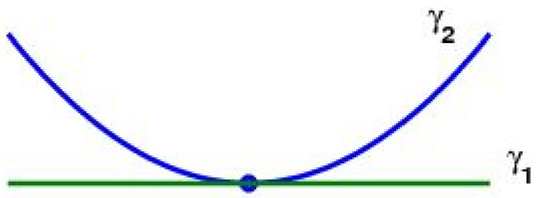

Figure 2: Two curves with simple tangent at point of intersection

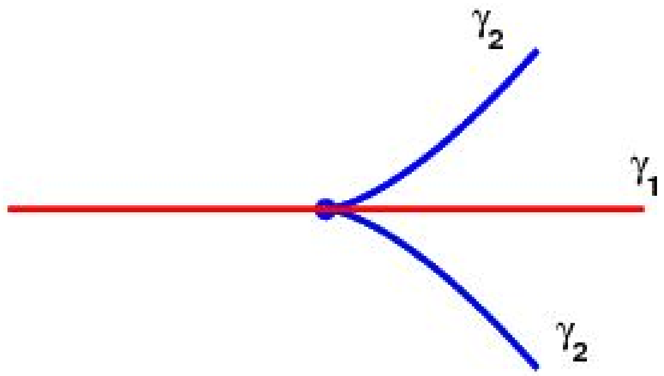

Figure 3: A line and a cusp

We claim that

$$P(x,y) := P_2(x,y) + (y - x^2)Q(x)$$

is an interpolant. Indeed, on γ_2, $y - x^2 = 0$ and so $(P - P_2)\big|_{\gamma_2} = 0$. Moreover, on γ_1, $y = 0$, so

$$\begin{aligned} (P - P_1)\big|_{\gamma_1} &= P(x,0) - P_1(x,0) \\ &= (P_2(x,0) + (0 - x^2)Q(x)) - P_1(x,0) \\ &= 0 \end{aligned}$$

by (11).

Remark. Note that these consistency conditions are not on the *entire* jet of a given order (as was the case for lines), just on the direction given by the common tangent. However, they are valid for data polynomials P_1 and P_2 of arbitrary degree at most n, not just the n given by (10) (which would however guarantee uniqueness).

Example 2. Consider $\Gamma = \{\gamma_1 = y, \gamma_2 = y^2 - x^3\}$. The two curves intersect (only) at the origin and are tangent there (see Figure 3), however, the curve γ_2 is a so-called cusp and is singular at the origin. Take the data polynomials $P_1(x,y) = \sum_{i+j\le n} a_{ij}x^i y^j$ and $P_2(x,y) = \sum_{i+j\le n} b_{ij}x^i y^j$. We claim that the consistency conditions are that $P_1(0,0) = P_2(0,0)$ (the function values agree at the point of intersection), $\dfrac{\partial P_1}{\partial x}(0,0) = \dfrac{\partial P_2}{\partial x}(0,0)$ (the derivatives in the direction of the common tangent agree) and $\dfrac{\partial^2 P_1}{\partial x^2}(0,0) = \dfrac{\partial^2 P_2}{\partial x^2}(0,0)$ (the second derivatives in the direction of the common tangent agree).

To see that these are necessary, just note that $(P-P_1)\big|_{\gamma_1}=0$ implies, by the factorization property, that

$$P-P_1=A\gamma_1=Ay$$

for some polynomial A. Similarly,

$$P-P_2=B\gamma_2=B(y^2-x^3)$$

for some polynomial B.

Subtracting these two equations yields

$$P_1-P_2=-Ay+B(y^2-x^3)$$

and then differentiating twice with respect to x and evaluating at the origin gives the consistency conditions listed above.

Conversely, to see that they are sufficient note that they hold iff $a_{00}=b_{00}$, $a_{10}=b_{10}$ and $a_{20}=b_{20}$. Then $P_1(x,0)-P_2(x,0)$ has a zero of order 3 at the origin and hence there exists a polynomial $Q(x)$ (of degree at most $n-3$) such that

$$P_1(x,0)-P_2(x,0)=x^3Q(x). \tag{12}$$

We claim that

$$P(x,y):=P_2(x,y)+(y^2-x^3)Q(x)$$

is an interpolant. Indeed, on γ_2, $y^2-x^3=0$ and so $(P-P_2)\big|_{\gamma_2}=0$. Moreover, on γ_1, $y=0$, so

$$\begin{aligned}(P-P_1)\big|_{\gamma_1} &= P(x,0)-P_1(x,0)\\ &= (P_2(x,0)+(0-x^3)Q(x))-P_1(x,0)\\ &= 0\end{aligned}$$

by (12).

Remark. The cusp has the effect of making the tangent a “double" tangent, a phenomenon that does not occur for lines.

Example 3. Consider $\Gamma=\{\gamma_1=y,\gamma_2=y^2-x^2(1-x^2).\}$. The two curves intersect at $(-1,0)$, $(0,0)$ and $(+1,0)$ (see Figure 4). The points $(\pm1,0)$ are simple intersections. The origin is a singular point for γ_2 where it has two distinct tangents $y=\pm x$. The temptation might be to think that the consistency conditions at the origin are those of three *lines* intersecting at a single point. However, this is *not* the case. In fact, we claim that the consistency conditions are: $P_1(\pm1,0)=P_2(\pm1,0)$ (simple intersections), $P_1(0,0)=P_2(0,0)$, and $\dfrac{\partial P_1}{\partial x}(0,0)=\dfrac{\partial P_2}{\partial x}(0,0)$. (What happens is that the derivatives along the two tangents define the gradient, and then the derivative along the line has to be consistent with this.)

To see that they are necessary, suppose that there does exist an interpolant. Then, as in the previous examples,

$$\begin{aligned}P_1(x,y)-P_2(x,y) &= -A\gamma_1+B\gamma_2\\ &= -Ay+B(y^2-x^2(1-x^2)).\end{aligned}$$

Evaluating at $(\pm1,0)$ gives the first two conditions, and at $(0,0)$, the third. Differentiating with respect to x, and then evaluating at $(0,0)$, easily gives the fourth.

To see that they are sufficient, suppose that they hold. Then $(P_1-P_2)(x,0)$ has simple zeros at $x=\pm1$ and a double zero at $x=0$. Hence

$$(P_1-P_2)(x,0)=x^2(1-x^2)Q(x)$$

for some polynomial $Q(x)$ of degree at most $n-4$. It is easy to check that

$$P(x,y):=P_2(x,y)+(y^2-x^2(1-x^2))Q(x)$$

is an interpolant (of degree at most n).

Example 4. Consider $\Gamma=\{\gamma_1=x^2-y^2-1,\gamma_2=x^2+y^2-3.\}$. In contrast to the previous examples, neither curve is a line. They have simple intersections at (only) the four points $(\pm\sqrt{2},\pm1)$ (see Figure 5). We claim that the consistency conditions are that the two polynomials agree at these four points of intersection, i.e., that $P_1(\pm\sqrt{2},\pm1)=P_2(\pm\sqrt{2},\pm1)$. Clearly they are necessary. To show that they are sufficient suppose then that they hold. We will need a somewhat more sophisticated argument than in the previous examples due to the fact that these four conditions are *not* enough to conclude a factorization of the type that we used above. Nevertheless, we claim that

$$P_1(x,y)-P_2(x,y)=A\gamma_1(x,y)+B\gamma_2(x,y) \tag{13}$$

for some polynomials A,B with $\deg(A)\le n-2$ and $\deg(B)\le n-2$. This is not entirely surprising as the consistency conditions mean that P_1-P_2 is zero on the intersection variety $V:=\gamma_1\cap\gamma_2$, i.e.,

$$P_1-P_2\in I(V):=\{P\in\mathbb{C}[x,y]\,:\,P(x,y)=0 \text{ for all } (x,y)\in V\}.$$

By the Hilbert Nullstellensatz

$$I(V)=\mathrm{Rad}(I(V))=\{P\in\mathbb{C}[x,y]\,:\,P^k\in I(V) \text{ for some integer } k\ge1\},$$

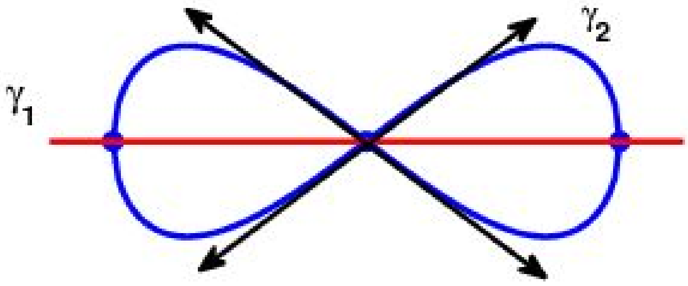

Figure 4: A figure eight, its two tangents at the origin, and a line

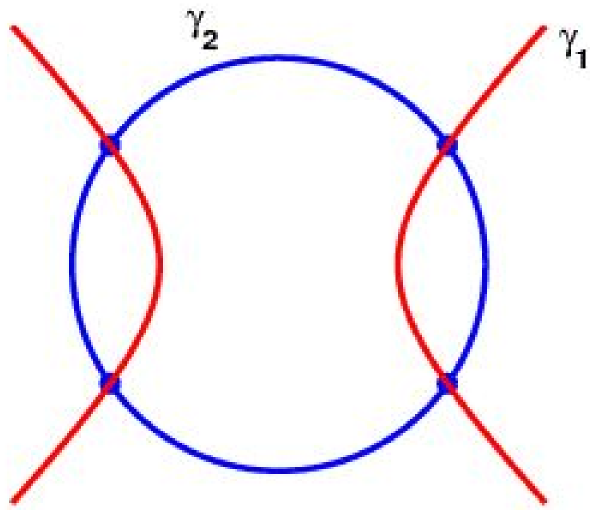

Figure 5: A circle and a hyperbola

the *radical* of the ideal. We are claiming first of all that $\mathrm{Rad}(I(V)) = \langle \gamma_1, \gamma_2 \rangle$, which implies that

$$P_1(x,y) - P_2(x,y) = A\gamma_1(x,y) + B\gamma_2(x,y)$$

for *some* polynomials A and B, and secondly, that there is a bound on the degrees of the factors,

$$\deg(A) \le n-2 \text{ and } \deg(B) \le n-2.$$

For this simple example we can show this by very elementary means. Let us write

$$P_1 - P_2 = \sum_{i+j\le n} c_{ij} x^i y^j.$$

If $P_1 - P_2 \equiv 0$ then we may take$A = B = 0$ and we are done. Otherwise, note that $\deg(P_1 - P_2) \ge 2$ since a polynomial of degree at most 1 cannot be zero at these four points without being identically zero. If $\deg(P_1 - P_2) = 2$ then it is easily concluded, by direct substitution, that

$$\begin{aligned} P_1 - P_2 &= \frac{c_{20} - c_{02}}{2}(x^2 - y^2 - 1) + \frac{c_{20} + c_{02}}{2}(x^2 + y^2 - 3) \\ &= \frac{c_{20} - c_{02}}{2}\gamma_1 + \frac{c_{20} + c_{02}}{2}\gamma_2 \end{aligned}$$

so that $A = (c_{20} - c_{02})/2$ and $B = (c_{20} + c_{02})/2$ are both constants, as desired. Otherwise, suppose that $\deg(P_1 - P_2) \ge 3$. Note that any monomial $x^i y^j$ of degree $i + j \ge 3$ must have either $i \ge 2$ or $j \ge 2$. Hence, the leading homogeneous term of $P_1 - P_2$ can be written

$$\sum_{i=0}^{n} c_{i,n-i} x^i y^{n-i} = \alpha x^2 + \beta y^2$$

(generally in many ways) for some homogeneous polynomials α and β of degree at most $n-2$. Then,

$$Q := (P_1 - P_2) - \left\{ \frac{\alpha - \beta}{2}(x^2 - y^2 - 1) + \frac{\alpha + \beta}{2}(x^2 + y^2 - 3) \right\}$$

is a polynomial of degree at most $n-1$ that is also zero at the four intersection points. By induction on the degree we may assume that $Q = A'\gamma_1 + B'\gamma_2$ for some polynomials A' and B' of degree at most $n-3$, so that

$$\begin{aligned} P_1 - P_2 &= Q + \left\{ \frac{\alpha - \beta}{2}(x^2 - y^2 - 1) + \frac{\alpha + \beta}{2}(x^2 + y^2 - 3) \right\} \\ &= Q + \left\{ \frac{\alpha - \beta}{2}\gamma_1 + \frac{\alpha + \beta}{2}\gamma_2 \right\} \\ &= A'\gamma_1 + B'\gamma_2 + \left\{ \frac{\alpha - \beta}{2}\gamma_1 + \frac{\alpha + \beta}{2}\gamma_2 \right\} \\ &= A\gamma_1 + B\gamma_2 \end{aligned}$$

with A and B having the right properties.

At this point it is easy to verify that

$$P := P_1 - A\gamma_1 = P_2 + B\gamma_2$$

is an interpolant. ■

The common feature of the above examples is that the listed point consistency conditions result in a relation of the form (13), from which it is easy to deduce the existence of an interpolant. We formalize this in a definition.

Definition 5. Given a degree $n \ge 0$ and real polynomials

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$$

we call the vector space

$$\mathrm{span}_H^n(\Gamma) := \{\sum_{i=1}^{s} p_i\gamma_i \,:\, \deg(p_i) \le n - \deg(\gamma_i)\},$$

the H-span of degree n of Γ.

Remark. Note that $\mathrm{span}_H^n(\Gamma)$ is a subspace of Π_n^2, the polynomials of degree at most n in two variables.

We will take the key condition (13) to be our *definition* of consistency.

Definition 6. Given a degree n and data $\Gamma = \{\gamma_1, \gamma_2\}$ with associated polynomials P_1, P_2 such that $\deg(P_1), \deg(P_2) \le n$, we say that the data are (pairwise) consistent for degree n if

$$P_1 - P_2 \in \mathrm{span}_H^n(\Gamma).$$

Proposition 3. Suppose that we are given a degree n and data $\Gamma = \{\gamma_1, \gamma_2\}$ with associated polynomials P_1, P_2 such that $\deg(P_1), \deg(P_1) \le n$. Then there exists an interpolant of degree at most n if and only if the data are consistent for degree n.

Proof. Suppose first that an interpolant exists, i.e., that there exists a polynomial $P \in \Pi_n^2$ such that

$$(P - P_1)\big|_{\gamma_1} = 0 \text{ and } (P - P_2)\big|_{\gamma_2} = 0.$$

Then, by the factorization property, there exist polynomials A and B with $\deg(A) \le n - \deg(\gamma_1)$ and $\deg(B) \le n - \deg(\gamma_2)$ such that

$$P = P_1 + A\gamma_1 \text{ and } P = P_2 + B\gamma_2.$$

Subtracting the second equation from the first results in

$$P_1 - P_2 = -A\gamma_1 + B\gamma_2 \in \mathrm{span}_H^n(\Gamma),$$

i.e., that the data are consistent for degree n.

Conversely, suppose that the data are consistent for degree n, i.e., that

$$P_1 - P_2 = A'\gamma_1 + B'\gamma_2$$

with $\deg(A') \le n - \deg(\gamma_1)$ and $\deg(B') \le n - \deg(\gamma_2)$. Then $P := P_1 - A'\gamma_1$ equals $P_2 + B'\gamma_2$ and hence is an interpolant.

2.1 Checking for (pairwise) Consistency

Proposition 3 shows that our definition of consistency (for two data curves) is correct. How does one check if consistency holds? There are several ways.

2.1.1 Linear Algebra Techniques

As noted above, the spaces $\mathrm{span}_H^n(\Gamma)$ are subspaces of the space of all polynomials of degree at most n, Π_n^2, and hence it is natural to use such techniques.

We first consider the *principal* spaces $W_\gamma := \mathrm{span}_H^n(\gamma)$, i.e., those for which Γ consists of a single element. Of course $W_\gamma = \{0\}$ if $n < \deg(\gamma) =: d_\gamma$ and hence we may suppose that $n \ge d_\gamma$. Then, if we write the polynomial $\gamma \in \Pi_n^2$ using standard multinomial notation as

$$\gamma(x) = \sum_{|\alpha| \le d_\gamma} \gamma_\alpha x^\alpha,$$

we have

$$W_\gamma := \mathrm{span}\{x^\beta \gamma(x) : |\beta| \le n - d_\gamma\}.$$

If we identify polynomials $q \in \Pi_n^2$ by the vector $\vec{q} := [q_\alpha] \in \mathbb{R}^N$, with $N := \binom{n+2}{2}$ (ordered in some degree-consistent manner), then we may represent the basis elements $p_\beta := x^\beta \gamma(x)$, $|\beta| \le n - d_\gamma$, by the "shifted down" vector

$$x^\beta \gamma(x) \equiv (p_\beta)_\alpha := \begin{cases} \gamma_{\alpha-\beta} & \text{if } \alpha \ge \beta \\ 0 & \text{if } \alpha \not\ge \beta \end{cases} \in \mathbb{R}^{N\times 1}. \tag{14}$$

we may combine all these "shifted down" vectors into a matrix, with the "βth" column corresponding to $p_\beta = x^\beta \gamma(x)$,

$$(A_\gamma)_{\alpha,\beta} := \begin{cases} \gamma_{\alpha-\beta} & \text{if } \alpha \ge \beta \\ 0 & \text{if } \alpha \not\ge \beta \end{cases} \tag{15}$$

for $|\alpha| \le n$ and $|\beta| \le n - d_\gamma$. It follows that $A_\gamma \in \mathbb{R}^{N\times M_\gamma}$ where

$$N = \binom{n+2}{2} = \dim(\Pi_n^2) \text{ and } M_\gamma := \binom{n-d_\gamma+2}{2} = \dim(\Pi_{n-d_\gamma}^2)$$

and A_γ has the property that

$$\mathrm{Im}(A_\gamma) = W_\gamma.$$

The general case of s curves follows immediately. Indeed, we have, for $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$,

$$\mathrm{span}_H^n(\Gamma) = W_{\gamma_1} + W_{\gamma_2} + \cdots + W_{\gamma_s} = \mathrm{Im}(A_\Gamma)$$

where

$$A_\Gamma := [A_{\gamma_1}\ A_{\gamma_2}\ \ldots\ A_{\gamma_s}] \in \mathbb{R}^{N\times(M_{\gamma_1}+\cdots+M_{\gamma_s})} \tag{16}$$

is the composite matrix constructed by placing the matrices A_{γ_j} side by side.

Remark. In one variable the "shifted down" vectors (14) are just the coefficient vectors of γ shifted down by β and hence the matrices, particularly in the case of $s = 2$, are multivariate (non-square) analogues of the classical resolvent matrix.

We thus have the following linear algebraic test for being an element of $\mathrm{span}_H^n(\Gamma)$.

Linear Algebraic Test for $\mathrm{span}_H^n(\Gamma)$

1. Fix a degree $n \geq \max\{d_{\gamma_1}, d_{\gamma_2}, \ldots, d_{\gamma_s}\}$ and consider a polynomial p of degree at most n which is to be tested to see if it is an element of $\mathrm{span}_H^n(\Gamma)$.
2. Construct the matrix $A_\Gamma \in \mathbb{R}^{N\times(M_{\gamma_1}+\cdots+M_{\gamma_s})}$.
3. Compute the vector of coefficients $[p_\alpha] \in \mathbb{R}^N$, where $p = \sum_{|\alpha|\leq n} p_\alpha x^\alpha$.
4. Check if $[p_\alpha]$ is in the column space of A_Γ.

Remark. A stable way to test if a vector is in the column space of A_Γ is to compute the QR factorization, $QR = A_\Gamma$ where $Q \in \mathbb{R}^{N\times N}$ is *orthogonal* and $R \in \mathbb{R}^{N\times(M_{\gamma_1}+\cdots+M_{\gamma_s})}$ is *upper triangular*. Then $p \in \mathrm{Im}(A_\Gamma)$ iff $Q^t p \in \mathrm{Im}(R)$. Since R is upper triangular, this is easily accomplished. For example, in the case that R is of full rank, then $Q^t p \in \mathrm{Im}(R)$ iff $(Q^t p)_j = 0$, $M_{\gamma_1} + \cdots + M_{\gamma_s} < j \leq N$.

Of course this gives us an immediate test for pairwise consistency.

Linear Algebraic Test for Consistency

1. Fix a degree $n \geq \max\{d_{\gamma_1}, d_{\gamma_2}\}$.
2. Construct the matrix $A_{\{\gamma_1,\gamma_2\}} \in \mathbb{R}^{N\times(M_{\gamma_1}+M_{\gamma_2})}$.
3. Take $p = P_1 - P_2$ where P_1 and P_2 are the data polynomials given on γ_1 and γ_2 respectively, and compute the vector of coefficients $[p_\alpha] \in \mathbb{R}^N$.
4. Check if $[p_\alpha]$ is in the column space of $A_{\{\gamma_1,\gamma_2\}}$.

It is of particular interest to note that for $\#\Gamma = 2$ we may also easily calculate the dimension of the space $\mathrm{span}_H^n(\Gamma)$.

Proposition 4. Suppose that $\Gamma = \{\gamma_1, \gamma_2\}$ with γ_1 and γ_2 having no common factors (i.e., are mutually prime). Suppose also that $n \geq d_{\gamma_1} + d_{\gamma_2} - 2$. Then

$$\dim(\mathrm{span}_H^n(\Gamma)) = \binom{n+2}{2} - d_{\gamma_1} d_{\gamma_2},$$

i.e., $\mathrm{span}_H^n(\Gamma)$ is of co-dimension $d_{\gamma_1} d_{\gamma_2}$ (in the space of polynomials of degree at most n).

Proof. We have

$$\begin{aligned}\dim(\mathrm{span}_H^n(\{\gamma_1,\gamma_2\})) &= \dim(W_{\gamma_1} + W_{\gamma_2})\\ &= \dim(W_{\gamma_1}) + \dim(W_{\gamma_2}) - \dim(W_{\gamma_1} \cap W_{\gamma_2}).\end{aligned}$$

But, $W_{\gamma_1} \cap W_{\gamma_2}$ is the space of polynomials of degree at most n that have both γ_1 and γ_2 as factors. Since by assumption, γ_1 and γ_2 are relatively prime, it follows that the elements of $W_{\gamma_1} \cap W_{\gamma_2}$ are divisible by the product $\gamma_1\gamma_2$, or, in other words, that

$$W_{\gamma_1} \cap W_{\gamma_2} = W_{\gamma_1\gamma_2}.$$

Hence,

$$\begin{aligned}&\dim(\mathrm{span}_H^n(\{\gamma_1,\gamma_2\}))\\ =\ &\dim(W_{\gamma_1}) + \dim(W_{\gamma_2}) - \dim(W_{\gamma_1} \cap W_{\gamma_2})\\ =\ &\dim(W_{\gamma_1}) + \dim(W_{\gamma_2}) - \dim(W_{\gamma_1\gamma_2})\\ =\ &M_{\gamma_1} + M_{\gamma_2} - M_{\gamma_1\gamma_2}\\ =\ &\binom{n-d_{\gamma_1}+2}{2} + \binom{n-d_{\gamma_2}+2}{2} - \binom{n-d_{\gamma_1}-d_{\gamma_2}+2}{2}\\ =\ &\binom{n+2}{2} - d_{\gamma_1} d_{\gamma_2},\end{aligned}$$

after some simple algebra. ■

In the case of Example 4, $d_{\gamma_1} = d_{\gamma_2} = 2$ so that $\mathrm{span}_H^n(\{x^2 - y^2 - 1, x^2 + y^2 - 3\})$ has co-dimension 4 and hence is determined by 4 orthogonality conditions. It is easy to verify that the evaluation conditions $q(\pm\sqrt{2}, \pm 1) = 0$ are independent over that span and hence, for $n \geq 2$,

$$\mathrm{span}_H^n(\{x^2 - y^2 - 1, x^2 + y^2 - 3\}) = \{q \in \Pi_n^2 \ :\ q(\pm\sqrt{2}, \pm 1) = 0\}.$$

The general case requires techniques from Algebraic Geometry and is the subject of the next section.

2.1.2 Ideal Theoretic Techniques

First note that

$$\mathrm{span}_H^\infty(\Gamma) := \{\sum_{i=1}^s p_i\gamma_i \,:\, p_i \text{ is a polynomial}\},$$

is the ideal generated by Γ which we will denote by $\langle\Gamma\rangle$. At times it will be necessary to distinguish, in its definition, the case of complex polynomials $p_i \in \mathbb{C}[x,y]$ from the case of real polynomials $p_i \in \mathbb{R}[x,y]$. We will then write $\langle\Gamma\rangle_{\mathbb{C}}$ and $\langle\Gamma\rangle_{\mathbb{R}}$. Note however that, if Γ consists of *real* polynomials only, then

$$\langle\Gamma\rangle_{\mathbb{R}} = \langle\Gamma\rangle_{\mathbb{C}} \cap \mathbb{R}[x,y].$$

The ideals $\langle\Gamma\rangle$ are infinite dimensional vector spaces and hence Linear Algebra techniques are more difficult to apply. Nevertheless, from a theoretical point of view, the algebra of ideals is simpler than this might lead one to suspect. Indeed they are one of the basic notions of Algebraic Geometry and and as such there is a well developed and beautiful theory for their analysis.

However, to pass from an ideal theoretic result to one about our spaces $\mathrm{span}_H^n(\Gamma)$ requires a bit of thought. In particular, it is *not* the case, in general, that

$$\mathrm{span}_H^n(\Gamma) = \langle\Gamma\rangle \cap \Pi_n^2$$

since Γ may fail to be an H-basis for $\langle\Gamma\rangle$. Here is an example.

Example 5. Take $\Gamma = \{\gamma_1,\gamma_2\}$ with $\gamma_1 = x^2+y^2+x$ and $\gamma_2 = x^2+y^2+y$. Then $p := x^2-xy = x(\gamma_1-\gamma_2) \in \langle\Gamma\rangle$ but $p \notin \mathrm{span}_H^2(\Gamma)$, as is easy to verify.

It is worthwhile to understand what goes wrong in this example. On one level the problem is that γ_1 and γ_2 have the same leading homogeneous term, x^2+y^2, which is canceled in the subtraction $\gamma_1-\gamma_2$ resulting in a polynomial $x-y$ of lower degree. There is another, geometric, way of understanding this. The curves γ_1 and γ_2 are two circles that intersect at the two points $(0,0)$ and $(-1/2,-1/2)$, as is easy to verify. However, in general two curves of degree 2 intersect at $2\times 2 = 4$ points. The other two points of intersection of γ_1 and γ_2 are at "infinity", in the following sense. We embed our problem in projective space $\mathbb{C}\mathrm{P}^2$. Any polynomial $\gamma(x,y)$ of degree at most n may be written

$$\gamma(x,y) = \sum_{j=0}^n g_j(x,y)$$

where $g_j(x,y)$ is a homogeneous polynomial of degree j (or possibly zero). Note that $\deg(\gamma) = n$ if $g_n \neq 0$. Then

$$\widetilde{\gamma}(x,y,z) := \sum_{j=0}^n z^{n-j} g_j(x,y) = z^n \gamma(x/z, y/z) \tag{17}$$

is the *homogenization* of γ. We note that γ can be recovered from $\widetilde{\gamma}$ by the relation

$$\gamma(x,y) = \widetilde{\gamma}(x,y,1).$$

The "line at infinity " in $\mathbb{C}\mathrm{P}^2$ corresponds to $z = 0$ and hence the points of intersection $\gamma_1 \cap \gamma_2$ are given by solving $\widetilde{\gamma_1}(x,y,0) = 0 = \widetilde{\gamma_2}(x,y,0)$. In this particular example $n = 2$ and $\widetilde{\gamma_1}(x,y,z) = x^2+y^2+xz$ and $\widetilde{\gamma_2}(x,y,z) = x^2+y^2+yz$, so that $\widetilde{\gamma_1}(x,y,0) = x^2+y^2$ and $\widetilde{\gamma_2}(x,y,z) = x^2+y^2$. It is easy to verify that the intersections $\widetilde{\gamma_1}(x,y,0) = 0 = \widetilde{\gamma_2}(x,y,0)$ are given by the two points $[1:i:0], [1:-i:0] \in \mathbb{C}\mathrm{P}^2$. Further, since $p = x^2-xy$ is already homogeneous, $\widetilde{p}(x,y,z) = x^2-xy$ and clearly $\widetilde{p}(1,\pm i,0) = 1\mp i \neq 0$. In other words, p is *not* zero at *all* the intersection points $\widetilde{\gamma_1}\cap\widetilde{\gamma_2}$. Consequently, even though $p \in \langle\Gamma\rangle$, it is *not* the case that $\widetilde{p} \in \langle\widetilde{\Gamma}\rangle$. This is a key consideration.

Lemma 4. Suppose that $\Gamma = \{\gamma_1,\gamma_2,\ldots,\gamma_s\}$ and fix a degree n with $n \geq \max\{d_{\gamma_1}, d_{\gamma_2}, \ldots, d_{\gamma_s}\}$. Then

$$p \in \mathrm{span}_H^n(\Gamma) \iff \widetilde{p} \in \langle\widetilde{\Gamma}\rangle.$$

Proof. First suppose that $p \in \mathrm{span}_H^n(\Gamma)$. Then we may write

$$p(x,y) = \sum_{j=1}^s a_j(x,y)\gamma_j(x,y)$$

with $\deg(a_j) \leq n - d_{\gamma_j}$. Hence (with homogenization degrees $\deg(p) = n$, $\deg(a_j) = n - d_{\gamma_j}$ and $\deg(\gamma_j) = d_{\gamma_j}$)

$$\begin{aligned}
\widetilde{p}(x,y,z) &= z^n p(x/z, y/z) \\
&= z^n \sum_{j=1}^{s} a_j(x/z,y/z)\gamma_j(x/z,y/z) \\
&= \sum_{j=1}^{s} \left\{ z^{n-d_{\gamma_j}} a_j(x/z,y/z) \right\} \left\{ z^{d_{\gamma_j}} \gamma_j(x/z,y/z) \right\} \\
&= \sum_{j=1}^{n} \widetilde{a_j}(x,y,z)\widetilde{\gamma_j}(x,y,z) \\
&\in \langle \widetilde{\Gamma} \rangle.
\end{aligned}$$

Conversely, suppose that $\widetilde{p} \in \langle \widetilde{\Gamma} \rangle$. Then we have

$$\widetilde{p} = \sum_{j=1}^{s} A_j \widetilde{\gamma_j} \tag{18}$$

for some polynomials $A_j = A_j(x,y,z)$. If we write

$$A_j(x,y,z) = \sum_{k=1}^{m_j} h_{jk}(x,y,z)$$

where $h_{jk}(x,y,z)$ is homogeneous of degree k, then, taking the degree n homogeneous part of (18), we have

$$\widetilde{p} = \sum_{j=1}^{s} h_{j,n-d_{\gamma_j}} \widetilde{\gamma_j}.$$

Putting $z = 1$ establishes that $p \in \mathrm{span}_H^n(\Gamma)$. ■

Hence our consistency condition reduces to an ideal membership problem, in one variable more. There is a situation when this is equivalent to an ordinary ideal membership problem.

Lemma 5. Suppose that $\Gamma = \{\gamma_1, \gamma_2\}$ and that these two curves have no points of intersection at infinity (in $\mathbb{C}\mathrm{P}^2$). Then

$$p \in \langle \Gamma \rangle \iff \widetilde{p} \in \langle \widetilde{\Gamma} \rangle.$$

Proof. Suppose first that $\widetilde{p} \in \langle \widetilde{\Gamma} \rangle$. Then we may write

$$\widetilde{p}(x,y,z) = \sum_{j=1}^{2} A_j(x,y,z)\widetilde{\gamma_j}(x,y,z)$$

for some polynomials $A_j(x,y,z)$. Setting $z = 1$ establishes that then $p \in \langle \Gamma \rangle$.

Conversely, suppose that $p \in \langle \Gamma \rangle$. Then we may write

$$p(x,y) = \sum_{j=1}^{2} a_j(x,y)\gamma_j(x,y)$$

for some polynomials $a_j(x,y)$. Homogenizing, we have

$$z^r \widetilde{p}(x,y,z) = \sum_{j=1}^{2} \widetilde{a_j}(x,y,z)\widetilde{\gamma_j}(x,y,z) \tag{19}$$

for some exponent $r \geq 0$. We claim that, in fact, we may actually take $r = 0$. To see this consider the case $r \geq 1$. Evaluating (19) at $z = 0$ we have

$$0 = \sum_{j=1}^{2} \widetilde{a_j}(x,y,0)\widetilde{\gamma_j}(x,y,0) \tag{20}$$

But since, by assumption, $\widetilde{\gamma_1}(x,y,0)$ and $\widetilde{\gamma_2}(x,y,0)$ have no common zeros (other than $x = y = 0$), $\widetilde{\gamma_1}(x,y,0)$ and $\widetilde{\gamma_2}(x,y,0)$ are coprime, and hence from (20) we may conclude that

$$\widetilde{a_1}(x,y,0) = -c(x,y)\widetilde{\gamma_2}(x,y,0) \text{ and } \widetilde{a_2}(x,y,0) = c(x,y)\widetilde{\gamma_1}(x,y,0) \tag{21}$$

for some polynomial $c(x,y)$. Now set

$$A_1(x,y,z) := \widetilde{a}_1(x,y,z) + c(x,y)\widetilde{\gamma}_2(x,y,z)$$

and

$$A_2(x,y,z) := \widetilde{a}_2(x,y,z) - c(x,y)\widetilde{\gamma}_1(x,y,z).$$

Then

$$\begin{aligned} A_1\widetilde{\gamma}_1 + A_2\widetilde{\gamma}_2 &= \{\widetilde{a}_1\widetilde{\gamma}_1 + \widetilde{a}_2\widetilde{\gamma}_2\} + c(x,y)\{\widetilde{\gamma}_2\widetilde{\gamma}_1 - \widetilde{\gamma}_1\widetilde{\gamma}_2\} \\ &= \widetilde{a}_1\widetilde{\gamma}_1 + \widetilde{a}_2\widetilde{\gamma}_2 \\ &= z^r\widetilde{p}. \end{aligned} \tag{22}$$

But

$$A_1(x,y,0) = \widetilde{a}_1(x,y,0) + c(x,y)\widetilde{\gamma}_2(x,y,0) = 0$$

and also

$$A_2(x,y,0) = \widetilde{a}_2(x,y,0) - c(x,y)\widetilde{\gamma}_1(x,y,0) = 0,$$

by (21). It follows that

$$A_1(x,y,z) = zA_1'(x,y,z) \text{ and } A_2(x,y,z) = zA_2'(x,y,z)$$

for some polynomials A_1' and A_2'. Substituting these into (22) we get

$$z^r\widetilde{p} = z\{A_1'\widetilde{\gamma}_1 + A_2'\widetilde{\gamma}_2\}.$$

Dividing by z and taking the appropriate homogeneous part of the right hand side, if necessary, we obtain a relation of the form (19) with r replaced by $r-1$. Repeating this as many times as necessary, we arrive at the $r=0$ case. Clearly, this implies that $\widetilde{p} \in \langle \widetilde{\Gamma} \rangle$, and we are done. ■

Remark. Lemmas 4 and 5 are basic facts from Algebraic Geometry that we have isolated due to their relevance for our interpolation problem. Indeed, the proofs we offer are extracted from the proof of Max Noether's Theorem in [7, p.120].

Remark. The condition that there are no intersections at infinity is actually not restrictive. Indeed, by assumption (and Bezout's Theorem) there are at most $d_{\gamma_1} \times d_{\gamma_2} < \infty$ many intersection points of $\widetilde{\gamma}_1 \cap \widetilde{\gamma}_2$ in $\mathbb{C}\mathrm{P}^2$ and hence, by means of a projective change of coordinates,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \ A \in \mathbb{C}^{3\times 3}, \tag{23}$$

we may always arrange that, in the new coordinates, there are *no* intersections at infinity.

Example 6. Consider the polynomials of Example 5,

$$\widetilde{\gamma}_1 = x^2 + y^2 + xz, \ \widetilde{\gamma}_2 = x^2 + y^2 + yz$$

and the projective change of variables given by

$$x = u, \ y = v \text{ and } z = u + v + w. \tag{24}$$

Then, in the new variables, we have

$$\widetilde{\gamma}_1 = 2u^2 + v^2 + uv + uw \text{ and } \widetilde{\gamma}_2 = u^2 + 2v^2 + uv + vw$$

which for $w = 1$ become

$$\widetilde{\gamma}_1 = 2u^2 + v^2 + uv + u \text{ and } \widetilde{\gamma}_2 = u^2 + 2v^2 + uv + v.$$

It is easy to verify that the four intersections are at $(0,0)$, $(-1/4,-1/4)$ and $((-1\pm i)/2, (-1\mp i)/2)$, i.e., none are at "infinity", $w = 0$.

We thus have the following ideal theoretic test for consistency.

Ideal Theoretic Test for Consistency

1. Choose a projective change of coordinates (23) so that, in the new coordinates, there are no points of intersection $\widetilde{\gamma}_1 \cap \widetilde{\gamma}_2$ at infinity ($z' = 0$).
2. Set $p = P_1 - P_2$ where P_1 and P_2 are the data polynomials given on γ_1 and γ_2, respectively, and compute $\widetilde{p}$.
3. Check if $\widetilde{p}(x',y',1) \in \langle \widetilde{\gamma}_1(x',y',1), \widetilde{\gamma}_2(x',y',1) \rangle$.

Example 7. We continue Examples 5 and 6. Suppose that we wish to check whether or not $p = x^2 - xy \in \mathrm{span}_H^2(\Gamma)$.
We homogenize with $n = 2$ to obtain

$$\widetilde{p}(x, y, z) = z^2 p(x/z, y/z) = x^2 - xy.$$

This, after the change of variables (24) becomes $u^2 - uv$ which remains unaltered in the chart $w = 1$. As we saw, in the chart $w = 1$ the two basis polynomials become

$$\widetilde{\gamma_1} = 2u^2 + v^2 + uv + u \text{ and } \widetilde{\gamma_2} = u^2 + 2v^2 + uv + v$$

and hence our test becomes whether or not

$$u^2 - uv \in \langle 2u^2 + v^2 + uv + u, u^2 + 2v^2 + uv + v\rangle.$$

This is easily seen to be *false* and hence $p = x^2 - xy \notin \mathrm{span}_H^2(\Gamma)$. We emphasize (see Example 5) that $p \in \langle\Gamma\rangle$ showing that conclusions must be drawn with care, only after a suitable projective change of variables. ■

2.1.3 The Differential Duality (Orthogonality) Conditions

It turns out that ideals whose zero sets are a finite number of points (so-called zero-dimensional ideals) can be described by differential orthogonality conditions such as we saw in Examples 1 through 4. These were evidently first described by Macaulay already in 1915 [11], put in a more modern framework by Gröbner [8, Chap. 4, §2] and subsequently described by various authors. We will follow the presentation of de Boor and Ron [5], which we find particularly clear, adjusting as needed for our particular interpolation problem. See also [6] for computational issues.

It is worthwhile to present this construction in general. In this section $\Pi^s = \mathbb{R}[x_1, x_2, \ldots, x_s]$ will denote the space of polynomials in s real variables. We will consider only *real* ideals $I \le \Pi^s$, i.e, such that $p \in I \iff \overline{p} \in I$. As before,

$$V_{\mathbb{C}}(I) := \{z \in \mathbb{C}^s \,:\, p(z) = 0,\ \forall p \in I\}$$

will denote the variety of I. The dimension of the factor space Π^s/I is known as the co-dimension of I, i.e.,

$$\mathrm{codim}(I) := \dim(\Pi^s/I).$$

It could be ∞. However, we have

Proposition 5. (Theorem 6, p. 232, of [2]) The variety $V_{\mathbb{C}}(I)$ is a finite point set if and only if $\mathrm{codim}(I) < \infty$.

We will henceforth consider only ideals for which $V_{\mathbb{C}}(I)$ is a finite set.

It is also useful to introduce an inner product on Π^s as follows. We may write a polynomial in Taylor form, using standard multinomial notation, as $p(x) = \sum_\alpha \frac{D^\alpha p(0)}{\alpha!} x^\alpha$. Then for $q \in \Pi^s$, take

$$(p, q) := \sum_\alpha \frac{(D^\alpha p(0))\overline{(D^\alpha q(0))}}{\alpha!}. \tag{25}$$

We note that we may also write

$$(p, q) = p(D)\overline{q}(0)$$

where

$$p(D) := \sum_\alpha \frac{D^\alpha p(0)}{\alpha!} D^\alpha$$

is the differential operator associated to p.

Definition 7. For $\theta \in \mathbb{C}^s$, the space

$$P_\theta := \{p \in \Pi^s \,:\, p(D)f(\theta) = 0,\ \forall f \in I\}$$

is called the multiplicity space of θ. Its dimension, $\dim(P_\theta)$, is called the multiplicity of θ (as an element of $V_{\mathbb{C}}(I)$), or equivalently, the intersection number of θ.

By making an appropriate translation we may assume, if we wish, that $\theta = 0 \in \mathbb{C}^s$. Then note that $P_\theta \cap I = \{0\}$ for if $0 \neq p \in P_\theta \cap I$,

$$p(D)\overline{p}(\theta) = (p, p) > 0$$

so that $p(D)f(\theta) \neq 0$, for $f = \overline{p} \in I$, a contradiction.

It follows that we may regard P_θ as a subspace of Π^s/I and consequently, $\dim(P_\theta) < \dim(\Pi^s/I) < \infty$, by assumption.

An important property is that these spaces are D*−invariant*, i.e.,

$$p \in P_\theta \implies D^\alpha p \in P_\theta$$

for all multi-indices α. Indeed, following [5], it is easy to verify that, for $p \in P_\theta$,

$$(D^\alpha p)f(\theta) = p(D)(x^\alpha f)(\theta) = 0$$

since $x^\alpha f \in I$ for $f \in I$, as I is an ideal.

A simple consequence is the following. If $\dim(P_\theta) \geq 1$, i.e., there exists a $0 \neq p \in P_\theta$, by judicious choice of derivative, we must have $1 \in P_\theta$. Consequently $0 = 1(D)f(\theta) = f(\theta)$, $\forall f \in I$ and hence $\theta \in V_{\mathbb{C}}(I)$. In other words, if $\theta \notin I$, $P_\theta = \{0\}$, and $\dim(P_\theta) = 0$. Conversely, if $\theta \in V_{\mathbb{C}}(I)$, then $1 \in P_\theta$ and hence $\dim(P_\theta) \geq 1$. In other words

$$\dim(P_\theta) \geq 1 \iff \theta \in V_{\mathbb{C}}(I). \tag{26}$$

When $\dim(P_\theta) = 1$, $\theta \in V_{\mathbb{C}}(I)$ is known as a *simple* zero.

These spaces give a decomposition of the ideal.

Theorem 2.1. *(cf. [5, §]) Set*

$$I_\theta := \{q \in \Pi^s : p(D)q(\theta) = 0, \forall p \in P_\theta\}.$$

Then,

$$I = \bigcap_{\theta \in V_{\mathbb{C}}(I)} I_\theta.$$

Moreover,

$$\sum_{\theta \in V_{\mathbb{C}}(I)} \dim(P_\theta) = \operatorname{codim}(I).$$

There is a more precise version for the case of I generated by two curves in two dimensions.

Theorem 2.2. *(Bezout's Theorem, see e.g. [7, p. 112]) Suppose that $I = \langle \gamma_1, \gamma_2 \rangle$ with $\gamma_1, \gamma_2 \in \mathbb{R}[x, y]$ where γ_1 and γ_2 have no common divisors, and also no common zeros at infinity. Let, as before, $d_{\gamma_j} = \deg(\gamma_j)$, $j = 1, 2$. Then*

$$\sum_{\theta \in V_{\mathbb{C}}(I)} \dim(P_\theta) = \operatorname{codim}(I) = d_{\gamma_1} d_{\gamma_1}.$$

Note that due to (26) we have the following corollary in case of only simple intersections.

Corollary 1. Suppose that $I = \langle \gamma_1, \gamma_2 \rangle$ with $\gamma_1, \gamma_2 \in \mathbb{R}[x, y]$. Suppose further that $V_{\mathbb{C}}(I)$ consists of $d_{\gamma_1} d_{\gamma_1}$ distinct points in $\mathbb{C}^2$. Then

$$p \in I \iff p(\theta) = 0, \forall \theta \in V_{\mathbb{C}}(I).$$

Proof. By (26), each space P_θ, $\theta \in V_{\mathbb{C}}(I)$ has dimension one, and since then $1 \in P_\theta$, we must have that each P_θ is the one dimensional space of constants. Consequently, for $\theta \in V_{\mathbb{C}}(I)$,

$$I_\theta = \{q \in \Pi^s : q(\theta) = 0\}$$

and the result follows from Theorem 2.1. ∎

This explains the consistency conditions of Example 4.

Example 8. We compute the spaces P_θ for Examples 1, 2 and 3.

For Example 1, $I = \langle y, y - x^2 \rangle = \langle y, x^2 \rangle$. There is only one point of intersection, $\theta = (0, 0)$. Any $f \in I$ can be written as $f = ya(x, y) + x^2 b(x, y)$ for arbitrary polynomials $a(x, y)$ and $b(x, y)$. More explicitly, we have $f \in I$ if and only if

$$f(x, y) = \sum_{i \geq 0, j \geq 1} a_{ij} x^i y^j + \sum_{s \geq 2, t \geq 0} b_{st} x^s y^t$$

for arbitrary coefficients a_{ij} and b_{st}. Hence

$$\begin{aligned} P_\theta &= \{p : p(D)f(0,0) = 0, \forall f \in I\} \\ &= \{p : f(D)p(0,0) = 0, \forall f \in I\} \\ &= \{p : \frac{\partial^{i+j}}{\partial x^i \partial y^j} p(0,0) = 0 = \frac{\partial^{s+t}}{\partial x^s \partial y^t} p(0,0),\ i \geq 0, j \geq 1, s \geq 2, t \geq 0\} \\ &= \{a + bx\}. \end{aligned}$$

Thus, since $\#(V_{\mathbb{C}}(I)) = 1$,

$$I = I_\theta = \{p : 0 = p(0,0) = \frac{\partial p}{\partial x}(0,0)\}.$$

For Example 2, $I = \langle y, y^2 - x^3 \rangle = \langle y, x^3 \rangle$ and again $\theta = (0, 0)$ is the only point of intersection. Just as above, we may easily calculate

$$P_\theta = \{a + bx + cx^2\}$$

and hence

$$I = I_\theta = \{p : 0 = p(0,0) = \frac{\partial p}{\partial x}(0,0) = \frac{\partial^2 p}{\partial x^2}(0,0)\}.$$

For Example 3, $I = \langle y, y^2 - x^2(1-x^2)\rangle = \langle y, x^2 - x^4\rangle$. In this case $V_{\mathbb{C}}(I) = \{(0,0),(1,0),(-1,0)\}$ consists of 3 points. It is easily seen that the intersections $(\pm 1, 0)$ are simple and that for both of them P_θ is the one-dimensional space of constants. Since by Bezout's Theorem the sum of the dimensions of the spaces P_θ is $1\times 4 = 4$, it follows that, for $\theta = (0,0)$, $\dim(P_\theta) = 2$. We always have $1 \in P_\theta$ and so if we can display one other independent element we will be done. But, if $q \in I$ then we may write $q = ya(x,y) + (x^2 - x^4)b(x,y)$ for some polynomials $a(x,y)$ and $b(x,y)$. It is easy to check that for such q,

$$\frac{\partial q}{\partial x}(0,0) = 0.$$

Hence $x \in P_\theta$ and $P_\theta = \{a + bx\}$. Consequently

$$\begin{aligned} I &= I_{(-1,0)} \cap I_{(1,0)} \cap I_{(0,0)} \\ &= \{p \,:\, 0 = p(-1,0) = p(1,0) = p(0,0) = \frac{\partial p}{\partial x}(0,0)\}. \end{aligned}$$

The results of this section allow us to test for consistency by checking certain derivative conditions at the points of intersection of the curves. Of course, such a test is not really practical, as it will not in general be possible to find even $V_{\mathbb{C}}(I)$ explicitly. However, it does provide a satisfying explanation of the consistency conditions we first encountered in the simple Examples, 1 through 4.

Differential Point Test for Consistency

1. Choose a projective change of coordinates (23) so that, in the new coordinates, there are no points of intersection $\widetilde{\gamma}_1 \cap \widetilde{\gamma}_2$ at infinity ($z' = 0$).
2. Set $p = P_1 - P_2$ where P_1 and P_2 are the data polynomials given on γ_1 and γ_2, respectively, and compute $\widetilde{p}$.
3. Compute the intersection points $0 = \widetilde{\gamma}_1(x', y', 1) = \widetilde{\gamma}_2(x', y', 1)$ and the corresponding P_θ and I_θ.
4. Check if $\widetilde{p}(x', y', 1) \in \langle \widetilde{\gamma}_1(x', y', 1), \widetilde{\gamma}_2(x', y', 1)\rangle$ by seeing if $\widetilde{p}(x', y', 1)$ satisfies the differential conditions that define the I_θ.

2.1.4 The notion of an H-Basis

We remark briefly that there is a notion (introduced evidently originally by Macaulay [11]) for when the H-span, $\mathrm{span}_H^n(\Gamma)$ (cf. Definition 5, above) is equal to the space of polynomials of degree at most n in a certain ideal, for all $n \geq 0$. Specifically, following [12, Definition 2.1], we have the following

Definition 8. The finite set $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ of non-zero polynomials is called an *H-basis* for the ideal $I := \langle \gamma_1, \ldots, \gamma_s\rangle$ if, for all $0 \neq p \in I$, there exist polynomials $h_1, \ldots, h_s$ such that

$$p = \sum_{i=1}^{s} h_i \gamma_i \quad \text{and} \quad \deg(h_i) + \deg(\gamma_i) \leq \deg(p),\ i = 1, \ldots, s.$$

2.2 Consistency between more than Two Curves; Aitken Interpolation

We have in the previous sections studied how to check for the consistency of data on two curves. We feel that this was a useful exercise as it explains the underlying geometry/algebra of the problem and how it relates to the simple case of lines. However, as it turns out, it is computationally more efficient to just go ahead and calculate the interpolant, by a procedure that we will describe in this section. If it can be carried out to completion, then a posteriori, the data were consistent, and if it fails, it means that the data were inconsistent.

For the reader's convenience we restate the interpolation problem.

Interpolation Problem. Suppose that we are given a set of s real algebraic curves

$$\Gamma := \{\gamma_1, \ldots, \gamma_2\}$$

such that each γ_j has the property that if a polynomial P is zero on $V_{\mathbb{R}}(\gamma_j)$ then there exists a divisor polynomial Q such that $P = \gamma_j Q$ (cf. Prop. 1). We will assume that no two of the curves have a common (complex) component (factor), i.e., are relatively prime over $\mathbb{C}$.

Suppose further that for each curve γ_j we are given an associated data polynomial P_j, $j = 1, \ldots, s$, each of degree at most n. The interpolation problem is to find a global polynomial P, of degree at most n, such that

$$P = P_j \text{ on the curve } \gamma_j,\ j = 1, \ldots, s.$$

Remark. Under our assumptions the products $\gamma_{jk} := \prod_{i=j}^{k} \gamma_i$ also have the same factorization property, as is easy to verify.

The procedure is a simple adaptation of the classical Aitken interpolation algorithm. We illustrate it below for $s = 3$.

$$\begin{tikzcd}[column sep=small]
(\gamma_1; P_1) \arrow[dr] & & (\gamma_2; P_2) \arrow[dl] \arrow[dr] & & (\gamma_3; P_3) \arrow[dl] \\
& (\gamma_1\gamma_2; P_{1,2}) \arrow[dr] & & (\gamma_2\gamma_3; P_{2,3}) \arrow[dl] & \\
& & (\gamma_1\gamma_2\gamma_3; P_{1,2,3}) & &
\end{tikzcd}$$

The general algorithm is of $s-1$ steps. The first is to construct *any* interpolants $P_{j,j+1}$ (of degree at most n) for the consecutive pairs $(\gamma_j; P_j)$, $(\gamma_{j+1}; P_{j+1})$, $j = 1, \ldots, s-1$. Note that if a global interpolant, P, for all the data exists then it is an interpolant of any subset of the data and hence $P_{j,j+1}$ exist. Conversely, if a $P_{j,j+1}$ did not exist, then no global interpolant P could also exist. In other words, if the algorithm fails at this step, the data are overall inconsistent and no global interpolant of degree at most n exists.

If so desired one may use the methods of §2.1 to test for the existence of a $P_{j,j+1}$.

Now, if the first step succeeds we are then left with an interpolation problem for the data

$$\{(\gamma_j\gamma_{j+1}; P_{j,j+1}) : j = 1, \ldots, s-1\},$$

i.e., with the same type of problem but with one fewer curve. Hence we may repeat.

Remark. After the first step, the curves (e.g. $\gamma_1\gamma_2$ and $\gamma_2\gamma_3$) will have common factors. This is not important (cf. Definition 6 and Proposition 3). What is important is that these product curves have the factorization property described in the Interpolation Problem above, which they always do, under our assumptions on the γ_j.

We finish this section with two simple examples.

Example 9. Consider the data

$$\gamma_1 = x^2 + y^2 - 1,\ \gamma_2 = y - x^3,\ \gamma_3 = x + y - 1$$

with data polynomials

$$\begin{aligned}
P_1 &= 1 - 2x^2y^2 + x^2 - 2x^3y - 2xy^3 + 2xy + y^2 \\
P_2 &= 1 + 2x^4 + y^4 - xy - y^2 + x^3y \\
P_3 &= 1 - 4x^3y + x^3 - 6x^2y^2 + 3x^2y - 4xy^3 + 3xy^2 + y^3.
\end{aligned}$$

We take $n = 4$.

One can easily verify that

$$\begin{aligned}
P_1 - P_2 &= -(x^2 + 2xy + y^2)\gamma_1 + (x + y)\gamma_2, \\
P_2 - P_3 &= (-1)\gamma_2 + (2x^3 + 3x^2y + 3xy^2 + y^3 - y)\gamma_3
\end{aligned}$$

and hence the two pairwise interpolation problems of the first step are consistent. In fact, we may take

$$\begin{aligned}
P_{1,2} &= P_1 + (x^2 + 2xy + y^2)\gamma_1 = 1 + x^4 + y^4, \\
P_{2,3} &= P_2 + \gamma_2 = 1 + 2x^4 + y^4 - xy - y^2 + x^3y + y - x^3.
\end{aligned}$$

Continuing, we check for consistency of the problems $(\gamma_1\gamma_2; P_{1,2})$, $(\gamma_2\gamma_3; P_{2,3})$. Indeed,

$$P_{1,2} - P_{2,3} = -x^4 + xy + y^2 - x^3y - y + x^3 = (0)\gamma_1\gamma_2 + (1)\gamma_2\gamma_3$$

and these data are also consistent. Finally, the global interpolant is

$$P := P_{1,2} - (0)\gamma_1\gamma_2 = 1 + x^4 + y^4.$$

Example 10. Consider the data

$$\gamma_1 = x^2 + y^2 - 1,\ \gamma_2 = y - x^3,\ \gamma_3 = x + y - 1$$

with data polynomials

$$\begin{aligned}
P_1 &= 1 - 2x^2y^2 + x^2 - 2x^3y - 2xy^3 + 2xy + y^2 \\
P_2 &= 1 + 2x^4 + y^4 - xy - y^2 + x^3y \\
P_3 &= 1 + x^4 - xy - y^2 - 3x^3y + x^3 - 6x^2y^2 + 3x^2y - 4xy^3 + 3xy^2 + y^3.
\end{aligned}$$

Except for P_3 these are the same as for Example 9. We again take $n = 4$.

As before, one can easily verify that

$$\begin{aligned} P_1 - P_2 &= -(x^2+2xy+y^2)\gamma_1 + (x+y)\gamma_2, \\ P_2 - P_3 &= (0)\gamma_2 + (x+y)^3\gamma_3 \end{aligned}$$

and hence the two pairwise interpolation problems of the first step are consistent. In fact, we may take

$$\begin{aligned} P_{1,2} &= P_1 + (x^2+2xy+y^2)\gamma_1 = 1 + x^4 + y^4, \\ P_{2,3} &= P_2 - (0)\gamma_2 = 1 + 2x^4 + y^4 - xy - y^2 + x^3y. \end{aligned}$$

Continuing, we check for consistency of the problems $(\gamma_1\gamma_2; P_{1,2})$, $(\gamma_2\gamma_3; P_{2,3})$. Indeed,

$$P_{1,2} - P_{2,3} = xy - x^4 + y^2 - x^3y.$$

But this cannot be written in the form $P_{1,2} - P_{2,3} = a\gamma_1\gamma_2 + b\gamma_2\gamma_3$ with $\deg(a) \le 4 - \deg(\gamma_1\gamma_2) = 4 - (2+3)$ and $\deg(b) \le 4 - \deg(\gamma_2\gamma_3) = 4 - (3+1)$ since would mean that $a = 0$ and b is a constant. However,

$$xy - x^4 + y^2 - x^3y = (y - x^3)(x+y) = \gamma_2(\gamma_3 + 1) \ne b\gamma_2\gamma_3, \ \forall b \in \mathbb{R}.$$

Hence these data are inconsistent and there is no global interpolant of degree at most 4.

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