Opuscula Mathematica • Vol. 31 • No. 3 • 2011

http://dx.doi.org/10.7494/OpMath.2011.31.3.433

POSITIVE SOLUTIONS OF ARBITRARY ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS

Sotiris K. Ntouyas, Guotao Wang, Lihong Zhang

Abstract. In this paper, we investigate the existence and uniqueness of positive solutions to arbitrary order nonlinear fractional differential equations with advanced arguments. By applying some known fixed point theorems, sufficient conditions for the existence and uniqueness of positive solutions are established.

Keywords: positive solution, advanced arguments, fractional differential equations, superlinear (sublinear) condition, uniqueness.

Mathematics Subject Classification: 34A08, 34B18, 34K37.

1. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order ([20, 21, 26, 27]). The interest in the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Fractional-order differential equations are also better for the description of hereditary properties of various materials and processes than integer-order differential equations. As a consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see ([1, 8, 10, 11, 13–15, 22, 23, 25, 32]) and the references therein. Some recent work on fractional differential equations can be seen in ([2–6, 9, 12, 24, 28]).

The theory of differential equations with deviated arguments is an important and significant branch of nonlinear analysis. It is worthwhile mentioning that differential equations with deviated arguments appear often in investigations connected with mathematical physics, mechanics, engineering, economics and so on (see [4,12]). One of the basic problems considered in the theory of differential equations with deviated arguments is to establish convenient conditions guaranteeing the existence of solutions of those equations. For the general theory and applications of differential equations with deviated arguments, we refer the reader to the references ([5, 17, 18, 28–31]). However, fractional differential equations with deviated arguments have not been much studied and many aspects of these equations are yet to be explored. For some recent work on equations of fractional order with deviated arguments, see [7], and the references therein.

Motivated by some recent work on advanced arguments and boundary value problems of fractional order, in this paper, we investigate the following nonlinear fractional-order differential equation with advanced arguments

$$\begin{cases} D_0^\alpha u(t) + a(t)f(u(\theta(t))) = 0, & 0 < t < 1,\ n-1 < \alpha \leq n, \\ u^{(i)}(0) = 0, & i = 0, 1, 2, \ldots, n-2, \\ [D_0^\beta u(t)]_{t=1} = 0, & 1 \leq \beta \leq n-2, \end{cases} \tag{1.1}$$

where $n > 3$ $(n \in \mathbb{N})$, D_0^α is the standard Riemann-Liouville fractional derivative of order α, $f : [0,\infty) \to [0,\infty)$, $a : [0,1] \to (0,\infty)$ and $\theta : (0,1) \to (0,1]$ are continuous functions.

By a positive solution of (1.1), one means a function $u(t)$ that is positive on $0 < t < 1$ and satisfies (1.1).

Our purpose here is to give existence and uniqueness results of positive solution to problem (1.1). We apply the Banach contraction principle and the well-known Guo-Krasnoselskii fixed point theorem:

Theorem 1.1 ([16])**.** *Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that:*

(i) $\|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. PRELIMINARIES

For the reader's convenience, we present some necessary definitions from fractional calculus theory and some known lemmas.

Definition 2.1. The Riemann-Liouville fractional integral of order q is defined as

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds, \qquad q > 0,$$

provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order q for a function y is defined by

$$D^q y(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} y(s)ds, \qquad n = [q]+1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3 ([15])**.** *Assume that $y(t) \in C[0,1]$. Then the following problem*

$$\begin{cases} D_0^\alpha u(t) + y(t) = 0, & 0 < t < 1, \ n-1 < \alpha \le n, \\ u^{(i)}(0) = 0, & i = 0, 1, 2, \ldots, n-2, \\ [D_0^\beta u(t)]_{t=1} = 0, & 1 \le \beta \le n-2, \end{cases} \tag{2.1}$$

has the unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

In the paper we will use the following lemma contained in [15].

Lemma 2.4. *There exists a constant $\gamma \in (0,1)$ such that*

$$\min_{t \in [\frac{1}{2}, 1]} G(t,s) \ge \gamma \max_{t \in [0,1]} G(t,s) = \gamma G(1,s),$$

where $G(t,s)$ is given by (2.2)*.*

Remark 2.5. In [15] it was proved that γ has the expression

$$\gamma = \min \left\{ \frac{\left(\frac{1}{2}\right)^{\alpha-\beta-1}}{2^\beta - 1}, \left(\frac{1}{2}\right)^{\alpha-1} \right\}. \tag{2.3}$$

3. MAIN RESULTS

Let $E = C[0,1]$ be the Banach space of all continuous real-valued functions on $[0,1]$ endowed with the usual sup-norm $\|\cdot\|$. Let us introduce the cone

$$P = \{u | u \in C[0,1], u \geq 0, \inf_{t\in[\frac{1}{2},1]} u(\theta(t)) \geq \gamma \|u\|\},$$

where $\gamma \in (0,1)$ is given by (2.3), and define the operator $T : C[0,1] \to C[0,1]$ as follows:

$$Tu(t) = \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds. \tag{3.1}$$

Using Lemma 2.3 with $y(t) = a(t)f(u(\theta(t)))$, the problem (1.1) reduces to a fixed point problem $u = Tu$, where T is given by (3.1). Thus the problem (1.1) has a solution if and only if the operator T has a fixed point.

By using the Ascoli-Arzela theorem, it is easy to prove that T is completely continuous. Since $t \leq \theta(t) \leq 1,\ t \in (0,1)$, we have

$$\inf_{t\in[\frac{1}{2},1]} u(\theta(t)) \geq \inf_{t\in[\frac{1}{2},1]} u(t) \geq \gamma\|u\|. \tag{3.2}$$

Thus Lemma 2.4 and (3.2) show that $TP \subset P$, i.e. $T : P \to P$.

Set

$$f_0 = \lim_{u\to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

Now we give our results.

Theorem 3.1. *Assume that:*

(H_1) $a \in C([0,1],[0,\infty))$ *and* a *does not vanish identically on any subinterval.*
(H_2) *The advanced arguments* θ *satisfy* $t \leq \theta(t) \leq 1$, *for all* $t \in (0,1)$.

Then the problem (1.1) *has at least one positive solution if:*

(i) $f_0 = \infty$ *and* $f_\infty = 0$ (*sublinear*)

or

(ii) $f_0 = 0$ *and* $f_\infty = \infty$ (*superlinear*).

Proof. (i) *Sublinear case* ($f_0 = \infty$ *and* $f_\infty = 0$).

Since $f_0 = \infty$, then there exists a constant $\rho_1 > 0$ such that $f(u) \geq \delta_1 u$ for $0 < u < \rho_1$, where $\delta_1 > 0$ satisfies

$$\delta_1\gamma \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)ds \geq 1. \tag{3.3}$$

Take $u \in P$ such that $\|u\| = \rho_1$. Then, we have:

$$\begin{aligned}\|Tu\| &\geq Tu\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, s\right) a(s) f(u(\theta(s)))ds = \\ &= \int_0^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) a(s) f(u(\theta(s)))ds + \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s) f(u(\theta(s)))ds \geq \\ &\geq \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s) f(u(\theta(s)))ds \geq \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)\delta_1 u(\theta(s))ds \geq \\ &\geq \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)\delta_1\gamma\|u\|ds = \delta_1\gamma \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)ds\|u\| \geq \|u\|.\end{aligned}$$

Let $\Omega_{\rho_1} = \{u \in C[0,1] \mid \|u\| < \rho_1\}$. Thus, (3.4) shows that $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_{\rho_1}$. Next, since $f_\infty = 0$, then there exists a constant $R > \rho_1$ such that $f(u) \leq \delta_2 u$ for $u \geq R$, where $\delta_2 > 0$ satisfies

$$\delta_2 \int_0^1 G(1,s)a(s)ds \leq 1. \tag{3.4}$$

We consider the following two cases.
Case 1. f is bounded. Then there exists a constant $R_1 > 0$ such that $f(u) \leq R_1$ for $u \in [0,\infty)$. Now, we may choose $u \in P$ such that $\|u\| = \rho_2$, where $\rho_2 \geq \max\{\mu, R\}$. Then,

$$Tu(t) = \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \leq R_1 \int_0^1 G(1,s)a(s)ds \triangleq \mu \leq \rho_2 = \|u\|. \tag{3.5}$$

Case 2. f is unbounded. Then there exists a constant $\rho_2 > \dfrac{R}{\gamma} > R$ such that $f(u) \leq f(\rho_2)$ for $0 < u \leq \rho_2$. (Note that $f \in C([0,\infty),[0,\infty))$). Let $u \in P$ such that $\|u\| = \rho_2$. We have

$$\begin{aligned}Tu(t) &= \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \leq \int_0^1 G(1,s)a(s)f(\rho_2)ds \leq \\ &\leq \delta_2\rho_2 \int_0^1 G(1,s)a(s)ds = \delta_2 \int_0^1 G(1,s)a(s)ds\|u\| \leq \|u\|.\end{aligned} \tag{3.6}$$

Hence, in either case, we may always let $\Omega_{\rho_2} = \{u \in C[0,1] \mid \|u\| < \rho_2\}$ such that $\|Tu\| \le \|u\|$ for $u \in P \cap \partial\Omega_{\rho_2}$. Thus, by the first part of the Guo-Krasnoselskii fixed point theorem, we can conclude that (1.1) has at least one positive solution.

(ii) *Superlinear case* ($f_0 = 0$ *and* $f_\infty = \infty$).

Now, in view of $f_0 = 0$, there exists a constant $r_1 > 0$ such that $f(u) \le \tau_1 u$ for $0 < u < r_1$, where $\tau_1 > 0$ satisfies

$$\tau_1 \int_0^1 G(1,s)a(s)ds \le 1. \tag{3.7}$$

Take $u \in P$ such that $\|u\| = r_1$. Then we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(\theta(s)))ds \le \int_0^1 G(1,s)a(s)f(u(\theta(s)))ds \le \\ &\le \int_0^1 G(1,s)a(s)\tau_1 u(\theta(s))ds \le \tau_1 \int_0^1 G(1,s)a(s)ds\|u\| \le \|u\|. \end{aligned} \tag{3.8}$$

Let $\Omega_{r_1} = \{u \in C[0,1] \mid \|u\| < r_1\}$. Thus, (3.8) shows $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_{r_1}$. Next, in view of $f_\infty = \infty$, there exists a constant $r > r_1$ such that $f(u) \ge \tau_2 u$ for $u \ge r$, where $\tau_2 > 0$ satisfies

$$\tau_2\gamma \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)ds \ge 1. \tag{3.9}$$

Let $\Omega_{r_2} = \{u \in C[0,1] \mid \|u\| < r_2\}$, where $r_2 > \dfrac{r}{\gamma} > r$. Then $u \in P$ and $\|u\| = r_2$ implies

$$\inf_{t \in [\frac{1}{2},1]} u(\theta(t)) \ge \gamma\|u\| > r,$$

and so

$$\begin{aligned} \|Tu\| \ge Tu\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) a(s)f(u(\theta(s)))ds \ge \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)\tau_2 u(\theta(s))ds \ge \\ &\ge \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)\tau_2\gamma\|u\|ds = \tau_2\gamma \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) a(s)ds\|u\| \ge \|u\|. \end{aligned} \tag{3.10}$$

This shows that $\|Tu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_{r_2}$. Therefore, by the second part of the Guo-Krasnoselskii fixed point theorem, we can conclude that (1.1) has at least one positive solution $u \in P \cap (\overline{\Omega}_{r_2} \setminus \Omega_{r_1})$. □

Theorem 3.2. *Assume that* (H_1) *holds. In addition we suppose that the following condition:*

(H_3) *There exists a non-negative bounded integrable function* $M(t)$ *such that*

$$|f(u) - f(v)| \leq M(t)|u - v|, \quad t \in (0,1),\ u, v \in C[0,1],$$

is satisfied. Then the problem (1.1) *has a unique solution provided*

$$L = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) M(s) ds \leq 1.$$

Proof. Let $u, v \in C[0,1]$. Then we can get

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \int_0^1 G(t,s)a(s)|f(u(\theta(s))) - f(v(\theta(s)))|ds \leq \\ &\leq \int_0^1 G(t,s)a(s)M(s)|u(\theta(s)) - v(\theta(s))|ds \leq \\ &\leq \int_0^1 G(1,s)a(s)M(s)ds\|u - v\| = \\ &= \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} a(s)M(s)ds\|u - v\|. \end{aligned}$$

Thus, $\|Tu - Tv\| \leq L\|u - v\|$. As $L < 1$, T is a contraction. Therefore, the conclusion of the theorem follows by the contraction mapping principle. □

4. EXAMPLE

Example 4.1. Consider the fractional differential equation with advanced arguments

$$\begin{cases} D_0^{8.6}u(t) + e^{-t^2} f(u(\theta(t))) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, & i = 0, 1, 2, \ldots, 7, \\ [D_0^{6.8}u(t)]_{t=1} = 0, & \end{cases} \tag{4.1}$$

where $\alpha = 8.6, \beta = 6.8,\ \theta(t) = t^\nu,\ 0 < \nu < 1$ and

$$f(u) = \begin{cases} \dfrac{e^u}{\sin^2 u}, & 0 \leq u \leq \frac{\pi}{2}, \\ e^{\frac{\pi}{2}} + \dfrac{\sin u + 2}{u}, & u > \frac{\pi}{2}. \end{cases}$$

Obviously, conditions (H_1) and (H_2) of Theorem 3.1 hold. Through a simple calculation we can get $f_0 = \infty$ and $f_\infty = 0$. Thus, by the first part of Theorem 3.1, we get that the problem (4.1) has at least one positive solution.

REFERENCES

[1] B. Ahmad, J.J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl. **58** (2009), 1838–1843.

[2] B. Ahmad, S. Sivasundaram, *Existence of solutions for impulsive integral boundary value problems of fractional order*, Nonlinear Anal.: Hybrid Syst. **4** (2010), 134–141.

[3] R.P. Agarwal, D. O'Regan, S. Stanek, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl. **371** (2010), 57–68.

[4] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[5] A. Augustynowicz, H. Leszczyński, W. Walter, *On some nonlinear ordinary differential equations with advanced arguments*, Nonlinear Anal. **53** (2003), 495–505.

[6] Z.B. Bai, *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear Anal. **72** (2010), 916–924.

[7] K. Balachandran, J.Y. Park, M.D. Julie, *On local attractivity of solutions of a functional integral equation of fractional order with deviating arguments*, Commun. Nonlinear Sci. Numer. Simulat. **15** (2010), 2809–2817.

[8] K. Balachandran, J.J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal. **72** (2010), 4587–4593.

[9] D. Băeanu, O.G. Mustafa, R.P. Agarwal, *An existence result for a superlinear fractional differential equation*, Appl. Math. Lett. **23** (2010), 1129–1132.

[10] M. Belmekki, J.J. Nieto, R. Rodriguez-Lopez, *Existence of periodic solution for a nonlinear fractional differential equation*, Bound. Value Probl. 2009, 18 p. Article ID 324561.

[11] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. **71** (2009), 2391–2396.

[12] T.A. Burton, *Differential inequalities for integral and delay differential equations*, Xinzhi Liu, David Siegel (Eds.), Comparison Methods and Stability Theory, [in:] Lecture Notes in Pure and Appl. Math., Dekker, New York, 1994.

[13] Y.K. Chang, J.J. Nieto, *Some new existence results for fractional differential inclusions with boundary conditions*, Math. Comput. Model. **49** (2009), 605–609.

[14] M.A. Darwish, S.K. Ntouyas, *Monotonic solutions of a perturbed quadratic fractional integral equation*, Nonlinear Anal. **71** (2009), 5513–5521.

[15] Ch.S. Goodrich, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett. **23** (2010), 1050–1055.

[16] D. Guo, V. Lakshmikantham, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, 1996.

[17] T. Jankowski, *Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions*, Nonlinear Anal. **73** (2010), 1289–1299.

[18] T. Jankowski, *First-order advanced difference equations*, Appl. Math. Comput. **216** (2010), 1242–1249.

[19] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, 1996.

[20] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[21] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[22] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. **69** (2008) 8, 2677–2682.

[23] M.P. Lazarević, A.M. Spasić, *Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach*, Math. Comput. Model. **49** (2009), 475–481.

[24] G.M. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations*, Nonlinear Anal. **72** (2010), 1604–1615.

[25] J.J. Nieto, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions*, Appl. Math. Lett. **23** (2010), 1248–1251.

[26] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[27] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[28] J.R. Yan, *Oscillation of first-order impulsive differential equations with advanced argument*, Comput. Math. Appl. **42** (2001), 1353–1363.

[29] Ch. Yang, Ch. Zhai, J. Yan, *Positive solutions of the three-point boundary value problem for second order differential equations with an advanced argument*, Nonlinear Anal. **65** (2006), 2013–2023.

[30] G. Wang, L. Zhang, G. Song, *Integral boundary value problems for first order integro-differential equations with deviating arguments*, J. Comput. Appl. Math. **225** (2009), 602–611.

[31] G. Wang, *Boundary value problems for systems of nonlinear integro-differential equations with deviating arguments*, J. Comput. Appl. Math. **234** (2010), 1356–1363.

[32] S.Q. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl. **59** (2010), 1300–1309.

Sotiris K. Ntouyas
sntouyas@uoi.gr

University of Ioannina
Department of Mathematics
451 10 Ioannina, Greece

Guotao Wang
wgt2512@163.com

Shanxi Normal University Linfen
School of Mathematics and Computer Science
Shanxi 041004, P.R. China

Lihong Zhang
zhanglih149@126.com

Shanxi Normal University Linfen
School of Mathematics and Computer Science
Shanxi 041004, P.R. China

Received: August 8, 2010.
Revised: November 12, 2010.
Accepted: November 16, 2010.