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SOLUTION OF THE STIELTJES TRUNCATED MATRIX MOMENT PROBLEM

Abstract. The truncated Stieltjes matrix moment problem consisting in the description of all matrix distributions $\boldsymbol{\sigma}(t)$ on $[0,\infty)$ with given first $2n+1$ power moments $(\mathbf{C}_j)_{j=0}^n$ is solved using known results on the corresponding Hamburger problem for which $\boldsymbol{\sigma}(t)$ are defined on $(-\infty,\infty)$. The criterion of solvability of the Stieltjes problem is given and all its solutions in the non-degenerate case are described by selection of the appropriate solutions among those of the Hamburger problem for the same set of moments. The results on extensions of non-negative operators are used and a purely algebraic algorithm for the solution of both Hamburger and Stieltjes problems is proposed.

Keywords: Stieltjes power moments, canonical solutions, Nevanlinna's formula.

Mathematics Subject Classification: Primary 30E05, 30E10.

1. INTRODUCTION

The *Hamburger truncated matrix moments problem* is formulated in the following way:

given a set of Hermitian $s \times s$ matrices

$$\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{2n}\}, \quad n = 0, 1, 2, \ldots. \tag{1}$$

find all non-negative matrix measures $d\boldsymbol{\sigma}(t)$ such that

$$\int_{-\infty}^{\infty} t^k d\boldsymbol{\sigma}(t) = \mathbf{C}_k, \quad k = 0, 1, 2, \ldots, 2n. \tag{2}$$

The additional demand: $d\boldsymbol{\sigma}(t) = 0$ for $t < 0$, transforms it into the *Stieltjes truncated matrix moment problem.* Classical results on the topic is reflected in the books

[5, 8, 9, 11, 12], more recent developments can be found in [1, 6, 7]. The aim of the present work is to solve the matrix variant of the Stieltjes truncated moment problem. The natural approach to this problem developed here consists in making clear, which additional conditions should be imposed on the given moments $(\mathbf{C}_j)_{j=0}^{2n}$ to provide the existence of the solutions $\boldsymbol{\sigma}(t)$ of the Hamburger problem with support on the positive half-axis and in singling out in the indefinite case among all solutions of the Hamburger problem those with support on the half-axis $t \geq 0$.

In Section 2 the criterion of solvability of the Stieltjes truncated matrix moment problem is established. The ascending to M. G. Krein [10] general approach to the truncated matrix moment problems based on fundamental results of the extension for symmetric operators is outlined here.

A special class of the so-called canonical solutions of the truncated Stieltjes problem is described in the next Section. For this class of solutions $\boldsymbol{\sigma}(t)$ the holomorphic matrix functions

$$\mathbf{K}(z) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d\boldsymbol{\sigma}(t), \quad \operatorname{Im} z \neq 0, \tag{3}$$

are such that $\det \mathbf{K}(z)$ is a rational function of the minimal degree $\leq ns$. An algebraic algorithm is given here for constructing of such solutions.

The description of all solutions of the matrix truncated moment problem in the non-degenerate case, where

$$\det\left(\mathbf{C}_{j+k}\right)_{j,k=0}^{n} > 0$$

is obtained in the last section of the paper.

The simplified scalar version of the present paper was considered earlier in [4].

2. EXISTENCE OF SOLUTIONS OF THE TRUNCATED STIELTJES PROBLEM FOR MATRIX MOMENTS

Any solution of the Stieltjes problem is evidently a special solution of the Hamburger problem, for which there are no points of growth of the sought non-decreasing matrix function $\boldsymbol{\sigma}$ on the half-axis $(-\infty, 0)$. Therefore a criterion of solvability of the Hamburger problem is only a necessary condition for the solvability of the Stieltjes problem.

Theorem 2.1. *A system of Hermitian matrices* $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{2n}\}$, $n = 0, 1, 2, \ldots$ *admits the representation*

$$\int_{0}^{\infty} t^k \, d\boldsymbol{\sigma}(t) = \mathbf{C}_k, \quad k = 0, 1, 2, \ldots, 2n, \tag{4}$$

if and only if:

a) *the block Hankel matrix* $\boldsymbol{\Gamma}_n := (\mathbf{C}_{k+j})_{k,j=0}^{n}$ *is non-negative;*

b) *for any set* $\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_r \in \mathbb{C}_s$, $0 \le r \le n-1$, *and with* $(\boldsymbol{\xi}, \eta)$ *being a standard scalar product in* $\mathbb{C}_s$, *the condition*

$$\sum_{j,k=0}^{r} (\mathbf{C}_{j+k}\boldsymbol{\xi}_k, \boldsymbol{\xi}_j) = 0 \tag{5}$$

implies

$$\sum_{j,k=0}^{r} (\mathbf{C}_{j+k+2}\boldsymbol{\xi}_k, \boldsymbol{\xi}_j) = 0; \tag{6}$$

c) *the block Hankel matrix* $\mathbf{\Gamma}^{(1)}_{n-1} := (\mathbf{C}_{k+j+1})_{k,j=0}^{n-1}$ *is non-negative and for any set* $\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_r \in \mathbb{C}_s$, $0 \le r \le n-1$, *the condition*

$$\sum_{j,k=0}^{r} (\mathbf{C}_{j+k+1}\boldsymbol{\xi}_k, \boldsymbol{\xi}_j) = 0 \tag{7}$$

implies (6).

Proof. Due to [6] and [1, 2, 3], the conditions a), b) of the theorem is the criterion of solvability of the truncated Hamburger problem for matrix moments. Therefore we need only to prove that the condition c), in addition to a) and b), is equivalent to the existence among the solutions of the Hamburger problem of those with $\boldsymbol{\sigma}(t < 0) = 0$.

1. Suppose that the relations (2) hold.

For an arbitrary set of s-dimensional complex vectors $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{n-1}\}$ we define

$$\mathbf{P}(t) = \boldsymbol{\xi}_0 + \boldsymbol{\xi}_1\ t + \boldsymbol{\xi}_2\ t^2 + \ldots + \boldsymbol{\xi}_{n-1}\ t^{n-1}. \tag{8}$$

By (2) and for $0 \le r \le n-1$

$$\sum_{k,j=0}^{r} \left(\mathbf{C}_{j+k+1}\ \boldsymbol{\xi}_k, \boldsymbol{\xi}_j\right) = \int_0^\infty t\,(d\boldsymbol{\sigma}(t)\mathbf{P}(t), \mathbf{P}(t)) \ge 0. \tag{9}$$

Hence the block matrix $(\mathbf{C}_{k+j+1})_{k,j=0}^{n-1}$ is non-negative.

If for some set $\{\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_r\}$, $\boldsymbol{\xi}_k \in \mathbb{C}_s$, $k = 0, 1, \ldots, r$, $0 \le r \le n-1$, (5) holds, then for the vector polynomial $\mathbf{P}(t)$ defined by (8) we have:

$$\int_0^\infty t\,(d\boldsymbol{\sigma}(t)\mathbf{P}(t), \mathbf{P}(t)) = 0$$

and hence,

$$\sum_{k,j=0}^{r} (\mathbf{C}_{j+k+2}\ \boldsymbol{\xi}_k, \boldsymbol{\xi}_j) = \int_0^\infty t^2\ \ (d\boldsymbol{\sigma}(t)\mathbf{P}(t), \mathbf{P}(t)) = 0.$$

2. Notice that due to the conditions a) and c) of the theorem, the quadratic forms of the matrices $\mathbf{C}_j$ are non-negative, or briefly $\mathbf{C}_j \geq 0$, $j = 0, \dots, 2n$.

From now on we can assume without loss of generality that all these matrices are invertible, i.e. that $\mathbf{C}_j > 0$, $j = 0, \dots, 2n$. Indeed, let $\mathbb{N}_j \subset \mathbb{C}_s$ be null-spaces of $\mathbf{C}_j$. Due to the condition a) the equality $\mathbf{C}_0 \boldsymbol{\eta} = 0$ for any vector $\boldsymbol{\eta} \in \mathbb{C}_s$ implies $\mathbf{C}_j \boldsymbol{\eta} = 0$, $j = 1, \dots, 2n$. Besides, since for any vector $\boldsymbol{\eta} \in \mathbb{C}_s$ all integrals

$$\int_0^\infty t^k \; d\left(\boldsymbol{\sigma}(t)\boldsymbol{\eta}, \boldsymbol{\eta}\right), \quad 1 \leq k \leq 2n$$

vanish simultaneously, any equality $\mathbf{C}_j \boldsymbol{\eta} = 0$ implies $\mathbf{C}_k \boldsymbol{\eta} = 0$, $1 \leq k \leq 2n$. Hence, $\mathbb{N}_0 \subset \mathbb{N}_1 = \mathbb{N}_k$, $k = 1, \dots, 2n$ and for a suitable basis in $\mathbb{C}_s$ the matrices $\mathbf{C}_j$ can be reduced to the form

$$\mathbf{C}_0 = \begin{pmatrix} \widetilde{\mathbf{C}}_0 & 0 \\ 0 & \widetilde{\widetilde{\mathbf{C}}}_0 \end{pmatrix}; \mathbf{C}_j = \begin{pmatrix} \widetilde{\mathbf{C}}_j & 0 \\ 0 & 0 \end{pmatrix}, \quad j = 1, \dots, 2n, \tag{10}$$

where by construction $\det \widetilde{\mathbf{C}}_j > 0$, $j = 0, 1, \dots, 2n$.

If the truncated Stieltjes problem for invertible matrix moments $\widetilde{\mathbf{C}}_j$ is solvable, then the initial problem is solvable as well, and its general solution $\boldsymbol{\sigma}(t)$ can be presented as

$$\boldsymbol{\sigma}(t) = \mathbf{U} \begin{pmatrix} \widetilde{\boldsymbol{\sigma}}\,(t) & 0 \\ 0 & \widetilde{\widetilde{\mathbf{C}}}_0 \vartheta\,(t) \end{pmatrix} \mathbf{U}^*,$$

where $\mathbf{U}$ is a fixed $s \times s$-unitary matrix,

$$\vartheta\,(t) = \begin{cases} 0, \; t \leq 0, \\ 1, \; t > 0, \end{cases}$$

and $\widetilde{\boldsymbol{\sigma}}(t)$ runs the set of solutions of the truncated Stieltjes matrix moment problem for the moments $\widetilde{\mathbf{C}}_j$.

3. Suppose now that a)–c) hold.

(a) In this case and for a given set of $s \times s$ positive definite matrices $\{\mathbf{C}_0, \dots, \mathbf{C}_{2n}\}$ by the conditions a), b) the truncated matrix Hamburger moment problem has at least one solution [7, 1, 3, 2]. Let a non-decreasing matrix function $\boldsymbol{\sigma}(t)$, $-\infty < t < \infty$, be such a solution, i.e.

$$\int_{-\infty}^{\infty} t^k \; d\boldsymbol{\sigma}(t) = \mathbf{C}_k \quad k = 0, 1, 2, \dots, 2n. \tag{11}$$

Consider the set of continuous vector functions $\mathbf{f}(t)$, $-\infty < t < \infty$, with values in $\mathbb{C}_s$, for which

$$\int_{-\infty}^{\infty} \left(d\boldsymbol{\sigma}(t)\mathbf{f}(t), \mathbf{f}(t)\right) < \infty. \tag{12}$$

Construct a pre-Hilbert space $\mathcal{L}$ of such vector functions taking the bilinear functional

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\infty}^{\infty} (d\boldsymbol{\sigma}(t)\mathbf{f}(t), \mathbf{g}(t)) \tag{13}$$

as the scalar product. Notice that by (2) the vector polynomials

$$\mathbf{f}(t) = \boldsymbol{\xi}_0 + t \cdot \boldsymbol{\xi}_1 + \ldots + t^r \cdot \boldsymbol{\xi}_r, \quad \boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_r \in \mathbb{C}_s, \tag{14}$$

of degree $r \leq n$ belong to $\mathcal{L}$. We will denote the linear subset of these polynomials by $\mathcal{P}_n$.

Let $\mathcal{L}_0$ be the subspace of $\mathcal{L}$ consisting of all vector functions $\mathbf{f}$ such that

$$\|\mathbf{f}\| := \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} = 0.$$

If $\mathbf{g} = \mathbf{f} + \mathbf{f}_0$, where $\mathbf{f} \in \mathcal{L}$, $\mathbf{f}_0 \in \mathcal{L}_0$, than, due to the Schwartz inequality $\langle \mathbf{f}, \mathbf{f}_0 \rangle = 0$ and hence $\|\mathbf{g}\| = \|\mathbf{f}\|$. Let us denote by $\mathcal{L}_1$ the factor — space $\mathcal{L} \setminus \mathcal{L}_0$. For the class of elements $\widehat{\mathbf{g}} = \mathbf{f} + \mathcal{L}_0$ of this factor space we set $\|\widehat{\mathbf{g}}\|_{\mathcal{L}_1} = \|\mathbf{f}\|$. Taking the closure of $\mathcal{L}_1$ with respect to this norm, we obtain the Hilbert space $\mathrm{L}^2_\sigma(\mathbb{C}_s)$. We keep the same symbol $\langle .,. \rangle$ for the scalar product in $\mathrm{L}^2_\sigma(\mathbb{C}_s)$. Let L_n be the subspace of $\mathrm{L}^2_\sigma(\mathbb{C}_s)$ generated by the subset of vector polynomials $\mathcal{P}_n$. By (11) and (13) for $\mathbf{f}, \mathbf{g} \in \mathcal{P}_n$,

$$\mathbf{f}(t) = \sum_{l=0}^{n} t^r \cdot \boldsymbol{\xi}_r, \;\; \mathbf{g}(t) = \sum_{l=0}^{n} t^r \cdot \boldsymbol{\eta}_r, \;\; \boldsymbol{\xi}_0, \ldots, \boldsymbol{\eta}_n \in \mathbb{C}_s,$$

we have

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{j,k=0}^{n} (\mathbf{C}_{j+k} \boldsymbol{\xi}_k, \eta_j). \tag{15}$$

Therefore for all non-decreasing matrix function $\boldsymbol{\sigma}(t)$ which satisfy (2), the restrictions onto L_n of the scalar products defined in the corresponding spaces $\mathrm{L}^2_\sigma(\mathbb{C}_s)$ must coincide.

The non-decreasing matrix functions $\boldsymbol{\sigma}(t)$ which satisfy (11) and for which $\mathrm{L}^2_\sigma(\mathbb{C}_s) = \mathrm{L}_n$, are called *canonical*. It was proven in [1, 3] that the set of canonical solutions of the truncated matrix Hamburger moment problem is non-empty whenever this problem is solvable, i.e. whenever the conditions a), b) of the theorem hold. Due to (15), a canonical $\boldsymbol{\sigma}(t)$ is a non-decreasing matrix function which has only a finite number of points of growth and the sum of the ranks of all jumps of $\boldsymbol{\sigma}$ at such points is $\leq ns$.

Take some canonical solution $\tilde{\sigma}(t)$ of the truncated Hamburger moment problem for the given set of moments and consider the self-adjoint operator $\widetilde{A}$ of multiplication by the independent variable t in the related space $\mathrm{L}^2_{\tilde{\sigma}}(\mathbb{C}_s) = \mathrm{L}_n$. Let us denote by L_{n-1} the subspace of L_n generated by vector polynomials of the degree $\leq n-1$. The restriction A_0 of the operator $\widetilde{A}$ onto L_{n-1} is a symmetric operator which by definition of $\widetilde{A}$ actually does not depend on the choice of a canonical solution of the truncated Hamburger moment problem. Therefore each canonical solution $\tilde{\sigma}(t)$ of this problem generates some self-adjoint extension $\widetilde{A}$ of A_0 in L_n.

Let $\widetilde{\mathbf{E}}_t$, $-\infty < t < \infty$, be the spectral function of some canonical extension $\widetilde{A}$. For the canonical orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ in $\mathbb{C}_s$ we introduce the set of classes $\{\widehat{\mathbf{e}}_{10}, \dots, \widehat{\mathbf{e}}_{s0}\} \subset \mathrm{L}_n$ which contain vector polynomials

$$\widehat{\mathbf{e}}_{10}(t) \equiv \mathbf{e}_1, \dots, \widehat{\mathbf{e}}_{s0}(t) \equiv \mathbf{e}_s,$$

respectively. Let us consider the non-decreasing $s \times s$ matrix function $\tilde{\sigma}(t) = (\widetilde{\sigma}_{\mu\nu}(t))_{\mu,\nu=1}^{s}$, $-\infty < t < \infty$,

$$\widetilde{\sigma}_{\mu\nu}(t) := \left\langle \widetilde{\mathbf{E}}_t \widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle_{\mathrm{L}_n}, \quad 1 \le \mu, \nu \le s. \tag{16}$$

By definition of A_0 and $\widetilde{A}$, for the classes which contain the vector monomials

$$\widehat{\mathbf{e}}_{1k}(t) \equiv t^k \mathbf{e}_1, \dots, \widehat{\mathbf{e}}_{sk}(t) \equiv t^k \mathbf{e}_s, \quad 0 \le k \le n$$

we have

$$\widehat{\mathbf{e}}_{\mu k} = A_0^k \widehat{\mathbf{e}}_{\mu 0} = \widetilde{A}^k \widehat{\mathbf{e}}_{\mu 0}, \quad 1 \le \mu \le s;\ 0 \le k \le n.$$

Hence,

$$\begin{aligned}(\mathbf{C}_{j+k})_{\mu,\nu} &= (\mathbf{C}_{j+k}\mathbf{e}_\nu, \mathbf{e}_\mu)_{\mathbb{C}_s} = \\ &= \langle \widehat{\mathbf{e}}_{\nu k}, \widehat{\mathbf{e}}_{\mu j} \rangle_{\mathrm{L}_n} = \left\langle \widetilde{A}^k \widehat{\mathbf{e}}_{\nu 0}, \widetilde{A}^j \widehat{\mathbf{e}}_{\mu 0} \right\rangle_{\mathrm{L}_n} = \\ &= \int_{-\infty}^{\infty} t^{j+k} d \left\langle \widetilde{\mathbf{E}}_t \widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle_{\mathrm{L}_n} = \int_{-\infty}^{\infty} t^{j+k} d\widetilde{\sigma}_{\mu\nu}(t), \quad 0 \le j, k \le n.\end{aligned}$$

Thus each canonical self-adjoint extension $\widetilde{A}$ of A_0 in L_n generates a certain solution $\tilde{\sigma}(t)$ of the truncated Hamburger matrix moment problem. Such a solution is at the same time a solution of the Stieltjes problem if and only if the corresponding spectral function $\widetilde{\mathbf{E}}_t$ has no points of growth on the half-axis $(-\infty, 0)$, i.e. if and only if $\widetilde{A}$ is a non-negative extension of A_0. Such extensions of A_0 might exist only if the operator A_0 is itself non-negative, i.e. the quadratic form of A_0 is non-negative. But this is the case, since by our assumptions

$$\langle A_0 \mathbf{f}, \mathbf{f} \rangle_{\mathrm{L}_n} = \sum_{j,k=0}^{n-1} (\mathbf{C}_{j+k+1} \boldsymbol{\xi}_k, \xi_j) \ge 0 \tag{17}$$

for a class $\mathbf{f} \in \mathrm{L}_{n-1}$ which contains a vector polynomial

$$\mathbf{f}(t) = \sum_{l=0}^{n-1} t^r \cdot \boldsymbol{\xi}_r.$$

(b) If $\mathrm{L}_n = \mathrm{L}_{n-1}$, then A_0 is a self-adjoint operator and in this case the truncated Hamburger problem has a unique solution $\boldsymbol{\sigma}_0(t)$. This solution is generated according to (16) by the spectral function $\mathbf{E}_t^0$ of A_0. Since $A_0 \geq 0$, then $\boldsymbol{\sigma}_0(t)$ is also the unique solution of the truncated Stieltjes problem.

(c) If $\mathrm{L}_n \neq \mathrm{L}_{n-1}$, then put $\mathcal{N} = \mathrm{L}_n \ominus \mathrm{L}_{n-1}$, $1 \leq \dim \mathcal{N} \leq s$.

(i) Let us assume first that

$$\det \mathbf{\Gamma}_{n-1}^{(1)} > 0. \tag{18}$$

With respect to the representation of L_n as the orthogonal sum $\mathrm{L}_{n-1} \oplus \mathcal{N}$, we can represent a self-adjoint extension $\widetilde{A}$ of A_0 as a 2×2 block operator matrix

$$\widetilde{A} = \begin{pmatrix} A_{00} & G^* \\ G & \widetilde{H} \end{pmatrix},$$

where A_{00} is a symmetric operator in L_{n-1}, the quadratic form of which coincides with that of A_0, $G = P_{\mathcal{N}} A_{0|\mathrm{L}_{n-1}}$, where $P_{\mathcal{N}}$ is the orthogonal projector onto the subspace $\mathcal{N}$ in L_n, and $\widetilde{H}$ is a self-adjoint operator in $\mathcal{N}$, which just defines the extension $\widetilde{A}$. Due to (17) and (18), A_{00} is a positive definite operator. We can use now the Schur–Frobenius factorization to represent $\widetilde{A}$ in the form

$$\widetilde{A} = \begin{pmatrix} I & 0 \\ GA_{00}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{00} & 0 \\ 0 & \widetilde{H} - GA_{00}^{-1}G^* \end{pmatrix} \begin{pmatrix} I & A_{00}^{-1}G^* \\ 0 & I \end{pmatrix}.$$

Then the extension $\widetilde{A} \geq 0$ if and only if $\widetilde{H} \geq GA_{00}^{-1}G^*$. Since the set of self-adjoint operators $\widetilde{H}$ in $\mathcal{N}$ which satisfy the last inequality is evidently non-empty, we conclude that the condition c) of Theorem 2.1 together with (18) guarantee the existence of non-negative extensions $\widetilde{A}$ of A_0.

(ii) Let us assume now that

$$\det \mathbf{\Gamma}_{n-1}^{(1)} = 0. \tag{19}$$

We will denote by $\mathcal{Z}$ the null-space of A_0 in L_{n-1} and by $\widetilde{\mathrm{L}}_{n-1}$ the subspace $\mathrm{L}_{n-1} \ominus \mathcal{Z}$. Notice that for a class $\mathbf{f} \in \mathcal{Z}$ which contains the vector polynomial

$$\mathbf{f}(t) = \sum_{l=0}^{n-1} t^r \cdot \boldsymbol{\xi}_r$$

and any class $\mathbf{g} \in \mathcal{N}$, the equality

$$\langle A_0 \mathbf{f}, \mathbf{f} \rangle = \sum_{j,k=0}^{n-1} (\mathbf{C}_{j+k+1} \boldsymbol{\xi}_k, \xi_j) = 0,$$

and the condition c) of the theorem yield:

$$\langle A_0 \mathbf{f}, \mathbf{g} \rangle \leq \sqrt{\langle A_0 \mathbf{f}, A_0 \mathbf{f} \rangle \, \langle \mathbf{g}, \mathbf{g} \rangle} = \left(\sum_{j,k=0}^{n-1} (\mathbf{C}_{j+k+2} \boldsymbol{\xi}_k, \xi_j) \right)^{\frac{1}{2}} \sqrt{\langle \mathbf{g}, \mathbf{g} \rangle} = 0.$$

Therefore

$$G_{|\mathcal{Z}} = P_{\mathcal{N}} A_{0|\mathcal{Z}} = 0, \quad P_{\mathcal{Z}} G^* = 0,$$

where $P_{\mathcal{Z}}$ is the orthogonal projector onto $\mathcal{Z}$ in L_n. With respect to the representation of L_n as the orthogonal sum $\mathcal{Z} \oplus \widetilde{\mathrm{L}}_{n-1} \oplus \mathcal{N}$ and since A_{00} is a symmetric operator, we can now represent a self-adjoint extension $\widetilde{A}$ of A_0 as a 3×3 block operator matrix

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{00}^{(1)} & G_1^* \\ 0 & G_1 & \widetilde{H} \end{pmatrix},$$

where $A_{00}^{(1)}$ is a positive definite operator in $\widetilde{\mathrm{L}}_{n-1}$, the quadratic form of which coincides with that of A_0 on $\widetilde{\mathrm{L}}_{n-1}$, $G_1 = P_{\mathcal{N}} A_{0|\widetilde{\mathrm{L}}_{n-1}}$, and $\widetilde{H}$ is a self-adjoint operator in $\mathcal{N}$, which defines the extension $\widetilde{A}$. As above, we can further factorize $\widetilde{A}$ as follows:

$$\begin{aligned} \widetilde{A} = & \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & G_1 A_{00}^{(1)-1} & I \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{00}^{(1)} & 0 \\ 0 & 0 & \widetilde{H} - G_1 A_{00}^{(1)-1} G_1^* \end{pmatrix} \times \\ & \times \begin{pmatrix} I & 0 & 0 \\ 0 & I & A_{00}^{(1)-1} G_1^* \\ 0 & 0 & I \end{pmatrix}. \end{aligned} \tag{20}$$

Due to (20) there are non-negative self-adjoint extensions $\widetilde{A}$ of A_0 in L_n. Such extensions are obtained if the operator $\widetilde{H}$ in $\mathcal{N}$ which defines $\widetilde{A}$ is such that

$$\widetilde{H} \geq G_1 A_{00}^{(1)-1} G_1^*. \tag{21}$$

□

3. CANONICAL SOLUTIONS

We call *canonical* the solutions of the truncated matrix Stieltjes problem given by the expression (16), where $\widetilde{\mathbf{E}}_t$ is the spectral function of some non-negative canonical self-adjoint extension $\widetilde{A}$ of A_0. The established correspondence between the set of such extensions of A_0 and the set of canonical solutions of the Stieltjes problem makes it possible to find, under the conditions of Theorem 2.1, an explicit algebraic formulas for the description of the sought canonical solutions. To this end we can use as a starting point (16) and the relation

$$\int_{-\infty}^{\infty} \frac{1}{t-z} d\widetilde{\sigma}_{\mu\nu}(t) = \int_{-\infty}^{\infty} \frac{1}{t-z} d\left\langle \widetilde{\mathbf{E}}_t \widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle = \left\langle \left(\widetilde{A} - z \right)^{-1} \widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle. \tag{22}$$

From now on we will assume that $\det \mathbf{\Gamma}_n > 0$, i.e. we will consider the *non-degenerate* case of the above problems.

Let $\mathbb{L}_n(\mathbb{C}_s)$ denote the $(n+1)s$-dimensional linear space of column vectors

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi}_0 & \cdots & \boldsymbol{\xi}_n \end{pmatrix}^{\dagger}, \quad \boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_n \in \mathbb{C}_s, \tag{23}$$

($\dagger$ stands for the transposition operation) with the scalar product

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{j=0}^{n} (\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)_{\mathbb{C}_s}.$$

We will denote as before by L_n the same linear vector space but with the scalar product

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = (\boldsymbol{\Gamma}_n \boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{j,k=0}^{n} \left(\mathbf{C}_{j+k} \boldsymbol{\xi}_k, \boldsymbol{\eta}_j\right)_{\mathbb{C}_s}.$$

L_n was considered above as the space of vector polynomials.

Let $\mathbb{L}_{n-1}(\mathbb{C}_s)$ be the subspace of $\mathbb{L}_n(\mathbb{C}_s)$ which consists of vectors (23) but with $\boldsymbol{\xi}_n = \mathbf{0}$ and let

$$\mathfrak{N} = \mathbb{L}_n(\mathbb{C}_s) \ominus \mathbb{L}_{n-1}(\mathbb{C}_s).$$

We denote by $P_{\mathfrak{N}}$ the orthogonal projector in $\mathbb{L}_n(\mathbb{C}_s)$ onto $\mathfrak{N}$. In the natural basis of subspaces of $\mathbb{L}_n(\mathbb{C}_s)$ this projector is evidently given as the following $(n+1)\times(n+1)$ block operator matrix

$$P_{\mathfrak{N}} = \begin{pmatrix} 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & I_s \end{pmatrix},$$

where I_s is the $s \times s$ unit matrix. Let us consider the linear operator Ψ given as the $(n+1)\times(n+1)$ block operator matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ I_s & 0 & 0 & \ldots & 0 \\ 0 & I_s & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I_s & 0 \end{pmatrix}.$$

The symmetric operator A_0 in L_n introduced above is the restriction of T onto $\mathbb{L}_{n-1}(\mathbb{C}_s)$. Let $\widetilde{\boldsymbol{\Gamma}}^{(1)}_{n-1}$ be the $(n+1)\times(n+1)$ block operator matrix

$$\widetilde{\boldsymbol{\Gamma}}^{(1)}_{n-1} = \begin{pmatrix} \boldsymbol{\Gamma}^{(1)}_{n-1} & 0_{n,s} \\ 0_{s,n} & 0_{s,s} \end{pmatrix}.$$

Here $0_{n,s}$, $0_{s,n}$, and $0_{s,s}$ are the $n \times s$, $s \times n$, and $s \times s$ null-matrices, respectively. Notice that for any $\boldsymbol{\xi} \in \mathbb{L}_{n-1}(\mathbb{C}_s)$ and any $\boldsymbol{\eta} \in \mathbb{L}_n(\mathbb{C}_s)$ we have

$$\langle A_0 \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \langle T \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \left(\widetilde{\boldsymbol{\Gamma}}^{(1)}_{n-1} \boldsymbol{\xi}, \boldsymbol{\eta}\right) + (P_{\mathfrak{N}} \boldsymbol{\Gamma}_n T \boldsymbol{\xi}, \boldsymbol{\eta}) =$$
$$= \left\langle \boldsymbol{\Gamma}_n^{-1} \widetilde{\boldsymbol{\Gamma}}^{(1)}_{n-1} \boldsymbol{\xi}, \boldsymbol{\eta} \right\rangle + \left\langle \boldsymbol{\Gamma}_n^{-1} P_{\mathfrak{N}} \boldsymbol{\Gamma}_n T \boldsymbol{\xi}, \boldsymbol{\eta} \right\rangle.$$

Hence,

$$A_{0|\mathbb{L}_{n-1}(\mathbb{C}_s)} = \mathbf{\Gamma}_n^{-1}\widetilde{\mathbf{\Gamma}}^{(1)}_{n-1|\mathbb{L}_{n-1}(\mathbb{C}_s)} + \mathbf{\Gamma}_n^{-1}P_{\mathfrak{N}}\mathbf{\Gamma}_n T_{|\mathbb{L}_{n-1}(\mathbb{C}_s)}. \tag{24}$$

Put $P_{\mathfrak{N}}^{\perp} = I - P_{\mathfrak{N}}$. Due to (24), any self-adjoint extension $\widetilde{A}$ of A in L_n can be represented as

$$\widetilde{A} = \mathbf{\Gamma}_n^{-1}\widetilde{\mathbf{\Gamma}}^{(1)}_{n-1}P_{\mathfrak{N}}^{\perp} + \mathbf{\Gamma}_n^{-1}P_{\mathfrak{N}}\mathbf{\Gamma}_n T P_{\mathfrak{N}}^{\perp} + \mathbf{\Gamma}_n^{-1}P_{\mathfrak{N}}^{\perp}T^*\mathbf{\Gamma}_n P_{\mathfrak{N}} + \mathbf{\Gamma}_n^{-1}\widetilde{H}, \tag{25}$$

where

$$\widetilde{H} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{H} \end{pmatrix}$$

and $\mathbf{H}$ is some $s \times s$ Hermitian matrix, which defines the extension $\widetilde{A}$. In a more detailed form,

$$\widetilde{A} = \mathbf{\Gamma}_n^{-1}\begin{pmatrix} & & & \mathbf{C}_{n+1} \\ & \mathbf{\Gamma}^{(1)}_{n-1} & & \vdots \\ & & & \mathbf{C}_{2n} \\ \mathbf{C}_{n+1} & \cdots & \mathbf{C}_{2n} & \mathbf{H} \end{pmatrix} = \tag{26}$$

$$= T + \mathbf{\Gamma}_n^{-1}\begin{pmatrix} & & & \mathbf{C}_{n+1} \\ & 0_{ns,ns} & & \vdots \\ & & & \mathbf{C}_{2n} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H} \end{pmatrix}. \tag{27}$$

Corollary 3.1. *The block Hankel matrix* $\mathbf{\Gamma}^{(1)}_{n-1}$ *is invertible.*

Proof. If $\mathbf{\Gamma}^{(1)}_{n-1}$ is not invertible, then due to the conditions of Theorem 2.1, the matrix

$$\mathbf{\Gamma}^{(2)}_{n-1} = (\mathbf{C}_{j+k+2})_{j,k=0}^{n-1}$$

is also not invertible. But $\mathbf{\Gamma}^{(2)}_{n-1}$ is a diagonal block of the positive definite matrix $\mathbf{\Gamma}_n$, a contradiction. □

Let us introduce now the inverse block matrices

$$\mathbf{\Gamma}^{(1)-1}_{n-1} = (\mathbf{b}_{jk})_{j,k=0}^{n-1}$$

and

$$\left(\widetilde{\mathbf{\Gamma}}^{(1)}_{n-1}\right)^{-1}_{\mathrm{cond}} = \begin{pmatrix} \mathbf{\Gamma}^{(1)-1}_{n-1} & 0_{ns,s} \\ 0_{s,ns} & 0_{s,s} \end{pmatrix}.$$

Remark 3.1. *The operator defined by the block matrix* (26) *is non-negative if and only if*

$$\widetilde{H} - P_{\mathfrak{N}}\mathbf{\Gamma}_n T P_{\mathfrak{N}}^{\perp}\left(\widetilde{\mathbf{\Gamma}}^{(1)}_{n-1}\right)^{-1}_{\mathrm{cond}} P_{\mathfrak{N}}^{\perp}T^*\mathbf{\Gamma}_n P_{\mathfrak{N}} \geq 0,$$

or, equivalently, if and only if

$$\mathbf{H} - \sum_{j,k=0}^{n-1} \mathbf{C}_{n+j+1}\mathbf{b}_{jk}\mathbf{C}_{n+k+1} \geq 0. \tag{28}$$

Since the $s \times s$ matrix

$$\mathbf{Q} := \sum_{j,k=0}^{n-1} \mathbf{C}_{n+j+1}\mathbf{b}_{jk}\mathbf{C}_{n+k+1}$$

is positive definite, then all matrices $\mathbf{H}$ *which generate the non-negative extensions* $\widetilde{A}$ *and hence the solutions of the Stieltjes problem, must be positive definite and, moreover, satisfy the inequality* $\mathbf{H} \geq \mathbf{Q}$. *Notice that the requirement* $\widetilde{A} \gg 0$ *excludes the equality in* (28).

Put

$$\mathbf{\Theta_H}(z) := \mathbf{\Gamma}_n \left(\mathbf{\Gamma}^{(1)}_{\mathbf{H};n} - z\mathbf{\Gamma}_n\right)^{-1} \mathbf{\Gamma}_n, \tag{29}$$

where

$$\mathbf{\Gamma}^{(1)}_{\mathbf{H};n} := \begin{pmatrix} & & & \mathbf{C}_{n+1} \\ & \mathbf{\Gamma}^{(1)}_{n-1} & & \vdots \\ & & & \mathbf{C}_{2n} \\ \mathbf{C}_{n+1} & \ldots & \mathbf{C}_{2n} & \mathbf{H} \end{pmatrix} \tag{30}$$

and let $\mathbf{\Delta_H}(z)$, $\mathrm{Im} z > 0$, be the upper left $s \times s$ block of $\mathbf{\Theta_H}(z)$,

$$\mathbf{\Delta_H}(z) := \mathbf{\Theta}_{\mathbf{H};00}(z). \tag{31}$$

The following theorem is an evident combination of the above arguments.

Theorem 3.1. *Let the conditions of Theorem 2.1 hold and* $\det \mathbf{\Gamma}_n > 0$. *Then the relation*

$$\int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}_{\mathbf{H}}(t) = \mathbf{\Delta_H}(z), \quad \mathrm{Im} z > 0,$$

establishes the one-to-one correspondence between the set of all canonical solutions of the truncated matrix Stieltjes moment problem with the given moments $\{\mathbf{C}_0, \ldots, \mathbf{C}_{2n}\}$ *and the set of positive definite matrices* $\mathbf{H}$ *such that*

$$\mathbf{H} - \sum_{j,k=0}^{n-1} \mathbf{C}_{n+j+1}\mathbf{b}_{jk}\mathbf{C}_{n+k+1} \geq 0. \tag{32}$$

Actually Theorem 3.1 with (29), (30) describes in the non-degenerate case an **algebraic algorithm** of construction of canonical solutions of the truncated Stieltjes matrix moment problem and with the omission of the condition (32) also the algorithm of construction of those of the Hamburger matrix moment problem.

Compare this algorithm with the method of construction of the solutions of the latter problem, described, in particular, in [1] for the Hamburger problem. To this end set

$$\mathbf{\Pi}_r = (\underbrace{0_{s,s},\dots,0_{s,s}}_{r}, I_s)^T, \mathbf{\Upsilon}_r(t) = (I_s, tI_s, \dots, t^r I_s), \quad r = n-1, n.$$

Since $\mathbf{\Gamma}_n$ is positive definite and invertible, the same is true for all

$$\mathbf{\Gamma}_r := (\mathbf{C}_{k+j})_{k,j=0}^r, \quad 0 \le r \le n-1.$$

Let us introduce $s \times s$ matrix polynomials

$$\mathbf{D}_r(t) = \mathbf{\Upsilon}_r(t)\mathbf{\Gamma}_r^{-1}\mathbf{\Pi}_r, \quad r = n-1, n \tag{33}$$

and the corresponding conjugate polynomials

$$\mathbf{E}_r(z) := \int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}(t) \left(\mathbf{D}_r(t) - \mathbf{D}_r(z)\right). \tag{34}$$

Let $\mathfrak{R}$ be the Nevanlinna class of holomorphic in the upper half-plane $s \times s$ dissipative matrix functions, i.e. matrix functions with non-negative imaginary parts and let

$$\mathfrak{R}_0 = \left\{ \mathbf{t} \in \mathfrak{R} | \lim_{y \uparrow \infty} \frac{1}{y} \mathbf{t}(iy) = 0 \right\}.$$

By [7, 1] and under all above assumptions, the Nevanlinna-type formula

$$\begin{aligned} \varphi(z) = \int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}(t) = -\left(\mathbf{E}_n(z)\left(\mathbf{R}(z) + zI\right) - \mathbf{E}_{n-1}(z)\right) \times \\ \times \left(\mathbf{D}_n(z)\left(\mathbf{R}(z) + zI\right) - \mathbf{D}_{n-1}(z)\right)^{-1}, \end{aligned} \tag{35}$$

$$\mathbf{R}(z) = \left(\mathbf{\Gamma}_n^{-1}\right)_{nn}^{-1} \mathbf{t}(z), \quad \mathrm{Im} z > 0,$$

establishes the one-to-one correspondence between the set of all non-decreasing matrix function $\boldsymbol{\sigma}(t)$, $-\infty < t < \infty$, satisfying (11) and the set Nevanlinna $s \times s$ matrix functions $\mathbf{t} \in \mathfrak{R}_0$.

The same formula with $\mathbf{t}(z)$ replaced by any $s \times s$ constant Hermitian matrices $\widehat{\mathbf{H}}$, establishes the one-to-one correspondence between the set of all non-decreasing canonical matrix functions $\boldsymbol{\sigma}_{\widehat{\mathbf{H}}}(t)$ which satisfy (11) and the set of all Hermitian $s \times s$ matrices $\widehat{\mathbf{H}}$. For a canonical solution $\boldsymbol{\sigma}_{\widehat{\mathbf{H}}}(t)$ of the truncated Hamburger problem for the given matrix moments, the expression on the right hand side of (35) is a rational matrix function of the Nevanlinna class $\mathfrak{R}_0$. The poles of this matrix function are the roots of the matrix polynomial

$$\mathfrak{P}_{\widehat{\mathbf{H}}}(z) := \left(\mathbf{D}_n(z)\left(\left(\mathbf{\Gamma}_n^{-1}\right)_{nn}^{-1} \widehat{\mathbf{H}} + zI\right) - \mathbf{D}_{n-1}(z)\right). \tag{36}$$

By [1] $\mathfrak{P}_{\widehat{\mathbf{H}}}(z)$ has only real roots. These roots are unique points of growth of $\boldsymbol{\sigma}_{\widehat{\mathbf{H}}}(t)$. Therefore a canonical solution $\boldsymbol{\sigma}_{\widehat{\mathbf{H}}}(t)$ of the Hamburger problem is at the same time a solution of the Stieltjes problem for the same set of matrix moments if and only if $\mathfrak{P}_{\widehat{\mathbf{H}}}(z)$ for the corresponding Hermitian matrix $\widehat{\mathbf{H}}$ has no roots on the half-axis $(-\infty, 0)$. By [1, 3] a Hermitian matrix $\mathbf{H}$ in (26), (27) which determines a canonical solution $\boldsymbol{\sigma}_{\mathbf{H}}(t)$ of the Stieltjes or Hamburger problem through the self-adjoint extensions $\widetilde{A}$ of A_0 given by (25), and the matrix $\widehat{\mathbf{H}}$ which replaces $\mathbf{t}(z)$ in (35) in order to obtain the same solution $\boldsymbol{\sigma}_{\widehat{\mathbf{H}}}(t)$, are connected by the relation

$$\begin{pmatrix} 0 & 0 \\ 0 & \widehat{\mathbf{H}} \end{pmatrix} = P_{\mathfrak{N}} \widetilde{A} \boldsymbol{\Gamma}_n^{-1} P_{\mathfrak{N}}.$$

Hence

$$\begin{aligned} \widehat{\mathbf{H}} &= \left(\boldsymbol{\Gamma}_n^{-1}\right)_{n-1,n} + \sum_{j=0}^{n-1} \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nj} \mathbf{C}_{j+1} \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn} + \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn} \mathbf{H} \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn} := \\ &:= \boldsymbol{\Lambda}_n + \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn} \mathbf{H} \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn} \end{aligned} \tag{37}$$

and

$$\mathbf{H} = (\boldsymbol{\Gamma}_n^{-1})_{nn}^{-1} \left(\widehat{\mathbf{H}} - \boldsymbol{\Lambda}_n\right) \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}^{-1}. \tag{38}$$

We see that

Theorem 3.2. *The formula*

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}_{\mathbf{H}}(t) = &- \left(\mathbf{E}_n(z) \left(\mathbf{R}_{\mathbf{H}} + zI\right) - \mathbf{E}_{n-1}(z)\right) \times \\ &\times \left(\mathbf{D}_n(z) \left(\mathbf{R}_{\mathbf{H}} + zI\right) - \mathbf{D}_{n-1}(z)\right)^{-1}, \end{aligned} \tag{39}$$

$$\mathbf{R}_{\mathbf{H}} = \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}^{-1} \boldsymbol{\Lambda}_n - \mathbf{H} \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}, \quad \mathrm{Im} z > 0, \tag{40}$$

establishes in the non-degenerate case the one-to-one correspondence between the set of all canonical solutions $\boldsymbol{\sigma}_{\mathbf{H}}(t)$ *of the truncated matrix Stieltjes problem and the set of positive definite* $s \times s$ *matrices* $\mathbf{H}$ *which satisfy* (28).

If we compare the last result with the assertion of Theorem 3.1, we can conclude that

$$\begin{aligned} &\left(\boldsymbol{\Gamma}_n \left(\boldsymbol{\Gamma}_{\mathbf{H};n}^{(1)} - z\boldsymbol{\Gamma}_n\right)^{-1} \boldsymbol{\Gamma}_n\right)_{00} = \\ &\quad = -\left(\mathbf{E}_n(z) \left(\mathbf{R}_{\mathbf{H}} + zI\right) - \mathbf{E}_{n-1}(z)\right) \left(\mathbf{D}_n(z) \left(\mathbf{R}_{\mathbf{H}} + zI\right) - \mathbf{D}_{n-1}(z)\right)^{-1}, \end{aligned} \tag{41}$$

$$\mathbf{R}_{\mathbf{H}} = \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}^{-1} \boldsymbol{\Lambda}_n - \mathbf{H} \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}, \quad \mathrm{Im} z \neq 0. \tag{42}$$

4. DESCRIPTION OF ALL SOLUTIONS OF THE TRUNCATED STIELTJES MATRIX NON-DEGENERATE MOMENT PROBLEM

By (22) we reduce the description of all solutions of the Stieltjes matrix moment problem to the construction of the upper left $s \times s$ block of the resolvent $(\widetilde{A} - z)^{-1}$ of the generalized non-negative self-adjoint extensions of A_0 with coming out the space L_n.

Certainly, each solution of the Stieltjes problem is at the same time the solution of the Hamburger problem for the same set of matrix moments. Hence, we can use (35) to describe the solutions of the Stieltjes problem, but we must restrict the set of "parameters" $\mathbf{t}(z)$ which reflect, according to (35), all matrix functions $\boldsymbol{\sigma}(t)$ which correspond to the non-negative extensions and only them.

To this end, let us consider a generalized non-negative self-adjoint extension A of A_0 with coming out L_n to a Hilbert space $\mathcal{H} = \mathrm{L}_n \oplus \mathcal{H}'$, $\dim \mathcal{H}' \le \infty$. In general, A is an unbounded operator, but since A is an extension of A_0, then $\mathrm{L}_{n-1} \subset \mathcal{D}_A$. Suppose first that $\mathrm{L}_n = (\mathrm{L}_{n-1} \oplus \mathcal{N}) \subset \mathcal{D}_A$. Then according to the splitting

$$\mathcal{H} = \mathrm{L}_{n-1} \oplus \mathcal{N} \oplus \mathcal{H}',$$

we can represent A in the form

$$A = \begin{pmatrix} A_{00} & G^* & 0 \\ G & H_A & G_1^* \\ 0 & G_1 & A_{11} \end{pmatrix}, \tag{43}$$

where A_{00}, G are defined as above, H_A is a non-negative operator in $\mathcal{N}$, G_1 is a bounded operator from $\mathcal{N}$ into $\mathcal{H}'$, and A_{11} is a non-negative self-adjoint operator in $\mathcal{H}'$. We can choose any $\lambda < 0$ and apply the Schur–Frobenius factorization to get:

$$\begin{aligned} A - \lambda = &\begin{pmatrix} I & 0 & 0 \\ G\,(A_{00}-\lambda)^{-1} & I & G_1^*(A_{11}-\lambda)^{-1} \\ 0 & 0 & I \end{pmatrix} \times \\ \times &\begin{pmatrix} A_{00}-\lambda & 0 & 0 \\ 0 & H_A - \lambda - G\,(A_{00}-\lambda)^{-1} G^* - G_1^*\,(A_{11}-\lambda)^{-1} G_1 & 0 \\ 0 & 0 & A_{11}-\lambda \end{pmatrix} \times \\ \times &\begin{pmatrix} I & (A_{00}-\lambda)^{-1} G^* & 0 \\ 0 & I & 0 \\ 0 & (A_{11}-\lambda)^{-1} G_1 & I \end{pmatrix}. \end{aligned} \tag{44}$$

Due to the formula (44) the assumption $A \ge 0$ is equivalent to the conditions

$$\begin{aligned} &A_{00} - \lambda \gg 0, \quad A_{11} - \lambda \gg 0, \\ &H_A - G\,(A_{00}-\lambda)^{-1} G^* - G_1^*\,(A_{11}-\lambda)^{-1} G_1 \ge 0 \end{aligned} \tag{45}$$

for any $\lambda < 0$. Further, we can take into account the representation $\mathcal{H} = \mathrm{L}_n \oplus \mathcal{H}'$ and rewrite $A - z$, $\mathrm{Im} z > 0$, as

$$\begin{aligned} A - z = &\begin{pmatrix} I & G_1^*(A_{11}-z)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} W(z) & 0 \\ 0 & A_{11}-z \end{pmatrix} \times \\ &\times \begin{pmatrix} I & 0 \\ (A_{11}-z)^{-1}G_1 & I \end{pmatrix}, \end{aligned} \tag{46}$$

where

$$W(z) = \begin{pmatrix} A_{00} - z & G^* \\ G & H_A - z - G_1^*(A_{11}-z)^{-1}G_1 \end{pmatrix}. \tag{47}$$

Thus, due to (46) and (22), the solution $\boldsymbol{\sigma}_A(t)$ of the truncated Stieltjes problem generated by the extension A is given by the expression

$$\int_{-\infty}^{\infty} \frac{1}{t-z} d\sigma_{A;\mu\nu}(t) = \left\langle (A-z)^{-1}\widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle = \left\langle W(z)^{-1}\widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle. \tag{48}$$

Introduce

$$\boldsymbol{\Theta}_A(z) := \boldsymbol{\Gamma}_n \left(\boldsymbol{\Gamma}_{A;n}^{(1)}(z) - z\boldsymbol{\Gamma}_n \right)^{-1} \boldsymbol{\Gamma}_n, \tag{49}$$

where

$$\boldsymbol{\Gamma}_{A;n}^{(1)}(z) := \begin{pmatrix} & & & \mathbf{C}_{n+1} \\ & \boldsymbol{\Gamma}_{n-1}^{(1)} & & \vdots \\ & & & \mathbf{C}_{2n} \\ \mathbf{C}_{n+1} & \dots & \mathbf{C}_{2n} & H_A - G_1^*(A_{11}-z)^{-1}G_1 \end{pmatrix}, \tag{50}$$

and let

$$\left(\Delta_{A;\mu\nu}(z)\right)_{\mu,\nu=1}^{s}, \quad \mathrm{Im} z > 0,$$

be the upper left $s \times s$ block of $\boldsymbol{\Theta}_A(z)$. The considerations similar to those in the proof of Theorem 3.1, now show that

$$\left\langle W(z)^{-1}\widehat{\mathbf{e}}_{\nu 0}, \widehat{\mathbf{e}}_{\mu 0} \right\rangle = \Delta_{A;\mu\nu}(z). \tag{51}$$

Put

$$\mathbf{t}_A(z) = G_1^*(A_{11}-z)^{-1}G_1$$

and compare the expressions (49)–(51) to (29)–(31).

We conclude that the replacement of the matrix $\mathbf{H}$ by the matrix function $H_A - \mathbf{t}_A(z)$ on both sides of (41) cannot violate this equality at least for $z \in (-\infty, 0)$.

Therefore

$$\int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}_A(t) =$$

$$= -\left(\mathbf{E}_n(z)\left(\mathbf{R}_A(z)+zI\right)-\mathbf{E}_{n-1}(z)\right)\left(\mathbf{D}_n(z)\left(\mathbf{R}_A(z)+zI\right)-\mathbf{D}_{n-1}(z)\right),$$

$$\mathbf{R}_A = \left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}^{-1}\boldsymbol{\Lambda}_n + \left(\mathbf{t}_A(z)-H_A\right)\left(\boldsymbol{\Gamma}_n^{-1}\right)_{nn}, \quad \mathrm{Im} z \neq 0.$$

If $\mathcal{N} \subsetneq \mathcal{D}_A$, then the representation (43) is not valid anymore. However, due to the representation $\mathcal{H} = \mathrm{L}_{n-1} \oplus \mathcal{H}''$, $\mathcal{H}'' = \mathcal{N} \oplus \mathcal{H}^{'}$, we can write

$$A = \begin{pmatrix} A_{00} & G^* \\ G & A_{11} \end{pmatrix}, \tag{52}$$

where A_{00} and G are defined as before and A_{11} is some non-negative self-adjoint operator in $\mathcal{H}^{'}$. The Schur–Frobenius factorization by virtue of (52) then yields:

$$(A-z)^{-1} = \begin{pmatrix} W_{A;00}(z)^{-1} & W_A(z)^{-1}G^*(A_{11}-z)^{-1} \\ (A_{11}-z)^{-1}GW_A(z)^{-1} & W_{A;11}(z)^{-1} \end{pmatrix}. \tag{53}$$

Here

$$W_{A;00}(z) = A_{00} - z - G^*(A_{11}-z)^{-1}G, \tag{54}$$

and

$$W_{A;11}(z)^{-1} = (A_{11}-z)^{-1} + (A_{11}-z)^{-1}GW_{A;00}(z)^{-1}G^*(A_{11}-z)^{-1}. \tag{55}$$

Let P_n be the orthogonal projector of L_n onto $\mathcal{H}$ and let

$$\Xi_A(z) = P_{\mathcal{N}}(A_{11}-z)^{-1}|_{\mathcal{N}}, \quad \mathrm{Im} z \neq 0. \tag{56}$$

Due to (53) and (55), the generalized resolvent

$$R_z(A) := P_n(A_{11}-z)^{-1}|_{\mathrm{L}_n}$$

of A can be represented in the form

$$R_z(A) = \begin{pmatrix} W_{A;00}(z)^{-1} & W_{A;00}(z)^{-1}G^*\Xi_A(z) \\ \Xi_A(z)GW_{A;00}(z)^{-1} & \Xi_A(z) + \Xi_A(z)GW_{A;00}(z)^{-1}G^*\Xi_A(z) \end{pmatrix}, \tag{57}$$

$\mathrm{Im} z \neq 0.$

Then the expressions (54) and (57) can be used to verify, by direct calculations, that

$$R_z(A) = \begin{pmatrix} A_{00} - z & G^* \\ G & \Xi_A(z)^{-1} \end{pmatrix}^{-1}, \quad \mathrm{Im} z \neq 0. \tag{58}$$

Next, we compare (58) and (47) to conclude that in the general non-degenerate case the solution $\boldsymbol{\sigma}_A(t)$ of the truncated Stieltjes moment problem generated by a non-negative self-adjoint extension A of A_0, is also defined as the upper left $s \times s$ block of the matrix function

$$\mathbf{\Gamma}_n \left(\mathbf{\Gamma}^{(1)}_{A;n}(z) - z\mathbf{\Gamma}_n\right)^{-1} \mathbf{\Gamma}_n,$$

where

$$\mathbf{\Gamma}^{(1)}_{A;n}(z) := \begin{pmatrix} & \mathbf{\Gamma}^{(1)}_{n-1} & & \begin{matrix}\mathbf{C}_{n+1} \\ \vdots \\ \mathbf{C}_{2n}\end{matrix} \\ \mathbf{C}_{n+1} & \dots & \mathbf{C}_{2n} & \Xi_A(z)^{-1} + z \end{pmatrix}. \tag{59}$$

Let $\mathfrak{S}$ be the subset of $\mathfrak{N}$ consisting of all Nevanlinna $s \times s$ matrix functions $t(z)$, $\mathrm{Im} z > 0$ which admit the integral representation

$$\mathbf{t}(z) = \int_0^\infty \frac{1}{t-z} d\boldsymbol{\rho}(t)$$

with a non-decreasing $s \times s$ matrix function ($\mathbb{C}_s$ operator) $\boldsymbol{\sigma}(t)$ such that

$$\int_0^\infty d\rho(t) < \infty.$$

It is evident that the operator functions $\mathbf{t}_A(z)$, $\Xi_A(z) \in \mathfrak{S}$.

On the other hand, we can make use of the usual constructions of the spectral theory of linear operators in the Hilbert spaces to verify that any function $\boldsymbol{\Xi}(z) \in \mathfrak{S}$ with the values on the set of linear operators acting in $\mathcal{N}$ admits *the realization* (56), i.e. for such a function there is a non-negative operator A_{11} in a Hilbert space $\mathcal{N} \oplus \mathcal{H}^{'}$ such that for $\boldsymbol{\Xi}(z)$ the equality (56) holds. However, the functions Ξ_A which are connected with non-negative extensions A which in turn generate the solutions of the Stieltjes problem, satisfy the additional condition: they are such that for any $\lambda < 0$ the block operator $R_\lambda(A)$ defined by (58) is positive.

Introduce now the block matrix

$$\left(\mathbf{\Gamma}^{(1)}_{n-1} - z\right)^{-1} = \left(\mathbf{b}_{jk}(z)\right)^{n-1}_{j,k=0}, \quad z \notin [0, \infty),$$

where $\mathbf{b}_{jk}(z)$ are $s \times s$ matrix functions. The latter requirement on $\boldsymbol{\Xi}$ is equivalent to the condition

$$\boldsymbol{\Xi}(\lambda)^{-1} - \sum_{j,k=0}^{n-1} \mathbf{C}_{n+j+1}\mathbf{b}_{jk}(\lambda)\mathbf{C}_{n+k+1} > 0, \quad \lambda < 0. \tag{60}$$

Notice that the matrix function on the left-hand side of (60) is non-increasing on the negative half-axis. Therefore, we can formally admit that some elements of $\Xi(0)^{-1}$ are $+\infty$, and substitute (60) by the condition

$$\Xi(-0)^{-1} - \sum_{j,k=0}^{n-1} \mathbf{C}_{n+j+1}\mathbf{b}_{jk}(\lambda)\mathbf{C}_{n+k+1} \geq 0. \tag{61}$$

We have thus proven the following theorems.

Theorem 4.1. *Let the conditions of Theorem 2.1 hold and* $\det \mathbf{\Gamma}_n > 0$*, and let* $\mathbf{\Delta}_\Xi(z)$ *be the upper-left block of the matrix function*

$$\mathbf{\Gamma}_n \left(\mathbf{\Gamma}^{(1)}_{\Xi;n}(z) - z\mathbf{\Gamma}_n\right)^{-1} \mathbf{\Gamma}_n$$

where

$$\mathbf{\Gamma}^{(1)}_{\Xi;n}(z) := \begin{pmatrix} & & & \mathbf{C}_{n+1} \\ & \mathbf{\Gamma}^{(1)}_{n-1} & & \vdots \\ & & & \mathbf{C}_{2n} \\ \mathbf{C}_{n+1} & \ldots & \mathbf{C}_{2n} & \Xi(z)^{-1} + z \end{pmatrix}.$$

Then the relation

$$\int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}_\Xi(t) = \mathbf{\Delta}_\Xi(z), \quad \mathrm{Im} z > 0,$$

establishes the one-to-one correspondence between the set of all solutions of the truncated Stieltjes matrix moment problem with the given moments $\{\mathbf{C}_0, \ldots, \mathbf{C}_{2n}\}$ *and the subset of the Nevanlinna matrix functions* Ξ *from* $\mathfrak{S}$ *which satisfy* (60)*.*

Theorem 4.2. *Let the conditions of Theorem 4.1 hold. Then the formula*

$$\int_{-\infty}^{\infty} \frac{1}{t-z} d\boldsymbol{\sigma}_\Xi(t) =$$

$$= -\left(\mathbf{E}_n(z)\left(\mathbf{R}_\Xi(z) + z\mathbf{I}\right) - \mathbf{E}_{n-1}(z)\right)\left(\mathbf{D}_n(z)\left(\mathbf{R}_\Xi(z) + z\mathbf{I}\right) - \mathbf{D}_{n-1}(z)\right),$$

$$\mathbf{R}_\Xi = \left(\mathbf{\Gamma}_n^{-1}\right)_{nn}^{-1} \mathbf{\Lambda}_n - \Xi(z)^{-1}\left(\mathbf{\Gamma}_n^{-1}\right)_{nn}, \quad \mathrm{Im} z > 0,$$

establishes the one-to-one correspondence between the set of all solutions $\boldsymbol{\sigma}_\Xi(t)$ *of the truncated matrix Stieltjes problem and the subset of the Nevanlinna matrix functions* $\Xi \in \mathfrak{S}$ *which satisfy* (60)*.*

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