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MULTIPOINT NORMAL DIFFERENTIAL OPERATORS OF FIRST ORDER

Abstract. In this paper we discuss all normal extensions of a minimal operator generated by a linear multipoint differential-operator expression of first order in the Hilbert space of vector-functions on the finite interval in terms of boundary and interior point values. Later on, we investigate the structure of the spectrum, its discreteness and the asymptotic behavior of the eigenvalues at infinity for these extensions.

Keywords: differential operator, formally normal and normal operator, multipoint minimal and maximal operators, extension, selfadjoint, accretive and unitary operators, class of compact operators, spectrum of an operators and its discreteness, asymptotics of eigenvalues, direct sum of spaces and operators.

Mathematics Subject Classification: 47A20.

1. INTRODUCTION

A densely defined closed operator N in a Hilbert space $\mathcal{H}$ is called formally normal if $D(N) \subset D(N^*)$ and $\|Nf\|_{\mathcal{H}} = \|N^* f\|_{\mathcal{H}}$ for all $f \in D(N)$. If a formally normal operator has no formally normal non-trivial extension, then it is called a maximal formally normal operator. If a formally normal operator N satisfies the condition $D(N) = D(N^*)$, then it is called a normal operator ([1]). The densely defined closed operator N is normal if and only if $NN^* = N^*N$ ([2]).

The first results in the area of normal extension of unbounded formally normal operators on a Hilbert space are due to Y. Kilpi ([3–5]) and R. H. Davis ([6]), furthermore E.A. Coddington ([1]), G. Biriuk and E.A. Coddington ([7]), J. Stochel and F.H. Szafraniec ([8–10]) established and developed it as a general theory. However, application of this theory to the theory of differential operators in a Hilbert space has not received the attention it deserves ([11–15]).

Fundamental results for the theory of two point ordinary differential operators has been studied by many authors (for example, see [16-17]).

The different characterization (calculus of adjoint operator, selfadjointness, lack of selfadjointness, differentiable properties, normally solvable and invertible properties etc.) of so-called multipoint ordinary differential operators has been investigated in many works (for example, see [18-26]).

In this paper $\mathcal{H}$ and $L^2 = L^2(\mathcal{H},(a,b))$, $-\infty < a < b < +\infty$, denote a separable Hilbert space of vector-functions from the interval $[a,b]$ into $\mathcal{H}$ with the inner product (norm) $(\cdot,\cdot)\,(\|\cdot\|)$ and $(\cdot,\cdot)_{L^2}\,(\|\cdot\|_{L^2})$, respectively. Moreover, let E be an identical operator in $\mathcal{H}$.

2. THE MINIMAL AND MAXIMAL OPERATORS

Let $[a,b]$, $-\infty < a < b < +\infty$, be an interval which is subdivided into n subintervals by points $c_0, c_1, c_2, \ldots, c_n$ such that $a = c_0 < c_1 < c_2 < \ldots < c_{n-1} < c_n = b$. Assume that

$$A(t) := A_k,\ t \in \Delta_k = (c_{k-1}, c_k),\ k = 1,2,\ldots,n$$

is a linear self-adjoint operator in $\mathcal{H}$ for every $k = 1,2,\ldots,n$ and operator A_k is a positive defined operator (for simplicity we assume that $A_k \geq E$) and does not depend on t.

In this work we consider in the space L^2 a linear multipoint differential-operator expression of first order of the following form

$$l(u) := l_k(u),\ t \in \Delta_k, \tag{2.1}$$

where

$$l_k(u) := u'(t) + A_k u(t),\ t \in \Delta_k,\ u \in L_k^2 := L^2(\mathcal{H}, \Delta_k),\ k = 1,2,\ldots,n. \tag{2.2}$$

The differential expression l_k is defined on all vector-valued functions absolutely continuous on $\triangle_k$ and $u' \in L_k^2$. It is clear that the formally adjoint expression to (2.2) in the Hilbert space L_k^2 for $k = 1,2,...,n$ is of the form

$$l_k^+(v) := -v'(t) + A_k v(t). \tag{2.3}$$

Let us define an operator $L_{k0}^{/}$ on the dense set of all vector-functions $D_{k0}^{/}$ in L_k^2,

$$D_{k0}^{/} := \Big\{ u(t) \in L_k^2 : u(t) = \sum_{k=1}^{m} \varphi_k(t) f_k,\ \varphi_k(t) \in C_0^\infty(\Delta_k), \\ f_k \in D(A_k),\ k = 1,2,\ldots,m,\ m \in \mathbb{N} \Big\}$$

as

$$L_{k0}^{/} u := l_k(u),\ k = 1,2,\ldots,n.$$

Since $A_k \geq E,\ k = 1,2,\ldots,n$, then from the formula

$$Re\,(L_{k0}^{/} u, u)_{L_k^2} = \int\limits_{c_{k-1}}^{c_k} \|A_k^{1/2} u(t)\|^{1/2} dt \geq 0, \quad u \in D_{k0}^{/},$$

we infer that operator $L^{/}_{k0}$ is accretive in L^2_k for every $k = 1, 2, \ldots, n$. Hence the operator $L^{/}_{k0}$, $k = 1, 2, \ldots, n$, has a closure in L^2_k. The closure $\overline{L}^{/}_{k0}$ of $L^{/}_{k0}$ is called the minimal operator generated by the differential-operator expression (2.2) and it is denoted by L_{k0}. On the other hand, the direct sum

$$L_0 := \bigoplus_{k=1}^{n} L_{k0}$$

of operators L_{k0}, $k = 1, 2, \ldots, n$, is called a minimal operator (multipoint) generated by the differential-operator expression (2.1) in L^2.

In a similar way the minimal operator L_0^+ in L^2 for the formally adjoint expression of (2.1)

$$l^+(v) = l_k^+(v), \; t \in \Delta_k, \; k = 1, 2, \ldots, n, \tag{2.4}$$

can be constructed, that is,

$$L_0^+ := \bigoplus_{k=1}^{n} L_{k0}^+,$$

where L_{k0}^+, $k = 1, 2, \ldots, n$, is a minimal operator which is generated by expression (2.3) in L_k^2.

The adjoint operator of $L_{k0}^+ \, (L_{k0})$ in L_k^2, $k = 1, 2, \ldots, n$ is called the maximal operator generated by (2.2) ((2.3)) and it is denoted by $L_k \left(L_k^+\right)$, i.e.

$$L_k = (L_{k0}^+)^*, \; L_k^+ = (L_{k0})^*, \; k = 1, 2, \ldots, n.$$

Similarly, direct sums of these operators, that is,

$$L := \bigoplus_{k=1}^{n} L_k \quad \text{and} \quad L^+ := \bigoplus_{k=1}^{n} L_k^+$$

are called maximal operators (multipoint) for the differential-expression (2.1) and (2.4), respectively. It is clear that $L_0 \subset L$, $L_0^+ \subset L^+$ (see [27, 28]).

It is evident that the following theorem is valid.

Theorem 2.1. *The domain $D(L_0)$ of the minimal operator L_0 is the set of all functions u absolutely continuous on each subinterval $[c_{k-1}, c_k]$ of $[a, b]$ such that*

$$u(c_{k-1}+) = u(c_k-) = 0,$$

$$u' + A_k u \in L^2(\mathcal{H}, \Delta_k)$$

and

$$L_0 u(t) = L_{k0} u(t), \, t \in \Delta_k, \, k = 1, 2, \ldots, n.$$

Now we investigate the domain of the maximal operator L in L^2 (an operator $L = (L_0^+)^*$ exists, because the linear manifold $D(L_0^+)$ is dense in L^2).

Theorem 2.2. *The domain $D(L)$ of the maximal operator L is the set of all functions u absolutely continuous on each subinterval $[c_{k-1}, c_k]$ of $[a,b]$ such that*

$$u' + A_k u \in L^2(\mathcal{H}, \Delta_k)$$

and

$$Lu(t) = u'(t) + A_k u(t),\ t \in \Delta_k,\ k = 1, 2, \dots, n.$$

Proof. Note that the analogous theorem for the minimal operator L_0^+ in L^2 is valid. On the other hand, it is easy to see that

$$L = \left(L_0^+\right)^* = \bigoplus_{k=1}^{n} \left(L_{k0}^+\right)^* = \bigoplus_{k=1}^{n} L_k.$$

From this it follows that to prove the claim it is sufficient to describe the domains of $(L_{k0}^+)^* = L_k$. Let u be any vector-function from $D(L_{k0}^+)$. Then for any $v \in D\left(L_k\right)$ we have

$$\begin{aligned}(u, L_k v)_{L_k^2} &= (u, (L_{k0}^+)^* v)_{L_k^2} = (L_{k0}^+ u, v)_{L_k^2} = \\ &= (-u + A_k u, v)_{L_k^2} = (-u', v)_{L_k^2} + (u, A_k v)_{L_k^2}\end{aligned}$$

From this relation we obtain

$$(u', v)_{L_k^2} = (u, (A_k v - L_k v))_{L_k^2}.$$

Since $u(c_k) = 0,\ k = 0, 1, \dots, n$, integrating by parts

$$\begin{aligned}(u', v)_{L_k^2} &= \left(u, \frac{d}{dt}\left(\int_{c_{k-1}}^{t} (A_k v - L_k v)\,dx\right)\right)_{L_k^2} = \\ &= \left(u, \int_{c_{k-1}}^{t} (A_k v - L_k v)\,dx\right)\Bigg|_{c_{k-1}}^{c_k} - \left(u', \int_{c_{k-1}}^{t} (A_k v - L_k v)\,dx\right) = \\ &= \left(u', \int_{c_{k-1}}^{t} (L_k v - A_k v)\,dx\right)\end{aligned}$$

and from this

$$\left(u', v + \int_{c_{k-1}}^{t} (A_k v - L_k v)\,dx\right)_{L_k^2} = 0,$$

which holds for any $u \in D(L_{k0}^+)$. Hence for any $k = 1, 2, \dots, n$ and $t \in (c_{k-1}, c_k)$

$$v\left(t\right) + \int_{c_{k-1}}^{t} (A_k v - L_k v)\,dx = C\left(v\right) = \text{constant}$$

is true. Then for $t \in (c_{k-1}, c_k)$, $k = 1, 2, \ldots, n$, v is differentiable and we have

$$L_k v = v'(t) + A_k v(t). \qquad \square$$

3. THE NORMAL EXTENSIONS OF THE MINIMAL OPERATOR

The main purpose of this section is to describe all normal extensions of the minimal operator L_0 in L^2 in terms of the boundary and interior points values.

Let us note that analogous problem under the condition

$$\int_a^b \|A(t)\|^2 \, dt < \infty$$

for the operator-coefficients $A(t)$, $a \le t \le b$, of a linear differential-operator expression of first order in L^2 has been established.

The condition

$$A_R(t) = \frac{1}{2}\overline{(A(t) + A^*(t))} = A_R = \text{constant a.e. in } (a, b)$$

is necessary and sufficient for the existence of the normal extensions of the minimal operator generated by this expression in L^2 (cf. [14]).

Here the main purpose is to generalize the obtained result in this theory (cf. [15]) to the family of unbounded operator-coefficients of the first-order multipoint linear differential operators.

First, let us define a Hilbert space $\mathcal{H}_j(T)$, $-\infty < j < +\infty$, constructed via the operator T^j. Let $\mathcal{H} = \mathcal{H}_0$ be a Hilbert space over the field of complex numbers with inner product $(\cdot, \cdot)_{\mathcal{H}_0}$ and norm $\|f\|_{\mathcal{H}_0} = (f, f)_{\mathcal{H}_0}^{1/2}$, $f \in \mathcal{H}_0$. Let T be a linear self-adjoint operator on the Hilbert space $\mathcal{H}$ such that $\|Tf\|_{\mathcal{H}_0} \ge \|f\|_{\mathcal{H}_0}$. The set $D(T^j)$, $0 < j < +\infty$, under the inner product

$$(f, g)_{\mathcal{H}+j} = \left(T^j f, T^j g\right)_{\mathcal{H}_0}, \quad f, g \in D\left(T^j\right),$$

is a Hilbert space. We define $\mathcal{H}_{+j} = \mathcal{H}_{+j}(T)$, $0 < j < +\infty$, and it is called a positive space.

In the similar way we define a Hilbert space, $\mathcal{H}_{-j} = \mathcal{H}_{-j}(T)$, $0 < j < +\infty$, and it is called a negative space.

It is clear that $\mathcal{H}_{+\tau} \subset \mathcal{H}_{+j}$, $0 < \tau < j < +\infty$, $\mathcal{H}_{+j} \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-j}$, $\mathcal{H}_{+j}^* = \mathcal{H}_{-j}$, $0 < j < +\infty$ and $\mathcal{H}_{+j}$, $0 < j < +\infty$, is dense in $\mathcal{H}$ (for more a detailed analysis of the space $\mathcal{H}_j$, $-\infty < j < \infty$, see [27,28]).

By $W_2^1(\mathcal{H}, (a, b))$ we define the Sobolev space of all vector-functions defined on the finite interval $[a, b]$ with values in $\mathcal{H}$ ([28]).

The following result characterizes the normal extension of the minimal operator L_0.

Theorem 3.1. *Let* $A_k^{1/2}\left[D\left(L_k\right)\cap D\left(L_k^{+}\right)\right]\subset W_2^1\left(\mathcal{H},\Delta_k\right)$ *for every* $k=1,2,\ldots,n$. *If* $\tilde{L}$ *is a normal extension of the minimal operator* L_0 *in* L^2, *then for every* $u\in D(\tilde{L})$, *the following relations*

$$\sum_{k=1}^{n}\left\|u\left(c_k-\right)\right\|^2=\sum_{k=1}^{n}\left\|u\left(c_{k-1}+\right)\right\|^2,$$

$$\sum_{k=1}^{n}\|A_k^{1/2}u\left(c_k-\right)\|^2=\sum_{k=1}^{n}\|A_k^{1/2}u\left(c_{k-1}+\right)\|^2$$

hold.

Proof. If $\tilde{L}$ is any normal extension of the minimal operator L_0 in L^2, then from the condition $D(\tilde{L})=D(\tilde{L}^*)$ we have

$$(\tilde{L}u,u)_{L^2}=(u,\tilde{L}^*u)_{L^2},\quad u\in D(\tilde{L}).$$

From this and conditions $A_k^{1/2}\left[D\left(L_k\right)\cap D\left(L_k^{+}\right)\right]\subset W_2^1\left(\mathcal{H},\Delta_k\right),\ k=1,2,\ldots,n$ we can write

$$(\tilde{L}u,u)_{L^2}-(u,\tilde{L}^*u)_{L^2}=\left(u',u\right)_{L^2}+\left(u,u'\right)_{L^2}=\sum_{k=1}^{n}(u,u)\Big|_{c_{k-1}}^{c_k}=$$

$$=\sum_{k=1}^{n}\left(\left\|u\left(c_k-\right)\right\|^2-\left\|u\left(c_{k-1}+\right)\right\|^2\right)=0,\ u\in D(\tilde{L}).$$

On the other hand, from the second property of normality of the extension $\tilde{L}$ we find

$$\|\tilde{L}u\|_{L^2}^2-\|\tilde{L}^*u\|_{L^2}^2=\sum_{k=1}^{n}\left(\left(u',A_ku\right)_{L_k^2}+\left(A_ku,u'\right)_{L_k^2}\right)=\sum_{k=1}^{n}(u,A_ku)\Big|_{c_{k-1}}^{c_k}=$$

$$=\sum_{k=1}^{n}\left(\|A_k^{1/2}u\left(c_k-\right)\|^2-\|A_k^{1/2}u\left(c_{k-1}+\right)\|^2\right)=0$$

for every $k=1,2,\ldots,n$. □

In general the following result is valid.

Theorem 3.2. *A necessary and sufficient condition for the normality of the extension* $\tilde{L}$ *of the minimal operator* L_0 *in* L^2 *is the normality of the extension* $\tilde{L}_k$ *of the minimal operator* L_{k0} *in* L_k^2 *for every* $k=1,2,\ldots,n$.

Proof. Let $\tilde{L}=\bigoplus_{k=1}^{n}\tilde{L}_k$ be a normal extension of the minimal operator $L_0=\bigoplus_{k=1}^{n}L_{k0}$ in L^2. Then from $D(\tilde{L})=D(\tilde{L}^*)$ we deduce that

$$D(\tilde{L}_1)\oplus D(\tilde{L}_2)\oplus\ldots\oplus D(\tilde{L}_k)\oplus\ldots\oplus D(\tilde{L}_n)=$$
$$=D(\tilde{L}_1^*)\oplus D(\tilde{L}_2^*)\oplus\ldots\oplus D(\tilde{L}_k^*)\oplus\ldots\oplus D(\tilde{L}_n^*).$$

From this we obtain

$$D(\tilde{L}_k) = D(\tilde{L}_k^*), \quad k = 1, 2, \dots, n.$$

On the other hand, since

$$D(L_{10}) \oplus D(L_{20}) \oplus \dots \oplus D(L_{(k-1)0}) \oplus D(\tilde{L}_k) \oplus D(L_{(k+1)0}) \oplus \dots \oplus D(L_{n0}) \subset D(\tilde{L})$$

for every $k = 1, 2, \dots, n$, then from the property $\|\tilde{L}u\|_{L^2} = \|\tilde{L}^* u\|_{L^2},\ u \in D(\tilde{L})$ of the normality it is established that

$$\|\tilde{L}_k u\|_{L_k^2} = \|\tilde{L}_k^* u\|_{L_k^2},$$

for every $u \in D(\tilde{L}_k)$ and $k = 1, 2, \dots, n$. Hence the operator $\tilde{L}_k$ is a normal extension of the minimal operator L_{k0} in L_k^2, $k = 1, 2, \dots, n$.

Conversely, if the operators $\tilde{L}_k,\ k = 1, 2, \dots, n$, are normal extensions of the minimal operators L_{k0} in the space L_k^2, then the normality of the extension $\tilde{L} := \bigoplus\limits_{k=1}^{n} \tilde{L}_k$ of the minimal operator $L_0 = \bigoplus\limits_{k=1}^{n} L_{k0}$ in the space L^2 is clear. □

Now we can describe all normal extensions $\tilde{L}$ of the minimal operator L_0 in the space L^2 in terms of boundary and interior points values.

Validity of the following main result of this section comes from Theorem 3.2 and Theorem 2.1 in work [15].

Theorem 3.3. *Let $A_k^{1/2}\left[D\left(L_k\right) \cap D\left(L_k^+\right)\right] \subset W_2^1\left(\mathcal{H}, \Delta_k\right)$ for every $k = 1, 2, \dots, n$. Then each normal extension $\tilde{L}$ of the minimal operator L_0 in the space L^2 is generated by the differential-operator expression* (2.1) *with the conditions*

$$u\left(c_k-\right) = W_k u\left(c_{k-1}+\right), \tag{3.1}$$

where W_k is a unitary operator in $\mathcal{H}$ and $A_k^{-1} W_k = W_k A_k^{-1}$ for every $k = 1, 2, \dots, n$. The unitary operators $W_1, W_2, \dots, W_n$ in $\mathcal{H}$ are determined uniquely by the extension, i.e. $\tilde{L} = L_W$, where $W = \bigoplus_{k=1}^{n} W_k$. Moreover, the restriction of the maximal operator L to the linear manifold of vector-functions $u \in D\left(L\right) \cap D\left(L^+\right)$ that satisfy conditions (3.1) *for some unitary operators $W_1, W_2, \dots, W_n$ with the corresponding properties in $\mathcal{H}$, is a normal extension of the minimal operator L_0 in the space L^2.*

Proof. Let $\tilde{L} = \bigoplus_{k=1}^{n} \tilde{L}_k$ be a normal extension of the minimal operator $L_0 = \bigoplus_{k=1}^{n} L_{k0}$ in the space L^2. Then by Theorem 3.2 the extension $\tilde{L}_k$ is a normal extension of the minimal operator L_{k0} in the space L_k^2 for every $k = 1, 2, \dots, n$. On the other hand, by Theorem 2.1 of [15] for each $k = 1, 2, \dots, n$ the normal extension $\tilde{L}_k$ of the minimal operator L_{k0} in the space L_k^2 is generated by the differential-operator expression (2.1) and boundary condition

$$u(c_k-) = W_k u(c_{k-1}+),$$

where W_k is a unitary operator in $\mathcal{H}$ and $A_k^{-1} W_k = W_k A_k^{-1}$. In addition, the unitary operator W_k is uniquely determined by the extension $\tilde{L}_k$, i.e $\tilde{L}_k = L_{W_k}$, for every $k =$

$1, 2, \ldots, n$. Consequently, the unitary operators $W_1, W_2, \ldots, W_n$ in $\mathcal{H}$ are uniquely generated by the extension $\tilde{L}$, i.e. $\tilde{L} = L_W$, where $W = \bigoplus_{k=1}^{n} W_k$.

On the contrary, now let $\tilde{L}_k$ be an extension generated by the differential-operator expression (2.1) with the boundary condition (3.1) in $L_k^2, k = 1, 2, \ldots, n$. In this case by Theorem 2.1 of [15] for every $k = 1, 2, \ldots, n$ the extension $\tilde{L}_k$ is normal in L_k^2. From this result and Theorem 3.2 we infer that the extension $\tilde{L} = \bigoplus_{k=1}^{n} \tilde{L}_k$ is a normal operator in the space L^2. □

Corollary 3.4. *Let $A_k^{1/2}\left[D\left(L_k\right) \cap D\left(L_k^{+}\right)\right] \subset W_2^1\left(\mathcal{H}, \Delta_k\right)$ for every $k = 1, 2, \ldots, n$. Then every normal extension $\tilde{L}$ of the minimal operator L_0 in the space L^2 is accretive.*

Proof. Indeed, from the validity of relation

$$2Re\,(\tilde{L}u, u)_{L^2} = \sum_{k=1}^{n}\left(\|u\left(c_k-\right)\|^2 - \|u\left(c_{k-1}+\right)\|^2\right) + 2\sum_{k=1}^{n}\int\limits_{\Delta_k}\|A_k^{1/2}u\left(t\right)\|^2 dt \geq 0,$$

which is true for all $u \in D(\tilde{L})$, and Theorem 3.1 we deduce that every normal extension $\tilde{L}$ is accretive. □

4. STRUCTURE AND DISCRETENESS OF THE SPECTRUM OF THE NORMAL EXTENSIONS

In this section we will study the structure and discreteness of the spectrum of the normal extensions of the minimal operator L_0 in L^2.

Let $C_p\left(\mathcal{H}\right),\ 1 \leq p \leq \infty$, denote the Schatten-von Neumann class of all linear operators in the Hilbert space $\mathcal{H}$ (see [17]). We set

$$h_k := c_k - c_{k-1},\ T_k := W_k^{*}\exp\left(-A_k h_k\right),\ k = 1, 2, \ldots, n.$$

Theorem 4.1. *The spectrum of the normal extension $\tilde{L} = \bigoplus\limits_{k=1}^{n} \tilde{L}_k$ has the form*

$$\sigma(\tilde{L}) = \bigcup_{k=1}^{n}\sigma(\tilde{L}_k),$$

where

$$\sigma(\tilde{L}_k) = \Big\{\lambda \in \mathbb{C} : \lambda = \lambda_0 + \frac{2p\pi i}{h_k},\ \text{where } \lambda_0 \text{ belongs to the set of solutions}$$
$$\text{of the equation } e^{-\lambda h_k} - \mu = 0 \text{ on } \lambda,\ \mu \in \sigma\left(T_k\right),\ p \in \mathbb{Z}\Big\},\ k = 1, 2, \ldots, n.$$

Proof. Let us consider a problem for the spectrum for the normal extension $\tilde{L}$, that is,

$$\tilde{L}u = \tilde{L}_k u = \lambda u + f, \quad \lambda \in \mathbb{C},\ f \in L_k^2$$

with the conditions

$$u\left(c_k-\right)=W_k u\left(c_{k-1}+\right),$$

where W_k is a unitary operator in $\mathcal{H}$ and

$$A_k^{-1}W_k=W_kA_k^{-1},\quad k=1,2,\ldots,n.$$

It is clear that a general solution of the considered differential-operator equation has the form

$$u_\lambda\left(t\right)=\exp\left(-\left(A_k-\lambda E\right)\left(t-c_{k-1}\right)\right)f_k+\int\limits_{c_{k-1}}^{t}\exp\left(-\left(A_k-\lambda E\right)\left(t-s\right)\right)f\left(s\right)ds,$$

where $f_k\in\mathcal{H}_{-1/2}\left(A_k\right),\ t\in\Delta_k,\ k=1,2,\ldots,n.$ From this and boundary condition in interval $\Delta_k,\ k=1,2,\ldots,n$ it follows

$$\left(W_k-\exp\left(-(A_k-\lambda E)\,h_k\right)\right)f_k=\int\limits_{c_{k-1}}^{c_k}\exp\left(-(A_k-\lambda E)\left(c_k-s\right)\right)f\left(s\right)ds.$$

Therefore, for $\lambda\in\mathbb{C}$ and $k=1,2,\ldots,n$, we have

$$\left(e^{-\lambda h_k}E-T_k\right)f_k=W_k^*\left(\int\limits_{c_{k-1}}^{c_k}\exp\left[-(A_k-\lambda E)\left(c_k-s\right)-\lambda h_kE\right]f\left(s\right)ds\right).$$

A necessary and sufficient condition for $\lambda\in\sigma(\tilde{L})$ is

$$e^{-\lambda h_k}-\mu=0,\ \mu\in\sigma\left(T_k\right)$$

for some $k=1,2,\ldots,n$. Therefore,

$$\lambda=\lambda_0+\frac{2p\pi i}{h_k},\ p\in\mathbb{Z},\ k=1,2,\ldots,n,$$

where $e^{-\lambda_0h_k}\in\sigma\left(T_k\right)$. This completes the proof. □

The following theorem describes discreteness of the spectrum of the normal extension $\tilde{L}$ of the minimal operator L_0.

Theorem 4.2. *If $A_k^{-1}\in C_\infty(\mathcal{H})$ for every $k=1,2,\ldots,n$, $\tilde{L}$ is a normal extension of minimal operator L_0 and $\lambda\in\rho(\tilde{L})$, then $R_\lambda(\tilde{L})\in C_\infty(L^2)$, where $R_\lambda(\tilde{L})$ is the resolvent of the operator $\tilde{L}$.*

Proof. First of all let us prove that if $A_k^{-1}\in C_\infty\left(\mathcal{H}\right)$, then $\tilde{L}^{-1}\in C_\infty\left(L^2\right)$. Let $\tilde{L}=\bigoplus\limits_{k=1}^{n}\tilde{L}_k$ and consider the operators

$$\tilde{L}_ku=u'+A_ku,\ u\in L_k^2$$

with boundary conditions

$$u\left(c_k-\right) = W_k u\left(c_{k-1}+\right), \quad k = 1, 2, \ldots, n.$$

It is known that the general solution of the equation $\tilde{L}_k u = f,\ f \in L_k^2$ is of the form

$$u\left(t\right) = \exp\left(-A_k\left(t - c_{k-1}\right)\right) f_k + \int\limits_{c_{k-1}}^{t} \exp\left(-A_k\left(t-s\right)\right) f\left(s\right) ds,$$

$t \in \Delta_k,\ f_k \in \mathcal{H}_{-1/2}\left(A_k\right),\ k = 1, 2, \ldots, n$. In this case from the boundary condition we have

$$\left(E - T_k\right) f_k = W_k^* \left(\int\limits_{\Delta_k} \exp\left(-A_k\left(c_k - s\right)\right) f\left(s\right) ds \right).$$

Since $\|T_k\| < 1$, then from the last relation we have

$$f_k = \left(E - T_k\right)^{-1} W_k^* \left(\int\limits_{\Delta_k} \exp\left(-A_k\left(c_k - s\right)\right) f\left(s\right) ds \right).$$

Therefore,

$$\tilde{L}_k^{-1} f\left(t\right) = \exp\left(-A_k\left(t - c_{k-1}\right)\right)\left(E - T_k\right)^{-1} W_k^* \left(\int\limits_{\Delta_k} \exp\left(-A_k\left(c_k - s\right)\right) f\left(s\right) ds \right) +$$

$$+ \int\limits_{c_{k-1}}^{t} \exp\left(-A_k\left(t-s\right)\right) f\left(s\right) ds,\ f \in L_k^2,\ k = 1, 2, \ldots, n.$$

From this and Theorem 3.4 of [15] it can be proved that

$$\tilde{L}_k^{-1} \in C_\infty\left(L_k^2\right),\ k = 1, 2, \ldots, n.$$

Furthermore note that since $\tilde{L}^{-1} = \bigoplus\limits_{k=1}^{\infty} \tilde{L}_k^{-1}$, then $\tilde{L}^{-1} \in C_\infty\left(L^2\right)$. Now let $\lambda \in \rho(\tilde{L})$. Then from the equality

$$R_\lambda(\tilde{L}) = \tilde{L}^{-1} + \lambda R_\lambda(\tilde{L})\tilde{L}^{-1}$$

it follows that $R_\lambda(\tilde{L}) \in C_\infty\left(L^2\right)$. □

Moreover, using the method established in Theorem 4.2, the following result is true in the general case.

Theorem 4.3. *If $A_k^{-1} \in C_p(\mathcal{H})$, $1 \le p \le \infty$, $k = 1, 2, \ldots, n$, $\tilde{L}$ is a normal extension of the minimal operator L_0 and $\lambda \in \rho(\tilde{L})$, then $R_\lambda(\tilde{L}) \in C_p(L^2)$.*

Furthermore, from the representation of the resolvent $R_\lambda(\tilde{L})$, $\lambda \in \rho(\tilde{L})$ of the normal extension it can be easily verified that the following result is true. It can be proved like in [15].

Theorem 4.4. *Let L_W and L_U be two normal extensions of the minimal operator L_0 with unitary operators $W := \bigoplus_{k=1}^n W_k$, $U := \bigoplus_{k=1}^n U_k$. Then for $R_\lambda(L_W) - R_\lambda(L_U) \in C_p(L^2)$, $1 \le p \le \infty$, $\lambda \in \rho(L_W) \cap \rho(L_U)$ if and only if*

$$W_k - U_k \in C_p(\mathcal{H}),\ 1 \le p \le \infty,\ k = 1, 2, \ldots, n.$$

Proof. In this case, we easily see that

$$R_\lambda(L_W) - R_\lambda(L_U) = \bigoplus_{k=1}^{n} [R_\lambda(L_{W_k}) - R_\lambda(L_{U_k})]$$

for $\lambda \in \rho(L_W) \cap \rho(L_U)$. From this relation it is evident that

$$R_\lambda(L_W) - R_\lambda(L_U) \in C_p(L^2), 1 \le p \le \infty$$

if and only if

$$R_\lambda(L_{W_k}) - R_\lambda(L_{U_k}) \in C_p(L_k^2),$$

for every $\lambda \in \rho(L_W) \cap \rho(L_U)$ and $k = 1, 2, \ldots, n$. But by Corollary 3.3 of [15] the last relation is true if and only if

$$W_k - U_k \in C_p(\mathcal{H}),\ 1 \le p \le \infty,\ k = 1, 2, \ldots, n. \qquad \square$$

Now we prove a result concerning the structure of the spectrum of the normal extension $\tilde{L} = L_W$, where $W = \bigoplus_{k=1}^n W_k$, in the case of the minimal operator L_0 has discrete spectrum.

Theorem 4.5. *Let $A_k^{-1} \in \sigma_p(\mathcal{H})$ for every $k = 1, 2, \ldots, n$. If L_W is any normal extension of the minimal operator L_0 in L^2, then the spectrum of L_W has the form*

$$\sigma(L_W) = \bigcup_{k=1}^{n} \sigma(L_{W_k}),$$

where

$$\sigma(L_{W_k}) = \left\{ \lambda \in \mathbb{C} : \lambda = \lambda_m(A_k) + \frac{(-i)}{h_k}(\arg \lambda_m(T_k) + 2p\pi),\ m \in \mathbb{N},\ p \in \mathbb{Z} \right\},$$

for every $k = 1, 2, \ldots, n$. In the special case, for Dirichlet's multipoint boundary value problem $(W_k = E, k = 1, 2, \ldots, n)$ $\arg \lambda_m(T_k) = 0$, $k = 1, 2, \ldots, n$, $m \in \mathbb{N}$.

Proof. By Theorem 4.1, $\sigma(L_W) = \bigcup\limits_{k=1}^{n} \sigma(L_{W_k})$ and

$$\sigma(L_{W_k}) = \left\{ \lambda \in \mathbb{C} : \lambda = \frac{-1}{h_k}(\ln|\mu| + i \arg \mu + 2p\pi i),\ \text{where } \mu \in \sigma(T_k),\ p \in \mathbb{Z} \right\}.$$

On the other hand, since $A_k^{-1} \in C_\infty(\mathcal{H})$, $k = 1, 2, \ldots, n$, then for every $k = 1, 2, \ldots, n$ there is $T_k = W_k^* \exp(-A_k h_k) \in C_\infty(\mathcal{H})$ (see also Theorem 4.2).

Now let $\mu \in \sigma_p(T_k)$, $k = 1, 2, \ldots, n$, be an eigenvalue of the operator T_k with any eigenvector $x(\mu) \in \mathcal{H}$, i.e. for any $k = 1, 2, \ldots, n$

$$W_k^* \exp(-A_k h_k)\, x(\mu) = \mu x(\mu).$$

In this case $\overline{\mu} \in \mathbb{C}$ is an eigenvalue number of the adjoint operator to $W_k^* \exp(-A_k h_k)$, that is, of the operator $\exp(-A_k h_k)\, W_k$ with the same eigenvector $x(\mu)$ in $\mathcal{H}$. Then the last relation implies that

$$\begin{aligned} &\exp(-A_k h_k)\, W_k W_k^* \exp(-A_k h_k)\, x(\mu) = \mu\left(W_k^* \exp(-A_k h_k)\right) \cdot \\ &\cdot \left(\exp(-A_k h_k)\, W_k x(\mu)\right) = \mu\left(W_k^* \exp(-A_k h_k)\right) \cdot \overline{\mu\left(W_k^* \exp(-A_k h_k)\right)} x(\mu) \end{aligned}$$

and from this we obtain

$$\exp(-2A_k h_k)\, x(\mu) = \left|\mu\left(W_k^* \exp(-A_k h_k)\right)\right|^2 x(\mu).$$

Hence

$$\left|\mu\left(W_k^* \exp(-A_k h_k)\right)\right| = \exp(-\lambda(A_k)\, h_k)$$

and from this relation we have

$$\ln|\mu| = (-h_k)\,\lambda(A_k), \ \ k = 1, 2, \ldots, n.$$

Thus

$$\sigma(L_{W_k}) = \left\{\lambda \in \mathbb{C} : \lambda = \lambda_m(A_k) + \frac{(-i)}{h_k}\left(\arg\lambda_m(T_k) + 2p\pi\right), \ m \in \mathbb{N}, \ p \in \mathbb{Z}\right\}$$

for every $k = 1, 2, \ldots, n$. □

5. ASYMPTOTICAL BEHAVIOR OF THE EIGENVALUES OF THE NORMAL EXTENSIONS

In this section we investigate the asymptotic behavior at infinity of eigenvalues of the normal extensions (having discrete spectrum) of the minimal operator L_0 in L^2.

We now state the following result for the special case which can be easily proved by Theorem 4.5.

Theorem 5.1. *If $\dim\mathcal{H} < +\infty$, then every normal extension L_W of the minimal operator L_0 in L^2 has a discrete spectrum and its eigenvalues at the infinity have the following asymptotics:*

$$|\lambda_n(L_W)| \sim \delta n, \ 0 < \delta < +\infty, \ \text{as } n \to +\infty.$$

Now we can prove the main result of this section.

Theorem 5.2. *If for each $k = 1, 2, \dots, n$, $A_k^{-1} \in C_\infty(\mathcal{H})$ and*

$$\lambda_m(A_k) \sim \beta_k m^{\alpha_k},\ 0 < \beta_k, \alpha_k < +\infty,\ \text{as } m \to \infty,$$

then the eigenvalues of any normal extension L_W of the minimal operator L_0 in L^2 have the following asymptotics:

$$|\lambda_q(L_W)| \sim \gamma q^{\frac{\alpha}{1+\alpha}},\ 0 < \gamma, \alpha < +\infty,\ \alpha = \max_{1 \le k \le n} \alpha_k,\ \text{as } q \to \infty.$$

Proof. Without loss of generality we can assume that for every $k, j = 1, 2, \dots, n$, $k \neq j$, the condition $\sigma(A_k) \cap \sigma(A_j) = \emptyset$ is satisfied.

We set

$$N(\lambda) := \sum_{|\lambda(S)| \le |\lambda|} 1, \quad \lambda \in \mathbb{C},$$

that is, the number of eigenvalues of a linear closed operator S in a Hilbert space which modules do not exceed $|\lambda|$, $\lambda \in \mathbb{C}$. This function takes values in the set of non-negative integer numbers and, in the case of an unbounded operator S, non-decreasing and tends to $+\infty$ as $|\lambda| \to +\infty$. Here let $L_W = \bigoplus_{k=1}^{n} L_{W_k}$ be any normal extension of the minimal operator L_0 in L^2,

$$N_k(\lambda) := \sum_{\substack{|\lambda(L)| \le |\lambda| \\ \lambda \in \sigma(L_{W_k})}} 1, \quad k = 1, 2, \dots, n$$

and

$$N(\lambda) := \sum_{|\lambda(L_W)| \le |\lambda|} 1, \quad \lambda \in \mathbb{C}.$$

Since $\sigma(A_k) \cap \sigma(A_j) = \emptyset$ for all $k, j = 1, 2, \dots, n$, $k \neq j$, then from Theorem 4.5 it is clear that

$$N(\lambda) = \sum_{k=1}^{n} N_k(\lambda), \quad \lambda \in \mathbb{C}.$$

By Theorem 4.5 for each $\lambda \in \sigma(L_{W_k})$, $k = 1, 2, \dots, n$, we have

$$|\lambda(L_{W_k})| = \left[\lambda_m^2(A_k) + \frac{1}{h_k^2}(\delta_m + 2p\pi)^2\right]^{1/2},$$

where $\delta_m := \arg \lambda_m(T_k)$, $k = 1, 2, \dots, n$, $m = 1, 2, \dots$. Since $0 \le \delta_m \le 2\pi$ for each $m \in \mathbb{N}$, then for $\lambda \in \sigma(L_{W_k})$, $k = 1, 2, \dots, n$ from the last equality we have

$$\left[\lambda_m^2(A_k) + \frac{4\pi^2}{h_k^2} p^2\right]^{1/2} \le |\lambda(L_{W_k})| \le \left[\lambda_m^2(A_k) + \frac{4\pi^2}{h_k^2}(p+1)^2\right]^{1/2},$$

$k = 1, 2, \ldots, n$, $m \in \mathbb{N}$, $p \in \mathbb{Z}$. Therefore the eigenvalues $\lambda \in \sigma(L_{W_k})$, $k = 1, 2, \ldots, n$, have the following asymptotical behavior:

$$|\lambda(L_{W_k})| \sim \left[\beta_k^2 m^{2\alpha_k} + \frac{4\pi^2}{h_k^2} p^2\right]^{1/2}, \quad k = 1, 2, \ldots, n \ , \ m \in \mathbb{N}, \ p \in \mathbb{Z}.$$

An argument similar to that used in [15] (or in [28]) shows that for the eigenvalues of L_{W_k}, $k = 1, 2, \ldots, n$, it can be found that the following asymptotic formula holds:

$$N_k(\lambda) \sim \gamma_k |\lambda|^{\frac{1+\alpha_k}{\alpha_k}}, \ 0 < \gamma_k < +\infty, \ \text{as} \ |\lambda| \to +\infty.$$

Hence

$$N(\lambda) \sim \sum_{k=1}^{n} \gamma_k |\lambda|^{\frac{1+\alpha_k}{\alpha_k}} = |\lambda|^{\frac{1+\alpha}{\alpha}} \left(\sum_{k=1}^{n} \gamma_k |\lambda|^{\frac{1+\alpha_k}{\alpha_k} - \frac{1+\alpha}{\alpha}}\right) \sim \gamma_* |\lambda|^{\frac{1+\alpha}{\alpha}},$$

where $0 < \gamma_* < +\infty$, $\alpha := \max_{1 \le k \le n} \alpha_k$, as $|\lambda| \to +\infty$. From this relation it is easy to see that

$$|\lambda_q(L_W)| \sim \gamma q^{\frac{\alpha}{1+\alpha}}, \quad 0 < \gamma, \alpha < +\infty, \ \text{as} \ q \to +\infty.$$

This completes the proof. □

Remark 5.1. In the case of $n = 1$, i.e. $A(t) = A$, $a \le t \le b$, the analogous problem has been investigated in [15].

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